17

FLEXURE OF THICK COMPOSITE PLATES

The mechanical behavior of a laminated plate as studied in Chapter 12 involves the definition of stress resultants N_x, N_y, T_{xy} and moment resultants M_x, M_y, M_{xy}. These resultants are obtained from the membrane stresses σ_x, σ_y, τ_{xy}. The other stress components, σ_z, τ_{xz}, τ_{yz} have not been taken into account until now.

In this chapter we note how these stresses exist and have influence on the mechanical behavior of the laminate. We will also examine the configurations of plates for which the influence of these stresses are significant (for example, plates with relatively high thicknesses, justifying the title given for this chapter). This study is based on the previous definition of the displacement parameters based on integral displacement forms, and constitutes an approach analogous (and also original) to that used in Chapter 15 for the description of the bending of composite beams.

17.1 PRELIMINARY REMARKS

17.1.1 Transverse Normal Stress σ_z

The plate is situated as in Chapter 12 from which the name of transverse normal stress σ_z. Such stress appears due to the application of a transverse load (concentrated or distributed) which will cause bending of the plate.

- A very local concentration of load in a very small zone cannot be examined with the theory of plates, which is not able to provide spatial distribution of the stresses in the neighborhood of the point of load application. This phenomenon is complex even in three-dimensional numerical modeling. Therefore, what will be presented will not be valid in the immediate surroundings of a very local transverse load (for example, an insert).
- A distributed load gives rise to the stresses σ_z with small amplitude as compared with the stresses σ_x and σ_y. This is the reason why σ_z is often neglected.

17.1.2 Transverse Shear Stresses τ_{xz} and τ_{yz}

Due to the assumption of perfect bonding between the plies, the stress vector remains continuous across an interfacial element with normal vector $\vec{n} = \vec{z}$, between two consecutive plies of the laminate. Then τ_{xz} and τ_{yz} remain continuous at the interfaces between plies (see Section 15.1.2). In addition, the upper face and lower face of the laminate are assumed to be free of tangential forces. The thickness of the laminate is denoted as h. One then has:

$$\tau_{xz} = \tau_{yz} = 0 \quad \text{for } z = \pm h/2$$

Assume stress and moment resultants to be constant in a given zone of the laminate:

$$N_x,\ N_y,\ T_{xy},\ M_y,\ M_x,\ M_{xy} \ \text{constants} \ \forall (x,\ y)$$

Then, by inversion of Equation 12.20, for example, one notes that the following global strain

$$\varepsilon_{ox},\ \varepsilon_{oy},\ \gamma_{oxy},\ \partial^2 w_0/\partial x^2,\ \partial^2 w_0/\partial y^2,\ 2\partial^2 w_0/\partial x \partial y$$

are constant in the zone under consideration. The local strains of Equation 12.12 then depend only on the coordinate z of the laminate. This is the same for the membrane stresses σ_x, σ_y, τ_{xy}.

With the above consideration, local equilibrium can be written as (in the absence of body forces):

$$\begin{aligned} \frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}'}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \end{aligned} \tag{17.1}$$

The transverse shear stresses then appear to be constant across the thickness of a ply. Being continuous at the interface and nil at the location $z = \pm h/2$, they are nil in all the thickness of the laminate.

From this, these stresses do not play an important role in all cases: they do not always exist, their existence being related to variable stresses and moment resultants. When they exist and depending on the composition of the laminate, they can have influence on the deformation in bending, and on the interlaminar adhesion (between layers).

We will assume that these stresses exist, associated with the hypotheses of the following paragraph.

17.1.3 Hypotheses

- The plate has midplane symmetry.
- The plies are orthotropic, the orthotropic axes coinciding with the x, y, z axes of the laminate.[1]
- The stress σ_z is negligible.

[1] For example, this is the case for a laminate made of layers of balanced fabric at 0°, 90°, or 45°, –45°, for unidirectional layers at 0° and 90°, or for mats. Instead of this hypothesis, one can also adopt the less restrictive hypothesis of a balanced laminate. In this case the following calculations are much more involved, without appreciable gain on the enlargement of the field of applications examined in Section 17.6.3.

