Chapter 7: Multiscale Modeling of Composites Using Analytical Methods

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7.1 Introduction

Both fiber and particulate composite materials provide applications in materials science where the multiscale microstructure leads to the need for multiscale modeling. The length scales encountered range from the fiber and particle sizes whose dimensions are measured in microns, to the individual plies in laminates whose thicknesses are measured in fractions of millimeters, to the laminates themselves whose thicknesses in the laboratory are measured in millimeters, e.g., 40–50 mm. The laminates then form parts of composite structures whose sizes are measured in meters, although modeling at this scale will not form part of this chapter. While conventional composites are based on essentially homogeneous matrices, which can be polymeric, metallic, or ceramic, advanced composites are also being considered to have matrices, which are themselves composites reinforced by submicron particles or whiskers, e.g., carbon nanotubes. Such developments lead to the need to be able to estimate the properties of composite laminates that have multiscale reinforcements, e.g., fibers in particulate/whisker-reinforced matrices. Also, there is a need to predict the onset of damage in the materials when they are operating in service conditions.

This chapter will focus on two aspects of the problem that complement work already published dealing with multiscale analytical modeling for damaged composite laminates [7]. The first aspect, considered in Sects. 7.2–7.4, is the development of new understanding with regard to the prediction of the properties of undamaged particulate and unidirectional fiber-reinforced composites. Recent work [8] has involved the study of a methodology first published in 1873 that was developed by Maxwell [3] when considering the effective electrical conductivity of a conducting medium in which a dilute distribution of particles having a different conductivity was dispersed. It has now been shown [8] that this methodology can be applied much more widely with the result that many other effective properties of both particulate and fiber-reinforced composites can be estimated. A description of the key results of this investigation will first be presented in this chapter together with a discussion of the relationship of the new results to existing formulae that can be used to estimate effective composite properties. The results given will be very useful when estimating undamaged properties for matrices if reinforced by submicron particles, and the undamaged properties of the individual plies of laminates that will become damaged when loaded in service conditions. Recent work to be reported in [8] has reconsidered the bounds on properties that arise from the use of variational methods [1, 2, 10]; and in Sect. 7.4, sets of conditions are given that identify whether the extreme values of properties are upper or lower bounds.

The onset of microstructural damage in the form of fiber and interface fracture for unidirectional composites, and of ply cracking and delamination in laminated composites, leads to a deterioration of thermoelastic properties. For structural applications of composites, such as in plates with bolt holes, stress concentrations lead to localized damage and to localized changes in modulus, Poisson's ratios, and thermal expansion coefficients that cause load to be transferred to other parts of the structure. Damage development in structures is, thus, a gradual inhomogeneous process of material deterioration that eventually culminates in the catastrophic failure of the structure. The local damage-induced load transfer can lead to composite components out performing their expected performance on the basis of laboratory coupon data. Sections 7.5 and 7.6 of this chapter are concerned with laminated composites for which ply cracking is the only damage mode, although results are expected to be valid more generally. The damage model for laminates will require the properties of undamaged plies, and use can be made of the results presented in Sects. 7.2–7.4 that relate to the fiber/matrix length scale rather than the ply/laminate length scales.

While a methodology for the prediction of damage formation has been described in [7], it involves the use of various interrelationships between the effective properties of damaged laminates. These relationships were derived from the development of an accurate stress-transfer model [4–6] that estimated the values of the effective thermoelastic properties of the laminate. The objective here is to show how the interrelationships developed can be derived independently of the stress-transfer model. In addition, the analysis is extended to demonstrate that the interrelationships are valid also for laminates having orthogonal sets of ply cracks.

Ply cracking damage is usually the first significant form of damage that occurs when a laminate is loaded. The occurrence of ply cracks leads to a degradation of laminate properties; and such property degradation needs to be taken into account when assessing the integrity of composite structures using numerical methods, such as finite element analysis (FEA). The use of FEA in a structural setting demands that the three-dimensional properties of laminates be available to the software. When considering the structural integrity of a component, account needs to be taken of the degradation of all local properties when damage occurs in the form of ply cracking. Such phenomena, leading to the redistribution of stress in the structure that will affect the onset of component failure during loading, need to be modeled realistically. The results given in Sects. 7.5 and 7.6 provide most of the necessary property degradation relationships. The prediction of damage formation in laminates is, thus, set on a firm basis; and the results of these sections provide a rigorous framework of general validity that can be used with confidence in design methodologies.

7.2 Application of Maxwell's Method to Particulate Composites

In the field of electricity and magnetism, Maxwell [3] (as early as 1873) developed a method of estimating the electrical conductivity of an isotropic cluster of spherical particles of the same size embedded in a matrix having different conduction properties. The method was based on the exact solution for an isolated sphere embedded in an infinite matrix subject to a uniform gradient of electrostatic potential. This solution is applied both to the individual particles in the cluster (which were assumed to be noninteracting), and to the effective composite material that can be used to replace the particle cluster without affecting the potential distribution in the matrix at large distances from the cluster.

The effective composite medium is taken to have a radius such that the matrix and particles enclosed have the same particle volume fraction as the composite for which properties are required. When observed at large distances from an isolated particle, the electrostatic potential has the form of the sum of the unperturbed potential distribution (that arises when the particle is not present) plus a term that is inversely proportional to the square of the radial distance from the center of the particle. The coefficient of the perturbation term depends on the particle radius and the electrical conductivities of both the particle and the matrix.

The isolated particle solution is applied both to the single sphere of effective composite material representing the cluster and to each particle in the cluster. Maxwell states that an assumption is made that the particles do not interact, in which case at large distances from a cluster of particles, the perturbing effect of the particles can be expressed as the sum of the perturbations that each particle would cause if isolated in the matrix at a given point. This assumption implies that results are likely to be valid only for low volume fractions of reinforcement, although evidence is presented suggesting much wider applicability.

The purpose of this section is to report results that have recently been derived [8] where Maxwell's methodology has been applied to the estimation of many other properties for both isotropic particulate composites and anisotropic fiber-reinforced composites.

7.2.1 Applying Maxwell's Approach to Multiphase Particulate Composites

Because of the use of the far field in Maxwell's methodology for estimating the properties of particulate composites, it is possible to consider multiple spherical reinforcements. Suppose, in a cluster of particulate reinforcements embedded in an infinite matrix that there are N different types such that for $i = 1,\ldots,N$ there are n_i spherical particles of radius a_i. The properties of the particles of type i are denoted by a superscript i and subscript p, where k will denote bulk moduli, μ will denote shear moduli, and α will denote thermal expansion coefficients. The cluster is assumed to be homogeneous regarding the distribution of particles and leads to isotropic effective properties. A suffix m will be used to denote matrix properties.

The cluster of all types of particle is now considered to be enclosed in a sphere of radius b such that the volume fraction of particles of type i within the sphere of radius b is given by $V_{\mathrm{p}}^i = n_i a_i^3 / b^3$. The volume fractions must satisfy the relation

$$V_{\mathrm{m}} + \sum_{i=1}^{N} V_{\mathrm{p}}^i = 1, \tag{7.1}$$

where V_{m} is the volume fraction of matrix material. For the case of multiple phases, it has been shown that the effective bulk modulus k_{eff}, shear modulus μ_{eff}, and thermal expansion coefficient α_{eff} are given by [8]

$$\frac{1}{k_{\text{eff}}} = \frac{1}{1+\Lambda}\left(\frac{1}{k_{\text{m}}} - \frac{3\Lambda}{4\mu_{\text{m}}}\right), \tag{7.2}$$

$$\mu_{\text{eff}} = \mu_{\text{m}} \frac{1-(7-5\nu_{\text{m}})\Gamma}{1+2(4-5\nu_{\text{m}})\Gamma}, \tag{7.3}$$

$$\alpha_{\text{eff}} = \alpha_{\text{m}} + \Omega\left(\frac{1}{k_{\text{eff}}} + \frac{3}{4\mu_{\text{m}}}\right), \tag{7.4}$$

where

$$\Lambda = \sum_{i=1}^{N} \frac{\dfrac{1}{k_{\text{m}}} - \dfrac{1}{k_{\text{p}}^{i}}}{\dfrac{1}{k_{\text{p}}^{i}} + \dfrac{3}{4\mu_{\text{m}}}} V_{\text{p}}^{i}, \tag{7.5}$$

$$\Gamma = \sum_{i=1}^{N} \frac{(\mu_{\text{m}} - \mu_{\text{p}}^{i})V_{\text{p}}^{i}}{2(4-5\nu_{\text{m}})\mu_{\text{p}}^{i} + (7-5\nu_{\text{m}})\mu_{\text{m}}}, \tag{7.6}$$

$$\Omega = \sum_{i=1}^{N} \frac{\alpha_{\text{p}}^{i} - \alpha_{\text{m}}}{\dfrac{1}{k_{\text{p}}^{i}} + \dfrac{3}{4\mu_{\text{m}}}} V_{\text{p}}^{i}. \tag{7.7}$$

Application to a two-phase system

Consider a cluster of n spherical particles, having the same properties and the same radius a, embedded in an infinite matrix of different properties. The cluster is just enclosed by a sphere of radius b and the particle distribution is sufficiently homogeneous for it to lead to isotropic properties for the composite formed by the cluster and the matrix lying within this sphere. If the particulate volume fraction of the composite is denoted by V_{p}, then

$$V_{\text{p}} = \frac{na^3}{b^3} = 1 - V_{\text{m}}, \tag{7.8}$$

where V_{m} is the volume fraction of the matrix. The suffices p and m will be used to refer properties k, μ, and α to the particles and matrix, respectively.

It can be shown from (7.2) to (7.7) that the effective bulk modulus, shear modulus, and thermal expansion coefficient are given by

$$\frac{1}{k_{\mathrm{eff}}}=\frac{\dfrac{4\mu_{\mathrm{m}}}{k_{\mathrm{p}}k_{\mathrm{m}}}+\dfrac{3V_{\mathrm{m}}}{k_{\mathrm{m}}}+\dfrac{3V_{\mathrm{p}}}{k_{\mathrm{p}}}}{4\mu_{\mathrm{m}}\left(\dfrac{V_{\mathrm{p}}}{k_{\mathrm{m}}}+\dfrac{V_{\mathrm{m}}}{k_{\mathrm{p}}}+\dfrac{3}{4\mu_{\mathrm{m}}}\right)}, \tag{7.9}$$

$$\mu_{\mathrm{eff}}=\mu_{\mathrm{m}}\left[1+\frac{15(1-\nu_{\mathrm{m}})(\mu_{\mathrm{p}}-\mu_{\mathrm{m}})V_{\mathrm{p}}}{2(4-5\nu_{\mathrm{m}})(V_{\mathrm{m}}\mu_{\mathrm{p}}+V_{\mathrm{p}}\mu_{\mathrm{m}})+(7-5\nu_{\mathrm{m}})\mu_{\mathrm{m}}}\right], \tag{7.10}$$

$$\alpha_{\mathrm{eff}}=\alpha_{\mathrm{m}}+\frac{\dfrac{1}{k_{\mathrm{eff}}}+\dfrac{3}{4\mu_{\mathrm{m}}}}{\dfrac{1}{k_{\mathrm{p}}}+\dfrac{3}{4\mu_{\mathrm{m}}}}V_{\mathrm{p}}(\alpha_{\mathrm{p}}-\alpha_{\mathrm{m}}). \tag{7.11}$$

It has been shown [5] that it is possible to express these relations as the sum of mixtures estimates plus correction terms so that

$$\frac{1}{k_{\mathrm{eff}}}=\frac{V_{\mathrm{p}}}{k_{\mathrm{p}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{m}}}-\frac{\left(\dfrac{1}{k_{\mathrm{p}}}-\dfrac{1}{k_{\mathrm{m}}}\right)^2 V_{\mathrm{p}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{p}}}{k_{\mathrm{m}}}+\dfrac{V_{\mathrm{m}}}{k_{\mathrm{p}}}+\dfrac{3}{4\mu_{\mathrm{m}}}}, \tag{7.12}$$

$$\mu_{\mathrm{eff}}=V_{\mathrm{p}}\mu_{\mathrm{p}}+V_{\mathrm{m}}\mu_{\mathrm{m}}-\frac{(\mu_{\mathrm{p}}-\mu_{\mathrm{m}})^2}{V_{\mathrm{p}}\mu_{\mathrm{m}}+V_{\mathrm{m}}\mu_{\mathrm{p}}+\dfrac{9k_{\mathrm{m}}+8\mu_{\mathrm{m}}}{6(k_{\mathrm{m}}+2\mu_{\mathrm{m}})}\mu_{\mathrm{m}}}V_{\mathrm{p}}V_{\mathrm{m}}, \tag{7.13}$$

$$\alpha_{\mathrm{eff}}=V_{\mathrm{p}}\alpha_{\mathrm{p}}+V_{\mathrm{m}}\alpha_{\mathrm{m}}-\frac{\left(\dfrac{1}{k_{\mathrm{p}}}-\dfrac{1}{k_{\mathrm{m}}}\right)(\alpha_{\mathrm{p}}-\alpha_{\mathrm{m}})V_{\mathrm{p}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{p}}}{k_{\mathrm{m}}}+\dfrac{V_{\mathrm{m}}}{k_{\mathrm{p}}}+\dfrac{3}{4\mu_{\mathrm{m}}}}. \tag{7.14}$$

The mixtures estimates are given by the first two terms on the right-hand side of (7.12)–(7.14), while the third term is the correction term that must be applied to the mixtures rule. The results (7.12) and (7.14) are identical to those derived by applying the spherical shell model of the particulate composite to a representative volume element comprising just one particle and a matrix region that is consistent with the volume fraction of the composite. The values of the results (7.12)–(7.14) are identical to one of the bounds obtained when using variational methods [1, 2, 10]. These results suggest that the assumption by Maxwell [3] of low volume fractions is not necessary; an issue that will be discussed in [5].

