11

ELASTIC CONSTANTS OF A PLY ALONG AN ARBITRARY DIRECTION

To study the behavior of a laminate made up of many plies with different orientations, it is necessary to know the behavior of each of the plies in directions that are different from the principal material directions of the ply. We propose to determine the elastic constants for this ply behavior using relatively simple calculations.

11.1 COMPLIANCE COEFFICIENTS

The ply is already defined in Chapter 3.[1] Let ℓ, t and z be the orthotropic axes of a ply shown in the Figure 11.1.[2] For a thin laminate made up by a superposition of many plies, we assume that the stresses σ_{zz} are zero. It is then possible, for an orthotropic material, to write the stress–strain relation in the plane ℓ,t starting from Equation 9.3 or 9.5 in the form:

$$\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_\ell} & -\dfrac{\nu_{t\ell}}{E_t} & 0 \\ -\dfrac{\nu_{\ell t}}{E_\ell} & \dfrac{1}{E_t} & 0 \\ 0 & 0 & \dfrac{1}{G_{\ell t}} \end{bmatrix} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} \tag{11.1}$$

Problem: How can one transform this relation expressed in the coordinates ℓ,t into a relation expressed in coordinates x,y inclined at an angle of θ with the ℓ,t coordinates (see Figure 11.1).[3]

First recall the following:

[1] See Section 3.2.

[2] The orthotropic axes 1,2,3 in Equation 9.3 are now called l,t,z, respectively.

[3] What follows is treated more globally and completely in Section 13.2.2.

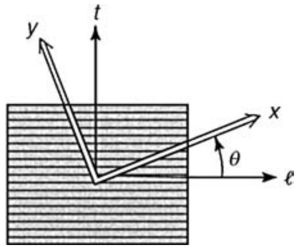


Figure 11.1 Orthotropic Axes and Arbitrary Direction in the Plane of a Ply

- **Recall 1:** The stress $\vec{\sigma}$ acting on a surface with a normal vector $\vec{n}$ is given by

$$\{\sigma\} = [\sigma_{ij}]\{n\} \tag{11.2}$$

where $\{\sigma\}$: *column matrix of stress components* $\vec{\sigma}$; $[\sigma_{ij}]$: *stress matrix*; $\{n\}$: *column matrix of direction cosines* $\vec{n}$.

- **Recall 2:** The coordinates of the same vector $\vec{V}$ in axes x,y as well as ℓ,t, such that $(\vec{x}.\vec{\ell}) = \cos\theta$, are

$$\vec{V} = V_\ell\vec{\ell} + V_t\vec{t} = V_x\vec{x} + V_y\vec{y}$$

with the relation:

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{Bmatrix} V_\ell \\ V_t \end{Bmatrix} \qquad \begin{cases} c = \cos\theta \\ c = \sin\theta \end{cases} \tag{11.3}$$

In axes ℓ,t the stress acting on the surface with a normal $\vec{x}$ can be expressed as follows, using Equation 11.2 above:

$$\{\sigma_{/x}\}_{\ell,t} = [\sigma_{ij}]_{\ell,t}\{x\}_{\ell,t} = [\sigma_{ij}]_{\ell,t}\begin{Bmatrix} c \\ s \end{Bmatrix}$$

where $\{\sigma_{/x}\}$ is the stress vector and $[\sigma_{ij}]$ is the stress matrix and in axes x,y, following Equation 11.3:

$$\{\sigma_{/x}\}_{x,y} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}[\sigma_{ij}]_{\ell,t}\begin{Bmatrix} c \\ s \end{Bmatrix}$$

In a similar manner, the stresses acting on the surface with the normal $\vec{y}$ are written in the x,y axes as:

$$\{\sigma_{/y}\}_{x,y} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} [\sigma_{ij}]_{\ell,t} \begin{Bmatrix} -s \\ c \end{Bmatrix}$$

Therefore, the matrix of stresses in the x,y axes is:

$$\{\sigma_{ij}\}_{x,y} = [\sigma_{/x}, \sigma_{/y}] = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} [\sigma_{ij}]_{\ell,t} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

in setting:

$$[P] = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

and observing that the matrix $[P]$ is orthogonal (${}^t[P] = [P]^{-1}$), one has[4]

$$[\sigma_{ij}]_{\ell,t} = {}^t[P][\sigma_{ij}]_{x,y}[P]$$

where ${}^t[P]$ is the transpose of matrix $[P]$.