Remarks:

- For each ply having orthotropic axes x, y, z, the constitutive Equation 13.3 can be written as, taking into account the simplification σ_z # 0:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}$$

or under inverse form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & 0 & 0 & 0 \\ \bar{E}_{21} & \bar{E}_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{E}_{33} = G_{xy} & 0 & 0 \\ 0 & 0 & 0 & \bar{E}_{44} = G_{xz} & 0 \\ 0 & 0 & 0 & 0 & \bar{E}_{55} = G_{yz} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (17.2)$$

where:

$$\bar{E}_{11} = \frac{E_x}{1 - \nu_{xy}\nu_{yx}}; \quad \bar{E}_{12} = \frac{\nu_{yx}E_x}{1 - \nu_{xy}\nu_{yx}}; \quad \bar{E}_{22} = \frac{E_y}{1 - \nu_{xy}\nu_{yx}}$$

- The transverse shear is at the origin of distortions as illustrated in Figure 17.1 for the shear stress τ_{yz}.

As a consequence, the displacements due to flexion discussed in Section 12.2.1 can be adapted as shown in Figure 17.2.

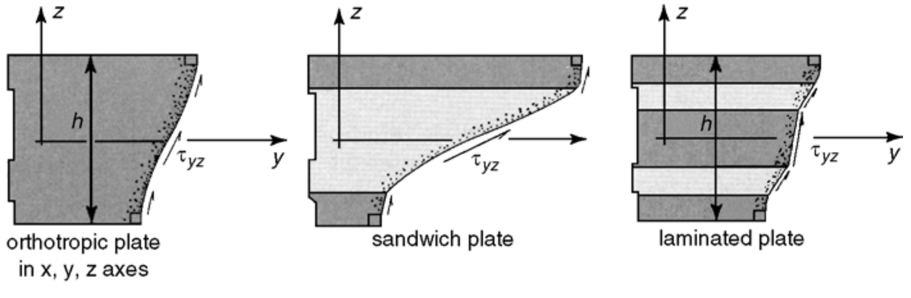

Figure 17.1 Distortion of Section due to Transverse Shear τ_{yz}

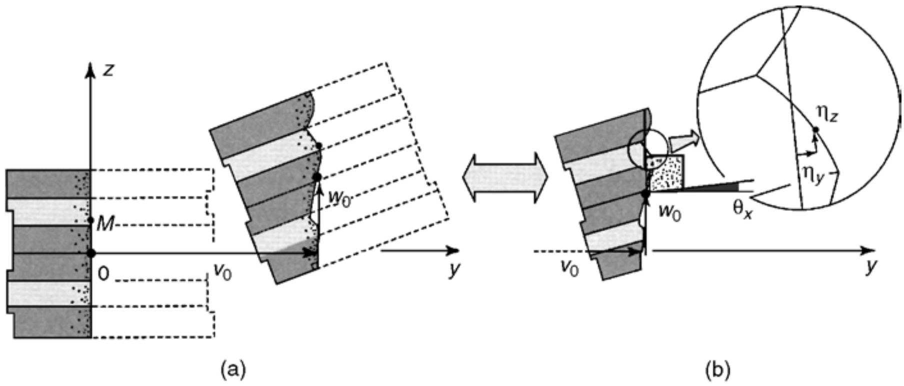

Figure 17.2 Flexural Displacements

In this figure, (a) represents a section before and after bending and (b) shows the evolution of the section as a rigid displacement (parameters v_0, w_0, and θ_x) to which are associated increments η_y and η_z in the plane y,z. Note: Due to the existence of midplane symmetry, the antisymmetric manner with respect to z, these increments are small, but they cannot be neglected *a priori*. (At this stage, we do not have a definition for the equivalent rotation, noted as θ_x in b).

