14

THE HILL–TSAI FAILURE CRITERION

There are many failure criteria for orthotropic materials. The most commonly used for design calculations is the so-called "Hill-Tsai" criterion.[1] This criterion can be interpreted as analogous to the Von Mises criterion which is applicable to isotropic material in elastic deformation. We will review at the beginning the principal aspects of the Von Mises criterion.

14.1 ISOTROPIC MATERIAL: VON MISES CRITERION

The material is elastic and isotropic. In Figure 14.1, one denotes by I,II,III the principal directions of the stress tensor Σ for a given point. The corresponding matrix is

$$\begin{bmatrix} \sigma_{\mathrm{I}} & 0 & 0 \\ 0 & \sigma_{\mathrm{II}} & 0 \\ 0 & 0 & \sigma_{\mathrm{III}} \end{bmatrix}$$

The general form of the deformation energy dW for an elementary volume dV surrounding the point considered can be written as:

$$dW_{\text{total}} = \frac{1}{2}\sum_i \sum_j \sigma_{ij}\,\varepsilon_{ij}\; dV$$

which can be reduced to

$$dW_{\text{total}} = \frac{1}{2}(\sigma_{\mathrm{I}}\varepsilon_{\mathrm{I}} + \sigma_{\mathrm{II}}\varepsilon_{\mathrm{II}} + \sigma_{\mathrm{III}}\varepsilon_{\mathrm{III}})\; dV$$

[1] See failure of composite materials in Section 5.3.2.

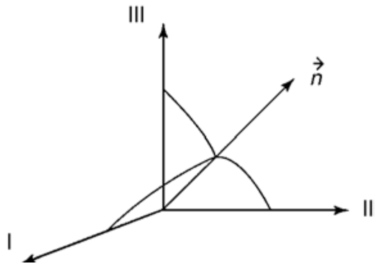

Figure 14.1 Principal Directions for the Stress Tensor

ε_I ε_{II} ε_{III} are the principal strains that one can express as functions of stresses using the constitutive Equation 10.1 as:

$$\varepsilon = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\text{trace}(\Sigma)\boldsymbol{I}$$

This leads to:

$$\left(\frac{dW}{dV}\right)_{\text{total}} = \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2) - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})^2\right\}$$

(Note that (dW/dV) represents energy per unit volume).

The total elastic deformation above is due to the dilatation and distortion of the material. The Von Mises criterion postulates that the material resists to a state of isotropic (or spherical) stress, but will plastify when the distortion energy per unit volume reaches a critical value. One notes that:

$$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \left(\frac{dW}{dV}\right)_{\text{total}} - \left(\frac{dW}{dV}\right)_{\text{spherical stress}}$$

The isotropic portion of the stress state here is written as: $\frac{\sigma_I + \sigma_{II} + \sigma_{III}}{3}$. It creates a state of isotropic dilatation (Equation 10.1):

$$\varepsilon = \frac{1+\nu}{E}\left(\frac{\sigma_I + \sigma_{II} + \sigma_{III}}{3}\right) - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})$$

then:

$$\left(\frac{dW}{dV}\right)_{\text{spherical stress}} = \frac{1}{2}\left\{3 \times \left(\frac{\sigma_I + \sigma_{II} + \sigma_{III}}{3}\right) \times \varepsilon\right\}$$

$$\left(\frac{dW}{dV}\right)_{\text{spherical stress}} = \frac{1}{2}\left\{\frac{1+\nu}{E}\frac{(\sigma_I + \sigma_{II} + \sigma_{III})^2}{3} - \frac{\nu}{E}(\sigma_I + \sigma_{II} + \sigma_{III})^2\right\}$$

One obtains then by replacing:

$$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \frac{1}{2}\left\{\frac{1+\nu}{E}(\sigma_{\text{I}}^2+\sigma_{\text{II}}^2+\sigma_{\text{III}}^2) - \frac{\nu}{E}(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2 \cdots\right.$$

$$\left.\cdots - \frac{1+\nu}{E}\frac{(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2}{3} + \frac{\nu}{E}(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2\right\}$$

then: $$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \frac{1}{4G}\left\{(\sigma_{\text{I}}^2+\sigma_{\text{II}}^2+\sigma_{\text{III}}^2) - \frac{(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2}{3}\right\} \tag{14.1}$$