The formulae (7.2)–(7.4) and (7.12)–(7.14) completely characterize the properties of an isotropic particulate composite and are expressed in the form of a mixtures estimate and a correction term. These formulae can be used to estimate the effective properties of a microreinforced matrix where the reinforcing phase is particulate in nature.

7.3 Application of Maxwell's Method to Fiber Composites

Because of the use of the far field in Maxwell's methodology for estimating the properties of composites, it is now possible to consider multiple fiber, rather than particulate reinforcements. Suppose in a cluster of fibers that there are N different types such that for $i = 1,\dots,N$ there are n_i fibers of radius a_i. The properties of the fibers of type i are denoted by a superscript i. Poisson's ratios are to be denoted by ν, and axial and transverse properties will be denoted by suffices A and T, respectively. The cluster is assumed to be homogeneous regarding the distribution of fibers and leads to transverse isotropic effective properties. The suffix or superscript m will be used to denote matrix properties.

The cluster of all types of fiber is now considered to be enclosed in a cylinder of radius b such that the volume fraction of fibers of type i within the cylinder of radius b is given by $V_{\mathrm{f}}^{i} = n_i a_i^2 / b^2$. The volume fractions must satisfy the relation

$$V_{\mathrm{m}} + \sum_{i=1}^{N} V_{\mathrm{f}}^{i} = 1. \tag{7.15}$$

7.3.1 Properties Derived from the Lamé Solution

By making use of Maxwell's methodology in conjunction with the Lamé solution for two bonded concentric cylinders, several properties of a fiber-reinforced composite can be estimated. It has been shown [8] that the following effective properties for the multiphase fiber-reinforced composite

Transverse bulk modulus: k_T^{eff}
Axial Poisson's ratio: ν_A^{eff}
Axial thermal expansion coefficient: α_A^{eff}
Transverse thermal expansion coefficient: α_T^{eff}

may be estimated using the formulae

$$\frac{1}{k_T^{eff}} = \frac{1}{1-\Lambda_1}\left(\frac{1}{k_T^m} + \frac{\Lambda_1}{\mu_T^m}\right), \tag{7.16}$$

$$\nu_A^{eff} = \nu_A^m - \Lambda_2\left(\frac{1}{\mu_T^m} + \frac{1}{k_T^{eff}}\right), \tag{7.17}$$

$$(\alpha_T^{eff} + \nu_A^{eff}\alpha_A^{eff}) = (\alpha_T^m + \nu_A^m\alpha_A^m) + \Lambda_3\left(\frac{1}{\mu_T^m} + \frac{1}{k_T^{eff}}\right), \tag{7.18}$$

where

$$\Lambda_1 = \sum_{i=1}^{N} V_f^i \frac{\dfrac{1}{k_T^i} - \dfrac{1}{k_T^m}}{\dfrac{1}{\mu_T^m} + \dfrac{1}{k_T^i}} = \frac{\dfrac{1}{k_T^{eff}} - \dfrac{1}{k_T^m}}{\dfrac{1}{\mu_T^m} + \dfrac{1}{k_T^{eff}}}, \tag{7.19}$$

$$\Lambda_2 = \sum_{i=1}^{N} V_f^i \frac{\nu_A^m - \nu_A^i}{\dfrac{1}{\mu_T^m} + \dfrac{1}{k_T^i}} = \frac{\nu_A^m - \nu_A^{eff}}{\dfrac{1}{\mu_T^m} + \dfrac{1}{k_T^{eff}}}, \tag{7.20}$$

$$\begin{aligned}\Lambda_3 &= \sum_{i=1}^{N} V_{\mathrm{f}}^{i} \frac{(\alpha_{\mathrm{T}}^{i} + \nu_{\mathrm{A}}^{i}\alpha_{\mathrm{A}}^{i}) - (\alpha_{\mathrm{T}}^{\mathrm{m}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}})}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{k_{\mathrm{T}}^{i}}} \\ &= \frac{(\alpha_{\mathrm{T}}^{\mathrm{eff}} + \nu_{\mathrm{A}}^{\mathrm{eff}}\alpha_{\mathrm{A}}^{\mathrm{eff}}) - (\alpha_{\mathrm{T}}^{\mathrm{m}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}})}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{k_{\mathrm{T}}^{\mathrm{eff}}}}.\end{aligned} \tag{7.21}$$

Application to a two-phase system

It follows from (7.16) to (7.21) that

$$\frac{1}{k_{\mathrm{T}}^{\mathrm{eff}}} \equiv \frac{2(1-\nu_{\mathrm{T}}^{\mathrm{eff}})}{E_{\mathrm{T}}^{\mathrm{eff}}} - \frac{4(\nu_{\mathrm{A}}^{\mathrm{eff}})^2}{E_{\mathrm{A}}^{\mathrm{eff}}} = \frac{\dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}k_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{f}}\mu_{\mathrm{T}}^{\mathrm{m}}}}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}} + \dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}}, \tag{7.22}$$

$$\nu_{\mathrm{A}}^{\mathrm{eff}} = \nu_{\mathrm{A}}^{\mathrm{m}} + V_{\mathrm{f}} \frac{\nu_{\mathrm{A}}^{\mathrm{f}} - \nu_{\mathrm{A}}^{\mathrm{m}}}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}} \left(\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \frac{1}{k_{\mathrm{T}}^{\mathrm{eff}}} \right), \tag{7.23}$$

$$\begin{aligned}\alpha_{\mathrm{T}}^{\mathrm{eff}} + \nu_{\mathrm{A}}^{\mathrm{eff}}\alpha_{\mathrm{A}}^{\mathrm{eff}} &= \alpha_{\mathrm{T}}^{\mathrm{m}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}} \\ &\quad + V_{\mathrm{f}} \frac{(\alpha_{\mathrm{T}}^{\mathrm{f}} + \nu_{\mathrm{A}}^{\mathrm{f}}\alpha_{\mathrm{A}}^{\mathrm{f}}) - (\alpha_{\mathrm{T}}^{\mathrm{m}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}})}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}} \left(\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \frac{1}{k_{\mathrm{T}}^{\mathrm{eff}}} \right),\end{aligned} \tag{7.24}$$

where $V_{\mathrm{f}} = 1 - V_{\mathrm{m}} = na^2/b^2$ is the volume fraction of n fibers of radius a embedded in matrix within a cylinder of radius b.

The relations (7.22)–(7.24) are now written as the sum of a mixtures term plus a correction term so that

$$\frac{1}{k_{\mathrm{T}}^{\mathrm{eff}}}=\frac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{m}}}-\frac{\left(\frac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\frac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)^2}{\frac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}}V_{\mathrm{f}}V_{\mathrm{m}}, \tag{7.25}$$

$$\nu_{\mathrm{A}}^{\mathrm{eff}}=V_{\mathrm{f}}\nu_{\mathrm{A}}^{\mathrm{f}}+V_{\mathrm{m}}\nu_{\mathrm{A}}^{\mathrm{m}}-\frac{(\nu_{\mathrm{A}}^{\mathrm{f}}-\nu_{\mathrm{A}}^{\mathrm{m}})\left(\frac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\frac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)}{\frac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}}V_{\mathrm{f}}V_{\mathrm{m}}, \tag{7.26}$$

$$\begin{aligned}\alpha_{\mathrm{T}}^{\mathrm{eff}}+\nu_{\mathrm{A}}^{\mathrm{eff}}\alpha_{\mathrm{A}}^{\mathrm{eff}}&=V_{\mathrm{f}}(\alpha_{\mathrm{T}}^{\mathrm{f}}+\nu_{\mathrm{A}}^{\mathrm{f}}\alpha_{\mathrm{A}}^{\mathrm{f}})+V_{\mathrm{m}}(\alpha_{\mathrm{T}}^{\mathrm{m}}+\nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}})\\&\quad-\frac{\frac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\frac{1}{k_{\mathrm{T}}^{\mathrm{m}}}}{\frac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}}[(\alpha_{\mathrm{T}}^{\mathrm{f}}+\nu_{\mathrm{A}}^{\mathrm{f}}\alpha_{\mathrm{A}}^{\mathrm{f}})-(\alpha_{\mathrm{T}}^{\mathrm{m}}+\nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}})]V_{\mathrm{f}}V_{\mathrm{m}}.\end{aligned} \tag{7.27}$$

These results correspond to one of the bounds derived using variational methods [1, 2, 10] which are identical to those obtained using the concentric cylinder model for a unidirectional, fiber-reinforced composite.

7.3.2 Axial Shear

It has been shown [8] that the effective transverse shear modulus for the multiphase fiber-reinforced composite is given by

$$\mu_{\mathrm{A}}^{\mathrm{eff}}=\mu_{\mathrm{A}}^{\mathrm{m}}\frac{1-\Lambda}{1+\Lambda}, \tag{7.28}$$

where

$$\Lambda\equiv\sum_{i=1}^{N}V_{\mathrm{f}}^{i}\frac{\mu_{\mathrm{A}}^{\mathrm{m}}-\mu_{\mathrm{A}}^{i}}{\mu_{\mathrm{A}}^{i}+\mu_{\mathrm{A}}^{\mathrm{m}}}=\frac{\mu_{\mathrm{A}}^{\mathrm{m}}-\mu_{\mathrm{A}}^{\mathrm{eff}}}{\mu_{\mathrm{A}}^{\mathrm{eff}}+\mu_{\mathrm{A}}^{\mathrm{m}}}. \tag{7.29}$$

Application to a two-phase system

When just one type of fiber is present, it follows from (7.28) and (7.29) that

$$\mu_{\mathrm{A}}^{\mathrm{eff}} = \mu_{\mathrm{A}}^{\mathrm{m}} \frac{(1+V_{\mathrm{f}})\mu_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{m}}\mu_{\mathrm{A}}^{\mathrm{m}}}{V_{\mathrm{m}}\mu_{\mathrm{A}}^{\mathrm{f}} + (1+V_{\mathrm{f}})\mu_{\mathrm{A}}^{\mathrm{m}}}. \quad (7.30)$$

The result (7.30) is now written in the form of the mixtures estimate plus a correction term as follows:

$$\mu_{\mathrm{A}}^{\mathrm{eff}} = V_{\mathrm{f}}\mu_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{m}}\mu_{\mathrm{A}}^{\mathrm{m}} - \frac{(\mu_{\mathrm{A}}^{\mathrm{f}} - \mu_{\mathrm{A}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{f}}\mu_{\mathrm{A}}^{\mathrm{m}} + V_{\mathrm{m}}\mu_{\mathrm{A}}^{\mathrm{f}} + \mu_{\mathrm{A}}^{\mathrm{m}}}. \quad (7.31)$$

This result corresponds to one of the bounds derived using variational methods [1, 2, 10] and is identical to the result obtained when using the concentric cylinder model for a unidirectional, fiber-reinforced composite.

7.3.3 Transverse Shear

It has been shown [8] that the effective transverse shear modulus for the multiphase composite is given by

$$\frac{1}{\mu_{\mathrm{T}}^{\mathrm{eff}}} = \frac{1}{1+\Lambda}\left[\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} - \Lambda\left(\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \frac{2}{k_{\mathrm{T}}^{\mathrm{m}}}\right)\right], \quad (7.32)$$

where

$$\Lambda \equiv \sum_{i=1}^{N} V_{\mathrm{f}}^{i} \frac{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} - \dfrac{1}{\mu_{\mathrm{T}}^{i}}}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{2}{k_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{\mu_{\mathrm{T}}^{i}}} = \frac{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} - \dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{eff}}}}{\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{2}{k_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{eff}}}}. \quad (7.33)$$

Application to a two-phase system

When just one type of fiber is present, it follows from (7.32) and (7.33) that

$$\mu_{\mathrm{T}}^{\mathrm{eff}} = \mu_{\mathrm{T}}^{\mathrm{m}} \frac{2\mu_{\mathrm{T}}^{\mathrm{f}}\mu_{\mathrm{T}}^{\mathrm{m}} + V_{\mathrm{m}}k_{\mathrm{T}}^{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{m}} + (1+V_{\mathrm{f}})\mu_{\mathrm{T}}^{\mathrm{f}}k_{\mathrm{T}}^{\mathrm{m}}}{V_{\mathrm{m}}(k_{\mathrm{T}}^{\mathrm{m}} + 2\mu_{\mathrm{T}}^{\mathrm{m}})\mu_{\mathrm{T}}^{\mathrm{f}} + (1+V_{\mathrm{f}})k_{\mathrm{T}}^{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{m}} + 2V_{\mathrm{f}}(\mu_{\mathrm{T}}^{\mathrm{m}})^2}. \quad (7.34)$$

The result (7.34) is now written as a mixtures estimate plus a correction term so that

$$\mu_{\mathrm{T}}^{\mathrm{eff}} = V_{\mathrm{f}}\mu_{\mathrm{T}}^{\mathrm{f}} + V_{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{m}} - \frac{(\mu_{\mathrm{T}}^{\mathrm{f}} - \mu_{\mathrm{T}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{f}}\mu_{\mathrm{T}}^{\mathrm{m}} + V_{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{f}} + \dfrac{k_{\mathrm{T}}^{\mathrm{m}}\mu_{\mathrm{T}}^{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{m}} + 2\mu_{\mathrm{T}}^{\mathrm{m}}}}. \tag{7.35}$$

This result corresponds to one of the bounds derived using variational methods [1, 2, 10].