This justifies the interest in the definition of the displacement field relating these increments. A supplementary interest rests in the possibility, during the study, to look after the necessary approximations more closely to lead to a useful technical formulation.[2]

17.2 DISPLACEMENT FIELD

The elastic displacement at each point of the laminate has the components $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$. With simplified description of Paragraph 12.2.1, one sees in Figure 17.2 (b), average translations denoted as v_0 and w_0, and a rotation of the section denoted as θ_x, to which one superimposes the supplementary displacements η_y and η_z. We will define these averages in integral forms as follows:

- **Translation along *x* direction:** By definition, this is u_0 such that:

$$u_0 = \frac{1}{h}\int_{-h/2}^{h/2} u(x, y, z)\,dz$$

- **Rotation about the *y* axis:** By definition, this is $\theta_y(x,y)$ such that[3]:

$$\theta_y = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)u(x, y, z) \times z\,dz$$

[2] Approximations that do not appear neatly in the specialized literature.
[3] Such a definition of the "average rotation" θ_y will be fundamental in the following to ensure the energetic coherence of the formulation for the transverse shears (see Section 17.6.6).

where one has reused the notations of Section 12.1.6 for the terms $\frac{1}{\overline{EI}_{ij}}$.[4]

The longitudinal displacement $u(x, y, z)$ then takes the form:

$$u(x,y,z) = u_0(x,y) + z\theta_y(x,y) + \eta_x(x,y,z)$$

with:

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)\eta_x \; z\,dz = 0$$

In effect, one can obtain starting from this expression:

$$\int_{-h/2}^{h/2} u dz = h \times u_o + \theta_y \int_{-h/2}^{h/2} z\,dz + \int_{-h/2}^{h/2} \eta_x\,dz$$

the integrals disappearing due to antisymmetry in z[5]:

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)uz\,dz = u_o\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)z\,dz \ldots$$

$$\cdots + \theta_y + \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)\eta_x \; z\,dz$$

In the second member, the first integral disappears due to midplane symmetry. In addition, taking into account the definition of θ_y written above, the second integral is also nil.

- **Translation along the *y* direction:** This is $v_0(x,y)$ such that:

$$v_0 = \frac{1}{h}\int_{-h/2}^{h/2} v(x,y,z)dz$$

- **Rotation about the *x* axis:** This is θ_x such that:

$$\theta_x = -\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{EI}_{22}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)v(x,y,z) \times z\,dz$$

The longitudinal displacement $v(x,y,z)$ then takes the form:

$$v(x,y) = v_0(x,y) - z\theta_x(x,y) + \eta_y(x,y,z)$$

[4] Recall that (Section 12.1.6):

$$\left[\frac{1}{\overline{EI}}\right] = [C]^{-1}, \quad \text{where} \quad C_{ij} = \sum_{k=1^{st}\text{ ply}}^{n^{th}\text{ ply}} \bar{E}_{ij}^{k}\left(\frac{z_k^3 - z_{k-1}^3}{3}\right)$$

[5] The coefficient of θ_y is 1 because one can note that:

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{EI}_{11}} + \frac{\bar{E}_{12}}{\overline{EI}_{12}}\right)z^2 dz = \frac{C_{11}}{\overline{EI}_{11}} + \frac{C_{12}}{\overline{EI}_{12}} = \frac{C_{11}C_{22}}{C_{11}C_{22} - C_{12}^2} - \frac{C_{12}^2}{C_{11}C_{22} - C_{12}^2} = 1$$

with:

$$\int_{-h/2}^{(h/2)}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\eta_y\ z\,dz = 0$$

- **Translation along the *z* direction:** This is $w_0(x,y)$ such that:

$$w_0(x,y) = \frac{1}{h}\int_{-h/2}^{(h/2)} w(x,y,z)dz$$

The vertical displacement takes the form:

$$w(x,y,z) = w_0(x,y)+\eta_z(x,y,z)$$

In summary, one obtains for the elastic displacement field:

$$\begin{aligned} u &= u_0 + z\theta_y + \eta_x(x,y,z) \\ v &= v_0 - z\theta_x + \eta_y(x,y,z) \\ w &= w_0 \quad\quad\;\; + \eta_z(x,y,z) \end{aligned} \tag{17.3}$$

$$\eta_x,\ \eta_y,\ \eta_z \ \text{ antisymmetric in } z. \tag{17.4}$$

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\eta_x\ z\,dz = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\eta_y\ z dz = 0 \tag{17.5}$$