One can rewrite as following the quantity in brackets:

$$\frac{2}{3}\{\sigma_{\text{I}}^2+\sigma_{\text{II}}^2+\sigma_{\text{III}}^2-\sigma_{\text{I}}\sigma_{\text{II}}-\sigma_{\text{II}}\sigma_{\text{III}}-\sigma_{\text{III}}\sigma_{\text{I}}\}$$

$$\frac{2}{3}\{(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2-3(\sigma_{\text{I}}\sigma_{\text{II}}+\sigma_{\text{II}}\sigma_{\text{III}}+\sigma_{\text{III}}\sigma_{\text{I}})\}$$

$$\boxed{\begin{aligned}\left(\frac{dW}{dV}\right)_{\text{distorsion}} &= \frac{1}{6G}\{(\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}})^2\cdots\\ &\cdots-3(\sigma_{\text{I}}\sigma_{\text{II}}+\sigma_{\text{II}}\sigma_{\text{III}}+\sigma_{\text{III}}\sigma_{\text{I}})\}\end{aligned}} \tag{14.2}$$

Remarks: If one denotes as $\vec{n}$ the direction making the same angle with each of the principal directions (following Figure 14.1), one observes on the face with the normal $\vec{n}$, a stress $\vec{\sigma}$ such that: $\vec{\sigma} = \Sigma(\vec{n})$

that is:

$$\{\sigma\} = \begin{Bmatrix}\sigma_{\text{I}}/\sqrt{3}\\ \sigma_{\text{II}}/\sqrt{3}\\ \sigma_{\text{III}}/\sqrt{3}\end{Bmatrix}$$

which can be decomposed as:

- A **normal stress:** $\sigma_n = \vec{\sigma}\cdot\vec{n}$

  then: $$\sigma_n = \frac{\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}}}{3}$$

  The above consists of the average or isotropic part of the stress tensor.[2]

- A **shear** stress:

$$\tau = \sqrt{\sigma^2-\sigma_n^{\,2}}$$

[2] Recall the expression $\sigma_{\text{I}}+\sigma_{\text{II}}+\sigma_{\text{III}}$ that constitutes the first scalar invariant of the stress tensor.

then:

$$\tau^2 = \frac{1}{3}\left\{\sigma_{\mathrm{I}}^2 + \sigma_{\mathrm{II}}^2 + \sigma_{\mathrm{III}}^2 - \left(\frac{\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}} + \sigma_{\mathrm{III}}}{3}\right)^2\right\}$$

which can be compared with Equation 14.1. Thus,

$$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \frac{1}{2G}\left(\frac{3}{2}\tau^2\right)$$

The shear stress τ also appears as the shear characteristic of the distortion energy.

- One recognizes in Equation 14.2 the presence of the first and second scalar invariants of the stress tensor independent of the coordinate system. In coordinate axes other than the principal directions, the second invariant can be written as:

$$(\sigma_{11}\sigma_{22} - \tau_{12}^2) + (\sigma_{22}\sigma_{33} - \tau_{23}^2) + (\sigma_{33}\sigma_{11} - \tau_{31}^2)$$

One then has for any coordinate system:

$$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \frac{1}{6G}\{(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 \cdots$$
$$\cdots - 3((\sigma_{11}\sigma_{22} - \tau_{12}^2) + (\sigma_{22}\sigma_{33} - \tau_{23}^2) + (\sigma_{33}\sigma_{11} - \tau_{31}^2))\}$$

then:

$$\left(\frac{dW}{dV}\right)_{\text{distorsion}} = \frac{1}{12G}\{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 \cdots$$
$$\cdots + (\sigma_{33} - \sigma_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)\}$$

The elastic domain (where the distortion energy is below a certain critical value) can then be characterized by the condition:

$$a\{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \cdots$$
$$\cdots + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)\} < 1 \qquad (14.3)$$

To determine the constant, a uniaxial test is sufficient; in effect if one denotes by σ_e the elastic limit obtained from a tension–compression test, one has:

$$a \times 2\sigma_e^2 = 1$$

then:

$$a = 1/2\sigma_e^2$$

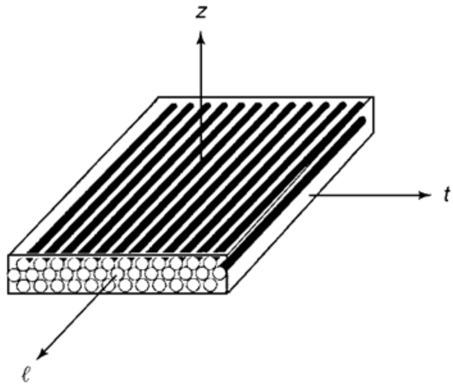

Figure 14.2 Principal Axes for an Orthotropic Ply

14.2 ORTHOTROPIC MATERIAL: HILL–TSAI CRITERION

14.2.1 Preliminary Remarks

A parallel with the Von Mises criterion can be seen with the following remarks:

