16

COMPOSITE BEAMS IN TORSION

As in the previous chapter, we consider here the composite beams made of isotropic phases. Extension to transversely isotropic phases is straightforward. The study of orthotropic phases with one principal direction parallel to the axis of the beam, the other two principal directions in the plane of a cross section, does not present fundamental difficulties.

16.1 UNIFORM TORSION

We will keep the conventions and notations of the previous chapter. On Figure 16.1, 0 is the **elastic center**, x,y, and z are the **principal axes**. The beam is slender and uniformly twisted, this means that every cross section is subjected to a pure and constant torsion moment, along the x axis, denoted as M_x.

Then, under the application of this moment, each line in the beam, initially parallel to the x axis, becomes a **helicoid curve**, including (in the absence of symmetry in the cross section) the line which, initially, was coinciding with the elastic x axis itself. The only line which remains rectilinear is cutting the plane of all sections at a point which will be called **torsion center** and denoted as C, with coordinates y_C and z_C in the principal axes (see Figure 16.1).

16.1.1 Torsional Degree of Freedom

By definition, this is the rotation of each section about the x axis, denoted as θ_x.[1] The torsional moment M_x being constant, the angle θ_x evolves along the x axis in such a manner that, for any pair of cross sections spaced with a distance dx, one can observe a same increment of rotation $d\theta_x$; then:

$$\boxed{\frac{d\theta_x}{dx} = \text{constant}}$$

[1] Here it is not necessary to define the rotation θ_x by means of an integral of displacements, as in the previous chapter relating to flexure. In effect, we will see in the following that the displacement field associated with this pure rotation of the sections leads to the exact solution of the problem in the elastic domain (at least for the case of uniform warping).

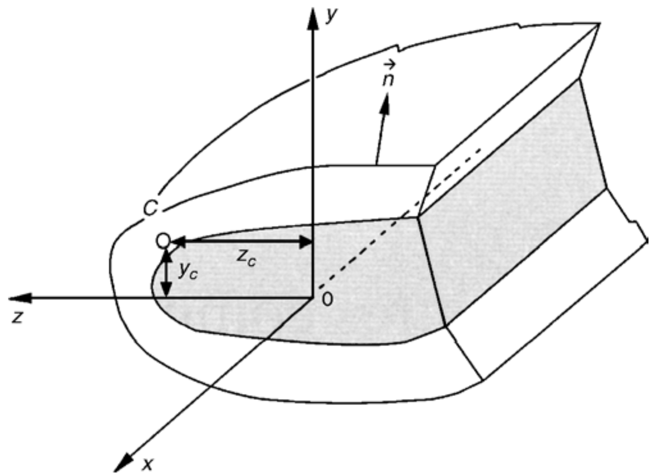

Figure 16.1 Elastic Center (O), Torsion Center (C), and Principal Axes

From this it comes that the angle of rotation of the sections varies linearly along the longitudinal axis x. As a consequence, we assume *a priori* the components of the displacement field u_x, u_y, u_z, to be written as:

$$\boxed{\begin{aligned} u_x &= \frac{d\theta_x}{dx} \times \varphi(y,z) \\ u_y &= -(z - z_c)\theta_x \\ u_z &= (y - y_c)\theta_x \end{aligned}} \tag{16.1}$$

where the function denoted as $\varphi(y,z)$ is characteristic of the cross section shape and of the materials that constitute the section. This is called the **warping function** for torsion.

16.1.2 Constitutive Relation

With the displacement field in Equation 16.1 the only nonzero strains are written as:

$$\gamma_{xy} = \frac{d\theta_x}{dx}\left(\frac{\partial\varphi}{\partial y} - (z - z_c)\right)$$

$$\gamma_{xz} = \frac{d\theta_x}{dx}\left(\frac{\partial\varphi}{\partial z} + (y - y_c)\right)$$

The only nonzero stresses are then the shear stresses τ_{xy} and τ_{xz}. The torsional moment can be deduced by integration over the domain of the straight section as:

$$M_x = \int_D (y\tau_{xz} - z\tau_{xy})dS = \frac{d\theta_x}{dx}\int_D G_i\left\{y\left(\frac{\partial\varphi}{\partial z} - y_c\right)\cdots\right.$$

$$\left.\cdots - z\left(\frac{\partial\varphi}{\partial y} + z_c\right) + y^2 + z^2\right\}dS$$