As results derived using Maxwell's methodology can correspond exactly to bounds obtained using variational methods, and to results obtained using the concentric cylinders model, it is clear that Maxwell's approach is not necessarily restricted to low fibre volume fractions where fibre interactions are negligible.

To date, it has not been possible to find a method of using Maxwell's methodology to estimate the axial modulus and axial thermal expansion coefficient for a fiber-reinforced composite. However, the concentric cylinder model for a composite [6] leads to the following relations, which are expressed in the form of a mixtures estimate plus a correction term,

$$E_{\mathrm{A}}^{\mathrm{eff}} = V_{\mathrm{f}}E_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{m}}E_{\mathrm{A}}^{\mathrm{m}} + \frac{4(\nu_{\mathrm{A}}^{\mathrm{f}} - \nu_{\mathrm{A}}^{\mathrm{m}})^2}{\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}} + \dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}} V_{\mathrm{f}} V_{\mathrm{m}}, \tag{7.36}$$

$$\begin{aligned} E_{\mathrm{A}}^{\mathrm{eff}}\alpha_{\mathrm{A}}^{\mathrm{eff}} &= V_{\mathrm{f}}E_{\mathrm{A}}^{\mathrm{f}}\alpha_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{m}}E_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}} \\ &+ \frac{4(\nu_{\mathrm{A}}^{\mathrm{f}} - \nu_{\mathrm{A}}^{\mathrm{m}})}{\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}} + \dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}} + \dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}} [\alpha_{\mathrm{T}}^{\mathrm{f}} + \nu_{\mathrm{A}}^{\mathrm{f}}\alpha_{\mathrm{A}}^{\mathrm{f}} - \alpha_{\mathrm{T}}^{\mathrm{m}} - \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}}] V_{\mathrm{f}} V_{\mathrm{m}}. \end{aligned} \tag{7.37}$$

The statement is now complete of all the analytical formulae that can be used to predict and completely characterize the thermoelastic properties of an undamaged unidirectional, fiber-reinforced composite. It only remains to state, in Sect. 7.4, formulae for the extreme values that arise from variational calculations [1, 2, 10] and to provide conditions that determine whether the extreme values are upper or lower bounds.

7.4 Bounds for Composite Properties

As already mentioned above, variational methods have been used [1, 2, 10] to estimate upper and lower bounds for the effective properties of both isotropic particulate and anisotropic fiber-reinforced composites. In many cases it is well known that the expressions for the bounds are such that interchanging fiber and matrix parameters merely interchanges the upper and lower bounds. One of the bounds corresponds to estimates that can be derived from spherical shell or concentric cylinder models, as described in [1]. An examination of the literature regarding the conditions that determine the type of bound (either upper or lower) has revealed that there is a good deal of ambiguity. The objective of this section is to state the bounds for a particular property and to identify unambiguously the conditions that determine whether the extreme value of the property is an upper or lower bound. It should be noted that the required bounds are most easily derived by considering formulae for properties that have (as in Sect. 7.3) been expressed as a mixtures estimate plus a correction term. It is easily seen that interchanging fiber and matrix parameters affects only one term in the denominator of the correction term. Many of the following conditions, determining the upper and lower bounds to be given, do not seem to appear in the literature. It should be noted that the correction terms all have a common structure involving proportionality to the product, $V_\mathrm{p}V_\mathrm{m}$ or $V_\mathrm{f}V_\mathrm{m}$, and to the product of two reinforcement and matrix property differences, and that interchanging the reinforcement and matrix properties does not change the value of these products. It is emphasized that although the expressions given for the bounds differ in form to those that are quoted in [1, 2, 10], their values are in fact identical to the corresponding published expressions.

7.4.1 Bounds for Properties of Particulate Composites

Bulk modulus

Bounds for the bulk modulus of a particulate composite may be written in the following form

$$\overline{\frac{1}{k}} - \frac{\left(\dfrac{1}{k_\mathrm{p}} - \dfrac{1}{k_\mathrm{m}}\right)^2 V_\mathrm{p}V_\mathrm{m}}{\dfrac{V_\mathrm{m}}{k_\mathrm{p}} + \dfrac{V_\mathrm{p}}{k_\mathrm{m}} + \dfrac{3}{4\mu_\mathrm{m}}} \le \frac{1}{k_\mathrm{eff}} \le \overline{\frac{1}{k}} - \frac{\left(\dfrac{1}{k_\mathrm{p}} - \dfrac{1}{k_\mathrm{m}}\right)^2 V_\mathrm{p}V_\mathrm{m}}{\dfrac{V_\mathrm{m}}{k_\mathrm{p}} + \dfrac{V_\mathrm{p}}{k_\mathrm{m}} + \dfrac{3}{4\mu_\mathrm{p}}}, \tag{7.38}$$

where

$$\overline{\frac{1}{k}} = \frac{V_\mathrm{p}}{k_\mathrm{p}} + \frac{V_\mathrm{m}}{k_\mathrm{m}}. \tag{7.39}$$

These bounds are valid only if

$$\mu_{\mathrm{p}} \leq \mu_{\mathrm{m}}, \tag{7.40}$$

and the bounds are reversed if $\mu_{\mathrm{p}} \geq \mu_{\mathrm{m}}$.

Shear modulus

Bounds for the shear modulus of a particulate composite may be written in the following form

$$\begin{aligned} \bar{\mu} - \frac{(\mu_{\mathrm{p}} - \mu_{\mathrm{m}})^2 V_{\mathrm{p}} V_{\mathrm{m}}}{V_{\mathrm{p}}\mu_{\mathrm{m}} + V_{\mathrm{m}}\mu_{\mathrm{p}} + \dfrac{(9k_{\mathrm{m}} + 8\mu_{\mathrm{m}})\mu_{\mathrm{m}}}{6(k_{\mathrm{m}} + 2\mu_{\mathrm{m}})}} \leq \mu_{\mathrm{eff}} \\ \leq \bar{\mu} - \frac{(\mu_{\mathrm{p}} - \mu_{\mathrm{m}})^2 V_{\mathrm{p}} V_{\mathrm{m}}}{V_{\mathrm{p}}\mu_{\mathrm{m}} + V_{\mathrm{m}}\mu_{\mathrm{p}} + \dfrac{(9k_{\mathrm{p}} + 8\mu_{\mathrm{p}})\mu_{\mathrm{p}}}{6(k_{\mathrm{p}} + 2\mu_{\mathrm{p}})}}, \end{aligned} \tag{7.41}$$

where

$$\bar{\mu} = V_{\mathrm{p}}\mu_{\mathrm{p}} + V_{\mathrm{m}}\mu_{\mathrm{m}}. \tag{7.42}$$

These bounds are valid for the practically important case

$$k_{\mathrm{p}} \geq k_{\mathrm{m}}, \quad \mu_{\mathrm{p}} \geq \mu_{\mathrm{m}}, \tag{7.43}$$

and the bounds are reversed if

$$k_{\mathrm{p}} \leq k_{\mathrm{m}}, \quad \mu_{\mathrm{p}} \leq \mu_{\mathrm{m}}.$$

Other cases can arise that are not considered here.

Thermal expansion

Bounds for the thermal expansion coefficient of a particulate composite may be written in the following form

$$\bar{\alpha} + \frac{(k_{\rm p} - k_{\rm m})(\alpha_{\rm p} - \alpha_{\rm m})V_{\rm p}V_{\rm m}}{k_{\rm p}V_{\rm p} + k_{\rm m}V_{\rm m} + \dfrac{3k_{\rm m}k_{\rm p}}{4\mu_{\rm m}}} \le \alpha_{\rm eff} \le \bar{\alpha} + \frac{(k_{\rm p} - k_{\rm m})(\alpha_{\rm p} - \alpha_{\rm m})V_{\rm p}V_{\rm m}}{k_{\rm p}V_{\rm p} + k_{\rm m}V_{\rm m} + \dfrac{3k_{\rm m}k_{\rm p}}{4\mu_{\rm p}}}, \tag{7.44}$$

where

$$\bar{\alpha} = V_{\rm p}\alpha_{\rm p} + V_{\rm m}\alpha_{\rm m}. \tag{7.45}$$

It should be noted that Rosen and Hashin [10, Eq. (2.27)] do in fact express their result in the form of a mixtures term plus a correction term. It follows that the bounds apply only if the following condition is satisfied

$$(k_{\rm p} - k_{\rm m})(\mu_{\rm p} - \mu_{\rm m})(\alpha_{\rm p} - \alpha_{\rm m}) \ge 0, \tag{7.46}$$

and the bounds are reversed if $(k_{\rm p} - k_{\rm m})(\mu_{\rm p} - \mu_{\rm m})(\alpha_{\rm p} - \alpha_{\rm m}) \le 0$.

7.4.2 Bounds for Properties of Fiber-Reinforced Composites

Axial modulus

The bounds are given by

$$\overline{E_{\rm A}} + \frac{4(\nu_{\rm A}^{\rm f} - \nu_{\rm A}^{\rm m})^2 V_{\rm f}V_{\rm m}}{\dfrac{V_{\rm m}}{k_{\rm T}^{\rm f}} + \dfrac{V_{\rm f}}{k_{\rm T}^{\rm m}} + \dfrac{1}{\mu_{\rm T}^{\rm m}}} \le E_{\rm A}^{\rm eff} \le \overline{E_{\rm A}} + \frac{4(\nu_{\rm A}^{\rm f} - \nu_{\rm A}^{\rm m})^2 V_{\rm f}V_{\rm m}}{\dfrac{V_{\rm m}}{k_{\rm T}^{\rm f}} + \dfrac{V_{\rm f}}{k_{\rm T}^{\rm m}} + \dfrac{1}{\mu_{\rm T}^{\rm f}}}, \tag{7.47}$$

where

$$\overline{E_{\rm A}} = V_{\rm f}E_{\rm A}^{\rm f} + V_{\rm m}E_{\rm A}^{\rm m}. \tag{7.48}$$

These bounds are valid only if

$$\mu_{\rm T}^{\rm f} \ge \mu_{\rm T}^{\rm m}, \tag{7.49}$$

and the bounds are reversed if $\mu_{\rm T}^{\rm f} \le \mu_{\rm T}^{\rm m}$.

Poisson's ratio

The bounds are given by

$$\overline{\nu_{\mathrm{A}}}-\frac{(\nu_{\mathrm{A}}^{\mathrm{f}}-\nu_{\mathrm{A}}^{\mathrm{m}})\left(\dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\dfrac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)V_{\mathrm{f}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}}\le\nu_{\mathrm{A}}^{\mathrm{eff}}\le\overline{\nu_{\mathrm{A}}}-\frac{(\nu_{\mathrm{A}}^{\mathrm{f}}-\nu_{\mathrm{A}}^{\mathrm{m}})\left(\dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\dfrac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)V_{\mathrm{f}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{f}}}},\tag{7.50}$$

where

$$\overline{\nu_{\mathrm{A}}}=V_{\mathrm{f}}\nu_{\mathrm{A}}^{\mathrm{f}}+V_{\mathrm{m}}\nu_{\mathrm{A}}^{\mathrm{m}}.\tag{7.51}$$

These bounds are valid only if

$$(\nu_{\mathrm{A}}^{\mathrm{f}}-\nu_{\mathrm{A}}^{\mathrm{m}})(k_{\mathrm{T}}^{\mathrm{f}}-k_{\mathrm{T}}^{\mathrm{m}})(\mu_{\mathrm{T}}^{\mathrm{f}}-\mu_{\mathrm{T}}^{\mathrm{m}})\ge 0,\tag{7.52}$$

and the bounds are reversed if $(\nu_{\mathrm{A}}^{\mathrm{f}}-\nu_{\mathrm{A}}^{\mathrm{m}})(k_{\mathrm{T}}^{\mathrm{f}}-k_{\mathrm{T}}^{\mathrm{m}})(\mu_{\mathrm{T}}^{\mathrm{f}}-\mu_{\mathrm{T}}^{\mathrm{m}})\le 0$.

Transverse bulk modulus

The bounds are given by

$$\overline{\frac{1}{k_{\mathrm{T}}}}-\frac{\left(\dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\dfrac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)^2V_{\mathrm{f}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}}}\le\frac{1}{k_{\mathrm{T}}^{\mathrm{eff}}}\le\overline{\frac{1}{k_{\mathrm{T}}}}-\frac{\left(\dfrac{1}{k_{\mathrm{T}}^{\mathrm{f}}}-\dfrac{1}{k_{\mathrm{T}}^{\mathrm{m}}}\right)^2V_{\mathrm{f}}V_{\mathrm{m}}}{\dfrac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\dfrac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{m}}}+\dfrac{1}{\mu_{\mathrm{T}}^{\mathrm{f}}}},\tag{7.53}$$

where

$$\overline{\frac{1}{k_{\mathrm{T}}}}=\frac{V_{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{f}}}+\frac{V_{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{m}}}.\tag{7.54}$$

These bounds are valid only if

$$\mu_{\mathrm{T}}^{\mathrm{f}}\le\mu_{\mathrm{T}}^{\mathrm{m}},\tag{7.55}$$

and the bounds are reversed if $\mu_{\mathrm{T}}^{\mathrm{f}}\ge\mu_{\mathrm{T}}^{\mathrm{m}}$.