17.3 STRAINS

One deduces from the previous displacements the strains:

$$\begin{aligned} \varepsilon_x &= \varepsilon_{0x} + z\frac{\partial\theta_y}{\partial x} + \frac{\partial\eta_x}{\partial x} \\ \varepsilon_y &= \varepsilon_{0y} - z\frac{\partial\theta_x}{\partial y} + \frac{\partial\eta_y}{\partial y} \\ \gamma_{xy} &= \gamma_{0xy} + z\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right) + \frac{\partial\eta_x}{\partial y} + \frac{\partial\eta_y}{\partial x} \\ \gamma_{xz} &= \frac{\partial w_0}{\partial x} + \theta_y + \frac{\partial\eta_x}{\partial z} + \frac{\partial\eta_z}{\partial x} \\ \gamma_{yz} &= \frac{\partial w_0}{\partial y} - \theta_x + \frac{\partial\eta_y}{\partial z} + \frac{\partial\eta_z}{\partial y} \end{aligned} \tag{17.6}$$

17.4 CONSTITUTIVE RELATIONS

17.4.1 Membrane Equations

Recall the method that was already used in Section 12.1.1.

- stress resultant $N_x = \int_{-h/2}^{h/2} \sigma_x dx$: from [17.2] et [17.6][6]:

$$N_x = \int_{-h/2}^{h/2} \bar{E}_{11}\left(\varepsilon_{0x} + z\frac{\partial \theta_y}{\partial x} + \frac{\partial \eta_x}{\partial x}\right)dz + \int_{-h/2}^{h/2} \bar{E}_{12}\left(\varepsilon_{0y} - z\frac{\partial \theta_x}{\partial y} + \frac{\partial \eta_y}{\partial y}\right)dz$$

$$N_x = A_{11}\varepsilon_{0x} + A_{12}\varepsilon_{0y} + \frac{\partial}{\partial x}\int_{-h/2}^{h/2} \bar{E}_{11}\eta_x dz + \frac{\partial}{\partial y}\int_{-h/2}^{h/2} \bar{E}_{12}\eta_y dz$$

- stress resultant $N_y = \int_{-h/2}^{h/2} \sigma_y \, dz$:

$$N_y = A_{21}\varepsilon_{0x} + A_{22}\varepsilon_{0y}$$

- stress resultant $T_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \, dz$:

$$T_{xy} = \int_{-h/2}^{h/2} \bar{E}_{33}\left(\gamma_{oxy} + z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x}\right) + \frac{\partial \eta_x}{\partial y} + \frac{\partial \eta_y}{\partial x}\right)dz$$

$$T_{xy} = A_{33}\gamma_{oxy}$$

In summary, one finds again the relations already established in Chapter 12 (Equations 12.5) as:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}$$

or, in inverse form, by using the notations in Equation 12.9:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \times \frac{1}{h}\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \frac{1}{h}\begin{bmatrix} 1/\bar{E}_x & -\bar{\nu}_{yx}/\bar{E}_y & 0 \\ -\bar{\nu}_{xy}/\bar{E}_x & 1/\bar{E}_y & 0 \\ 0 & 0 & 1/\bar{G}_{xy} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} \quad (17.7)$$

17.4.2 Bending Behavior

One has again the already known moment resultants (see Section 12.2.1).

- Moment resultant $M_y = \int_{-h/2}^{h/2} \sigma_x z dz$:
  with [17.2] and [17.5]:

$$M_y = \int_{-h/2}^{h/2} \bar{E}_{11}\left(z\varepsilon_{ox} + z^2\frac{\partial \theta_y}{\partial x} + z\frac{\partial \eta_x}{\partial x}\right)dz \cdots$$

$$\cdots + \int_{-h/2}^{h/2} \bar{E}_{12}\left(z\varepsilon_{oy} - z^2\frac{\partial \theta_x}{\partial y} + z\frac{\partial \eta_y}{\partial y}\right)dz$$

$$M_y = C_{11}\frac{\partial \theta_y}{\partial x} + C_{12} \times -\frac{\partial \theta_x}{\partial y} + \frac{\partial}{\partial x}\int_{-h/2}^{h/2} \bar{E}_{11}\eta_x z dz + \frac{\partial}{\partial y}\int_{-h/2}^{h/2} \bar{E}_{12}\eta_y z dz$$

[6] The simplifications are due to the antisymmetry of the integrated functions (midplane symmetry).

