15

COMPOSITE BEAMS IN FLEXURE

Due to their slenderness, a number of composite elements (mechanical components or structural pieces) can be considered as beams. A few typical examples are shown schematically in Figure 15.1. The study of the behavior under loading of these elements (evaluation of stresses and displacements) becomes a very complex problem when one gets into three-dimensional aspects. In this chapter, we propose a monodimensional approach to the problem in an original method. It consists of the definition of displacements corresponding to the traditional stress and moment resultants for the applied loads. This leads to a **homogenized** formulation for the flexure—and for torsion. This means that the equilibrium and behavior relations are formally identical to those that characterize the behavior of classical homogeneous beams. Utilization of these relations for the calculation of stresses and displacements then leads to expressions that are analogous to the common beams.

We will limit ourselves to the composite beams with constant characteristics (geometry, materials) in any cross section, made of different materials—which we call **phases**—that are assumed to be perfectly bonded to each other.

To clarify the procedure and for better simplicity in the calculations, we will limit ourselves in this chapter to the case of composite beams with isotropic phases. The extension to the transversely isotropic materials is immediate. When the phases are orthotropic, with eventually orthotropic directions that are changing from one point to another in the section, the study will be analogous, with a much more involved formulation.[1]

15.1 FLEXURE OF SYMMETRIC BEAMS WITH ISOTROPIC PHASES

In the following, D symbolizes the domain occupied by the cross section, in the y,z plane. The **external** frontier is denoted as ∂D. One distinguishes also (see Figure 15.2) the internal frontiers which limit the phases, denoted by l_{ij} for two contiguous phases i and j. The area of the phase i is denoted as S_i; its moduli of elasticity are denoted by E_i and G_i. The elastic displacement at any point of the beam has the components: u_x (x,y,z); u_y (x,y,z); u_z (x,y,z).

The beam is bending in the plane of symmetry x,y under external loads which are also symmetric with respect to this plane.

[1] The only restrictive condition lies in the fact that one of the orthotropic directions is supposed to remain parallel to the longitudinal axis of the beam.

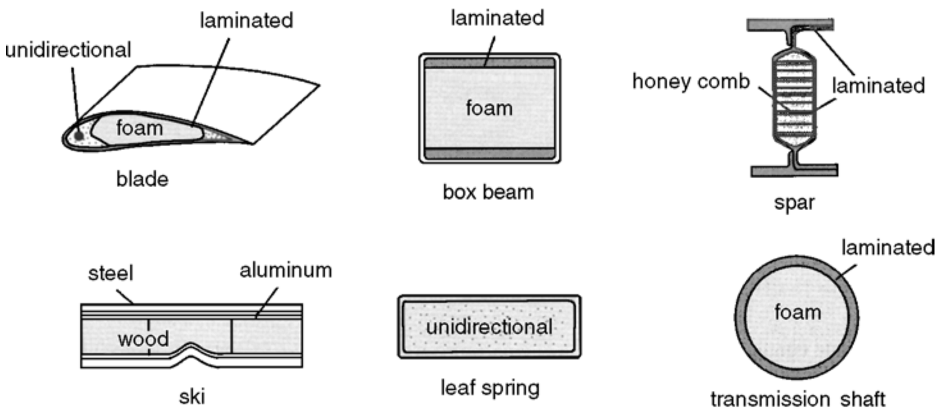


Figure 15.1 Composite Beams

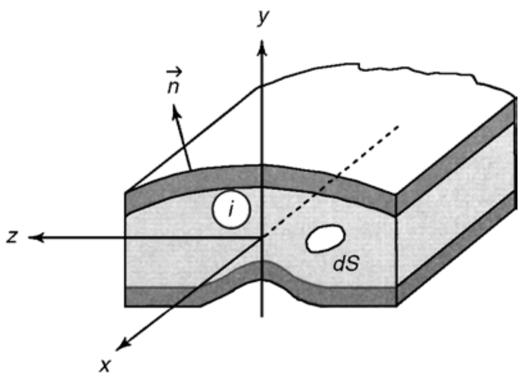


Figure 15.2 Composite Beam, with Plane of Symmetry

15.1.1 Degrees of Freedom

15.1.1.1 Equivalent Stiffness

One writes in condensed form the following integrals taken over the total cross section[2]:

$$\langle ES \rangle = \int_D E_i dS \quad \text{or} \quad = \sum_{i}^{\text{number of phases}} E_i S_i$$

$$\langle EI_z \rangle = \int_D E_i y^2 dS \quad \text{or} \quad = \sum_{i}^{\text{number of phases}} E_i I_{zi} \qquad (15.1)$$

$$\langle GS \rangle = \int_D G_i dS \quad \text{or} \quad = \sum_{i}^{\text{number of phases}} G_i S_i$$

[2] I_{zi} is the quadratic moment of the Phase i with respect with z axis.

15.1.1.2 Longitudinal Displacement

By definition, longitudinal displacement is denoted by $u(x)$ and written as:

$$u(x) = \frac{1}{\langle ES \rangle}\int_D E_i u_x(x, y, z)\, dS$$

which consists of a mean displacement $u(x)$ and an incremental displacement Δu_x as:

$$u_x(x, y, z) = u(x) + \Delta u_x(x, y, z)$$

where one notes that:

$$\int_D E_i \Delta u_x \, dS = 0 \qquad (15.2)$$

15.1.1.3 Rotations of the Sections

By definition, this is the fictitious rotation—or equivalent—given by the following expression:

$$\theta_z(x) = \frac{-1}{\langle EI_z \rangle}\int_D E_i u_x(x, y, z) \times y \, dS$$

Or, with the above:

$$\theta_z(x) = \frac{-1}{\langle EI_z \rangle}\left\{ u(x)\int_D E_i y\, dS + \int_D E_i \Delta u_x(x, y, z) y \, dS \right\}$$

15.1.1.4 Elastic Center

Origin 0 of the coordinate y is chosen such that the following integral is zero:

$$\int_D E_i y \, dS = 0$$

We call **elastic center** the corresponding point 0 that is located as in the expression above. Then Δu_x takes the form:

$$\Delta u_x(x, y, z) = -y\theta_z(x) + \eta_x(x, y, z)$$

with[3]:

$$\int_D E_i \eta_x y \, dS \quad \text{and} \quad \int_D E_i \eta_x \, dS = 0 \quad (*)$$

The displacement u_x (x,y,z) can then take the form:

$$u_x(x, y, z) = u(x) - y\theta_z(x) + \eta_x(x, y, z).$$

[3] In what follows, the second property is the consequence of Equation 15.2.