Transverse shear modulus

The bounds are given by

$$\overline{\mu_{\mathrm{T}}} - \frac{(\mu_{\mathrm{T}}^{\mathrm{f}} - \mu_{\mathrm{T}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{m}} \mu_{\mathrm{T}}^{\mathrm{f}} + V_{\mathrm{f}} \mu_{\mathrm{T}}^{\mathrm{m}} + \dfrac{k_{\mathrm{T}}^{\mathrm{m}} \mu_{\mathrm{T}}^{\mathrm{m}}}{k_{\mathrm{T}}^{\mathrm{m}} + 2\mu_{\mathrm{T}}^{\mathrm{m}}}} \le \mu_{\mathrm{T}}^{\mathrm{eff}} \le \overline{\mu_{\mathrm{T}}} - \frac{(\mu_{\mathrm{T}}^{\mathrm{f}} - \mu_{\mathrm{T}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{m}} \mu_{\mathrm{T}}^{\mathrm{f}} + V_{\mathrm{f}} \mu_{\mathrm{T}}^{\mathrm{m}} + \dfrac{k_{\mathrm{T}}^{\mathrm{f}} \mu_{\mathrm{T}}^{\mathrm{f}}}{k_{\mathrm{T}}^{\mathrm{f}} + 2\mu_{\mathrm{T}}^{\mathrm{f}}}}, \tag{7.56}$$

where

$$\overline{\mu_{\mathrm{T}}} = V_{\mathrm{f}} \mu_{\mathrm{T}}^{\mathrm{f}} + V_{\mathrm{m}} \mu_{\mathrm{T}}^{\mathrm{m}}. \tag{7.57}$$

These bounds are valid for the practically important case

$$k_{\mathrm{T}}^{\mathrm{f}} \ge k_{\mathrm{T}}^{\mathrm{m}}, \quad \mu_{\mathrm{T}}^{\mathrm{f}} \ge \mu_{\mathrm{T}}^{\mathrm{m}}, \tag{7.58}$$

and the bounds are reversed if

$$k_{\mathrm{T}}^{\mathrm{f}} \le k_{\mathrm{T}}^{\mathrm{m}}, \quad \mu_{\mathrm{T}}^{\mathrm{f}} \le \mu_{\mathrm{T}}^{\mathrm{m}}.$$

Other cases can arise that are not considered here.

Axial shear modulus

The bounds are given by

$$\overline{\mu_{\mathrm{A}}} - \frac{(\mu_{\mathrm{A}}^{\mathrm{f}} - \mu_{\mathrm{A}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{m}} \mu_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{f}} \mu_{\mathrm{A}}^{\mathrm{m}} + \mu_{\mathrm{A}}^{\mathrm{m}}} \le \mu_{\mathrm{A}}^{\mathrm{eff}} \le \overline{\mu_{\mathrm{A}}} - \frac{(\mu_{\mathrm{A}}^{\mathrm{f}} - \mu_{\mathrm{A}}^{\mathrm{m}})^2 V_{\mathrm{f}} V_{\mathrm{m}}}{V_{\mathrm{m}} \mu_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{f}} \mu_{\mathrm{A}}^{\mathrm{m}} + \mu_{\mathrm{A}}^{\mathrm{f}}}, \tag{7.59}$$

where

$$\overline{\mu_{\mathrm{A}}} = V_{\mathrm{f}} \mu_{\mathrm{A}}^{\mathrm{f}} + V_{\mathrm{m}} \mu_{\mathrm{A}}^{\mathrm{m}}. \tag{7.60}$$

These bounds are valid only if

$$\mu_{\mathrm{A}}^{\mathrm{m}} \le \mu_{\mathrm{A}}^{\mathrm{f}}, \tag{7.61}$$

and the bounds are reversed if $\mu_{\mathrm{A}}^{\mathrm{m}} \ge \mu_{\mathrm{A}}^{\mathrm{f}}$.

Axial thermal expansion

The bounds are given by

$$\overline{E_A\alpha_A} + \frac{4(\nu_A^f - \nu_A^m)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)V_f V_m}{\dfrac{V_m}{k_T^f} + \dfrac{V_f}{k_T^m} + \dfrac{1}{\mu_T^m}} \le E_A^{eff}\alpha_A^{eff} \le \overline{E_A\alpha_A} + \frac{4(\nu_A^f - \nu_A^m)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)V_f V_m}{\dfrac{V_m}{k_T^f} + \dfrac{V_f}{k_T^m} + \dfrac{1}{\mu_T^f}}, \tag{7.62}$$

where

$$\overline{E_A\alpha_A} = V_f E_A^f \alpha_A^f + V_m E_A^m \alpha_A^m, \quad \hat{\alpha}_T = \alpha_T + \nu_A \alpha_A. \tag{7.63}$$

These bounds are valid only if

$$(\nu_A^f - \nu_A^m)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)(\mu_T^f - \mu_T^m) \ge 0, \tag{7.64}$$

and the bounds are reversed if $(\nu_A^f - \nu_A^m)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)(\mu_T^f - \mu_T^m) \le 0$.

Transverse thermal expansion

The bounds are given by

$$\overline{\hat{\alpha}_T} - \frac{\left(\dfrac{1}{k_T^f} - \dfrac{1}{k_T^m}\right)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)V_f V_m}{\dfrac{V_m}{k_T^f} + \dfrac{V_f}{k_T^m} + \dfrac{1}{\mu_T^m}} \le \hat{\alpha}_T^{eff} \le \overline{\hat{\alpha}_T} - \frac{\left(\dfrac{1}{k_T^f} - \dfrac{1}{k_T^m}\right)(\hat{\alpha}_T^f - \hat{\alpha}_T^m)V_f V_m}{\dfrac{V_m}{k_T^f} + \dfrac{V_f}{k_T^m} + \dfrac{1}{\mu_T^f}}, \tag{7.65}$$

where

$$\overline{\hat{\alpha}_T} = V_f \hat{\alpha}_T^f + V_m \hat{\alpha}_T^m. \tag{7.66}$$

These bounds are valid only if

$$(k_{\mathrm{T}}^{\mathrm{f}} - k_{\mathrm{T}}^{\mathrm{m}})(\hat{\alpha}_{\mathrm{T}}^{\mathrm{f}} - \hat{\alpha}_{\mathrm{T}}^{\mathrm{m}})(\mu_{\mathrm{T}}^{\mathrm{f}} - \mu_{\mathrm{T}}^{\mathrm{m}}) \leq 0, \tag{7.67}$$

and the bounds are reversed if $(k_{\mathrm{T}}^{\mathrm{f}} - k_{\mathrm{T}}^{\mathrm{m}})(\hat{\alpha}_{\mathrm{T}}^{\mathrm{f}} - \hat{\alpha}_{\mathrm{T}}^{\mathrm{m}})(\mu_{\mathrm{T}}^{\mathrm{f}} - \mu_{\mathrm{T}}^{\mathrm{m}}) \geq 0$.

7.5 General Framework for Prediction of Ply Cracking in Laminates

Having considered, in Sects. 7.2–7.4, methods of estimating the effective properties of both particulate and fiber-reinforced composites, and showing how the expression of the results in a special form enables the determination of the conditions for the extreme values being upper and lower bounds, it is now appropriate to consider laminated composites where results already derived can be used to estimate the properties of undamaged laminates. This section is concerned with the development of a generalized theoretical framework that enables the prediction of microstructural damage formation in any symmetric crossply laminate subject to in-plane loading involving both biaxial and through-thickness loading modes.

In McCartney [4–6], the details are given for a high-quality stress-transfer model that has been shown to be capable of deriving accurate stress and displacement distributions at all points within any symmetrical multiple-ply laminate containing an equally spaced array of cracks in some or all of the 90° plies. The analysis accounts for the presence of residual stresses arising from thermal expansion mismatch effects. On formation, cracks are assumed (when applying stress-transfer models) to be fully developed so that each ply crack intersects with two edges of the laminate. It is assumed that the crack faces are always stress-free. This latter assumption clearly restricts the validity of the analysis to certain loading states for which cracks are either open or just closed such that no compressive or shear stresses (possibly arising from friction) are transmitted across the crack surfaces.

Having developed a stress-transfer model that can predict both stress and displacement distributions at all points in a cracked symmetric crossply laminate, together with the effective thermoelastic constants, it is necessary to be able to use such results in a way that the conditions for ply crack initiation and progressive formation can be determined, as described in [7]. The theoretical framework to be developed in this chapter will, however, put aside the important aspect of stress-transfer modeling and address the problem of predicting ply crack formation using other extremely useful principles. It is found that a great deal of progress can be made without having to resort to the use of a stress-transfer model. In addition, the approach is likely to be valid for more general damage distributions that cannot, as yet, be analyzed using a stress-transfer model. The analysis proposed here is based upon an assumption of damage homogeneity at the macroscopic laminate level. This means that bending modes arising from nonuniform damage formation are not taken into account.

The important results that are derived progressively during the development of the general framework have been checked by making use of the detailed solutions that can be obtained from the application of the stress-transfer model [4, 5]. This reinforces the validity of the proposed framework and confirms that the stress-transfer model is indeed one of high quality. The results presented here, while developed for crossply laminates subject to general in-plane loading, can be extended [5, 6] to general symmetric laminates. In such cases, in-plane biaxial loading and in-plane shear loading effects are not then separable.

7.5.1 Definition of Effective Stresses and Strains for Damaged Laminates

Consider a multiple-ply crossply laminate subject to multiaxial loading without shear. During deformation, cracks parallel to the fibers in both 0° and 90° plies in the laminate may form progressively at locations that will lead to nonuniform crack distributions because of the statistical variability of fiber distribution and of the fiber, matrix, and interface properties. The laminate is assumed to be uncracked at the commencement of loading. The state of deformation in the laminate at any stage of loading and cracking is assumed to be governed by the field equations of linear elasticity theory. When the cracks that form are open during loading, the tractions on crack surfaces are zero. The outer faces of the laminate are assumed to be subject to the same uniform applied stress denoted by σ_t.

Boundary conditions need to be imposed on the external edges of the laminate. It is assumed that cracks in plies never form on the outer edges of the laminate and that in-plane loading is applied to the laminate by imposing uniform axial and transverse displacements on the outer edges. The laminate is assumed to have length $2L$ in the y-direction and width $2W$ in the z-direction. The normal displacement components in the y- and z-directions on the edges $y = \pm L$ and $z = \pm W$ are uniform and are denoted by $\pm v_0$ and $\pm w_0$, respectively, defining, as in [4–7], effective axial and transverse applied in-plane strains as follows:

$$\varepsilon = \frac{v_0}{L}, \quad \varepsilon_{\mathrm{T}} = \frac{w_0}{W}. \tag{7.68}$$

Because uniform displacements have been imposed on the edges of the laminate, the stresses σ_{yy} and σ_{zz} resulting on the edges are nonuniform while the shear stress σ_{yz} is zero. Effective applied axial and transverse stresses are defined, as in [4–7], by

$$\begin{aligned} \sigma &= \frac{1}{4hW}\int_{-W}^{W}\int_{-h}^{h}\sigma_{yy}(x,L,z)\mathrm{d}x\mathrm{d}z, \\ \sigma_{\mathrm{T}} &= \frac{1}{4hL}\int_{-L}^{L}\int_{-h}^{h}\sigma_{zz}(x,y,W)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{7.69}$$

where

$$h = \sum_{i=1}^{N+1} h_i, \tag{7.70}$$

so that $2h$ is the total thickness of the laminate. The effective out-of-plane transverse strain is defined by

$$\varepsilon_{\mathrm{t}} = \frac{1}{4hLW}\int_{-L}^{L}\int_{-W}^{W} u(h,y,z)\mathrm{d}y\mathrm{d}z. \tag{7.71}$$

7.5.2 Stress–Strain Relations for Damaged Laminates

The following theoretical developments are intended to apply to any symmetric crossply laminate that may be subject to microcracking involving the formation of fully developed cracks that traverse the entire width or length of some or all of the individual 90° plies in the laminate. The approach to be followed, while it will be developed for nonuniform distributions of fully developed ply cracks in some or all of the 90° plies, is in fact valid also for much more general damage patterns where small ply cracks (not traversing the full width of the laminate) are present. The approach to be described will apply to these damage modes provided that all cracks are free of compressive or frictional shear loading and that the damage formed is effectively homogeneous when viewed at the macroscopic scale. For such cases, the prediction of the effective thermoelastic constants of the damaged laminate will be much more complex than is the case for fully developed ply cracks.

Consider a multiple-ply crossply laminate whose damage is in the form of stress-free ply cracks. It is *assumed* that the distribution of damage is uniform on average such that the nonshear effective stress/strain relations of the damaged laminate may be expressed in the form

$$\varepsilon_{\mathrm{t}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}} - \frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\sigma - \frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}\Delta T, \tag{7.72}$$

$$\varepsilon = -\frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\sigma_{\mathrm{t}} + \frac{\sigma}{E_{\mathrm{A}}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{A}}\Delta T, \tag{7.73}$$

$$\varepsilon_{\mathrm{T}} = -\frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\sigma + \frac{\sigma_{\mathrm{T}}}{E_{\mathrm{T}}} + \alpha_{\mathrm{T}}\Delta T, \tag{7.74}$$

where, for specified values of σ_{t}, σ, σ_{T}, and ΔT, the parameters ε_{t}, ε, and ε_{T} are, respectively, the effective out-of-plane, axial, and in-plane transverse strains of the damaged laminate, and where E_{A}, E_{T}, E_{t}, ν_{A}, ν_{a}, ν_{t}, α_{A}, α_{T}, and α_{t} denote the effective thermoelastic constants of the cracked laminate. It is assumed that the values of the thermoelastic constants in (7.72)–(7.74) may be determined by detailed stress analysis for any distribution of ply cracks. In McCartney [4–6], it is shown that the form (7.72)–(7.74) assumed for the stress–strain relations is in fact obtained from the use of stress-transfer models when ply cracks having only a single orientation are fully developed in each cracked ply. It is also shown how the thermoelastic constants may be determined for the special case of generalized plane strain conditions where ply cracking is uniformly distributed in the cracked 90° plies of the laminate. In McCartney and Schoeppner [9], it is shown how to account for nonuniform ply cracking in a laminate, assuming the same crack pattern occurs in each cracked ply.