In the last two terms there appear the nonzero integrals of even functions. If one neglects the contribution of the rates of variation along the x and y direction, respectively, of these terms, the previous equation is reduced to[7]

$$M_y = C_{11}\frac{\partial\theta_y}{\partial x} + C_{12}\times -\frac{\partial\theta_x}{\partial y}$$

- Moment resultant $M_x = -\int_{-h/2}^{h/2} \sigma_y z\,dz$:

$$-M_x = \int_{-h/2}^{h/2} \bar{E}_{12}\left(z\varepsilon_{ox} + z^2\frac{\partial\theta_y}{\partial x} + z\frac{\partial\eta_x}{\partial x}\right)dz\ldots$$

$$\cdots + \int_{-h/2}^{h/2} \bar{E}_{22}\left(z\varepsilon_{oy} - z^2\frac{\partial\theta_x}{\partial y} + z\frac{\partial\eta_y}{\partial y}\right)dz$$

which is reduced to:

$$-M_x = C_{12}\frac{\partial\theta_y}{\partial x} + C_{22}\times -\frac{\partial\theta_x}{\partial y} + \frac{\partial}{\partial x}\int_{-h/2}^{h/2} \bar{E}_{12}\eta_x z\,dz + \frac{\partial}{\partial y}\int_{-h/2}^{h/2} \bar{E}_{22}\eta_y z\,dz$$

and in neglecting the contribution of the last two terms[7]:

$$-M_x = C_{12}\frac{\partial\theta_y}{\partial x} + C_{22}\times -\frac{\partial\theta_x}{\partial y}$$

- Moment resultant $M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z dz$

$$-M_{xy} = \int_{-h/2}^{h/2} \bar{E}_{33}\left(z\gamma_{oxy} + z^2\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right) + z\frac{\partial\eta_x}{\partial y} + z\frac{\partial\eta_y}{\partial x}\right)dz$$

which is reduced to:

$$-M_{xy} = C_{33}\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right) + \frac{\partial}{\partial y}\int_{-h/2}^{h/2} \bar{E}_{33}\eta_x z dz + \frac{\partial}{\partial x}\int_{-h/2}^{h/2} \bar{E}_{33}\eta_y z dz$$

and in neglecting the contribution of the variations of the differences η_x and η_y[7]:

$$-M_{xy} = C_{33}\left(\frac{\partial\theta_y}{\partial y} - \frac{\partial\theta_x}{\partial x}\right)$$

[7] The existence of such approximation does not appear if one neglects *a priori* the increments η_x, η_y, η_z in Equation 17.3.

In summary, one finds again a form analogous to Equation 12.16 (with $C_{13} = C_{23} = 0$ due to the orientation of the plies (see Hypotheses in Section 17.1.3):

$$\begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_y}{\partial x} \\ -\dfrac{\partial \theta_x}{\partial y} \\ \left(\dfrac{\partial \theta_y}{\partial y} - \dfrac{\partial \theta_x}{\partial x}\right) \end{Bmatrix} \tag{17.8}$$

Or in the inverse form, by reusing the notations of Section 12.1.6.

$$\begin{Bmatrix} \dfrac{\partial \theta_y}{\partial x} \\ -\dfrac{\partial \theta_x}{\partial y} \\ \left(\dfrac{\partial \theta_y}{\partial y} - \dfrac{\partial \theta_x}{\partial x}\right) \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{\overline{EI}}_{11}} & \dfrac{1}{\overline{\overline{EI}}_{12}} & 0 \\ \dfrac{1}{\overline{\overline{EI}}_{21}} & \dfrac{1}{\overline{\overline{EI}}_{22}} & 0 \\ 0 & 0 & \dfrac{1}{C_{33}} \end{bmatrix} \begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} \tag{17.9}$$