This expression can be developed to become

$$\begin{bmatrix} \sigma_\ell & \tau_{\ell t} \\ \tau_{\ell t} & \sigma_t \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

One can also rearrange the equation to be

$$\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ sc & -sc & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{11.4}$$

Then:

$$[\sigma]_{\ell,t} = [T][\sigma]_{x,y}$$

[4] One has: $\sigma_{x,y} = P\sigma_{\ell,t}\,{}^tP$; $\sigma_{\ell t}\,{}^tP = {}^tP\sigma_{xy}$; $P\sigma_{\ell,t} = \sigma_{x,y}P$; $\sigma_{\ell,t} = {}^tP\sigma_{x,y}P$.

with[5]

$$[T] = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & (c^2 - s^2) \end{bmatrix}$$

In a similar manner, the strain components can be transformed as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \varepsilon_{\ell t} \end{Bmatrix}$$

or:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix}$$

then:
$$\begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{x,y} = [T'] \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{\ell,t}$$

with:
$$[T'] = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & (c^2 - s^2) \end{bmatrix}$$

The stress–strain Equation 11.1 can then be expressed in the axes x,y since we have written:

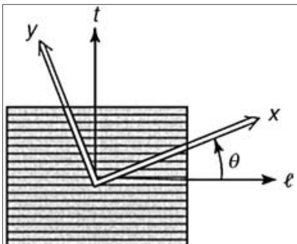
$$\begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{x,y} = [T'] \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{\ell,t} ; \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{\ell,t} = \begin{bmatrix} \frac{1}{E_\ell} & \frac{-\nu_{t\ell}}{E_t} & 0 \\ \frac{-\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} \{\sigma\}_{\ell,t}; \; \{\sigma\}_{\ell,t} = [T]\{\sigma\}_{x,y}$$

[5] This $[T]$ matrix is readily established if one knows the relation that allows one to express the components of a tensor in one system in terms of the components of the same tensor in another system. Here this relation is: $\sigma_{IJ} = \cos_I^m \cos_J^m \sigma_{mn}$ with $\cos_I^m = \cos(\vec{m}, \vec{I})$; see Section 13.1.

where after substitution:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \underbrace{[T'] \begin{bmatrix}\frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0\\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0\\ 0 & 0 & \frac{1}{G_{\ell t}}\end{bmatrix} [T]}_{\text{new matrix of elastic coefficients in } x,y \text{ axes}} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

When all calculations are performed, one obtains the following constitutive relation, written in the coordinates x,y that make an angle θ with the axes ℓ,t. The elastic moduli and Poisson coefficients appear in these relations. One can also see the existence of the coupling coefficients η and μ,[6] which demonstrates that a normal stress can produce a distortion.[7]



$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy}}{G_{xy}}\\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\mu_{xy}}{G_{xy}}\\ \frac{\eta_x}{E_x} & \frac{\mu_y}{E_y} & \frac{1}{G_{xy}}\end{bmatrix} \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

with:

$$E_x(\theta) = \frac{1}{\frac{c^4}{E_\ell} + \frac{s^4}{E_t} + c^2 s^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)}$$

$$E_y(\theta) = \frac{1}{\frac{s^4}{E_\ell} + \frac{c^4}{E_t} + c^2 s^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)} \qquad (11.5)$$

$$G_{xy}(\theta) = \frac{1}{4c^2 s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} + 2\frac{\nu_{t\ell}}{E_t}\right) + \frac{(c^2 - s^2)^2}{G_{\ell t}}}$$

$$\frac{\nu_{yx}}{E_y}(\theta) = \frac{\nu_{t\ell}}{E_t}(c^4 + s^4) - c^2 s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} - \frac{1}{G_{\ell t}}\right)$$