15.1.1.5 Transverse Displacement along y Direction

By definition, this is $v(x)$ that is given by the following expression:

$$v(x) = \frac{1}{\langle GS \rangle}\int_D G_i u_y(x,y,z)\,dS$$

It follows from this definition that:

$$u_y(x,y,z) = v(x) + \eta_y(x,y,z)$$

where one notes that:

$$\int_D G_i \eta_y\,dS = 0.$$

15.1.1.6 Transverse Displacement along z Direction

By definition, this is $w(x)$ given by

$$w(x) = \frac{1}{\langle GS \rangle}\int_D G_i u_z(x,y,z)\,dS$$

It follows from this definition and from the existence of the plane of symmetry x,y of the beam a zero average transverse displacement, as: $w(x) = 0$.

$$u_z(x,y,z) = 0 + \eta_z(x,y,z), \qquad \text{with} \qquad \int_D G_i \eta_z\,dS = 0.$$

In summary, we obtain the following elastic displacement field:

$$\begin{cases} u_x = u(x) - y\theta_z(x) + \eta_x(x,y,z) \\ u_y = v(x) + \eta_y(x,y,z) \\ u_z = \eta_z(x,y,z) \end{cases} \tag{15.3}$$

The origin of the axes is the elastic center such that:

$$\int_D E_i y\,dS = 0 \tag{15.4}$$

The three-dimensional incremental displacements η_x, η_y, η_z with respect to the unidimensional approximation u, v, θ_z verify the following:

$$\begin{aligned} &\int_D E_i \eta_x\,dS = \int_D E_i y \eta_x\,dS = 0 \\ &\int_D G_i \eta_y\,dS = 0 \\ &\int_D G_i \eta_z\,dS = 0 \end{aligned} \tag{15.5}$$

Remarks:

- η_x represents the **longitudinal distortion** of a cross section, that is, the quantity that this section displaces **out of the plane** which characterizes it if it moves truly as a rigid plane body.
- η_y and η_z represent the displacements that characterize the variations of the form of the cross section in its initial plane.

15.1.2 Perfect Bonding between the Phases

15.1.2.1 Displacements

The bonding is assumed to be perfect. Then the displacements are continuous when crossing through the interface between two phases in contact. For two phases in contact i and j, one has:

$$u_x \textcircled{i} = u_x \textcircled{j}$$
$$u_y \textcircled{i} = u_y \textcircled{j}$$
$$u_z \textcircled{i} = u_z \textcircled{j}$$

15.1.2.2 Strains

For the phases i and j in Figure 15.3, in the plane of an elemental interface with a normal vector of $\vec{n}$, the relations between the strain tensors ε are:

$$\vec{x} \cdot \varepsilon(\vec{x})_{\textcircled{i}} = \vec{x} \cdot \varepsilon(\vec{x})_{\textcircled{j}}$$

$$\vec{t} \cdot \varepsilon(\vec{x})_{\textcircled{i}} = \vec{t} \cdot \varepsilon(\vec{x})_{\textcircled{j}}$$

$$\vec{t} \cdot \varepsilon(\vec{t})_{\textcircled{i}} = \vec{t} \cdot \varepsilon(\vec{t})_{\textcircled{j}}$$

which can also be written as:

$$\underset{\textcircled{i}}{\varepsilon_{xx}} = \underset{\textcircled{j}}{\varepsilon_{xx}}$$
$$-\underset{\textcircled{i}}{\varepsilon_{xy}} n_z + \underset{\textcircled{i}}{\varepsilon_{xz}} n_y = -\underset{\textcircled{j}}{\varepsilon_{xy}} n_z + \underset{\textcircled{j}}{\varepsilon_{xz}} n_y$$
$$\underset{\textcircled{i}}{\varepsilon_{yy}} n_z^2 - 2\underset{\textcircled{i}}{\varepsilon_{yz}} n_y n_z + \underset{\textcircled{i}}{\varepsilon_{zz}} n_y^2 = \underset{\textcircled{j}}{\varepsilon_{yy}} n_z^2 - 2\underset{\textcircled{j}}{\varepsilon_{yz}} n_y n_z + \underset{\textcircled{j}}{\varepsilon_{zz}} n_y^2$$

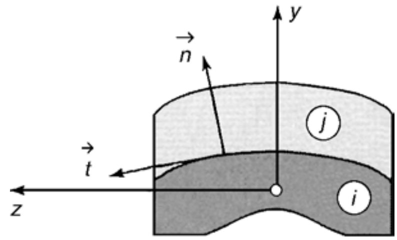


Figure 15.3 Interface between Two Phases

15.1.2.3 Stresses

The stress vector $\vec{\sigma} = \Sigma(\vec{n})$, where Σ represents the stress tensor, remains continuous across an element of the interface with normal $\vec{n}$ as:

$$\begin{array}{|c|}\hline \underset{(i)}{\tau_{xy}}n_y + \underset{(i)}{\tau_{xz}}n_z = \underset{(j)}{\tau_{xy}}n_y + \underset{(j)}{\tau_{xz}}n_z \\ \underset{(i)}{\sigma_{yy}}n_y + \underset{(i)}{\tau_{yz}}n_z = \underset{(j)}{\sigma_{yy}}n_y + \underset{(j)}{\tau_{yz}}n_z \\ \underset{(i)}{\tau_{yz}}n_y + \underset{(i)}{\sigma_{zz}}n_z = \underset{(j)}{\tau_{yz}}n_y + \underset{(j)}{\sigma_{zz}}n_z \\ \hline \end{array} \tag{15.6}$$

15.1.3 Equilibrium Relations

Starting from the local equilibrium, in the absence of body forces, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

By integrating over the cross section:

$$\frac{d}{dx}\int_D \sigma_{xx} dS + \int_D \left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dS = 0$$

where **the normal stress resultant** N_x appears as:

$$N_x = \int_D \sigma_{xx}\, dS \,.$$

Then, transforming the second integral to an integral over the frontier ∂D of D[4]:

$$\frac{dN_x}{dx} + \int_{\partial D} (\tau_{xy} n_y + \tau_{xz} n_z) d\Gamma = 0$$

