10

ELASTIC CONSTANTS OF UNIDIRECTIONAL COMPOSITES

In this chapter we examine a distinct combination of two materials (matrix and fiber), with simple geometry and loading conditions, in order to estimate the elastic properties of the equivalent material, i.e., of the composite.

10.1 LONGITUDINAL MODULUS E_ℓ

The two materials are shown schematically in Figure 10.1 where

m stands for matrix.
f stands for fiber.

- **Hypothesis:** The two materials are bonded together. More precisely, one makes the following assumptions:
- Both the matrix m and the fiber f have the same longitudinal strain ε_l.
- The interface between the two materials allows the z normal strains in the two materials to be different.

$$\underset{(m)}{\varepsilon_z} \neq \underset{(f)}{\varepsilon_z}$$

The state of stresses resulting from a force F can therefore be written as:

$$\underset{(m)}{\Sigma} \rightarrow \begin{bmatrix} \sigma_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(m)} \qquad \underset{(f)}{\Sigma} \rightarrow \begin{bmatrix} \sigma_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(f)}$$

and the corresponding state of strains:

$$\underset{(m)}{\varepsilon} \rightarrow \begin{bmatrix} \varepsilon_\ell & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}_{(m)} \qquad \underset{(f)}{\varepsilon} \rightarrow \begin{bmatrix} \varepsilon_\ell & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}_{(f)}$$

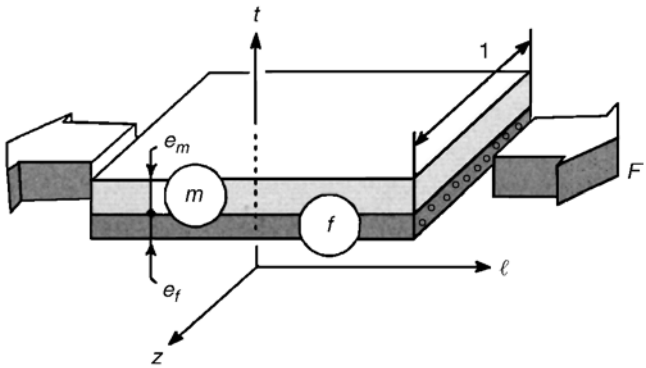

Figure 10.1 Longitudinal Modulus E_ℓ

Each material is assumed to be linear elastic and isotropic, with the following stress–strain relation:

$$\varepsilon = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\,\text{trace}\,(\Sigma)\mathbf{I} \tag{10.1}$$

in which $\boldsymbol{\varepsilon}$ represents the strain tensor, Σ represents the stress tensor, and I the unity tensor. E and ν are the elastic constants of the considered material.

For the composite $(m+f)$, one uses Equation 9.5 with restriction to the plane l,t. It reduces to:

$$\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0 \\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix}$$

The stress $\sigma_{\ell(m+f)}$ can be written as (see Figure 10.1 above):

$$\underset{(m)+(f)}{\sigma_\ell} = \frac{F}{S} = \frac{F}{(e_m+e_f)\times 1} = \underset{(m)}{\sigma_\ell}\frac{e_m}{e_m+e_f} + \underset{(f)}{\sigma_\ell}\frac{e_f}{e_m+e_f}$$

which can be written in terms of the volume fractions of the fiber and the matrix as[1]

$$\underset{(m)+(f)}{\sigma_\ell} = \underset{(m)}{\sigma_\ell} V_m + \underset{(f)}{\sigma_\ell} V_f$$

[1] See Section 3.2.2.

Expressing the stresses in terms of the strains for each material yields

$$E_\ell \varepsilon_\ell = E_m \varepsilon_\ell V_m + E_f \varepsilon_\ell V_f$$

then:

$$\boxed{E_\ell = E_m V_m + E_f V_f} \tag{10.2}$$

Note: Among the real phenomena that are not taken into account in the expression of E_l is the lack of perfect straightness of the fibers in the matrix. Also, the modulus E_l depends on the sign of the stress (tension or compression). In rigorous consideration, the material is "**bimodulus**."