$$\frac{\eta_{xy}}{G_{xy}}(\theta) = -2cs\left\{\frac{c^2}{E_\ell} - \frac{s^2}{E_t} + (c^2 - s^2)\left(\frac{\nu_{t\ell}}{E_t} - \frac{1}{2G_{\ell t}}\right)\right\}$$

$$\frac{\mu_{xy}}{G_{xy}}(\theta) = -2cs\left\{\frac{s^2}{E_\ell} - \frac{c^2}{E_t} - (c^2 - s^2)\left(\frac{\nu_{t\ell}}{E_t} - \frac{1}{2G_{\ell t}}\right)\right\}$$

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[6] Recall that the matrix of elastic coefficients is symmetric, meaning in particular: $\eta_{xy}/G_{xy} = \eta_x/E_x$ and $\mu_{xy}/G_{xy} = \mu_y/E_y$.

[7] See example described in Section 3.1.

11.2 STIFFNESS COEFFICIENTS

When one inverts Equation 11.1 written in the coordinate axes *l,t* of a ply, one obtains

$$
\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} = \begin{bmatrix} \dfrac{E_\ell}{(1-\nu_{\ell t}\nu_{tl})} & \dfrac{\nu_{t\ell}E_\ell}{(1-\nu_{\ell t}\nu_{t\ell})} & 0 \\ \dfrac{\nu_{\ell t}E_t}{(1-\nu_{\ell t}\nu_{t\ell})} & \dfrac{E_t}{(1-\nu_{\ell t}\nu_{t\ell})} & 0 \\ 0 & 0 & G_{\ell t} \end{bmatrix} \begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix}
$$

where the "**stiffness**" coefficients appear, as opposed to the Equation 11.1 where the "**compliance**" coefficients appear. To simplify the writing, one can denote

$$
\begin{Bmatrix} \sigma_l \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} = \begin{bmatrix} \bar{E}_\ell & \nu_{t\ell}\bar{E}_\ell & 0 \\ \nu_{\ell t}\bar{E}_t & \bar{E}_t & 0 \\ 0 & 0 & G_{\ell t} \end{bmatrix} \begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} \tag{11.6}
$$

An identical procedure can be followed to arrive at the stress–strain relation:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & (c^2-s^2) \end{bmatrix}}_{[T_1]} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix};
$$

$$
\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \underbrace{\begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & (c^2-s^2) \end{bmatrix}}_{[T_1']} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{11.7}
$$

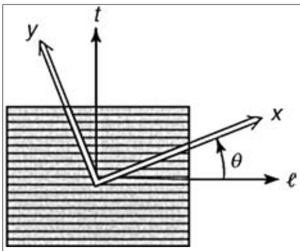
Recall that axes x,y are derived from the axes ℓ,t by a rotation θ about the third axis z. Substituting Equations 11.7 into 11.6, one obtains

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_1] \begin{bmatrix} \bar{E}_\ell & \nu_{t\ell}\bar{E}_\ell & 0 \\ \nu_{\ell t}\bar{E}_t & \bar{E}_t & 0 \\ 0 & 0 & G_{\ell t} \end{bmatrix} [T_1'] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}
$$

which can be rewritten as:

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}
$$

Once the calculations are performed, one obtains the following expressions for the stiffness coefficients $\bar{E}_{ij}$, where $c = \cos\theta$ and $s = \sin\theta$.



$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

with:

$$\begin{aligned}
\bar{E}_{11}(\theta) &= c^4\bar{E}_\ell + s^4\bar{E}_t + 2c^2s^2(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t}) \\
\bar{E}_{22}(\theta) &= s^4\bar{E}_\ell + c^4\bar{E}_t + 2c^2s^2(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t}) \\
\bar{E}_{33}(\theta) &= c^2s^2(\bar{E}_\ell + \bar{E}_t - 2\nu_{t\ell}\bar{E}_\ell) + (c^2 - s^2)^2 G_{\ell t} \\
\bar{E}_{12}(\theta) &= c^2s^2(\bar{E}_\ell + \bar{E}_t - 4G_{\ell t}) + (c^4 + s^4)\nu_{t\ell}\bar{E}_\ell \\
\bar{E}_{13}(\theta) &= -cs\{c^2\bar{E}_\ell - s^2\bar{E}_t - (c^2 - s^2)(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t})\} \\
\bar{E}_{23}(\theta) &= -cs\{s^2\bar{E}_\ell - c^2\bar{E}_t + (c^2 - s^2)(\nu_{t\ell}\bar{E}_\ell + 2G_{\ell t})\}
\end{aligned} \qquad (11.8)$$