- For an orthotropic material, the principal directions for the stresses do not coincide with the orthotropic directions, unlike the isotropic case.
- A uniaxial test is not enough to determine all the terms of the equation for the criterion because the mechanical behavior changes with the direction of loading.
- For the fiber/resin composites, the elastic limit corresponds with the rupture limit.
- The rupture strengths are very different when loading is applied along the l direction or along the t direction.
- The rupture strengths are different in tension as compared with in compression.

One can then write in the orthotropic coordinates ℓ,t,z — shown in Figure 14.2 — an expression similar to Equation 14.3, as:

$$a(\sigma_\ell - \sigma_t)^2 + b(\sigma_t - \sigma_z)^2 + c(\sigma_z - \sigma_\ell)^2 + d\tau_{\ell z}^2 + e\tau_{tz}^2 + f\tau_{\ell t}^2 \le 1 \qquad (14.4)$$

14.2.2 Case of a Transversely Isotropic Material

In the following, we will limit ourselves, for the purpose of simplification, to the case of a transversely isotropic material.[3] The constants a, b, c, d, e, f in Equation 14.4 above will be determined using the results of the following tests:

- **Test along the longitudinal direction ℓ:**

$$a + c = \frac{1}{\sigma_{\ell\ \text{rupture}}^2}$$

[3] For an orthotropic material, the procedure is identical.

- **Test along the transverse direction *t*:**

$$a+b = \frac{1}{\sigma_{t\ \text{rupture}}^2}$$

- **Test along the transverse direction *z*: due to transverse isotropy:**

$$b+c = \frac{1}{\sigma_{t\ \text{rupture}}^2}$$

then:

$$a = c = \frac{1}{2\sigma_{\ell\ \text{rupture}}^2}$$

$$b = \frac{1}{\sigma_{t\ \text{rupture}}^2} - \frac{1}{2\sigma_{\ell\ \text{rupture}}^2}$$

- **Shear tests:**

$$\blacksquare\ \tau_{\ell t} \rightarrow f = \frac{1}{\tau_{\ell t\ \text{rupture}}^2}$$

$$\blacksquare\ \tau_{tz} \rightarrow e = \frac{1}{\tau_{tz\ \text{rupture}}^2}$$

$$\blacksquare\ \tau_{\ell z} \rightarrow d = \frac{1}{\tau_{\ell t\ \text{rupture}}^2}$$

due to transverse isotropy.
Replacing in Equation 14.4:

$$\frac{1}{2\sigma_{\ell\ \text{rupture}}^2}\{(\sigma_\ell - \sigma_t)^2 + (\sigma_\ell - \sigma_z)^2\}\cdots$$

$$\cdots - \left(\frac{1}{2\sigma_{\ell\ \text{rupture}}^2} - \frac{1}{\sigma_{t\ \text{rupture}}^2}\right)(\sigma_t - \sigma_z)^2 + \frac{1}{\tau_{\ell t\ \text{rupture}}^2}(\tau_{\ell t}^2 + \tau_{\ell z}^2) + \frac{\tau_{tz}^2}{\tau_{tz\ \text{rupture}}^2} \leq 1$$

and in developing[4]:

$$\frac{\sigma_\ell^2}{\sigma_{\ell\,\text{rupture}}^2} + \frac{\sigma_t^2 + \sigma_z^2}{\sigma_{t\,\text{rupture}}^2} - \frac{\sigma_\ell}{\sigma_{\ell\,\text{rupture}}^2}(\sigma_t + \sigma_z) + \sigma_z\sigma_t\left(\frac{1}{\sigma_{\ell\,\text{rupture}}^2} - \frac{2}{\sigma_{t\,\text{rupture}}^2}\right)\cdots$$
$$\cdots + \frac{\tau_{\ell t}^2 + \tau_{\ell z}^2}{\tau_{\ell t\,\text{rupture}}^2} + \frac{\tau_{tz}^2}{\tau_{tz\,\text{rupture}}^2} \le 1 \qquad (14.5)$$