Corresponding to (7.72)–(7.74), the stress/strain relations for an uncracked laminate subject to the same applied stresses σ_{t}, σ, σ_{T} and temperature difference ΔT are written as

$$\varepsilon_{\mathrm{t}}^{\mathrm{o}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma - \frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}^{\mathrm{o}}\Delta T, \tag{7.75}$$

$$\varepsilon^{\mathrm{o}} = -\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{t}} + \frac{\sigma}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{A}}^{\mathrm{o}}\Delta T, \tag{7.76}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{o}} = -\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma + \frac{\sigma_{\mathrm{T}}}{E_{\mathrm{T}}^{\mathrm{o}}} + \alpha_{\mathrm{T}}^{\mathrm{o}}\Delta T, \tag{7.77}$$

where the parameters $\varepsilon_{\mathrm{t}}^{\mathrm{o}}$, ε^{o}, and $\varepsilon_{\mathrm{T}}^{\mathrm{o}}$ are, respectively, the effective out-of-plane, axial, and in-plane transverse strains of the undamaged laminate, where $E_{\mathrm{A}}^{\mathrm{o}}$ is the axial Young's modulus of an uncracked laminate and, similarly, for the other thermoelastic constants. It is shown in [6], for example, how the effective thermoelastic constants for an undamaged crossply laminate may be calculated.

7.5.3 Case of One Damage Mode: Cracking in 90° Plies

Consider now the special case when ply cracks can form only in the 90° plies of the laminate. When the applied loading is such that cracks are *just* closed, the stress and displacement distributions in the laminate correspond to those in an undamaged laminate subject to the same loading and uniform temperature. The stress–strain relations for undamaged 0° plies are given by

$$\varepsilon_{\mathrm{t}}^{\mathrm{o}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}^{(0)}} - \frac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\sigma - \frac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}^{(0)}\Delta T, \tag{7.78}$$

$$\varepsilon^{\mathrm{o}} = -\frac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\sigma_{\mathrm{t}} + \frac{\sigma}{E_{\mathrm{A}}^{(0)}} - \frac{\nu_{\mathrm{A}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{A}}^{(0)}\Delta T, \tag{7.79}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{o}} = -\frac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\sigma + \frac{\sigma_{\mathrm{T}}}{E_{\mathrm{T}}^{(0)}} + \alpha_{\mathrm{T}}^{(0)}\Delta T, \tag{7.80}$$

while the stress–strain relations for undamaged 90° plies are given by

$$\varepsilon_{\mathrm{t}}^{\mathrm{o}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}^{(90)}} - \frac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma - \frac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}^{(90)}\Delta T, \tag{7.81}$$

$$\varepsilon^{\mathrm{o}} = -\frac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{t}} + \frac{\sigma}{E_{\mathrm{A}}^{(90)}} - \frac{\nu_{\mathrm{A}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{A}}^{(90)}\Delta T, \tag{7.82}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{o}} = -\frac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma + \frac{\sigma_{\mathrm{T}}}{E_{\mathrm{T}}^{(90)}} + \alpha_{\mathrm{T}}^{(90)}\Delta T. \tag{7.83}$$

The stress–strain relations (7.81)–(7.83) are based on the selection of an axial direction that is the same as that of the 0° plies, i.e., a global axial direction has been defined rather than local directions that depend on the fiber directions in a given ply. Most crossply laminates use the same material for both 0° and 90° plies in which case the stress–strain relations for both 0° and 90° plies of an undamaged laminate have the form

$$\varepsilon_{\mathrm{t}}^{\mathrm{o}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}^{*}} - \frac{\nu_{\mathrm{a}}^{*}}{E_{\mathrm{A}}^{*}}\sigma - \frac{\nu_{\mathrm{t}}^{*}}{E_{\mathrm{T}}^{*}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{t}}^{*}\Delta T, \tag{7.84}$$

$$\varepsilon^{\mathrm{o}} = -\frac{\nu_{\mathrm{a}}^{*}}{E_{\mathrm{A}}^{*}}\sigma_{\mathrm{t}} + \frac{\sigma}{E_{\mathrm{A}}^{*}} - \frac{\nu_{\mathrm{A}}^{*}}{E_{\mathrm{A}}^{*}}\sigma_{\mathrm{T}} + \alpha_{\mathrm{A}}^{*}\Delta T, \tag{7.85}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{o}} = -\frac{\nu_{\mathrm{t}}^{*}}{E_{\mathrm{T}}^{*}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{*}}{E_{\mathrm{A}}^{*}}\sigma + \frac{\sigma_{\mathrm{T}}}{E_{\mathrm{T}}^{*}} + \alpha_{\mathrm{T}}^{*}\Delta T, \tag{7.86}$$

where the axial direction for starred properties is defined as being in the fiber direction. It can then be shown that

$$\begin{aligned} &E_{\mathrm{t}}^{(0)} = E_{\mathrm{t}}^{*}, \quad E_{\mathrm{A}}^{(0)} = E_{\mathrm{A}}^{*}, \quad E_{\mathrm{T}}^{(0)} = E_{\mathrm{T}}^{*}, \\ &\nu_{\mathrm{t}}^{(0)} = \nu_{\mathrm{t}}^{*}, \quad \nu_{\mathrm{A}}^{(0)} = \nu_{\mathrm{A}}^{*}, \quad \nu_{\mathrm{a}}^{(0)} = \nu_{\mathrm{a}}^{*}, \\ &\alpha_{\mathrm{t}}^{(0)} = \alpha_{\mathrm{t}}^{*}, \quad \alpha_{\mathrm{A}}^{(0)} = \alpha_{\mathrm{A}}^{*}, \quad \alpha_{\mathrm{T}}^{(0)} = \alpha_{\mathrm{T}}^{*}, \end{aligned} \tag{7.87}$$

and that

$$\begin{aligned} &E_{\mathrm{t}}^{(90)} = E_{\mathrm{t}}^{*}, \quad E_{\mathrm{A}}^{(90)} = E_{\mathrm{T}}^{*}, \qquad E_{\mathrm{T}}^{(90)} = E_{\mathrm{A}}^{*}, \\ &\nu_{\mathrm{t}}^{(90)} = \nu_{\mathrm{a}}^{*}, \quad \nu_{\mathrm{A}}^{(90)} = \nu_{\mathrm{A}}^{*}\frac{E_{\mathrm{T}}^{*}}{E_{\mathrm{A}}^{*}}, \quad \nu_{\mathrm{a}}^{(90)} = \nu_{\mathrm{T}}^{*}, \\ &\alpha_{\mathrm{t}}^{(90)} = \alpha_{\mathrm{t}}^{*}, \quad \alpha_{\mathrm{A}}^{(90)} = \alpha_{\mathrm{T}}^{*}, \qquad \alpha_{\mathrm{T}}^{(90)} = \alpha_{\mathrm{A}}^{*}. \end{aligned} \tag{7.88}$$

The results of Sect. 7.3 can be used to provide estimates of the various ply properties if the fiber volume fraction and fiber and matrix properties are known. The analysis given in this chapter will not use the identifications (7.87) and (7.88) in order that results can be applied to hybrid laminates where the 0° and 90° plies are made of different materials.

Since the stress $\sigma^{(90)}$ in the 90° plies, in the direction of the axial loading of the laminate, is everywhere zero at the point of closure, it then follows that the in-plane strains ε^{c} and $\varepsilon_{\mathrm{T}}^{\mathrm{c}}$ at the point of closure satisfy the following relations for any value of the through-thickness stress at closure, denoted by σ_{t}, and for any temperature difference ΔT,

$$\varepsilon^{\mathrm{c}} = -\frac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{T}}^{(90)} + \alpha_{\mathrm{A}}^{(90)}\Delta T, \tag{7.89}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{c}} = -\frac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}}\sigma_{\mathrm{t}} + \frac{\sigma_{\mathrm{T}}^{(90)}}{E_{\mathrm{T}}^{(90)}} + \alpha_{\mathrm{T}}^{(90)}\Delta T, \tag{7.90}$$

where $\sigma_{\mathrm{T}}^{(90)}$ is the in-plane transverse stress in the 90° plies. It follows from (7.90) that

$$\sigma_{\mathrm{T}}^{(90)} = E_{\mathrm{T}}^{(90)}\varepsilon_{\mathrm{T}}^{\mathrm{c}} + \nu_{\mathrm{t}}^{(90)}\sigma_{\mathrm{t}} - E_{\mathrm{T}}^{(90)}\alpha_{\mathrm{T}}^{(90)}\Delta T. \tag{7.91}$$

On substituting (7.91) into (7.89) it can be shown that

$$\varepsilon^{\mathrm{c}} = -A\sigma_{\mathrm{t}} - B\varepsilon_{\mathrm{T}}^{\mathrm{c}} + C\Delta T, \tag{7.92}$$

where

$$A = \frac{\nu_a^{(90)} + \nu_t^{(90)}\nu_A^{(90)}}{E_A^{(90)}} = \frac{\nu_t^*}{E_T^*} + \frac{\nu_a^*\nu_A^*}{E_A^*},$$

$$B = \nu_A^{(90)}\frac{E_T^{(90)}}{E_A^{(90)}} = \nu_A^*, \tag{7.93}$$

$$C = \alpha_A^{(90)} + \nu_A^{(90)}\frac{E_T^{(90)}}{E_A^{(90)}}\alpha_T^{(90)} = \alpha_T^* + \nu_A^*\alpha_A^*,$$

where the starred quantities denote the ply properties for a laminate made of just one type of ply material (see (7.87) and (7.88)). The relation (7.92) implies that the axial closure strain depends on the transverse strain ε_T^c in addition to the through-thickness stress σ_t and the temperature difference ΔT.

7.5.4 Ply Crack Closure for Uniaxial Loading in Axial Direction

Consider now the critical point for the first closure of the ply cracks for the case of uniaxial loading such that $\sigma_t = \sigma_T = 0$. It follows from (7.72) to (7.74), on setting $\varepsilon_t = \bar{\varepsilon}_t^c$, $\varepsilon = \bar{\varepsilon}^c$, $\varepsilon_T = \bar{\varepsilon}_T^c$, $\sigma = \bar{\sigma}^c$, that

$$\bar{\varepsilon}_t^c = -\frac{\nu_a}{E_A}\bar{\sigma}^c + \alpha_t\Delta T, \tag{7.94}$$

$$\bar{\varepsilon}^c = \frac{\bar{\sigma}^c}{E_A} + \alpha_A\Delta T, \tag{7.95}$$

$$\bar{\varepsilon}_T^c = -\frac{\nu_A}{E_A}\bar{\sigma}^c + \alpha_T\Delta T. \tag{7.96}$$

For undamaged laminates, it follows that at the critical closure stress $\bar{\sigma}^c$ the laminate is subject to the same strains as in the corresponding cracked laminate, with the result that from (7.75) to (7.77)

$$\bar{\varepsilon}_t^c = -\frac{\nu_a^o}{E_A^o}\bar{\sigma}^c + \alpha_t^o\Delta T, \tag{7.97}$$

$$\bar{\varepsilon}^{c} = \frac{\bar{\sigma}^{c}}{E_{A}^{o}} + \alpha_{A}^{o}\Delta T, \tag{7.98}$$

$$\bar{\varepsilon}_{T}^{c} = -\frac{\nu_{A}^{o}}{E_{A}^{o}}\bar{\sigma}^{c} + \alpha_{T}^{o}\Delta T. \tag{7.99}$$

On subtracting (7.97) to (7.99) from (7.94) to (7.96) to eliminate the closure strains $\bar{\varepsilon}_{t}^{c}$, $\bar{\varepsilon}^{c}$, and $\bar{\varepsilon}_{T}^{c}$, it then follows that

$$\bar{\sigma}^{c} = -\frac{\alpha_{t} - \alpha_{t}^{o}}{\dfrac{\nu_{a}}{E_{A}} - \dfrac{\nu_{a}^{o}}{E_{A}^{o}}}\Delta T = -\frac{\alpha_{A} - \alpha_{A}^{o}}{\dfrac{1}{E_{A}} - \dfrac{1}{E_{A}^{o}}}\Delta T = -\frac{\alpha_{T} - \alpha_{T}^{o}}{\dfrac{\nu_{A}^{o}}{E_{A}^{o}} - \dfrac{\nu_{A}}{E_{A}}}\Delta T. \tag{7.100}$$

It follows from (7.92) that for the case of uniaxial loading under discussion

$$\bar{\varepsilon}^{c} = -B\bar{\varepsilon}_{T}^{c} + C\Delta T. \tag{7.101}$$

Thus on substituting (7.98) and (7.99) into (7.101) and solving for $\bar{\sigma}^{c}$, the following alternative expression for the crack closure stress is obtained

$$\bar{\sigma}^{c} = -\frac{E_{A}^{o}[\alpha_{A}^{o} + B\alpha_{T}^{o} - C]}{1 - \nu_{A}^{o}B}\Delta T. \tag{7.102}$$

Thus, it follows from (7.100) and (7.102) that the following inter-relationships are valid

$$\frac{\alpha_{t} - \alpha_{t}^{o}}{\dfrac{\nu_{a}}{E_{A}} - \dfrac{\nu_{a}^{o}}{E_{A}^{o}}} = \frac{\alpha_{A} - \alpha_{A}^{o}}{\dfrac{1}{E_{A}} - \dfrac{1}{E_{A}^{o}}} = \frac{\alpha_{T} - \alpha_{T}^{o}}{\dfrac{\nu_{A}^{o}}{E_{A}^{o}} - \dfrac{\nu_{A}}{E_{A}}} = k_{1}, \tag{7.103}$$

where

$$k_{1} = \frac{E_{A}^{o}[\alpha_{A}^{o} + B\alpha_{T}^{o} - C]}{1 - \nu_{A}^{o}B}, \quad \bar{\sigma}^{c} = -k_{1}\Delta T. \tag{7.104}$$

From (7.93) it is clear that the parameter k_1 is a constant for an undamaged laminate.