Substituting to the function $\varphi(y,z)$ the function $\Phi(y,z)$ such that:

$$\boxed{\Phi(y,\ z)\ =\ \varphi(y,z) + yz_c - zy_c} \tag{16.2}$$

it becomes:

$$M_x\ =\ \frac{d\theta_x}{dx} \times \int_D G_i\left(y\frac{\partial\Phi}{\partial z} - z\frac{\partial\Phi}{\partial y} + y^2 + z^2\right)dS$$

In this expression, it is possible to define an **equivalent stiffness in torsion** with the form:

$$\boxed{\langle GJ\rangle\ =\ \int_D G_i\left(y\frac{\partial\Phi}{\partial z} - z\frac{\partial\Phi}{\partial y} + y^2 + z^2\right)\ dS} \tag{16.3}$$

One obtains then for the constitutive relation:

$$M_x\ =\ \langle GJ\rangle\frac{\partial\theta_x}{\partial x}$$

16.1.3 Determination of the Function $\Phi(y,z)$

16.1.3.1 Local Equilibrium

Local equilibrium is written as:

$$\frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\ =\ 0$$

then with the displacement field in Equation 16.1:

$$\nabla^2\varphi = 0$$

and with form 16.2 of the function Φ:

$$\nabla^2\Phi = 0$$

16.1.3.2 Conditions at the External Boundary

The lateral surface being free of stresses, one can write along the external boundary ∂D:

$$\vec{\tau}\ .\ \vec{n}\ =\ 0\,.$$

With the displacement field in Equation 16.1:

$$\left\{\frac{\partial\phi}{\partial y} - (z - z_c)\right\}n_y + \left\{\frac{\partial\phi}{\partial z} + (y - y_c)\right\}n_z\ =\ 0$$

then again:

$$\frac{\partial\Phi}{\partial y}n_y + \frac{\partial\Phi}{\partial z}n_z = zn_y - yn_z$$

16.1.3.3 Conditions at the Internal Boundaries

The continuity conditions of Section 15.1.2 are verified for u_y and u_z. At an interfacial line l_{ij} between two phases i and j, the continuity of u_x leads to

$$\Phi_i = \Phi_j$$

The continuity relations in Equation 15.6 lead to the continuity of $(\tau_{xy}n_y + \tau_{xz}\, n_z)$ when crossing the lines l_{ij}, that produces the continuity of

$$G_i\left(\frac{\partial\Phi_i}{\partial y} - z\right)n_y + G_i\left(\frac{\partial\Phi_i}{\partial z} + y\right)n_z$$

16.1.3.4 Uniqueness of the Function Φ

If one superimposes torsion and bending, by using the degrees of freedom for flexure defined in the previous chapter, the displacement component u_x becomes:

$$u_x = u - y\theta_z + z\theta_y + \frac{d\theta_x}{dx}\varphi + \eta_x$$

The longitudinal displacement $u(x)$ has to respond to its definition (Section 15.1.1), meaning

$$u = \frac{1}{\langle ES\rangle}\int_D E_i u_x\ dS$$

$$u = \frac{1}{\langle ES\rangle}\left\{u\int_D E_i\ dS - \theta_z\int_D \cancel{E_i y\ dS} + \theta_y\int_D \cancel{E_i z\ dS}\cdots\right.$$

$$\left.\cdots + \frac{d\theta_x}{dx}\int_D E_i\varphi\ dS + \int_D \cancel{E_i\eta_x\ dS}\right\}$$

This requires that:

$$\int_D E_i\varphi\ dS = 0$$

Then, taking into account the form in Equation 16.2 of **Φ** and the properties of the elastic center:

$$\int_D E_i\Phi\ dS = 0$$

In summary, the function $\Phi(y,z)$ is the solution of the problem:

$$\begin{cases} \nabla^2\Phi = 0 & \text{in domain } D \text{ of the section} \\ \dfrac{\partial\Phi}{\partial n} = zn_y - yn_z & \text{on the external boundary} \partial D \end{cases}$$

with the internal continuity:

$$\left.\begin{aligned} \Phi_i &= \Phi_j \\ G_i\left(\frac{\partial\Phi_i}{\partial n} - (zn_y + yn_z)\right) &= G_j\left(\frac{\partial\Phi_j}{\partial n} - (zn_y + yn_z)\right) \end{aligned}\right\} \text{along the internal boundaries } \ell_{ij}$$