Example: Unidirectional layers with 60% fiber volume fraction ($V_f = 0.60$) with epoxy matrix:

	Kevlar	"HR" Carbon	"HM" Carbon
E_ℓ tension (MPa)	85,000	134,000	180,000
E_ℓ compression (MPa)	80,300	134,000	160,000

10.2 POISSON COEFFICIENT

Considering again the loading defined in the previous paragraph, the transverse strain for the matrix *m* and fiber *f* can be written as:

$$\varepsilon_t = -\frac{\nu}{E}\sigma_\ell = -\nu\varepsilon_\ell$$

and for the composite ($m + f$):

$$\underset{(m)+(f)}{\varepsilon_t} = \frac{\nu_{\ell t}}{E_\ell} \times \underset{(m)+(f)}{\sigma_\ell} = -\nu_{\ell t}\varepsilon_\ell$$

The strain in the transverse direction can also be written as:

$$\underset{(m)+(f)}{\varepsilon_t} = \frac{\Delta(e_m + e_f)}{e_m + e_f} = \frac{\Delta e_m}{e_m} V_m + \frac{\Delta e_f}{e_f} V_f$$

$$\underset{(m)+(f)}{\varepsilon_t} = \underset{(m)}{\varepsilon_t} V_m + \underset{(f)}{\varepsilon_t} V_f$$

Because ε_ℓ has the same value in *m* and *f*:

$$-\nu_{\ell t}\varepsilon_\ell = -\nu_m \varepsilon_\ell V_m - \nu_f \varepsilon_\ell V_f$$

$$\boxed{\nu_{\ell t} = \nu_m V_m + \nu_f V_f} \tag{10.3}$$

10.3 TRANSVERSE MODULUS E_t

To evaluate the modulus along the transverse direction E_t, the two materials are shown in the Figure 10.2. In addition, one uses the following simplifications:

- **Hypothesis:** At the interface between the two materials, assume the following:
  - Freedom of movement in the l direction allows for different strains in the two materials:

$$\underset{(m)}{\varepsilon_{\ell}} \neq \underset{(f)}{\varepsilon_{\ell}}$$

  - Freedom of movement in the z direction allows for different strains in the two materials:

$$\underset{(m)}{\varepsilon_{z}} \neq \underset{(f)}{\varepsilon_{z}}$$

Then, the state of stress created by a load F (see Figure 10.2), can be reduced for each material to the following:

$$\Sigma \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

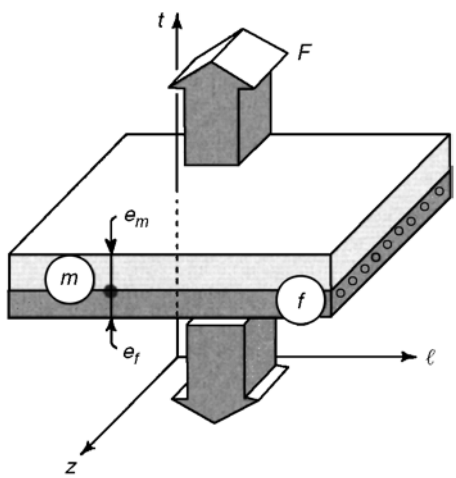

Figure 10.2 Transverse Modulus E_t

The strains can be written as:

$$\varepsilon \rightarrow \begin{bmatrix} \varepsilon_\ell & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \underset{\textcircled{f}}{\overset{\textcircled{m}}{\text{or}}}$$