7.5.5 Ply Crack Closure for Uniaxial Loading in In-Plane Transverse Direction

Consider now ply crack closure for the case of uniaxial loading such that $\sigma_t = \sigma = 0$. By following the approach of Sect. 7.5.4, it can be shown, on setting $\varepsilon_t = \hat{\varepsilon}_t^c$, $\varepsilon = \hat{\varepsilon}^c$, $\varepsilon_T = \hat{\varepsilon}_T^c$, $\sigma_T = \hat{\sigma}_T^c$, that

$$\hat{\sigma}^c = -\frac{\alpha_t - \alpha_t^o}{\frac{\nu_t^o}{E_T^o} - \frac{\nu_t}{E_T}}\Delta T = -\frac{\alpha_A - \alpha_A^o}{\frac{\nu_A^o}{E_A^o} - \frac{\nu_A}{E_A}}\Delta T = -\frac{\alpha_T - \alpha_T^o}{\frac{1}{E_T} - \frac{1}{E_T^o}}\Delta T. \tag{7.105}$$

It follows from (7.92) that for the case of transverse uniaxial loading under discussion

$$\hat{\varepsilon}^c = -B\hat{\varepsilon}_T^c + C\Delta T. \tag{7.106}$$

Thus, on using the stress–strain relations (7.75)–(7.77) for an undamaged laminate applied to uniaxial transverse loading, the following alternative expression for the crack closure stress is obtained

$$\hat{\sigma}^c = -\frac{E_T^o[\alpha_A^o + B\alpha_T^o - C]}{B - \nu_A^o E_T^o / E_A^o}\Delta T. \tag{7.107}$$

It then follows from (7.105) and (7.107) that

$$\frac{\alpha_t - \alpha_t^o}{\frac{\nu_t^o}{E_T^o} - \frac{\nu_t}{E_T}} = \frac{\alpha_A - \alpha_A^o}{\frac{\nu_A^o}{E_A^o} - \frac{\nu_A}{E_A}} = \frac{\alpha_T - \alpha_T^o}{\frac{1}{E_T} - \frac{1}{E_T^o}} = k_2, \tag{7.108}$$

where

$$k_2 = \frac{E_T^o[\alpha_A^o + B\alpha_T^o - C]}{B - \nu_A^o E_T^o / E_A^o}. \tag{7.109}$$

The constant k_2 is thus another laminate constant for an undamaged laminate. It follows from (7.104) and (7.109) that

$$k = \frac{k_1}{k_2} = \frac{E_A^o}{E_T^o}\frac{B - \nu_A^o E_T^o / E_A^o}{1 - \nu_A^o B}, \tag{7.110}$$

where the constant k is independent of the thermal expansion coefficients. It then follows from (7.103), (7.108), and (7.110) that the following interrelationships must also be valid

$$\frac{\dfrac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}}{\dfrac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}}=\frac{\dfrac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}}{\dfrac{1}{E_{\mathrm{A}}}-\dfrac{1}{E_{\mathrm{A}}^{\mathrm{o}}}}=\frac{\dfrac{1}{E_{\mathrm{T}}}-\dfrac{1}{E_{\mathrm{T}}^{\mathrm{o}}}}{\dfrac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}}=\frac{\alpha_{\mathrm{T}}-\alpha_{\mathrm{T}}^{\mathrm{o}}}{\alpha_{\mathrm{A}}-\alpha_{\mathrm{A}}^{\mathrm{o}}}=k, \tag{7.111}$$

where k is a constant for an undamaged laminate and is independent of the thermal expansion coefficients.

7.5.6 Ply Crack Closure for Uniaxial Loading in Out-of-Plane Direction

Consider now ply crack closure for the case of uniaxial out-of-plane loading such that $\sigma=\sigma_{\mathrm{T}}=0$. Again, following the approach of Sect. 7.5.4, it follows from (7.72) to (7.74), on setting $\varepsilon_{\mathrm{t}}=\tilde{\varepsilon}_{\mathrm{t}}^{\mathrm{c}}$, $\varepsilon=\tilde{\varepsilon}^{\mathrm{c}}$, $\varepsilon_{\mathrm{T}}=\tilde{\varepsilon}_{\mathrm{T}}^{\mathrm{c}}$, and $\sigma_{\mathrm{T}}=\tilde{\sigma}_{\mathrm{T}}^{\mathrm{c}}$, that

$$\tilde{\sigma}_{\mathrm{t}}^{\mathrm{c}}=-\frac{\alpha_{\mathrm{t}}-\alpha_{\mathrm{t}}^{\mathrm{o}}}{\dfrac{1}{E_{\mathrm{t}}}-\dfrac{1}{E_{\mathrm{t}}^{\mathrm{o}}}}\Delta T=-\frac{\alpha_{\mathrm{A}}-\alpha_{\mathrm{A}}^{\mathrm{o}}}{\dfrac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}}\Delta T=-\frac{\alpha_{\mathrm{T}}-\alpha_{\mathrm{T}}^{\mathrm{o}}}{\dfrac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}}\Delta T. \tag{7.112}$$

It follows from (7.92) that for the case of uniaxial loading under discussion

$$\tilde{\varepsilon}^{\mathrm{c}}=-A\tilde{\sigma}_{\mathrm{t}}^{\mathrm{c}}-B\tilde{\varepsilon}_{\mathrm{T}}^{\mathrm{c}}+C\Delta T. \tag{7.113}$$

Thus, on using the stress–strain relations (7.75)–(7.77) for an undamaged laminate subject to a uniaxial through-thickness loading, the following alternative expression for the crack closure stress is obtained

$$\tilde{\sigma}_{\mathrm{t}}^{\mathrm{c}}=-\frac{\alpha_{\mathrm{A}}^{\mathrm{o}}+B\alpha_{\mathrm{T}}^{\mathrm{o}}-C}{A-\dfrac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\dfrac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}B}\Delta T. \tag{7.114}$$

It then follows from (7.112) and (7.114) that

$$\frac{\alpha_{\rm t}-\alpha_{\rm t}^{\rm o}}{\dfrac{1}{E_{\rm t}}-\dfrac{1}{E_{\rm t}^{\rm o}}}=\frac{\alpha_{\rm A}-\alpha_{\rm A}^{\rm o}}{\dfrac{\nu_{\rm a}^{\rm o}}{E_{\rm A}^{\rm o}}-\dfrac{\nu_{\rm a}}{E_{\rm A}}}=\frac{\alpha_{\rm T}-\alpha_{\rm T}^{\rm o}}{\dfrac{\nu_{\rm t}^{\rm o}}{E_{\rm T}^{\rm o}}-\dfrac{\nu_{\rm t}}{E_{\rm T}}}=k_3, \tag{7.115}$$

where

$$k_3=\frac{\alpha_{\rm A}^{\rm o}+B\alpha_{\rm T}^{\rm o}-C}{A-\dfrac{\nu_{\rm a}^{\rm o}}{E_{\rm A}^{\rm o}}-\dfrac{\nu_{\rm t}^{\rm o}}{E_{\rm T}^{\rm o}}B}. \tag{7.116}$$

k_3 is another constant for an undamaged laminate.

It follows from (7.104) and (7.116) that

$$k'=\frac{k_1}{k_3}=\frac{E_{\rm A}^{\rm o}A-\nu_{\rm a}^{\rm o}-\nu_{\rm t}^{\rm o}\dfrac{E_{\rm A}^{\rm o}}{E_{\rm T}^{\rm o}}B}{1-\nu_{\rm A}^{\rm o}B}, \tag{7.117}$$

where the constant k' is independent of the thermal expansion coefficients.

It then follows from (7.103), (7.115), and (7.117) that the following interrelationships must also be valid

$$\frac{\dfrac{1}{E_{\rm t}}-\dfrac{1}{E_{\rm t}^{\rm o}}}{\dfrac{\nu_{\rm a}^{\rm o}}{E_{\rm A}^{\rm o}}-\dfrac{\nu_{\rm a}}{E_{\rm A}}}=\frac{\dfrac{\nu_{\rm a}^{\rm o}}{E_{\rm A}^{\rm o}}-\dfrac{\nu_{\rm a}}{E_{\rm A}}}{\dfrac{1}{E_{\rm A}}-\dfrac{1}{E_{\rm A}^{\rm o}}}=\frac{\dfrac{\nu_{\rm t}^{\rm o}}{E_{\rm T}^{\rm o}}-\dfrac{\nu_{\rm t}}{E_{\rm T}}}{\dfrac{\nu_{\rm A}^{\rm o}}{E_{\rm A}^{\rm o}}-\dfrac{\nu_{\rm A}}{E_{\rm A}}}=\frac{\alpha_{\rm t}-\alpha_{\rm t}^{\rm o}}{\alpha_{\rm A}-\alpha_{\rm A}^{\rm o}}=k', \tag{7.118}$$

where k' is another constant for an undamaged laminate and is independent of the thermal expansion coefficients.

7.5.7 Useful Independent Interrelationships

The relations (7.103), (7.108), (7.111), (7.115), and (7.118) are not, of course, all independent. It will, therefore, be very useful to identify a set of independent interrelationships. To achieve this objective, it is convenient to introduce the parameter D defined by

$$D=\frac{1}{E_{\rm A}}-\frac{1}{E_{\rm A}^{\rm o}}. \tag{7.119}$$

The quantity D can be regarded as a definition of a damage parameter arising in the field of continuum damage mechanics [6]. It should be noted that as the axial modulus of a damaged laminate is nearly always measured as a function of crack density, the parameter D may be estimated from experimental data. Also, the dependence of the axial modulus on crack density is nearly always predicted by mathematical models of stress transfer, especially the elementary models based on shear lag theory. It can be shown, from (7.103), (7.108), and (7.118), that the following relations form the required set of independent interrelationships:

$$\frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}} = kD, \tag{7.120}$$

$$\frac{1}{E_{\mathrm{T}}} - \frac{1}{E_{\mathrm{T}}^{\mathrm{o}}} = k^2 D, \tag{7.121}$$

$$\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}} = k' D, \tag{7.122}$$

$$\frac{1}{E_{\mathrm{t}}} - \frac{1}{E_{\mathrm{t}}^{\mathrm{o}}} = (k')^2 D, \tag{7.123}$$

$$\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}} = kk' D, \tag{7.124}$$

$$\alpha_{\mathrm{A}} - \alpha_{\mathrm{A}}^{\mathrm{o}} = k_1 D, \tag{7.125}$$

$$\alpha_{\mathrm{T}} - \alpha_{\mathrm{T}}^{\mathrm{o}} = kk_1 D, \tag{7.126}$$

$$\alpha_{\mathrm{t}} - \alpha_{\mathrm{t}}^{\mathrm{o}} = k' k_1 D. \tag{7.127}$$

In the above interrelationships, the parameters k, k', and k_1 are constants for undamaged laminates defined by the relations (7.110), (7.117), and (7.104), respectively. Thus, the values of these constants are readily calculated using (7.87), (7.88), and (7.93) and the results given in [6] for undamaged laminate properties. On using the relations (7.120)–(7.127), the effective thermoelastic constants of a damaged laminate can be calculated in terms of the damage-dependent parameter D defined by (7.119).

The importance of these relationships must be emphasized. First of all, they enable an exact and simple means of fully characterizing the effects of microstructural damage on the effective thermoelastic constants of multiple-ply crossply laminates, requiring only the value of the parameter D which depends upon the axial modulus that is frequently measured in practice or can be calculated from a suitable model. A reasonable estimate of the axial modulus of a damaged laminate can be obtained using a wide variety of models. The relation (7.119) can then be used to estimate the damage parameter D, and the relations (7.120)–(7.127) provide a rigorous method of extending such results so that estimates can be made of many other properties of damaged laminates that are needed by finite element analyses of damaged composite structures. Secondly, the relationships enable an efficient method of calculating the effective thermoelastic constants of damaged laminates using stress-transfer models, avoiding a great deal of time-consuming numerical computation if using state-of-the-art stress-transfer models that are capable of predicting highly accurate results.