and the condition of uniqueness:

$$\int_D E_i\Phi\, dS = 0$$

16.1.4 Energy Interpretation

The strain energy of an elementary segment of a beam with thickness dx is written as:

$$dW = \frac{1}{2}\int 2(\tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz})dV = \left\{\frac{1}{2}\int_D G_i(\gamma_{xy}^2 + \gamma_{xz}^2)\, dS\right\} dx$$

then, taking into account the displacement field in Equation 16.1:

$$\frac{dW}{dx} = \frac{1}{2}\left(\frac{d\theta_x}{dx}\right)^2 \int_D G_i\left\{\left(\frac{\partial\Phi}{\partial y} - z\right)^2 + \left(\frac{\partial\Phi}{\partial z} + y\right)^2\right\} dS$$

which can be rewritten as[2]:

$$\frac{dW}{dx} = \frac{1}{2}\left(\frac{d\theta_x}{dx}\right)^2 \left\{\int_D G_i\left\{y\frac{\partial\Phi}{\partial z} - z\frac{\partial\Phi}{\partial y} + y^2 + z^2\right\} dS - \int_D G_i\Phi\cancel{\nabla^2\Phi}\, dS \ldots\right.$$

$$\left.\cdots + \int_{\partial D} G_i\Phi\left\{\cancel{\left(\frac{\partial\Phi}{\partial y} - z\right)n_y + \left(\frac{\partial\Phi}{\partial z} + y\right)n_z}\right\} d\Gamma\right\}$$

[2] In effect, one has, for example:

$$\left(\frac{\partial\Phi}{\partial y}\right)^2 - z\frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial y}\left(\frac{\partial\Phi}{\partial y} - z\right) = \frac{\partial}{\partial y}\left\{\Phi\left(\frac{\partial\Phi}{\partial y} - z\right)\right\} - \Phi\frac{\partial^2\Phi}{\partial y^2}$$

where we note the presence of the stiffness in torsion $\langle GJ \rangle$ defined by Equation 16.3. Thus,

$$\frac{dW}{dx} = \frac{1}{2}\langle GJ \rangle \left(\frac{d\theta_x}{dx}\right)^2 \quad \text{or} = \frac{1}{2}\frac{M_x^2}{\langle GJ \rangle}$$

16.2 LOCATION OF THE TORSION CENTER

Consider the cantilever beam that is clamped at its left end as shown schematically in Figure 16.2, and more particularly the segment limited by the cross sections denoted by D_0 and D_1. In the section D_1,0 is the elastic center and C is the torsion center the position of which we wish to determine.

With this objective, we will apply on the cross section D_1 the two following successive loadings:

- **Loading No. 1:** One applies on the torsion center C of the cross section D_1 a force $\vec{F}$ situated in the plane of the section.
- **Loading No. 2:** One applies on the same cross section D_1 a torsional moment denoted as M_x (see Figure 16.2).

When one applies these two loads successively, the final state is independent of the order of the application. As a consequence for the external forces acting on the isolated segment $(D_0 D_1)$, the work corresponding to loading No. 1 on the displacements created by loading No. 2 is equal to the work corresponding to loading No. 2 on the displacements created by loading No. 1. This can be written in the following form:

$$W_{(\text{loading } 1 \times \text{displacement } 2)} = W_{(\text{loading } 2 \times \text{displacement } 1)}$$

Now we evaluate these works:

a) W (loading 1 × displacement 2)

- On D_0: $\vec{F}$ creates the bending moments M_z and M_y, thus a normal stress distribution given in the principal axes by Equation 15.17 as:

$$(\sigma_{xx})_1 = -E_i \frac{M_z}{\langle EI_z \rangle} \times y + E_i \frac{M_y}{\langle EI_y \rangle} \times z$$