Then for the composite $(m + f)$, one has

$$\varepsilon_t = \frac{1}{E_t}\sigma_t$$

On the other hand, using direct calculation leads to (see Figure 10.2)

$$\varepsilon_t = \frac{\Delta(e_m + e_f)}{e_m + e_f} = \underset{\textcircled{m}}{\varepsilon_t V_m} + \underset{\textcircled{f}}{\varepsilon_t V_f}$$

then:

$$\frac{1}{E_t}\sigma_t = \frac{1}{E_m}\sigma_t V_m + \frac{1}{E_f}\sigma_t V_f$$

$$\boxed{\frac{1}{E_t} = \frac{V_m}{E_m} + \frac{V_f}{E_f} \quad \text{or} \quad E_t = E_m\left[\frac{1}{(1 - V_f) + \frac{E_m}{E_f}V_f}\right]} \qquad (10.4)$$

Remarks:

- Due to the above simplifications that allow the possibility for relative sliding along the l and z directions at the interface, the transverse modulus E_t above may not be accurate.
- One finds in the technical literature many more complex formulae giving E_t. However, none can provide guaranteed good result.
- Taking into consideration the load applied (see Figure 10.2),: the modulus E_f that appears in Equation 10.4 is the modulus of elasticity of the fiber in a direction that is perpendicular to the fiber axis. This modulus can be very different from the modulus along the axis of the fiber, due to the anisotropy that exists in fibers.[2]

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[2] This point was discussed in Paragraph 3.3.1.

10.4 SHEAR MODULUS $G_{\ell t}$

Load application that can be used to evaluate the shear modulus $G_{\ell t}$ is shown schematically in the Figure 10.3, both with the angular deformations that are produced. The state of stress, identical for both the matrix and fiber material, can be written as:

$$\Sigma \rightarrow \begin{bmatrix} 0 & \tau_{\ell t} & 0 \\ \tau_{\ell t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding strains can be written as:

$$\underset{(m) \text{ or } (f)}{\varepsilon} \rightarrow \begin{bmatrix} 0 & \varepsilon_{\ell t} & 0 \\ \varepsilon_{\ell t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the constitutive equation, one has

$$\varepsilon_{\ell t} = \frac{1+\nu}{E}\tau_{\ell t} = \frac{\tau_{\ell t}}{2G}$$

then:

$$\gamma_{\ell t} = \frac{\tau_{\ell t}}{G}$$

Also, from Figure 10.3, one has

$$\underset{(m)+(f)}{\gamma_{\ell t}} (e_m + e_f) = \underset{(m)}{\gamma_{\ell t}} e_m + \underset{(f)}{\gamma_{\ell t}} e_f$$

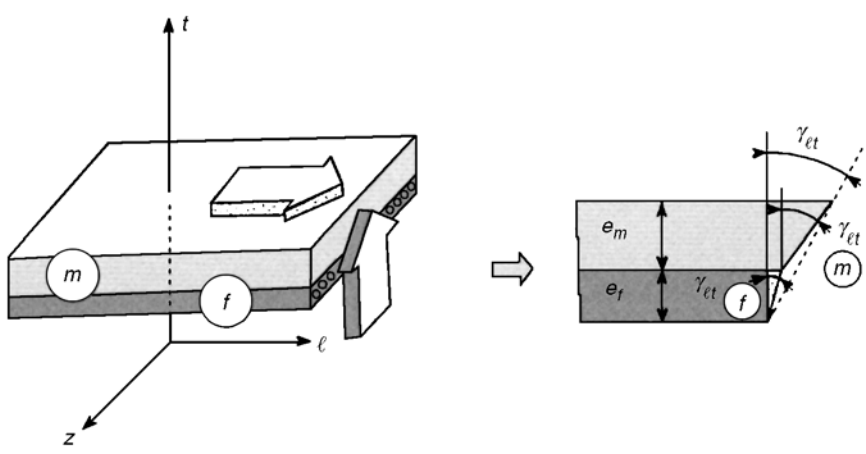

Figure 10.3 Shear modulus $G_{\ell t}$

which can be rewritten as:

$$\underset{(m)+(f)}{\gamma_{\ell t}} = \underset{(m)}{\gamma_{\ell t}} V_m + \underset{(f)}{\gamma_{\ell t}} V_f$$

$$\frac{\tau_{\ell t}}{G_{\ell t}} = \frac{\tau_{\ell t}}{G_m} V_m + \frac{\tau_{\ell t}}{G_f} V_f$$

$$\frac{1}{G_{\ell t}} = \frac{V_m}{G_m} + \frac{V_f}{G_f}$$

$$\boxed{G_{\ell t} = G_m \left[\frac{1}{(1 - V_f) + \frac{G_m}{G_f} V_f} \right]} \qquad (10.5)^{3}$$

10.5 THERMOELASTIC PROPERTIES

10.5.1 Isotropic Material: Recall

When the influence of temperature is taken into consideration, Hooke's law for the case of no temperature influence:

$$\varepsilon = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\,\text{trace}(\mathbf{\Sigma})\mathbf{I}$$

is replaced by the *Hooke–Duhamel* law:

$$\varepsilon = \frac{1+\nu}{E}\Sigma - \frac{\nu}{E}\,\text{trace}(\mathbf{\Sigma})\,\mathbf{I} + \alpha\Delta T\,\mathbf{I} \qquad (10.6)$$

where

ε = Strain tensor
Σ = Stress tensor
I = Unity tensor
E, ν = Elastic constants for the considered material
α = Coefficient of thermal expansion[4]
ΔT = Change in temperature with respect to a reference temperature at which the stresses and strains are nil

10.5.2 Case of Unidirectional Composite

The coefficient of thermal expansion of the matrix is usually much larger (more than ten times) than that of the fiber.[4] In Figure 10.4, one can imagine that even in the absence of mechanical loading, a change in temperature ΔT will produce a

[3] A few values of the shear modulus G_f are shown in Section 3.3.1.
[4] See Section 1.6, "Principal Physical Properties."

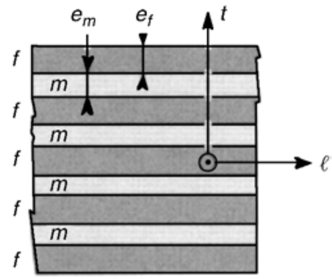

Figure 10.4 Unidirectional Composite

longitudinal strain in the composite. This longitudinal strain has a value that is intermediate between the strain of the fiber alone and that of the matrix alone. Then, in the composite one finds internal stresses along the direction *l* (along the direction *t*, the fiber and matrix can expand differently). One then has:

- For the stresses:

$$\underset{(m)}{\Sigma} \rightarrow \begin{bmatrix} \sigma_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(m)} \qquad \underset{(f)}{\Sigma} \rightarrow \begin{bmatrix} \sigma_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{(f)}$$

- For the strains:

$$\underset{(m)}{\varepsilon} \rightarrow \begin{bmatrix} \varepsilon_\ell & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}_{(m)} \qquad \underset{(f)}{\varepsilon} \rightarrow \begin{bmatrix} \varepsilon_\ell & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}_{(f)}$$

10.5.2.1 Coefficient of Thermal Expansion along the Direction l

One has for the fiber and the matrix, respectively:

$$\underset{(m)}{\varepsilon_\ell} = \frac{\sigma_\ell}{E_m}(m) + \alpha_m \Delta T = \underset{(f)}{\varepsilon_\ell} = \frac{\sigma_\ell}{E_f}(f) + \alpha_f\,\Delta T$$

The external equilibrium can be written as (see Figure 10.4):

$$\underset{(m)}{\sigma_\ell} \times e_m + \underset{(f)}{\sigma_\ell} \times e_f = 0$$

where, taking into account the equality of the strains:

$$\frac{\sigma_\ell}{E_m}\textcircled{m} + \alpha_m \Delta T = -\underset{\textcircled{m}}{\sigma_\ell} \times \frac{e_m}{e_f} \times \frac{1}{E_f} + \alpha_f \,\Delta T$$

$$\underset{\textcircled{m}}{\sigma_\ell} = \frac{(\alpha_f - \alpha_m)\Delta T}{\frac{1}{E_m} + \frac{e_m}{e_f} \times \frac{1}{E_f}} = \frac{(\alpha_f - \alpha_m)\Delta T}{\frac{1}{E_m} + \frac{V_m}{V_f} \times \frac{1}{E_f}}$$