Remark: For the case of a "three-dimensional" orthotropic material, an analogous reasoning to the previous presentation leads to a more general criterion, which can be written as:

$$\frac{\sigma_\ell^2}{\sigma_{\ell\,\text{rupt.}}^2} + \frac{\sigma_t^2}{\sigma_{t\,\text{rupt.}}^2} + \frac{\sigma_z^2}{\sigma_{z\,\text{rupt.}}^2} - \left(\frac{1}{\sigma_{\ell\,\text{rupt.}}^2} + \frac{1}{\sigma_{t\,\text{rupt.}}^2} - \frac{1}{\sigma_{z\,\text{rupt.}}^2}\right)\sigma_\ell\sigma_t\cdots$$
$$\cdots - \left(\frac{1}{\sigma_{t\,\text{rupt.}}^2} + \frac{1}{\sigma_{z\,\text{rupt.}}^2} - \frac{1}{\sigma_{\ell\,\text{rupt.}}^2}\right)\sigma_t\sigma_z - \left(\frac{1}{\sigma_{z\,\text{rupt.}}^2} + \frac{1}{\sigma_{\ell\,\text{rupt.}}^2} - \frac{1}{\sigma_{t\,\text{rupt.}}^2}\right)\sigma_z\sigma_\ell\cdots$$
$$\cdots + \frac{\tau_{\ell t}^2}{\tau_{\ell t\,\text{rupt.}}^2} + \frac{\tau_{tz}^2}{\tau_{tz\,\text{rupt.}}^2} + \frac{\tau_{z\ell}^2}{\tau_{z\ell\,\text{rupt.}}^2} \le 1$$

14.2.3 Case of a Unidirectional Ply Under In-Plane Loading

When the stress state is plane-stress, in the plane defined by the axes ℓ,t (see Figure 14.2), one has

$$\sigma_z = \tau_{\ell z} = \tau_{tz} = 0$$

Equation 14.5 is simplified, and one obtains what is called "the Hill–Tsai criterion" for a ply subject to stresses within its plane:

$$\frac{\sigma_\ell^2}{\sigma_{\ell\,\text{rupture}}^2} + \frac{\sigma_t^2}{\sigma_{t\,\text{rupture}}^2} - \frac{\sigma_\ell\sigma_t}{\sigma_{\ell\,\text{rupture}}^2} + \frac{\tau_{\ell t}^2}{\tau_{\ell t\,\text{rupture}}^2} < 1 \qquad (14.6)$$

Remarks:

- The rupture strengths of the "fiber/matrix" plies are different in tension and in compression.[5] Do not forget to place in the denominator of each of the first three terms of Equation 14.6 the values of the rupture strengths

[4] Attention, this is not valid for a fabric that is not transversely isotropic! (see Application 18.2.10).

[5] See values in Section 3.3.3.

corresponding to the type of loadings in the numerators (tension or compression).

- **Safety factor:** Let $\alpha^2 < 1$ the Hill–Tsai expression found for a state of stress σ_ℓ, σ_t, $\tau_{\ell t}$. One can then increase the load by means of a multiplication coefficient k to reach the limit as:

$$\frac{(k\sigma_\ell)^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(k\sigma_t)^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{(k\sigma_\ell)(k\sigma_t)}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(k\tau_{\ell t})^2}{\underset{\text{rupture}}{\tau_{\ell t}^2}} = k^2\alpha^2 = 1$$

The margin of safety can then be defined as the expression:

$$\frac{(k\sigma_\ell) - \sigma_\ell}{\sigma_\ell} = k - 1$$

which can also be written as:

$$\boxed{\text{safety factor} = \frac{1}{\alpha} - 1}$$

14.3 VARIATION OF RESISTANCE OF A UNIDIRECTIONAL PLY WITH RESPECT TO THE DIRECTION OF LOADING

14.3.1 Tension and Compression Resistance

We propose to evaluate the maximum stress σ_x that one can apply on a ply in the direction x in Figure 14.3. The stresses σ_ℓ, σ_t, $\tau_{\ell t}$ in the orthotropic axes are given by Equation 11.4 as:

$$\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix}$$

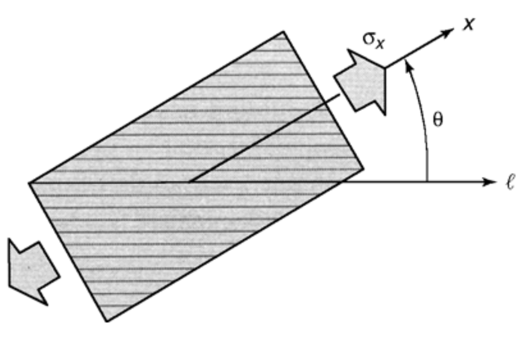

Figure 14.3 Direction of Loading Distinct from Orthotropic Axes

where one recalls that $c = \cos\theta$ and $s = \sin\theta$. Thus,

$$\sigma_\ell = c^2\sigma_x$$
$$\sigma_t = s^2\sigma_x$$
$$\tau_{\ell t} = cs\sigma_x$$