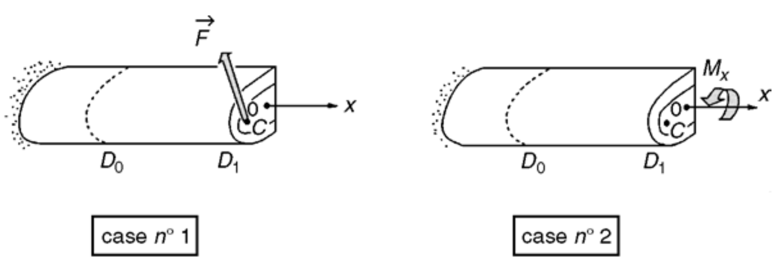

Figure 16.2 Cantilever Beam with Two Successive Loadings

Then, taking into account the displacement field in Equation 16.1, the work done on D_0 is

$$\int_D (\sigma_{xx})_1 \times (u_x)_2 \, dS = \int_D \left\{ -E_i \frac{M_z}{\langle EI_z \rangle} \times y + E_i \frac{M_y}{\langle EI_y \rangle} \times z \right\} \frac{d\theta_x}{dx} \varphi \, dS$$

$$= \frac{d\theta_x}{dx} \int_D \left\{ -E_i \frac{M_z}{\langle EI_z \rangle} \times y + E_i \frac{M_y}{\langle EI_y \rangle} \times z \right\} (\Phi - y z_c + z y_c) dS$$

- On D_1: The torsion center C does not move in the plane of the cross section during torsion. The work done by the force $\vec{F}$ in the displacement field of torsion is nil.

b) W (loading 2 × displacement 1)

Force $\vec{F}$ as applied to the torsion center C does not lead to the rotation of the cross sections around the longitudinal axis x. From this the torsional moment M_x does not work on the bending displacement field due to $\vec{F}$.

The equality of the two works is then written as:

$$\frac{d\theta_x}{dx} \int_D \left\{ -E_i \frac{M_z}{\langle EI_z \rangle} \times y + E_i \frac{M_y}{\langle EI_y \rangle} \times z \right\} (\Phi - y z_c + z y_c) dS = 0$$

then:

$$\frac{M_z}{\langle EI_z \rangle} \int_D (E_i y \; \Phi - E_i y^2 z_c + \cancel{E_i y z y_c}) dS \cdots$$

$$\cdots + \frac{M_y}{\langle EI_y \rangle} \int_D (E_i z \; \Phi + E_i z^2 y_c - \cancel{E_i y z z_c}) ds = 0$$

This relation has to be verified when the force applied at C varies in magnitude and direction in the plane of the section. One can deduce from there that the relation is valid no matter what the values of M_z and M_y are. Both the above integrals are then nil. One extracts from this property the coordinates of the torsion center:

$$y_c = -\frac{1}{\langle EI_y \rangle} \int_D E_i z \, \Phi \, dS$$

$$z_c = \frac{1}{\langle EI_z \rangle} \int_D E_i y \; \Phi \; dS$$

In summary, the uniform torsion of a cylindrical composite beam made of perfectly bonded isotropic phases can be characterized by a homogenized formulation—equivalent to that of a classical homogeneous beam—in the following manner:

- ***degree of freedom:*** *about x axis:* θ_x
- ***elastic center 0:*** *it is such that* $\int_D E_i y \, dS = \int_D E_i z \, dS = 0$
- ***principal axes:*** *they are such that* $\int_D E_i yz \, dS = 0$
- ***equivalent stiffnesses:***

$$\langle EI_z \rangle = \sum_i E_i I_{zi} \quad ; \quad \langle EI_y \rangle = \sum_i E_i I_{yi}$$

$$\langle GJ \rangle = \int_D G_i \left(y \frac{\partial \Phi}{\partial z} - z \frac{\partial \Phi}{\partial y} + y^2 + z^2 \right) dS$$

- ***torsion center:*** *coordinates in principal axes:*

$$y_c = -\frac{1}{\langle EI_y \rangle} \int_D E_i z \, \Phi \, dS$$

$$z_c = \frac{1}{\langle EI_z \rangle} \int_D E_i y \, \Phi \, dS$$

- ***equilibrium relation:*** $\dfrac{dM_x}{dx} = 0 \quad (M_x = constant)$
- ***constitutive relation:*** $M_x = \langle GJ \rangle \dfrac{d\theta_x}{dx}$
- ***shear stresses***

$$\tau_{xy} = G_i \frac{d\theta_x}{dx} \left(\frac{\partial \Phi}{\partial y} - z \right)$$

$$\tau_{xz} = G_i \frac{d\theta_x}{dx} \left(\frac{\partial \Phi}{\partial z} + y \right)$$