[4] Note that equality $\int_D (\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}) dS = \int_{\partial D} (\tau_{xy} n_y + \tau_{xz} n_z) d\Gamma$ is made possible due to the continuity of the expression $(\tau_{xy}\ n_y + \tau_{xz}\ n_z)$ across the interfaces between the different phases (see Equation 15.6).

in which n_y and n_z are the cosines of the outward normal $\vec{n}$, and $d\Gamma$ represents element of frontier ∂D. If one assumes the absence of shear stresses applied over the lateral surface of the beam, then $\tau_{xy}\, n_y + \tau_{xz}\, n_z = 0$ along the external frontier ∂D. Then for longitudinal equilibrium we have[5]

$$\frac{dN_x}{dx} = 0$$

$$\frac{d}{dx}\int_D \tau_{xy} dS + \int_D \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) dS = 0$$

where one recognizes the **shear stress resultant:**

$$T_y = \int_D \tau_{xy} dS.$$

Then transforming the second integral into an integral over the external frontier ∂D of the domain D of the cross section[6]:

$$\frac{\partial T_y}{\partial x} + \int_{\partial D} (\sigma_{yy} n_y + \tau_{yz} n_z) d\Gamma = 0$$

if one remarks that:

$$\int_{\partial D} (\sigma_{yy} n_y + \tau_{yz} n_z) d\Gamma = \int_{\partial D} \vec{y} \cdot \Sigma(\vec{n}) d\Gamma = \vec{y} \cdot \int_{\partial D} \vec{\sigma} d\Gamma = p_y \,(\text{N/m})$$

which is the transverse density of loading on the lateral surface of the beam, transverse equilibrium can be written as:

$$\frac{dT_y}{dx} + p_y = 0$$

$$\frac{d}{dx}\int_D -y\sigma_{xx} dS + \int_D -y\left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) dS = 0$$

5 We have neglected the body forces which appear in the local equation of equilibrium in the form of a function f_x. If these exist (inertia forces, centrifugal forces, or vibration inertia, for example), one obtains for the equilibrium: $\frac{dN_x}{dx} + p_x = 0$ in which $p_x = \int_D f_x dS$ represents the longitudinal load density.

6 Note that the equality $\int_D (\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}) dS = \int_{\partial D} (\sigma_{yy} n_y + \tau_{yz} n_z) d\Gamma$ is made possible due to the continuity of the expression $(\sigma_{yy}\, n_y + \tau_{yz}\, n_z)$ across the frontier lines between different phases (see Equation 15.6).

where appears the **moment resultant:**

$$M_z = \int_D -y\sigma_{xx} dS.$$

Then transforming the second integral[7]:

$$\frac{dM_z}{dx} + \int_{\partial D} -y(\tau_{xy}n_y + \tau_{xz}n_z)d\Gamma + \int_D \tau_{xy} dS = 0$$

where one notes that:

$$\int_{\partial D} -y(\tau_{xy}n_y + \tau_{xz}n_z)d\Gamma = \int_{\partial D} -y\vec{x}\cdot\Sigma(\vec{n})d\Gamma\cdots = \int_{\partial D} -y(\vec{\sigma}\cdot\vec{x})d\Gamma = \mu_{z(\mathrm{mN/m})}$$

which can be called a density moment on the beam. Then one obtains the equilibrium relation:

$$\frac{dM_z}{dx} + T_y + \mu_z = 0$$

The case where a density moment could exist in statics is practically nil, we therefore assume that $\mu_z = 0$. In summary, one obtains for the equations of equilibrium:

$$\boxed{\begin{aligned} \frac{dN_x}{dx} &= 0 \\ \frac{dT_y}{dx} + p_y &= 0 \\ \frac{dM_z}{dx} + T_y &= 0 \end{aligned}} \tag{15.7}$$

15.1.4 Constitutive Relations

Taking into account the isotropic nature of the different phases, the constitutive relation can be written in tensor form for Phase i as:

$$\varepsilon = \frac{1+\nu_i}{E_i}\Sigma - \frac{\nu_i}{E_i}\mathrm{tr}(\Sigma)I \qquad (I = \text{unity tensor})$$

[7] Analogous note for the continuity of the expression $(\tau_{xy}\, n_y + \tau_{xz}\, n_z)$ across the lines of internal interfaces (Equation 15.6).

One deduces, in integrating over the domain occupied by the cross section of the beam:

$$\int_D \varepsilon_{xx} E_i dS = \int_D \sigma_{xx} dS - \int_D \nu_i(\sigma_{yy} + \sigma_{zz}) dS$$

Taking into account the form of the displacements in Equation 15.3, one can write

$$\int_D \varepsilon_{xx} E_i \, dS = \int_D \frac{\partial u_x}{\partial x} E_i \, dS = -\frac{d\theta_z}{dx}\int_D yE_i \, dS + \frac{du}{dx}\int_D E_i \, dS + \frac{\partial}{\partial x}\int_D E_i \eta_x \, dS$$

which leads, with the notation in Equation 15.1, to the relation:

$$N_x = \langle ES \rangle \frac{du}{dx} + \int_D \nu_i(\sigma_{yy} + \sigma_{zz}) dS \tag{15.8}$$

$$\int_D -y\varepsilon_{xx} E_i \, dS = \int_D -y\sigma_{xx} \, dS + \int_D \nu_i y(\sigma_{yy} + \sigma_{zz}) \, dS$$

Taking into account the form of the displacements in Equation 15.3 one can write

$$\int_D -y\varepsilon_{xx} E_i \, dS = \frac{d\theta_z}{dx}\int_D E_i y^2 \, dS - \frac{du}{dx}\int_D E_i y \, dS - \frac{\partial}{\partial x}\int_D E_i y \eta_x \, dS$$

This leads, with the notation in Equation 15.1, to the relation:

$$M_z = \langle EI_z \rangle \frac{d\theta_z}{dx} - \int_D \nu_i y(\sigma_{yy} + \sigma_{zz}) \, dS$$

$$\int_D 2\varepsilon_{xy} G_i dS = \int_D \tau_{xy} \, dS \tag{15.9}$$

Taking into account the form of the displacements in Equation 15.3, one can write

$$\int_D 2\varepsilon_{xy} G_i dS = \int_D \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) G_i dS = -\theta_z \int_D G_i dS \cdots$$

$$\cdots + \int_D G_i \frac{\partial \eta_x}{\partial y} dS + \frac{dv}{dx}\int_D G_i dS + \frac{\partial}{\partial x}\int_D \eta_y G_i dS$$

which gives with the notation in Equation 15.1, the relation:

$$T_y = \langle GS \rangle \left(\frac{dv}{dx} - \theta_z\right) + \int_D G_i \frac{\partial \eta_x}{\partial y} dS \tag{15.10}$$