expressions in which:

$$\bar{E}_\ell = E_\ell/(1 - \nu_{\ell t}\nu_{t\ell}) : \bar{E}_t = E_t/(1 - \nu_{\ell t}\nu_{t\ell})$$

The rate of variation of stiffness coefficients $\bar{E}_{ij}$ as functions of the angle θ is repre sented in Figure 11.2 for a ply characterized by moduli E_ℓ and E_t with very different values, for example the case of unidirectional layers of fiber/resin.[8]

11.3 CASE OF THERMOMECHANICAL LOADING

11.3.1 Compliance Coefficients

When considering the temperature variations,[9] one must substitute the stress–strain Equation 11.1 with Equation 10.9:

$$\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0 \\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_\ell \\ \alpha_t \\ 0 \end{Bmatrix}$$

[8] See characteristics of the fiber/resin unidirectionals in Paragraph 3.3.3.
[9] See Section 10.5.

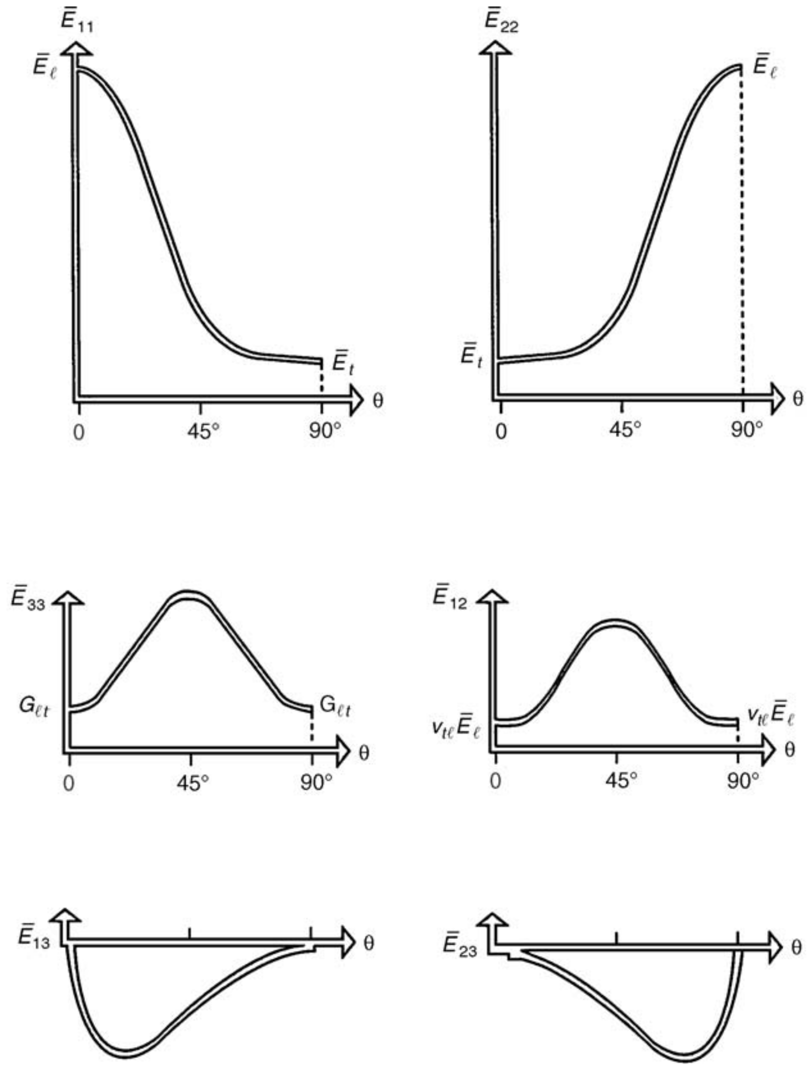


Figure 11.2 Variation of Stiffness Coefficients for a Misaligned Ply

in which α_ℓ and α_t are the coefficients of thermal expansion of the unidirectional layer along the longitudinal direction ℓ and transverse direction t, respectively. Following the same procedure as in Section 11.1 with the same notations, one can write