7.6 Ply Crack Closure for Orthogonal Ply Cracking

In Sect. 7.5, the special case was considered where cracks were allowed to form only in the 90° plies of a crossply laminate. A consequence of this restriction is the validity of the very useful relations (7.119)–(7.127). The more general case of orthogonal cracking is now addressed. Consider the special loading case when all the ply cracks in the 0° and 90° plies of the laminate *just* close during multiaxial loading, where the through-thickness stress may have any value σ_t. For these crack closure conditions, let $\sigma = \sigma^c$ and $\sigma_T = \sigma_T^c$ so that from (7.72) to (7.74)

$$\varepsilon_t^c = \frac{\sigma_t}{E_t} - \frac{\nu_a}{E_A}\sigma^c - \frac{\nu_t}{E_T}\sigma_T^c + \alpha_t \Delta T, \tag{7.128}$$

$$\varepsilon^c = -\frac{\nu_a}{E_A}\sigma_t + \frac{\sigma^c}{E_A} - \frac{\nu_A}{E_A}\sigma_T^c + \alpha_A \Delta T, \tag{7.129}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{c}} = -\frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\sigma^{\mathrm{c}} + \frac{\sigma_{\mathrm{T}}^{\mathrm{c}}}{E_{\mathrm{T}}} + \alpha_{\mathrm{T}}\Delta T, \tag{7.130}$$

and from (7.75) to (7.77) that

$$\varepsilon_{\mathrm{t}}^{\mathrm{c}} = \frac{\sigma_{\mathrm{t}}}{E_{\mathrm{t}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma^{\mathrm{c}} - \frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}\sigma_{\mathrm{T}}^{\mathrm{c}} + \alpha_{\mathrm{t}}^{\mathrm{o}}\Delta T, \tag{7.131}$$

$$\varepsilon^{\mathrm{c}} = -\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{t}} + \frac{\sigma^{\mathrm{c}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma_{\mathrm{T}}^{\mathrm{c}} + \alpha_{\mathrm{A}}^{\mathrm{o}}\Delta T, \tag{7.132}$$

$$\varepsilon_{\mathrm{T}}^{\mathrm{c}} = -\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}\sigma_{\mathrm{t}} - \frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}\sigma^{\mathrm{c}} + \frac{\sigma_{\mathrm{T}}^{\mathrm{c}}}{E_{\mathrm{T}}^{\mathrm{o}}} + \alpha_{\mathrm{T}}^{\mathrm{o}}\Delta T, \tag{7.133}$$

where $\varepsilon_{\mathrm{t}}^{\mathrm{c}}$, ε^{c}, and $\varepsilon_{\mathrm{T}}^{\mathrm{c}}$ are the laminate strains when cracks in the 0° and 90° plies just close for any values of σ_{t} and ΔT. At the point of closure, both cracked and uncracked laminates will have the same stress and strain distributions. It is shown in Appendix A how the values of the crack closure stresses and strains appearing in (7.128)–(7.133) may be calculated from the properties of the 0° and 90° plies. On subtracting (7.131)–(7.133) from (7.128)–(7.130), it follows that

$$\begin{aligned} &\left[\frac{1}{E_{\mathrm{t}}} - \frac{1}{E_{\mathrm{t}}^{\mathrm{o}}}\right]\sigma_{\mathrm{t}} + \left[\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\right]\sigma^{\mathrm{c}} \\ &\quad + \left[\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\right]\sigma_{\mathrm{T}}^{\mathrm{c}} + [\alpha_{\mathrm{t}} - \alpha_{\mathrm{t}}^{\mathrm{o}}]\Delta T = 0, \end{aligned} \tag{7.134}$$

$$\begin{aligned} &\left[\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\right]\sigma_{\mathrm{t}} + \left[\frac{1}{E_{\mathrm{A}}} - \frac{1}{E_{\mathrm{A}}^{\mathrm{o}}}\right]\sigma^{\mathrm{c}} \\ &\quad + \left[\frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\right]\sigma_{\mathrm{T}}^{\mathrm{c}} + [\alpha_{\mathrm{A}} - \alpha_{\mathrm{A}}^{\mathrm{o}}]\Delta T = 0, \end{aligned} \tag{7.135}$$

$$\left[\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\right]\sigma_{\mathrm{t}}+\left[\frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\right]\sigma^{\mathrm{c}} + \left[\frac{1}{E_{\mathrm{T}}}-\frac{1}{E_{\mathrm{T}}^{\mathrm{o}}}\right]\sigma_{\mathrm{T}}^{\mathrm{c}}+[\alpha_{\mathrm{T}}-\alpha_{\mathrm{T}}^{\mathrm{o}}]\Delta T=0. \tag{7.136}$$

These relations must be satisfied for all possible damage states in the form of orthogonal ply cracks and for all values of σ_{t} and ΔT. When there are only ply cracks in the 90° plies, use can be made of the relations (7.119)–(7.127) in which case the relations (7.134)–(7.136) all reduce to the following single relationship

$$k'\sigma_{\mathrm{t}}+\sigma^{\mathrm{c}}+k\sigma_{\mathrm{T}}^{\mathrm{c}}+k_1\Delta T=0. \tag{7.137}$$

From (7.104) it follows that (7.137) may be written as

$$k'\sigma_{\mathrm{t}}+\sigma^{\mathrm{c}}+k\sigma_{\mathrm{T}}^{\mathrm{c}}=\bar{\sigma}^{\mathrm{c}}, \tag{7.138}$$

showing how the stresses σ_{t}, σ^{c}, and $\sigma_{\mathrm{T}}^{\mathrm{c}}$ are related to the closure stress $\bar{\sigma}^{\mathrm{c}}$ for a uniaxially loaded laminate for the special case when ply cracks form only in the 90° plies.

For orthogonal cracking, it can be shown on using (A16) that (7.134)–(7.136) can be satisfied for all damage states, and for all values of σ_{t} and ΔT, only if relations of the following type are satisfied

$$\left[\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\right]R+\left[\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\right]R_{\mathrm{T}}+\left[\frac{1}{E_{\mathrm{t}}}-\frac{1}{E_{\mathrm{t}}^{\mathrm{o}}}\right]=0, \tag{7.139}$$

$$\left[\frac{1}{E_{\mathrm{A}}}-\frac{1}{E_{\mathrm{A}}^{\mathrm{o}}}\right]R+\left[\frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\right]R_{\mathrm{T}}+\left[\frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}\right]=0, \tag{7.140}$$

$$\left[\frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\right]R+\left[\frac{1}{E_{\mathrm{T}}}-\frac{1}{E_{\mathrm{T}}^{\mathrm{o}}}\right]R_{\mathrm{T}}+\left[\frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}}-\frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}\right]=0, \tag{7.141}$$

$$\left[\frac{\nu_a^o}{E_A^o}-\frac{\nu_a}{E_A}\right]P+\left[\frac{\nu_t^o}{E_T^o}-\frac{\nu_t}{E_T}\right]P_T+[\alpha_t-\alpha_t^o]=0, \tag{7.142}$$

$$\left[\frac{1}{E_A}-\frac{1}{E_A^o}\right]P+\left[\frac{\nu_A^o}{E_A^o}-\frac{\nu_A}{E_A}\right]P_T+[\alpha_A-\alpha_A^o]=0, \tag{7.143}$$

$$\left[\frac{\nu_A^o}{E_A^o}-\frac{\nu_A}{E_A}\right]P+\left[\frac{1}{E_T}-\frac{1}{E_T^o}\right]P_T+[\alpha_T-\alpha_T^o]=0, \tag{7.144}$$

where the quantities P, P_T, R, and R_T are laminate constants defined in Appendix A. The relations (7.139)–(7.144) must be satisfied for all possible damage states and this includes the case where there are only ply cracks in the 90° plies of the laminate for which the interrelationships (7.119)–(7.127) are valid. On substituting them in (7.139)–(7.144), it follows that the relations (7.139)–(7.141) reduce to the single relation

$$k' + R + kR_T = 0, \tag{7.145}$$

and the relations (7.142)–(7.144) reduce to the single relation

$$P + kP_T + k_1 = 0. \tag{7.146}$$

The validity of these relations has been verified numerically and this implies that the interrelationships (7.119)–(7.127), derived for laminates having ply cracking damage only in the 90° plies, are valid also for the case of orthogonal cracking where ply cracks are present in both 0° and 90° plies of a laminate. Appendix B provides further analysis to support this statement. Although the analysis in this section has been derived for sets of orthogonal fully developed ply cracks in a crossply laminate, the approach described should be valid for more general damage states involving partially formed ply cracks, i.e., cracks that do not traverse the entire section of a ply. The assumption for such damage states is that biaxial effective stresses can be applied to the laminate, for a given through-thickness stress σ_t and temperature difference ΔT that lead to the precise closure of any cracks that have formed. This condition must be satisfied for the analysis given in this section to have validity.

As described in [4, 6, 7], the interrelationships (7.119)–(7.127) enable a very simple expression for the change in the Gibbs energy (or complementary energy) per unit volume as a result of ply cracking in the laminate, namely

$$\bar{g} - \bar{g}_0 = -\frac{1}{2}\left(\frac{1}{E_{\mathrm{A}}} - \frac{1}{E_{\mathrm{A}}^{\mathrm{o}}}\right)[s - \bar{\sigma}^{\mathrm{c}}]^2 = -\frac{1}{2}D[s - \bar{\sigma}^{\mathrm{c}}]^2, \qquad (7.147)$$

where s is an effective stress that is defined by

$$s = k'\sigma_{\mathrm{t}} + \sigma + k\sigma_{\mathrm{T}}, \qquad (7.148)$$

and where use has been made of the relation (7.119). This is the stress that controls ply crack formation and takes full account of the combined effects of biaxial loading and through-thickness loading of the laminate. Thermal residual stresses are accounted for through the axial ply crack closure stress $\bar{\sigma}^{\mathrm{c}}$, which, from (7.104), is proportional to the temperature difference ΔT. It is remarkable that such a simple relationship can be derived that takes account of the complex stress-transfer mechanisms arising from multiaxial loads and thermal residual stresses. The relations (7.147) and (7.148) have been used to derive [6, 7] simple expressions for the ply cracking stresses that have very good potential for use in design methodologies.

7.7 Conclusions

The following conclusions are drawn from the theoretical analysis presented in this chapter.

7.7.1 Undamaged Properties of Composites (at Ply Level)

1. A methodology developed by Maxwell in 1873 for estimating the electrical conductivity of isotropic particulate composites has been shown in another publication to be of much wider applicability. Results are presented in this chapter giving formulae for many thermoelastic constants for both isotropic particulate composites and fiber-reinforced composites.

2. The results are shown to be identical with results that can be derived using spherical shell and concentric cylinder models, which, in turn, are identical to one of the bounds derived using variational methods, implying that Maxwell's methodology is not necessarily restricted to low volume fractions.

3. The estimates of the various thermoelastic constants for both particulate and fiber-reinforced composites derived using Maxwell's method, or other nonvariational methods, can all be expressed in the form of a mixtures estimate of the property plus a correction term. These results have a common structure; and they correspond to one of the bounds derived using variational methods, where the other bound can be found simply by interchanging reinforcement and matrix properties and volume fractions.

4. Definitive conditions for an extreme property value being an upper bound, and those for it being a lower bound, have easily been determined (based on results described in [8]). The conditions given do not seem to have been given before in the literature.

7.7.2 Damaged Properties of Composites (at Laminate Level)

1. By making a limited number of reasonable assumptions, it is possible to develop, without the use of a stress-transfer model, a detailed theoretical framework that can be used to characterize the properties of laminates when ply cracks form in crossply laminates subject to thermal residual stresses and to multiaxial loading involving combined in-plane biaxial and through-thickness applied stresses. Results derived have been confirmed by an accurate stress-transfer model.

2. By considering ply crack closure, it is possible to develop a set of interrelationships that must be satisfied by the effective thermoelastic constants of cracked crossply laminates and which are derived without the use of a stress-transfer model. The interrelationships indicate that the effective constants of a cracked laminate can be characterized by just one damage function D that is defined in terms of the axial modulus of the laminate for both damaged and undamaged states. The interrelationships provide a rational and rigorous method of degrading many laminate properties due to damage formation. The interrelationships have been shown to be valid also for laminates with orthogonal ply cracking. The interrelationships should be applied to structural analyses when modeling the local stress redistribution that can occur due to the local formation of laminate damage in the form of ply cracking.

3. The interrelationships enable the change of Gibbs free energy (or complementary energy) resulting from ply crack formation to be written in a very simple form that involves macroscopic quantities and which is exact within the assumptions made when developing the theoretical framework. One key feature is the identification of an effective applied stress that takes account of the combined effects on ply crack formation of biaxial and through-thickness loading. Another key feature is the demonstration that the effects of thermal residual stresses can be taken into account by introducing the uniaxial crack closure stress.

4. Stress-transfer models predicting the stress and displacement distributions in cracked laminates are needed only to predict the actual values of the effective thermoelastic constants of cracked laminates. All other aspects of the prediction of ply crack formation can be accounted for using the derived theoretical framework.

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Appendix A: Crack Closure Stresses for Orthogonal Cracking

Consider an undamaged crossply laminate where the 0° plies are made of the same material and the 90° plies are made of the same material that could differ from that of the 0° plies. The stresses in each ply, made of the same material and having the same orientation, must have the same values. This arises because each ply experiences the same values ε and ε_{T} for the axial and transverse in-plane strains, respectively, and the same value for the through-thickness stress σ_{t}. It is useful to denote the axial and transverse in-plane stresses in the 0° plies by $\sigma^{(0)}$ and $\sigma_{\mathrm{T}}^{(0)}$, respectively, and to denote the axial and transverse in-plane stresses in the 90° plies by $\sigma^{(90)}$ and $\sigma_{\mathrm{T}}^{(90)}$, respectively.