15.1.5 Technical Formulation

15.1.5.1 Simplifications

We extend to the composite beams the simplifications made for the homogeneous beams as:

1. σ_{yy} and $\sigma_{zz} << \sigma_{xx}$ at almost all points of the cross section.[8]
2. We neglect the variation of warping $\{\eta_x, \eta_y, \eta_z\}$ between two neighboring infinitely near sections in order to calculate the flexure stresses, that are

$$\sigma_{xx}, \; \tau_{xy}, \quad \text{and} \quad \tau_{xz}.^{9}$$

15.1.5.2 Expression for the Normal Stresses

With the previous simplifications, one extracts from the constitutive relation:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_i} - \frac{\nu_i}{E_i}(\sigma_{yy} + \sigma_{zz})$$

the following simplified form:

$$\frac{\sigma_{xx}}{E_i} \# \frac{\partial u_x}{\partial x} = -y\frac{d\theta_z}{dx} + \frac{du}{dx} + \cancel{\frac{\partial \eta_x}{\partial x}}$$

Then with $M_z \# < EI_z > \dfrac{d\theta_z}{dx}$ (Equation 15.9) and $N_x \# < ES > \dfrac{du}{dx}$ (Equation 15.8):

$$\boxed{\sigma_{xx} = \underbrace{-E_i\frac{M_z}{\langle EI_z \rangle}y}_{\text{bending}} + \underbrace{E_i\frac{N_x}{\langle ES \rangle}}_{\text{extension}}} \tag{15.11}$$

Remark: The continuity $(\varepsilon_{xx})_i = (\varepsilon_{xx})_j$[10] at the interface between the Phases i and j leads to

$$\frac{(\sigma_{xx})_i}{E_i} = \frac{(\sigma_{xx})_j}{E_j}$$

[8] This hypothesis is better verified by the fact that the Poisson coefficients of the different phases have similar values.

[9] This hypothesis is known in the literature for the homogeneous beams as the generalized "Navier–Bernoulli" hypothesis.

[10] See Section 15.1.2.

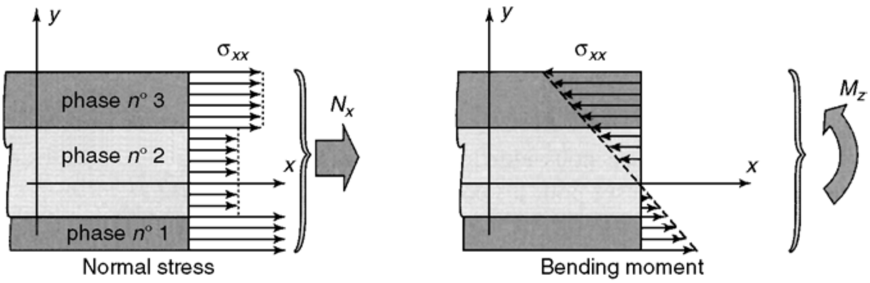


Figure 15.4 Normal and Bending Stresses

which shows the **discontinuity of normal stresses** due to the difference in the longitudinal moduli, as illustrated in Figure 15.4.

15.1.5.3 Shear Stress Expression

15.1.5.3.1 Characterization of Warping

Starting from the local equilibrium described by the relation:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0$$

we study the flexure shear stress in the plane of the cross section, noted as:

$$\vec{\tau} = \tau_{xy}\vec{y} + \tau_{xy}\vec{z}$$

Taking into consideration Equations 15.11 and 15.7, and the simplification 2, at the beginning of Section 15.1.5.1, one can write

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} = \frac{E_i}{\langle EI_z \rangle}\frac{dM_z}{dx}y - \frac{E_i}{\langle ES \rangle}\frac{dN_x}{dx} = -\frac{E_i}{\langle EI_z \rangle}T_y \times y$$

and with the displacement field in Equation 15.3:

$$G_i\left(\frac{\partial^2 \eta_x}{\partial y^2} + \frac{\partial^2 \eta_x}{\partial z^2}\right) = -T_y\frac{E_i}{\langle EI_z \rangle} \times y$$

Putting η_x in the form:

$$\eta_x = \frac{T_y}{\langle GS \rangle} \times g(y,z) \tag{15.12}$$

leads to

$$\nabla^2 g = -\frac{E_i}{G_i}\frac{\langle GS\rangle}{\langle EI_z\rangle}\times y \tag{15.13}$$

and Equation 15.10 becomes

$$T_y = \langle GS\rangle\left(\frac{dv}{dx} - \theta_z\right) + \int_D G_i \frac{T_y}{\langle GS\rangle}\frac{\partial g}{\partial y} dS$$

$$T_y\left(1 - \frac{1}{\langle GS\rangle}\int_D G_i \frac{\partial g}{\partial y} dS\right) = \langle GS\rangle\left(\frac{dv}{dx} - \theta_z\right)$$

or:

$$\boxed{T_y = \frac{\langle GS\rangle}{k}\left(\frac{dv}{dx} - \theta_z\right)} \tag{15.14}$$

In the above relation appears a k coefficient which is analogous to the shear coefficient for homogeneous beams.

15.1.5.3.2 External Limit Condition

We have supposed that the lateral surface of the beam is free from shear. This gives, along the external contour ∂D of the cross section, the relation:

$$\vec{\tau}\cdot\vec{n} = \tau_{xy} n_y + \tau_{xz} n_z = 0,$$

and, with the displacement field in Equation 15.3 and the simplifications described above:

$$\left(\frac{dv}{dx} - \theta_z\right) n_y + \overrightarrow{\text{grad}}\,\eta_x \cdot \vec{n} = 0$$

Introducing the function $g(y,z)$ (Equation 15.12) and with 15.14, one obtains

$$\overrightarrow{\text{grad}}\; g \cdot \vec{n} = \frac{\partial g}{\partial n} = -k n_y$$

Substituting the function $g(y,z)$ with the function $g_o\,(y,z)$ such that:

$$g_o(y,z) = g(y,z) + k \times y \tag{15.15}$$

one verifies that g_o is solution of the problem:

$$\begin{cases} \nabla^2 g_o = -\dfrac{E_i}{G_i}\dfrac{\langle GS\rangle}{\langle EI_z\rangle} \times y \text{ in domain D} \\ \dfrac{\partial g_o}{\partial n} = 0 \text{ on the boundary } \partial D \end{cases}$$

We denote $g_o(y,z)$ as the **longitudinal warping function** for the cross section.