$$\begin{Bmatrix}\varepsilon \\ \gamma\end{Bmatrix}_{x,y} = [T']\begin{Bmatrix}\varepsilon \\ \gamma\end{Bmatrix}_{\ell,t};\ \{\sigma\}_{\ell,t} = [T]\{\sigma\}_{x,y}$$

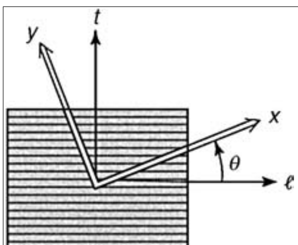
Then, upon substituting,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [T'] \begin{bmatrix} \frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0 \\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \Delta T [T'] \begin{Bmatrix} \alpha_\ell \\ \alpha_t \\ 0 \end{Bmatrix}$$

One finds again in the first part of the second term a matrix of compliance coefficients, the terms of which are described in details in Equation 11.5. The second part of the second term is written as:

$$\Delta T \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \alpha_\ell \\ \alpha_t \\ 0 \end{Bmatrix} = \Delta T \begin{Bmatrix} c^2\alpha_\ell + s^2\alpha_t \\ s^2\alpha_\ell + c^2\alpha_t \\ 2cs(\alpha_t - \alpha_\ell) \end{Bmatrix}$$

Therefore, the thermomechanical relation for a unidirectional layer written in the axes $x.y$, different from the ℓ,t coordinates, can be summarized as follows:



$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\mu_{xy}}{G_{xy}} \\ \frac{\eta_x}{E_x} & \frac{\mu_y}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{Bmatrix} \tag{11.9}$$

$E_x, E_y, G_{xy}, \nu_{xy}, \nu_{yx}, \eta_{xy}, \mu_{xy}$ are given by the relations [11.5]

$$\alpha_x = c^2\alpha_\ell + s^2\alpha_t$$

$$\alpha_y = s^2\alpha_\ell + c^2\alpha_t$$

$$\alpha_{xy} = 2cs(\alpha_t - \alpha_\ell)$$

$$c = \cos\theta; \quad s = \sin\theta$$

11.3.2 Stiffness Coefficients

Inverting Equation 10.9 gives

$$
\begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} =
\begin{bmatrix}
\dfrac{E_\ell}{(1-\nu_{\ell t}\nu_{t\ell})} & \dfrac{\nu_{t\ell}E_\ell}{(1-\nu_{\ell t}\nu_{t\ell})} & 0 \\
\dfrac{\nu_{\ell t}E_t}{(1-\nu_{\ell t}\nu_{t\ell})} & \dfrac{E_t}{(1-\nu_{\ell t}\nu_{t\ell})} & 0 \\
0 & 0 & G_{\ell t}
\end{bmatrix}
\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix}
$$

$$
-\Delta T \begin{Bmatrix}
\dfrac{E_\ell}{(1-\nu_{\ell t}\nu_{t\ell})}\alpha_\ell + \dfrac{\nu_{t\ell}E_\ell}{(1-\nu_{\ell t}\nu_{t\ell})}\alpha_t \\
\dfrac{\nu_{\ell t}E_t}{(1-\nu_{\ell t}\nu_{t\ell})}\alpha_l + \dfrac{E_t}{(1-\nu_{\ell t}\nu_{t\ell})}\alpha_t \\
0
\end{Bmatrix}
$$