V represents the volume fraction. The longitudinal strain can then be written as:

$$\underset{\textcircled{m}}{\varepsilon_\ell} = \underset{\textcircled{f}}{\varepsilon_\ell} = \frac{(\alpha_f \, E_f V_f + \alpha_m \, E_m V_m)(\Delta T)}{E_f V_f + E_m V_m}$$

It is also the longitudinal strain that is created only by the effect of temperature:

$$\underset{\textcircled{m}+\textcircled{f}}{\varepsilon_\ell} = \alpha_\ell \Delta T$$

where α_ℓ is the longitudinal coefficient of thermal expansion. One can then equate the above expressions to obtain:

$$\boxed{\alpha_\ell = \frac{\alpha_f \, E_f V_f + \alpha_m \, E_m V_m}{E_f V_f + E_m V_m}} \qquad (10.7)$$

10.5.2.2 Coefficient of Thermal Expansion along the Transverse Direction t

The global thermal strain can be written as (see Figure 10.4):

$$\underset{\textcircled{m}+\textcircled{f}}{\varepsilon_t} = \frac{\Delta(e_m + e_f)}{e_m + e_f} = \underset{\textcircled{m}}{\varepsilon_t}\frac{e_m}{e_m + e_f} + \underset{\textcircled{f}}{\varepsilon_t}\frac{e_f}{e_m + e_f}$$

then:

$$\underset{\textcircled{m}+\textcircled{f}}{\varepsilon_t} = \underset{\textcircled{m}}{\varepsilon_t} \times V_m + \underset{\textcircled{f}}{\varepsilon_t} \times V_f$$

Using the Hooke and Duhamel law (Equation 10.6)[5]:

$$\underset{\textcircled{m}+\textcircled{f}}{\varepsilon_t} = \left(-\frac{V_m}{E_m}\underset{\textcircled{m}}{\sigma_\ell} + \alpha_m \Delta T\right)V_m + \left(-\frac{\nu_f}{E_f}\underset{\textcircled{f}}{\sigma_\ell} + \alpha_f \,\Delta T\right)V_f$$

[5] For the Poisson coefficients of common fibers, see Section 3.3.1.

Using the expressions for stresses obtained before, one obtains:

$$\underset{\textcircled{m}+\textcircled{f}}{\varepsilon_t} = \left\{(\alpha_m V_m + \alpha_f V_f) + \frac{(\nu_f E_m - \nu_m E_f)}{E_m V_m + E_f V_f} V_m V_f(\alpha_f - \alpha_m)\right\}\Delta T$$

The quantity between the brackets represent the coefficient of thermal expansion along the transverse direction t, α_t, which can be written as:

$$\alpha_t = \alpha_m V_m + \alpha_f V_f + \frac{(\nu_f E_m - \nu_m E_f)}{\frac{E_m}{V_f} + \frac{E_f}{V_m}} \times (\alpha_f - \alpha_m) \tag{10.8}$$

10.5.3 Thermomechanical Behavior of a Unidirectional Layer

Under the combined effect of the stresses and temperature, the global thermomechanical strains of a unidirectional layer can be obtained using the following relation:

$$\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & 0 \\ -\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_\ell \\ \alpha_t \\ 0 \end{Bmatrix} \tag{10.9}$$

in which the coefficients $E_\ell, E_t, \nu_{\ell t}$, $G_{\ell t}$, α_ℓ and α_t have the values given by the Equations 10.2 to 10.8, respectively.