15.1.5.3.3 Conditions at the Interfaces

The continuity conditions already described in Section 15.1.2 lead for the warping function, at the interface between two phases i and j,

$$g_{oi} = g_{oj}$$
$$\tau_{xyi} n_y + \tau_{xzi} n_z = \tau_{xyj} n_y + \tau_{xzj} n_z$$

then:

$$G_i \frac{\partial g_{oi}}{\partial n} = G_j \frac{\partial g_{oj}}{\partial n}$$

15.1.5.3.4 Uniqueness of the Solution

This is given by Equation 15.5 which is interpreted here as:

$$\int_D E_i g_o dS = 0$$

15.1.5.3.5 Form of the Shear Stresses

One can easily verify the following expressions:

$$\tau_{xy} = G_i \frac{T_y}{\langle GS\rangle} \frac{\partial g_o}{\partial y}$$
$$\tau_{xz} = G_i \frac{T_y}{\langle GS\rangle} \frac{\partial g_o}{\partial z}$$

then again:

$$\vec{\tau} = G_i \frac{T_y}{\langle GS\rangle} \overrightarrow{\text{grad}}\ g_o$$

15.1.5.3.6 Shear Coefficient for the Section

The shear coefficient for the section is obtained starting from the Equation 15.5:

$$\int_D E_i \eta_x dS = 0$$

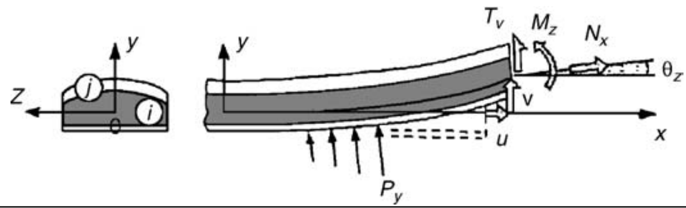
This necessitates the knowledge of the warping function g_o. Then the above relation can be rewritten as:

$$\int_D E_i \frac{T_y}{\langle GS \rangle}(g_o - k \times y) \times y\, dS = 0$$

which leads to:

$$k = \frac{1}{\langle EI_z \rangle}\int_D E_i g_o y dS$$

In summary, in the absence of body forces (inertia forces for example), the bending of a composite beam in its plane of symmetry can be characterized by a homogenized formulation—equivalent to a classical homogeneous beam solution—in the following manner:



- **Elastic center 0:** it is such that

$$\int_D E_i y dS = 0$$

- **Equivalent stiffnesses:**

$$\langle ES \rangle = \sum_i E_i S_i; \quad \langle EI_z \rangle = \sum_i E_i I_{zi}; \quad \frac{\langle GS \rangle}{k} = \sum_i G_i S_i \times \frac{1}{k} \qquad (15.16)$$

- **Equilibrium relations:** (stress resultants calculated at elastic center)

$$\frac{dN_x}{dx} = 0; \quad \frac{dT_y}{dx} + p_y = 0; \quad \frac{dM_z}{dx} + T_y = 0$$

- **Constitutive relations:**

$$N_x = \langle ES \rangle \frac{du}{dx}; \quad T_y = \frac{\langle GS \rangle}{k}\left(\frac{dv}{dx} - \theta_z\right); \quad M_z = \langle EI_z \rangle \frac{d\theta_z}{dx}$$

(Contd.)

Stresses:

normal stresses $\sigma_{xx} = -E_i\frac{M_z}{\langle EI_z\rangle}y + E_i\frac{N_x}{\langle ES\rangle}$

shear stresses

$$\left.\begin{aligned}\tau_{xy} &= G_i\frac{T_y}{\langle GS\rangle}\frac{\partial g_o}{\partial y}\\ \tau_{xz} &= G_i\frac{T_y}{\langle GS\rangle}\frac{\partial g_o}{\partial z}\end{aligned}\right\}\ \vec{\tau} = \frac{G_i}{\langle GS\rangle}T_y\overrightarrow{\text{grad}}g_o$$

- longitudinal warping function $g_o(y, z)$: it is the solution to the problem

$$\begin{cases}\dfrac{\partial^2 g_o}{\partial y^2} + \dfrac{\partial^2 g_o}{\partial z^2} = -\dfrac{E_i}{G_i}\dfrac{\langle GS\rangle}{\langle EI_z\rangle}y \text{ in domain } D \text{ of the section.}\\ \dfrac{\partial g_o}{\partial n} = 0 \text{ on the boundary } \partial D\end{cases} \tag{15.16}$$

with internal continuity:

$$\left.\begin{aligned}g_{oi} &= g_{oj}\\ G_i\frac{\partial g_{oi}}{\partial n} &= G_j\frac{\partial g_{oj}}{\partial n}\end{aligned}\right\}\text{ along internal boundaries } \ell_{ij}$$

and the uniqueness condition:

$$\int_D E_i g_o\, dS = 0$$

Shear coefficient *k*: it is given by the formula:

$$k = \frac{1}{\langle EI_z\rangle}\int_D E_i g_o y\ dS$$

15.1.6 Energy Interpretation

15.1.6.1 Energy Due to Normal Stresses σ_{xx}

Denoting by dW_σ as the deformation energy of an elementary portion of a beam with length dx due to the application of normal stresses σ_{xx}, one has

$$dW_\sigma = \frac{1}{2}\int \sigma_{xx}\varepsilon_{xx}\, dV = \left\{\frac{1}{2}\int_D \frac{\sigma_{xx}^2}{E_i}dS\right\}dx$$