For a laminate containing cracks in both 0° and 90° plies, the crack closure stresses are such that $\sigma^{(90)} = 0$ and $\sigma_{\mathrm{T}}^{(0)} = 0$ for a given value of the through-thickness stress σ_{t} and for a given value of the temperature difference ΔT. The resulting stress state in the laminate corresponds to the case when orthogonal cracks in the laminate just close. For this special case, mechanical equilibrium asserts that

$$\sigma^{(0)} = \frac{h\sigma^{\mathrm{c}}}{h^{(0)}}, \quad \sigma_{\mathrm{T}}^{(90)} = \frac{h\sigma_{\mathrm{T}}^{\mathrm{c}}}{h^{(90)}}, \tag{A1}$$

where $h^{(0)}$ and $h^{(90)}$ denote the total thicknesses in the laminate of all 0° plies and all 90° plies, respectively, and where σ^{c} and $\sigma_{\mathrm{T}}^{\mathrm{c}}$ are, respectively, the effective applied axial stress and the in-plane transverse stress at the point of crack closure. Substitution of (A1) into the stress–strain equations for both 0° and 90° plies leads to the relations

$$
\begin{aligned}
\varepsilon^{\mathrm{c}} &= -\frac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\sigma_{\mathrm{t}} + \frac{h}{h^{(0)}E_{\mathrm{A}}^{(0)}}\sigma^{\mathrm{c}} + \alpha_{\mathrm{A}}^{(0)}\Delta T \\
&= -\frac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{t}} - \frac{h\nu_{\mathrm{A}}^{(90)}}{h^{(90)}E_{\mathrm{A}}^{(90)}}\sigma_{\mathrm{T}}^{\mathrm{c}} + \alpha_{\mathrm{A}}^{(90)}\Delta T,
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
\varepsilon_{\mathrm{T}}^{\mathrm{c}} &= -\frac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}}\sigma_{\mathrm{t}} - \frac{h\nu_{\mathrm{A}}^{(0)}}{h^{(0)}E_{\mathrm{A}}^{(0)}}\sigma^{\mathrm{c}} + \alpha_{\mathrm{T}}^{(0)}\Delta T \\
&= -\frac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}}\sigma_{\mathrm{t}} + \frac{h}{h^{(90)}E_{\mathrm{T}}^{(90)}}\sigma_{\mathrm{T}}^{\mathrm{c}} + \alpha_{\mathrm{T}}^{(90)}\Delta T,
\end{aligned}
\tag{A3}
$$

where the superscripts 0 and 90 refer the thermoelastic constants to the properties of the 0° and 90° plies, respectively. The parameters ε^{c} and $\varepsilon_{\mathrm{T}}^{\mathrm{c}}$ are, respectively, the uniform axial and transverse in-plane strains in the laminate when the cracks are just closed.

Regarding (A2) and (A3) as simultaneous algebraic equations for the unknowns σ^{c} and $\sigma_{\mathrm{T}}^{\mathrm{c}}$, it can be shown when $\sigma_{\mathrm{t}} = 0$ that

$$
\sigma^{\mathrm{c}} = P\Delta T, \quad \sigma_{\mathrm{T}}^{\mathrm{c}} = P_{\mathrm{T}}\Delta T, \tag{A4}
$$

where

$$
P = \frac{h^{(0)}E_{\mathrm{A}}^{(0)}}{h}\,\frac{\alpha_{\mathrm{A}}^{(90)} - \alpha_{\mathrm{A}}^{(0)} + \nu_{\mathrm{A}}^{(90)}\dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}(\alpha_{\mathrm{T}}^{(90)} - \alpha_{\mathrm{T}}^{(0)})}{1 - \nu_{\mathrm{A}}^{(0)}\nu_{\mathrm{A}}^{(90)}\dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}, \tag{A5}
$$

$$
P_{\mathrm{T}} = \frac{h^{(90)}E_{\mathrm{T}}^{(90)}}{h}\,\frac{\alpha_{\mathrm{T}}^{(0)} - \alpha_{\mathrm{T}}^{(90)} + \nu_{\mathrm{A}}^{(0)}(\alpha_{\mathrm{A}}^{(0)} - \alpha_{\mathrm{A}}^{(90)})}{1 - \nu_{\mathrm{A}}^{(0)}\nu_{\mathrm{A}}^{(90)}\dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}. \tag{A6}
$$

The corresponding values of the ply strains given by (A2) and (A3) are

$$
\varepsilon^{\mathrm{c}} = Q\Delta T, \quad \varepsilon_{\mathrm{T}}^{\mathrm{c}} = Q_{\mathrm{T}}\Delta T, \tag{A7}
$$

where

$$Q = \frac{\alpha_{\mathrm{A}}^{(90)} + \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}} (\alpha_{\mathrm{T}}^{(90)} - \alpha_{\mathrm{T}}^{(0)} - \nu_{\mathrm{A}}^{(0)} \alpha_{\mathrm{A}}^{(0)})}{1 - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}, \tag{A8}$$

$$Q_{\mathrm{T}} = \frac{\alpha_{\mathrm{T}}^{(0)} + \nu_{\mathrm{A}}^{(0)} (\alpha_{\mathrm{A}}^{(0)} - \alpha_{\mathrm{A}}^{(90)}) - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}} \alpha_{\mathrm{T}}^{(90)}}{1 - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}. \tag{A9}$$

Similarly, when $\Delta T = 0$, it can be shown that

$$\sigma^{\mathrm{c}} = R \sigma_{\mathrm{t}}, \quad \sigma_{\mathrm{T}}^{\mathrm{c}} = R_{\mathrm{T}} \sigma_{\mathrm{t}}, \tag{A10}$$

where

$$R = \frac{h^{(0)} E_{\mathrm{A}}^{(0)}}{h} \frac{\dfrac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}} - \dfrac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}} + \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}} \left(\dfrac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}} - \dfrac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}} \right)}{1 - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}, \tag{A11}$$

$$R_{\mathrm{T}} = \frac{h^{(90)} E_{\mathrm{T}}^{(90)}}{h} \frac{\dfrac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}} - \dfrac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}} + \nu_{\mathrm{A}}^{(0)} \left(\dfrac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}} - \dfrac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}} \right)}{1 - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}. \tag{A12}$$

The corresponding values of the ply strains given by (A2) and (A3) are

$$\varepsilon^{\mathrm{c}} = S \sigma_{\mathrm{t}}, \quad \varepsilon_{\mathrm{T}}^{\mathrm{c}} = S_{\mathrm{T}} \sigma_{\mathrm{t}}, \tag{A13}$$

where

$$S = \frac{\nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}} \left(\dfrac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}} - \dfrac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}} + \dfrac{\nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}} \right) - \dfrac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}{1 - \nu_{\mathrm{A}}^{(0)} \nu_{\mathrm{A}}^{(90)} \dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}, \tag{A14}$$

$$S_{\mathrm{T}} = \frac{\nu_{\mathrm{A}}^{(0)}\left(\dfrac{\nu_{\mathrm{a}}^{(90)}}{E_{\mathrm{A}}^{(90)}} - \dfrac{\nu_{\mathrm{a}}^{(0)}}{E_{\mathrm{A}}^{(0)}}\right) - \dfrac{\nu_{\mathrm{t}}^{(0)}}{E_{\mathrm{T}}^{(0)}} + \nu_{\mathrm{A}}^{(0)}\nu_{\mathrm{A}}^{(90)}\dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}\dfrac{\nu_{\mathrm{t}}^{(90)}}{E_{\mathrm{T}}^{(90)}}}{1 - \nu_{\mathrm{A}}^{(0)}\nu_{\mathrm{A}}^{(90)}\dfrac{E_{\mathrm{T}}^{(90)}}{E_{\mathrm{A}}^{(90)}}}. \tag{A15}$$

In general, it follows from (A2–A4), (A7), (A10), and (A13) that

$$\sigma^{\mathrm{c}} = P\Delta T + R\sigma_{\mathrm{t}}, \quad \sigma_{\mathrm{T}}^{\mathrm{c}} = P_{\mathrm{T}}\Delta T + R_{\mathrm{T}}\sigma_{\mathrm{t}}, \tag{A16}$$

$$\varepsilon^{\mathrm{c}} = Q\Delta T + S\sigma_{\mathrm{t}}, \quad \varepsilon_{\mathrm{T}}^{\mathrm{c}} = Q_{\mathrm{T}}\Delta T + S_{\mathrm{T}}\sigma_{\mathrm{t}}. \tag{A17}$$

Thus, having specified the values of σ_{t} and ΔT, the results (A16) and (A17) show how the ply crack closure stresses and strains may be calculated.

Appendix B: Interrelationships for Orthogonal Ply Cracks

On using the analysis given in Appendix A, the relations (7.139)–(7.144) are now expressed in the form

$$Ra + R_{\mathrm{T}}b + f = 0, \tag{B1}$$

$$Rc + R_{\mathrm{T}}d + a = 0, \tag{B2}$$

$$Rd + R_{\mathrm{T}}e + b = 0, \tag{B3}$$

$$Pa + P_{\mathrm{T}}b + x = 0, \tag{B4}$$

$$Pc + P_{\mathrm{T}}d + y = 0, \tag{B5}$$

$$Pd + P_{\mathrm{T}}e + z = 0, \tag{B6}$$

where

$$\begin{aligned} a &= \frac{\nu_{\mathrm{a}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{a}}}{E_{\mathrm{A}}}, \quad b = \frac{\nu_{\mathrm{t}}^{\mathrm{o}}}{E_{\mathrm{T}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{t}}}{E_{\mathrm{T}}}, \quad f = \frac{1}{E_{\mathrm{t}}} - \frac{1}{E_{\mathrm{t}}^{\mathrm{o}}}, \\ c &= \frac{1}{E_{\mathrm{A}}} - \frac{1}{E_{\mathrm{A}}^{\mathrm{o}}}, \quad d = \frac{\nu_{\mathrm{A}}^{\mathrm{o}}}{E_{\mathrm{A}}^{\mathrm{o}}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}, \quad e = \frac{1}{E_{\mathrm{T}}} - \frac{1}{E_{\mathrm{T}}^{\mathrm{o}}}, \\ x &= \alpha_{\mathrm{t}} - \alpha_{\mathrm{t}}^{\mathrm{o}}, \quad y = \alpha_{\mathrm{A}} - \alpha_{\mathrm{A}}^{\mathrm{o}}, \quad z = \alpha_{\mathrm{T}} - \alpha_{\mathrm{T}}^{\mathrm{o}}. \end{aligned} \tag{B7}$$

It is useful to define the following parameters:

$$P' = \frac{P}{PR_T - P_T R}, \quad P_T' = \frac{P_T}{PR_T - P_T R},$$
$$R' = \frac{R}{PR_T - P_T R}, \quad R_T' = \frac{R_T}{PR_T - P_T R}. \tag{B8}$$

Solving (B1) and (B4) for the parameters a and b leads to

$$a = P_T' f - R_T' x, \quad b = R' x - P' f. \tag{B9}$$

Solving (B2) and (B5) for the parameters c and d leads to

$$c = P_T' a - R_T' y, \quad d = R' y - P' a. \tag{B10}$$

Solving (B1) and (B4) for the parameters d and e leads to

$$d = P_T' b - R_T' z, \quad e = R' z - P' b. \tag{B11}$$

On eliminating x, y, and z using (B9)–(B11), respectively, it can be shown that

$$R + R_T \frac{b}{a} + \frac{f}{a} = 0,$$
$$R + R_T \frac{d}{c} + \frac{a}{c} = 0, \tag{B12}$$
$$R + R_T \frac{e}{d} + \frac{b}{d} = 0.$$

The relations (B12) must be satisfied for all possible damage states, where R and R_T are known laminate constants (see (A11) and (A12)); and in view of the relation (7.145), it is deduced that

$$\frac{b}{a} = \frac{d}{c} = \frac{e}{d} = k, \quad \frac{f}{a} = \frac{a}{c} = \frac{b}{d} = k', \tag{B13}$$

where the parameters k and k' are given by (7.110) and (7.117), respectively.

Since from (B13) $b = ka$, it follows from (B9) that

$$R' x - P' f = kP_{\mathrm{T}}' f - kR_{\mathrm{T}}' x, \tag{B14}$$

so that on using (B8)

$$(P + kP_{\mathrm{T}}) f = (R + kR_{\mathrm{T}}) x. \tag{B15}$$

It then follows from (7.145) and (7.146) that

$$x = \frac{k_1}{k'} f, \tag{B16}$$

where k_1 is a laminate constant defined by (7.104). Similarly, it can be shown, using (B10), (B11), and (B13), that

$$y = \frac{k_1}{k'} a, \tag{B17}$$

$$z = \frac{k_1}{k'} b. \tag{B18}$$

It follows from (B13) and (B16–B18) that

$$\frac{b}{a} = \frac{d}{c} = \frac{e}{d} = \frac{z}{y} = k, \quad \frac{f}{a} = \frac{a}{c} = \frac{b}{d} = \frac{x}{y} = k', \tag{B19}$$

and it then follows that

$$\frac{x}{a} = \frac{y}{c} = \frac{z}{d} = k_1. \tag{B20}$$

On using (B7), it is clear that the relations (B19) and (B20) are precisely the relations (7.103), (7.111), and (7.118) that were derived for the case of ply cracking only in the 90° plies. The analysis of this appendix thus demonstrates that the interrelationships (7.103), (7.111), and (7.118) are valid also for orthogonal systems of ply cracks in crossply laminates.