- ***function Φ(y, z):*** *it is the solution to the problem:*

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial y^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 & \text{in domain } D \text{ of the section.} \\ \dfrac{\partial \Phi}{\partial n} = z n_y - y n_z & \text{on the external boundary } \partial D. \end{cases}$$

with internal continuity:

$$\left. \begin{aligned} \Phi_i &= \Phi_j \\ G_i \left(\frac{\partial \Phi_i}{\partial n} - z n_y + y n_z \right) &= G_j \left(\frac{\partial \Phi_j}{\partial n} - z n_y + y n_z \right) \end{aligned} \right\} \text{along internal boundaries } \ell_{ij}$$

and the uniqueness condition: $\int_D E_i \Phi \, dS = 0$

- ***strain energy density***

$$\frac{dW}{dx} = \frac{1}{2} \frac{M_x^2}{\langle GJ \rangle}$$

(16.4)

Remarks:

- A finite element computer program for the classical homogeneous beams is usable[3] with the condition that the equivalent rigidity in torsion $\langle GJ \rangle$ can be available. This requires the numerical calculation of the function Φ.[4] The latter is the solution of a Laplace type problem, which can be noted in relations 16.4. An equivalent functional is possible to define, which leads to the calculation of Φ by the finite element method, by means of discretization of the cross section.
- **Flexion-torsion coupling:** When, due to the loads applied on the beam, there exists simultaneously bending and torsion of the beam, the approach of the previous chapter is always valid. Keeping the definitions in Sections 15.1.1 and 15.2 for the degrees of freedom u, v, θ_x, θ_y, one arrives at the following displacement field:

$$\begin{cases} u_x = u - y\theta_z + z\theta_y + \varphi \dfrac{d\theta_x}{dx} + \eta_x \\ u_y = v - z\theta_x + \eta_y \\ u_z = w + y\theta_x + \eta_z \end{cases}$$

The torsion being uniform, the equilibrium relations in 15.19 become more restrictive and can be reduced to:

$$\boxed{\begin{array}{lll} \dfrac{dN_x}{dx} = 0; & \dfrac{dT_y}{dx} = 0; & \dfrac{dT_z}{dx} = 0 \\ \dfrac{dM_x}{dx} = 0; & \dfrac{dM_z}{dx} + T_y = 0; & \dfrac{dM_y}{dx} - T_z = 0 \end{array}} \qquad (16.5)$$

[3] Except if the considered application requires the calculation of the shear stresses in a cross section.

[4] One has to solve an analogous problem for the homogeneous beams, when one desires to calculate the torsional Saint-Venant stiffness:

$$J = \int_D \left(y\frac{\partial \Phi}{\partial z} - z\frac{\partial \Phi}{\partial y} + y^2 + z^2 \right) dS .$$

Taking into account six degrees of freedom also leads to six constitutive relations. One finds[5]:

$$
\boxed{\begin{aligned}
N_x &= \langle ES\rangle \frac{du}{dx} \\
T_y &= \frac{\langle GS\rangle}{k_y}\left(\frac{dv}{dx} - \theta_z - z_c\frac{d\theta_x}{dx}\right)(*) \\
T_z &= \frac{\langle GS\rangle}{k_z}\left(\frac{dw}{dx} + \theta_y + y_c\frac{d\theta_x}{dx}\right)(*) \\
M_x &= \langle GJ\rangle \frac{d\theta_x}{dx} - z_c T_y + y_c T_z \\
M_y &= \langle EI_y\rangle \frac{d\theta_y}{dx} \\
M_z &= \langle EI_z\rangle \frac{d\theta_z}{dx}
\end{aligned}} \tag{16.6}
$$

[5] In each of the relations marked with (*), there appears a supplementary coupling term connected to the existence of a third coefficient denoted as k_{yz}. The complete form is then:

$$
\begin{aligned}
k_y T_y + k_{yz} T_z &= \langle GS\rangle\left(\frac{dv}{dx} - \theta_z - z_c\frac{d\theta_x}{dx}\right) \\
k_{yz} T_y + k_z T_z &= \langle GS\rangle\left(\frac{dw}{dx} + \theta_y + y_c\frac{d\theta_x}{dx}\right)
\end{aligned}
$$

This secondary coupling has been neglected in the indicated form.