Taking into account Equation 15.11 for the normal stresses:

$$\begin{aligned}\frac{dW_\sigma}{dx} &= \frac{1}{2}\int_D \frac{1}{E_i}\left[-\frac{E_i}{\langle EI_z\rangle}M_z y + \frac{E_i}{\langle ES\rangle}N_x\right]^2 dS \\ &= \frac{1}{2}\int_D E_i \frac{M_z^2}{\langle EI_z\rangle^2}y^2\, dS + \frac{1}{2}\int_D E_i \frac{N_x^2}{\langle ES\rangle^2}dS \cdots \\ &\cdots + \int_D E_i \frac{M_z N_x}{\langle EI_z\rangle\langle ES\rangle} y\, dS\end{aligned}$$

(the above expression simplifies due to the definition of the elastic center 0 in 15.16); therefore:

$$\frac{dW_\sigma}{dx} = \frac{1}{2}\frac{M_z^2}{\langle EI_z\rangle} + \frac{1}{2}\frac{N_x^2}{\langle ES\rangle}$$

15.1.6.2 Energy Due to Shear Stresses $\vec{\tau}$

Denoting dW_τ as the deformation energy of an elementary portion of a beam with length dx due to the application of shear stresses $\vec{\tau}$, one has:

$$dW_\tau = \frac{1}{2}\int_D 2(\tau_{xy}\varepsilon_{xy} + \tau_{xz}\varepsilon_{xz})dV = \frac{1}{2}\left\{\int_D \frac{1}{G_i}(\tau_{xy}^2 + \tau_{xz}^2)dS\right\}dx$$

then, taking into account the form of the shear stresses in 15.16:

$$\frac{dW_\tau}{dx} = \frac{1}{2}\int_D G_i \frac{T_y^2}{\langle GS\rangle^2}\left\{\left(\frac{\partial g_o}{\partial y}\right)^2 + \left(\frac{\partial g_o}{\partial y}\right)^2\right\}dS$$

$$\frac{dW_\tau}{dx} = \frac{1}{2}\frac{T_y^2}{\langle GS\rangle^2}\int_D G_i\left\{\frac{\partial}{\partial y}\left(g_o\frac{\partial g_o}{\partial y}\right) + \frac{\partial}{\partial z}\left(g_o\frac{\partial g_o}{\partial z}\right) - g_o\cdot\nabla^2 g_o\right\}dS$$

with the value from 15.16 of the Laplacian of the warping function g_o[11]:

$$\frac{dW_\tau}{dx} = \frac{1}{2}\frac{T_y^2}{\langle GS\rangle^2}\left\{\int_D G_i g_o \frac{E_i}{G_i}\frac{\langle GS\rangle}{\langle EI_z\rangle}y\,dS + \int_{\partial D} G_i g_o \frac{\partial g_o}{\partial n}d\Gamma\right\}$$

[11] The equality $\int_D G_i\{\frac{\partial}{\partial y}(g_o\frac{\partial g_o}{\partial y}) + \frac{\partial}{\partial z}(g_o\frac{\partial g_o}{\partial y})\}dS = \int_{\partial D} G_i g_o \frac{\partial g_o}{\partial n}d\Gamma$ is made possible due the continuity of the quantities $G_i g_o \frac{\partial g_o}{\partial n}$ at the interfaces l_{ij} (See Section 15.1.5.3.3, "Conditions at the Interfaces").

One knows in the above the expression of the shear coefficient k for the section (see 15.16). Then:

$$\frac{dW_\tau}{dx} = \frac{1}{2}k\frac{T_y^2}{\langle GS\rangle}$$

In summary, the strain energy density can be written as:

$$\boxed{\frac{dW}{dx} = \frac{1}{2}\frac{N_x^2}{\langle ES\rangle} + \frac{1}{2}\frac{M_z^2}{\langle EI_z\rangle} + \frac{1}{2}k\frac{T_y^2}{\langle GS\rangle}} \tag{15.17}$$

Remarks:

- Note the analogy between this expression and that for the strain energy of a classical homogeneous beam, which should be written here as:

$$\frac{dW}{dx} = \frac{1}{2}\frac{N_x^2}{ES} + \frac{1}{2}\frac{M_z^2}{EI_z} + \frac{1}{2}k\frac{T_y^2}{GS}$$

- As a practical consequence of this homogenization, it becomes possible to determine the **equivalent characteristics** which are necessary for the entry of data into a computer program utilizing finite elements of classical homogeneous beams. The problem then comes to the numerical evaluation of the following values:

Equivalent moduli: $E_{\text{equivalent}}$, $G_{\text{equivalent}}$, (or $\nu_{\text{equivalent}}$)

Geometric characteristics: $S_{\text{equivalent}}$, $I_{z\ \text{equivalent}}$, and k

By taking $S_{\text{equivalent}} = S$ (real area of the cross section), one can easily verify that:

$$E_{\text{equivalent}} = \frac{\langle ES\rangle}{S}; \quad G_{\text{equivalent}} = \frac{\langle GS\rangle}{S}$$

$$I_{z\ \text{equivalent}} = \frac{\langle EI_z\rangle}{E_{\text{equivalent}}}; \quad \nu_{\text{equivalent}} = \frac{1}{2}\frac{\langle ES\rangle}{\langle GS\rangle} - 1$$

15.1.7 Extension to the Dynamic Case

The equilibrium relations of Section 15.1.3. were written in the absence of body forces. In vibratory motions, these body forces exist in the form of inertia forces. One then has

$$\frac{\partial\sigma_{ij}}{\partial x_j} - \rho\ddot{u}_i = 0$$

Following the main steps of the calculations in Section 15.1.3, for a beam under free vibration,[12] one obtains:

$$\frac{\partial N_x}{\partial x} = \frac{\partial^2}{\partial t^2}\int_D \rho_i u_x \, dS$$

which leads, with Equation 15.3, to the following expression:

$$\frac{\partial N_x}{\partial x} = \langle \rho S \rangle \frac{\partial^2 u}{\partial t^2} - y_G \langle \rho S \rangle \frac{\partial^2 \theta_z}{\partial t^2}$$

in which we denote

$$\langle \rho S \rangle = \int_D \rho_i \, dS \quad \text{and} \quad y_G \langle \rho S \rangle = \int_D \rho_i y \, dS.$$

y_G appears here as the ordinate of the mass center (center of gravity) of the section. One has neglected the secondary coupling due to η_x:

$$\frac{\partial T_y}{\partial x} = \frac{\partial^2}{\partial t^2}\int_D \rho_i u_y \, dS$$

with Equation 15.3 and neglecting the secondary coupling due to η_x:

$$\frac{\partial T_y}{\partial x} = \langle \rho S \rangle \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial M_z}{\partial x} + T_y = \frac{\partial^2}{\partial t^2}\int_D -y \rho_i u_x \, dS$$

with Equation 15.3, posing $\langle \rho I_z \rangle = \int_D \rho_i y^2 dS$, and neglecting the secondary coupling due to η_x:

$$\frac{\partial M_z}{\partial x} + T_y = \langle \rho I_z \rangle \frac{\partial^2 \theta_z}{\partial t^2} - y_G \langle \rho S \rangle \frac{\partial^2 u}{\partial t^2}$$

The above relations are to be joined with the constitutive relations in 15.16. However one must note these constitutive relations were written in the absence of body forces. Nevertheless, we will consider them to be valid, with the condition that the concerned frequencies are not too high. Generally this case corresponds to the mechanical frequencies, and one denotes this as the "*quasi static*" domain.