Replacing in the expression of the Hill–Tsai criterion of Equation 14.6, we have

$$\boxed{\sigma_x^2\left\{\frac{c^4}{\sigma^2_{\ell\,\text{rupt.}}} + \frac{s^4}{\sigma^2_{t\,\text{rupt.}}} - \frac{c^2s^2}{\sigma^2_{\ell\,\text{rupt.}}} + \frac{c^2s^2}{\tau^2_{\ell t\,\text{rupt.}}}\right\} \le 1}$$

then:

$$\boxed{\sigma_{x_\text{rupture}} = \frac{1}{\sqrt{\dfrac{c^4}{\sigma^2_{\ell\ \text{rupt.}}} + \dfrac{s^4}{\sigma^2_{t\ \text{rupt.}}} + c^2s^2\left(\dfrac{1}{\tau^2_{\ell t\ \text{rupt.}}} - \dfrac{1}{\sigma^2_{\ell\ \text{rupt.}}}\right)}}}$$

Remarks:

- If σ_x is in tension, then $\sigma_{\ell\ \text{rupture}}$ and $\sigma_{t\ \text{rupture}}$ are the limit stresses in tension (tensile strengths). In effect, when $\theta = 0$:

  $$\sigma_{x\ \text{rupture}} = \sigma_{\ell\ \text{rupture}}$$

  and when $\theta = 90°$:

  $$\sigma_{x\ \text{rupture}} = \sigma_{t\ \text{rupture}}$$

- The evolution of the $\sigma_{x\ \text{rupture}}$, when θ varies, was discussed in Section 3.3.2.

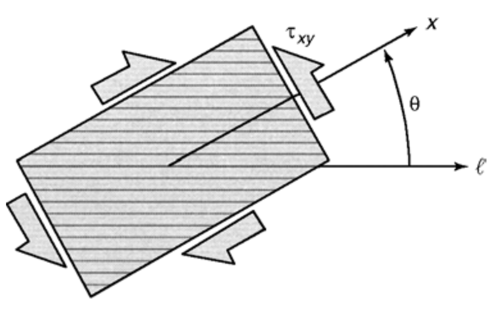

Figure 14.4 Pure Shear in *x,y* Axes

14.3.2 Shear Strength

For a state of pure shear represented in Figure 14.4, one will have in an analogous manner:

$$\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{aligned} \sigma_\ell &= -2cs\tau_{xy} \\ \sigma_t &= 2cs\tau_{xy} \\ \tau_{\ell t} &= (c^2 - s^2)\tau_{xy} \end{aligned} \tag{14.7}$$

Using this in the Hill–Tsai criterion in Equation 14.6:

$$\tau_{xy}^2 \left\{ \frac{4c^2s^2}{\sigma^2_{\ell\ \text{rupture}}} + \frac{4c^2s^2}{\sigma^2_{t\ \text{rupture}}} + \frac{4c^2s^2}{\sigma^2_{\ell\ \text{rupture}}} + \frac{(c^2 - s^2)^2}{\tau^2_{\ell t\ \text{rupture}}} \right\} \leq 1$$

then:

$$\tau_{xy\ \text{rupture}} = \frac{1}{\sqrt{4c^2s^2\left(\dfrac{2}{\sigma^2_{\ell\ \text{rupture}}} + \dfrac{1}{\sigma^2_{t\ \text{rupture}}}\right) + \dfrac{(c^2 - s^2)^2}{\tau^2_{\ell t\ \text{rupture}}}}}$$

Remarks: Here, taking into account Figure 14.4 ($\tau_{xy} > 0$) and Equations 14.7, $\sigma_{\ell\ \text{rupture}}$ will be the limit stress in compression, and $\sigma_{t\ \text{rupture}}$ the limit stress in tension for $0° \leq \theta \leq 90°$.