Following the procedure of Section 11.2, with the same notations, one can write:

$$
\{\sigma\}_{x,y} = [T_1]\{\sigma\}_{\ell,t}\ ; \quad \begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{\ell,t} = [T_1']\begin{Bmatrix} \varepsilon \\ \gamma \end{Bmatrix}_{x,y}
$$

where after substitution:

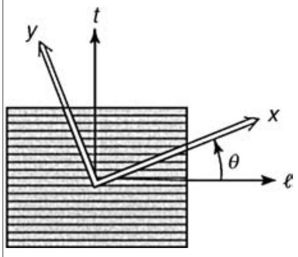
$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_1]
\begin{bmatrix}
\bar{E}_\ell & \nu_{t\ell}\bar{E}_\ell & 0 \\
\nu_{\ell t}\bar{E}_t & \bar{E}_t & 0 \\
0 & 0 & G_{\ell t}
\end{bmatrix}
[T_1']\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}
-\Delta T[T_1]\begin{Bmatrix} \bar{E}_\ell\alpha_\ell + \nu_{t\ell}\bar{E}_\ell\alpha_t \\ \nu_{\ell t}\bar{E}_t\alpha_\ell + \bar{E}_t\alpha_t \\ 0 \end{Bmatrix}
$$

One finds again, in the first part of the second term, the matrix detailed in Equation 11.8. The second part of the second term can be developed as follows:

$$
-\Delta T\begin{bmatrix}
c^2 & s^2 & 2cs \\
s^2 & c^2 & -2cs \\
-cs & cs & (c^2-s^2)
\end{bmatrix}
\begin{Bmatrix} \bar{E}_\ell\alpha_\ell + \nu_{t\ell}\bar{E}_\ell\alpha_t \\ \nu_{\ell t}\bar{E}_t\alpha_\ell + \bar{E}_t\alpha_t \\ 0 \end{Bmatrix} =
$$

$$
-\Delta T\begin{Bmatrix}
c^2\bar{E}_\ell(\alpha_\ell + \nu_{t\ell}\alpha_t) + s^2\bar{E}_t(\nu_{\ell t}\alpha_\ell + \alpha_t) \\
s^2\bar{E}_\ell(\alpha_\ell + \nu_{t\ell}\alpha_t) + c^2\bar{E}_t(\nu_{\ell t}\alpha_\ell + \alpha_t) \\
cs[\bar{E}_t(\nu_{\ell t}\alpha_\ell + \alpha_t) - \bar{E}_\ell(\alpha_\ell + \nu_{t\ell}\alpha_t)]
\end{Bmatrix}
$$

Therefore, the thermomechanical behavior of a unidirectional layer in the coordinate axes x,y can be written in the following form, in terms of the properties in the ℓ,t coordinates.



$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}\bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13}\\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23}\\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}-\Delta T\begin{Bmatrix}\overline{\alpha E_1}\\ \overline{\alpha E_2}\\ \overline{\alpha E_3}\end{Bmatrix} \quad (11.10)$$

$\bar{E}_{11}\ \bar{E}_{22}\ \bar{E}_{33}\ \bar{E}_{12}\ \bar{E}_{13}\ \bar{E}_{23}$ are given by the relations [11.8]

$$\overline{\alpha E_1} = c^2\bar{E}_\ell(\alpha_\ell+\nu_{t\ell}\alpha_t)+s^2\bar{E}_t(\nu_{\ell t}\alpha_\ell+\alpha_t)$$

$$\overline{\alpha E_2} = s^2\bar{E}_\ell(\alpha_\ell+\nu_{t\ell}\alpha_t)+c^2\bar{E}_t(\nu_{\ell t}\alpha_\ell+\alpha_t)$$

$$\overline{\alpha E_3} = cs[\bar{E}_t(\nu_{\ell t}\alpha_\ell+\alpha_t)-\bar{E}_\ell(\alpha_\ell+\nu_{t\ell}\alpha_t)]$$

$$c = \cos\theta\,;\ s = \sin\theta$$

$$\bar{E}_\ell = E_\ell/(1-\nu_{\ell t}\nu_{t\ell})$$

$$\bar{E}_t = E_t/(1-\nu_{\ell t}\nu_{t\ell})$$