[12] One removes all the forces and moments on the beam except inertial forces and moments.

In summary, in dynamic regime, one has to replace the equilibrium relations and the constitutive behavior which appear in 15.16 with the following relations:

- Governing equations (stress resultants calculated at elastic center):

$$\begin{aligned}\frac{\partial N_x}{\partial x} &= \langle \rho S\rangle \frac{\partial^2 u}{\partial t^2} - y_G \langle \rho S\rangle \frac{\partial^2 \theta_z}{\partial t^2}\\ \frac{\partial T_x}{\partial x} &= \langle \rho S\rangle \frac{\partial^2 v}{\partial t^2}\\ \frac{\partial M_z}{\partial x} + T_y &= \langle \rho I_z\rangle \frac{\partial^2 \theta_z}{\partial t^2} - y_G \langle \rho S\rangle \frac{\partial^2 u}{\partial t^2}\end{aligned} \qquad (15.18)$$

with

$$\langle \rho S\rangle = \sum_i \rho_i S_i; \quad \langle \rho I_z\rangle = \sum_i \rho_i I_{zi}; \quad y_G \langle \rho S\rangle = \int_D \rho_i \; y\,dS$$

- Constitutive relations:

$$N_x = \langle ES\rangle \frac{\partial u}{\partial x}; \qquad T_y = \frac{\langle GS\rangle}{k}\left(\frac{\partial v}{\partial x} - \theta_z\right); \qquad M_z = \langle EI_z\rangle \frac{\partial \theta_z}{\partial x}$$

Remark: We note in the above relations a nonclassical coupling between the longitudinal oscillations $u(x,t)$ and flexural oscillations. This coupling disappears if the elastic center is mixed with the center of gravity.[13]

15.2 CASE OF ANY CROSS SECTION (ASYMMETRIC)

Now, the cross section of the beam does not present any particular symmetry (see Figure 15.5). One can consider for this general case the procedure adopted in the previous paragraph for beams with symmetry. One notes the supplementary equivalent stiffness:

$$\langle EI_y\rangle = \int_D E_i z^2 dS = \sum_{\substack{i \\ \text{number of phases}}} E_i I_{yi}$$

It also appears an equivalent rotation $\theta_y(x)$ defined by the expression:

$$\theta_y(x) = \frac{1}{\langle EI_y\rangle}\int_D E_i u_x(x,y,z) \times z\,dS$$

Then, from the definitions of θ_y, u, and θ_z (Section 15.1.1):

$$\theta_y(x) = \frac{1}{\langle EI_y\rangle}\int_D E_i\{u - y\theta_z + \eta_{ox}\} \times z\,dS$$

[13] See Chapter 18, Application 18.3.9.

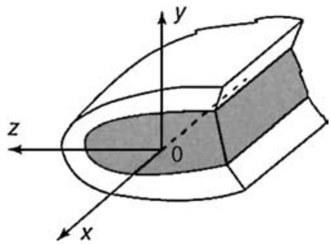


Figure 15.5 Composite Beam with any Cross Section

this expression simplifies if one chooses the origin of the coordinate z such that:

$$\int_D E_i z\, dS = 0; \quad \int_D E_i yz\, dS = 0$$

These relations, joined with the condition already established in the previous paragraph: $\int_D E_i y\ ds = 0$, allow one to define the position of the **elastic center** 0 of the section, as well as the orientation of the axes y and z that we will call **the principal axes of the section. Then, in summary**:

elastic center:	$\int_D E_i y\, dS = 0$
	$\int_D E_i z\, dS = 0$
principal axes:	$\int_D E_i yz\, dS = 0$

One can then rewrite the contribution η_{ox} to the longitudinal displacement u_x, which appeared above, in the form:

$$\eta_{ox}(x, y, z) = z \times \theta_y + \eta_x(x, y, z)$$

and one verifies that, by definition of the degree of freedom θ_y:

$$\int_D E_i \eta_x z\, dS = 0$$

The displacement u_x (x, y, z) then takes the form:

$$u_x(x, y, z) = u(x) - y\theta_z(x) + z\theta_y(x) + \eta_x(x, y, z)$$

In addition, due to the disappearance of symmetry in the section, the transverse displacement $w(x)$ (Section 15.1.1.6) is not zero.

We then obtain for the elastic displacement field:

$$\boxed{\begin{aligned} u_x &= u(x) - y\theta_z + z\theta_y + \eta_x \\ u_y &= v(x) + \eta_y \\ u_z &= w(x) + \eta_z \end{aligned}}$$

The three-dimensional incremental displacements η_x, η_y, η_z verify (see Section 15.1.1.6):

$$\int_D E_i \eta_x \, dS = \int_D E_i \eta_x y \, dS = \int_D E_i \eta_x z \, dS = 0$$
$$\int_D G_i \eta_y \, dS = 0$$
$$\int_D G_i \eta_z \, dS = 0$$

Starting from the above and following the same procedure as in the previous paragraph, successively for the bending in the plane x,y, with identical results, then in the plane x,z, we obtain results summarized in the following relations:

• **degree of freedom:** along x: $u(x)$ along y: $v(x); \theta_y(x)$ along z: $w(x); \theta_z(x)$
• **elastic center:** it is such that: $\int_D E_i y dS = \int_D E_i z \, dS = 0$
• **principal axes:** they are such that: $\int_D E_i yz \, dS = 0$
• **equivalent stiffnesses:** $\langle ES \rangle = \sum_i E_i S_i$ $\langle EI_z \rangle = \sum_i E_i I_{zi}; \quad \langle EI_y \rangle = \sum_i E_i I_{yi}$ $\frac{\langle GS \rangle}{k_y} = \sum_i G_i S_i \times \frac{1}{k_y}; \quad \frac{\langle GS \rangle}{k_z} = \sum_i G_i S_i \times \frac{1}{k_z}$ (Continued)

(Contd.)

- **equilibrium relations:** (stress resultants calculated at elastic center)

$$\frac{dN_x}{dx} = 0$$

$$\frac{dT_y}{dx} + p_y = 0; \quad \frac{dT_z}{dx} + p_z = 0$$

$$\frac{dM_z}{dx} + T_y = 0; \quad \frac{dM_y}{dx} - T_z = 0$$

- **constitutive relations:**

$$N_x = \langle ES \rangle \frac{du}{dx}$$

$$T_y = \frac{\langle GS \rangle}{k_y}\left(\frac{dv}{dx} - \theta_z\right); \quad T_z = \frac{\langle GS \rangle}{k_z}\left(\frac{dw}{dx} + \theta_y\right) \qquad (15.19)^{14}$$

$$M_z = \langle EI_z \rangle \frac{d\theta_z}{dx}; \quad M_y = \langle EI_y \rangle \frac{d\theta_y}{dx}$$

- **normal stresses:**

$$\sigma_{xx} = -E_i \frac{M_z}{\langle EI_z \rangle} y + E_i \frac{M_y}{\langle EI_y \rangle} z + E_i \frac{N_x}{\langle ES \rangle}$$

- **shear stresses:**

$$\tau_{xy} = \frac{G_i}{\langle GS \rangle}\left(T_y \frac{\partial g_o}{\partial y} + T_z \frac{\partial h_o}{\partial y}\right)$$

$$\tau_{xz} = \frac{G_i}{\langle GS \rangle}\left(T_y \frac{\partial g_o}{\partial z} + T_z \frac{\partial h_o}{\partial z}\right)$$

then

$$\vec{\tau} = \frac{G_i}{\langle GS \rangle}(T_y \overrightarrow{\text{grad}} \; g_o + T_z \overrightarrow{\text{grad}} \; h_o)$$ (Continued)

Remark: As already mentioned in Section 15.1.6 for a beam with a plane of symmetry, it is possible to evaluate the equivalent characteristics that one has to introduce in data form in order to utilize computer programs for the calculation by

[14] In fact, in place of the constitutive relation $T_y = \frac{\langle GS \rangle}{k_y}(\frac{dv}{dx} - \theta_z)$ it comes to a form such as: $k_y T_y + k_{yz} T_z = \langle GS \rangle (\frac{dv}{dz} - \theta_z)$ where appears a coupling coefficient k_{yz}. This means that a unique shear resultant T_z leads to flexure in the x,y plane. This secondary effect has been neglected here. Analogous remark yields for the constitutive relation $T_z = \frac{\langle GS \rangle}{k_z}(\frac{dw}{dx} + \theta_y)$.

- **Longitudinal warping functions:**

function $g_o(y,z)$: It is the solution to the problem

$$\begin{cases} \dfrac{\partial^2 g_o}{\partial y^2} + \dfrac{\partial^2 g_o}{\partial z^2} = -\dfrac{E_i}{G_i}\dfrac{\langle GS\rangle}{\langle EI_z\rangle} y \text{ in domain D of the section} \\ \dfrac{\partial g_o}{\partial n} = 0 \text{ on the boundary } \partial D \end{cases}$$

with internal continuity:

$$\left.\begin{array}{r} g_{oi} = g_{oj} \\ G_i \dfrac{\partial g_{oi}}{\partial n} = G_j \dfrac{\partial g_{oj}}{\partial n} \end{array}\right\} \text{along internal boundaries } \ell_{ij}$$

and the uniqueness condition: $\int_D E_i g_o \, dS = 0$

function $h_o(y,z)$: It is the solution to the problem :

$$\begin{cases} \dfrac{\partial^2 h_o}{\partial y^2} + \dfrac{\partial^2 h_o}{\partial z^2} = -\dfrac{E_i}{G_i}\dfrac{\langle GS\rangle}{\langle EI_y\rangle} z \text{ in domain D of the section} \\ \dfrac{\partial h_o}{\partial n} = 0 \text{ on the boundary } \partial D \end{cases}$$

with internal continuity:

$$\left.\begin{array}{r} h_{oi} = h_{oj} \\ G_i \dfrac{\partial h_{oi}}{\partial n} = G_j \dfrac{\partial h_{oj}}{\partial n} \end{array}\right\} \text{along internal boundaries } \ell_{ij}$$

and the uniqueness condition: $\int_D E_i h_o \, dS = 0$

- **Shear coefficients**

coefficient k_y: it is given by the formula: $k_y = \dfrac{1}{\langle EI_z\rangle}\int_D E_i g_o y \, dS$

coefficient k_z: it is given by the formula: $k_z = \dfrac{1}{\langle EI_y\rangle}\int_D E_i h_o z \, dS$

- **Strain energy**

$$\frac{dW}{dx} = \frac{1}{2}\frac{N_x^2}{\langle ES\rangle} + \frac{1}{2}\frac{M_z^2}{\langle EI_z\rangle} + \frac{1}{2}\frac{M_y^2}{\langle EI_z\rangle} + \frac{1}{2}k_y\frac{T_y^2}{\langle GS\rangle} + \frac{1}{2}k_z\frac{T_z^2}{\langle GS\rangle}$$

finite elements for the classical beams.[15] The characteristics $E_{\text{equivalent}}$, $G_{\text{equivalent}}$, $I_{z\ \text{equivalent}}$, $I_{y\ \text{equivalent}}$, can be obtained right away. On the contrary, the calculation of the shear coefficients k_y and k_z is not direct. At first it is necessary to know the values of the functions g_o and h_o, solutions in the domain occupied by the cross section of Poisson type problem, as one can take note in the preceding table. The nature of these problems makes it possible for each of the functions g_o and h_o to write an equivalent functional which allows the calculation of the function considered by means of discretization of the cross section using finite elements.

[15] It is convenient to note that a computer program based on elements of homogeneous beams cannot provide correct values for the stresses in a cross section, because these stresses are of particular formulation for composite beams (see 15.16).