17.4.3 Transverse Shear Equation

We define here new stress resultants starting from the transverse shear stresses, which are denoted as transverse shear stress resultants:

- Shear stress resultant $Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz$
  Using Equations 17.2 and 17.6:

$$Q_x = \int_{-h/2}^{h/2} G_{xz}\left(\frac{\partial w_0}{\partial x} + \theta_y + \frac{\partial \eta_x}{\partial z} + \frac{\partial \eta_z}{\partial x}\right) dz$$

  in setting

$$\langle hG_{xz} \rangle = \int_{-h/2}^{h/2} G_{xz}\, dz$$

  yields

$$Q_x = \langle hG_{xz} \rangle \left(\frac{\partial w_0}{\partial x} + \theta_y\right) + \int_{-h/2}^{h/2} G_{xz} \frac{d\eta_x}{\partial z}\, dz \tag{17.10}$$

  where one can note the presence of the integral of an even function:

- Shear stress resultant $Q_y = \int_{-h/2}^{h/2} \tau_{yz}\, dz$

$$Q_y = \int_{-h/2}^{h/2} G_{yz}\left(\frac{\partial w_0}{\partial y} - \theta_x + \frac{\partial \eta_y}{\partial z} + \frac{\partial \eta_z}{\partial y}\right) dz$$

in setting

$$\langle hG_{yz}\rangle = \int_{-h/2}^{h/2} G_{yz}\, dz$$

yields

$$Q_y = \langle hG_{yz}\rangle\left(\frac{\partial w_0}{\partial y} - \theta_x\right) + \int_{-h/2}^{h/2} G_{yz}\frac{\partial \eta_y}{\partial z}\, dz \tag{17.11}$$

17.5 EQUILIBRIUM EQUATIONS

These are the same for the plates in general, no matter what are their compositions and are, therefore, classical.

One recalls here only the equilibrium equations for bending.

17.5.1 Transverse Equilibrium

- local equilibrium relation $\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$
  In integrating across the thickness, one reveals the transverse shear stresses Q_x and Q_y:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + [\sigma_z]_{-h/2}^{h/2} + \int_{-h/2}^{h/2} f_z\, dz = 0$$

Then denoted by p_z, the transverse stress density:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p_z = 0$$

17.5.2 Equilibrium in Bending

- local equilibrium relation $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$
  After multiplication with z, the integration over the thickness leads to

$$\frac{\partial M_y}{\partial x} - \frac{\partial M_{xy}}{\partial y} + \int_{-h/2}^{h/2}\left[\frac{\partial}{\partial z}(z\tau_{xz}) - \tau_{xz}\right]dz + \int_{-h/2}^{h/2} zf_x\, dz = 0$$

$$\frac{\partial M_y}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x + [z\tau_{xz}]_{-h/2}^{h/2} + \int_{-h/2}^{h/2} zf_x\, dz = 0$$

In neglecting the moment density:

$$\frac{\partial M_y}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{17.12}$$

- local equilibrium relation $\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$
  An analogous calculation leads to

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_x}{\partial y} + Q_y = 0 \tag{17.13}$$

17.6 TECHNICAL FORMULATION FOR BENDING

- One can note in the preceding that midplane symmetry always leads to the decoupling of the membrane behavior from the bending behavior. As a consequence, in what follows, one will consider uniquely the stresses due to bending (one will make $N_x = N_y = T_{xy} = 0$).
- In addition to the hypotheses in Section 17.1.3, one will neglect, for the calculation of the stresses, the variations of increments $\{\eta_x, \ \eta_y, \ \eta_z\}$ as functions of x and y.[8]

17.6.1 Plane Stresses Due to Bending

One can write successively for a ply number k:

- $\sigma_x = \bar{E}_{11}^k \ \varepsilon_x + \bar{E}_{12}^k \ \varepsilon_y$

  Then with [17.6]:

$$\sigma_x = \bar{E}_{11}^k \left(\varepsilon_{ox} + z\frac{\partial \theta_y}{\partial x} + \frac{\partial \eta'_x}{\partial x} \right) + \bar{E}_{12}^k \left(\varepsilon_{oy} - z\frac{\partial \theta_x}{\partial y} + \frac{\partial \eta'_y}{\partial y} \right)$$

  and with [17.7] and [17.9]:

$$\sigma_x = \bar{E}_{11}^k \left[\frac{N_x}{hE_x} - \frac{\nu_{yx}}{hE_y} N_y + z\left(\frac{M_y}{\overline{\overline{EI}}_{11}} - \frac{M_x}{\overline{\overline{EI}}_{12}} \right) \right] \cdots$$

$$\cdots + \bar{E}_{12}^k \left[-\frac{\nu_{xy}}{hE_x} N_x + \frac{N_y}{hE_y} + z\left(\frac{M_y}{\overline{\overline{EI}}_{12}} - \frac{M_x}{\overline{\overline{EI}}_{22}} \right) \right]$$

$$\sigma_x = z \left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}} \right) M_y + z \left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{12}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{22}} \right) \times -M_x \qquad (17.14)$$

- $\sigma_y = \bar{E}_{12}^k \ \varepsilon_x + \bar{E}_{22}^k \ \varepsilon_y$

  An analogous calculation leads to

$$\sigma_y = z \left(\frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{22}^k}{\overline{\overline{EI}}_{12}} \right) M_y + z \left(\frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}} + \frac{\bar{E}_{22}^k}{\overline{\overline{EI}}_{22}} \right) \times -M_x \qquad (17.15)$$

- $\tau_{xy} = \bar{E}_{33}^k \gamma_{xy} = G_{xy}^k \gamma_{xy}$

  Then with Equation 17.6:

$$\tau_{xy} = G_{xy}^k \left(\gamma_{oxy} + z\left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \right) + \frac{\partial \eta'_x}{\partial y} + \frac{\partial \eta'_y}{\partial x} \right)$$

---

[8] This simplification constitutes here the extension to plates of the generalized Navier-Bernoulli principle for beams (see Section 15.1.5).

and with Equations 17.7 and 17.9 and $T_{xy} = 0$:

$$\tau_{xy} = -z\frac{G_{xy}^k}{C_{33}}M_{xy} \tag{17.16}$$

17.6.2 Transverse Shear Stresses in Bending

- $\tau_{xz} = \bar{E}_{44}^k \gamma_{xz} = G_{xz}^k \gamma_{xz}$ from [17.2]

and with Equation 17.6 and neglecting the variation $\partial\eta_z/\partial x$:

$$\tau_{xz} = G_{xz}^k\left(\frac{\partial w_0}{\partial x} + \theta_y\right) + G_{xz}^k\frac{\partial \eta_x}{\partial z} \tag{17.17}$$

- $\tau_{yz} = \bar{E}_{55}^k \gamma_{yz} = G_{yz}^k \gamma_{yz}$

which leads in an analogous manner to

$$\tau_{yz} = G_{yz}^k\left(\frac{\partial w_0}{\partial y} - \theta_x\right) + G_{yz}^k\frac{\partial \eta_y}{\partial z} \tag{17.18}$$

Knowledge of the transverse shears, then, requires the previous calculation of the warping increments η_x and η_y.

17.6.3 Characterization of the Bending, Warping Increments η_x and η_y

- **Warping** $\eta_x\,(x, y, z)$:
  Starting from the first equation of local equilibrium:

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xy}}{\partial y}$$

  Then with Equations 17.14, 17.16, and 17.17:

$$G_{xz}^k\frac{\partial^2 \eta_x}{\partial z^2} = -z\left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right)\frac{\partial M_y}{\partial x} + z\left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{12}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{22}}\right)\frac{\partial M_x}{\partial x} + z\frac{G_{xy}^k}{C_{33}}\frac{\partial M_{xy}}{\partial y}$$

  Taking into account the equilibrium Equation 17.12, one can rewrite

$$G_{xz}^k\frac{\partial^2 \eta_x}{\partial z^2} = -z\left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right)Q_x + z\left(\frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{12}} + \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{22}}\right)\frac{\partial M_x}{\partial x}\cdots$$

$$\cdots + z\left(\frac{G_{xy}^k}{C_{33}} - \frac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}} - \frac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right)\frac{\partial M_{xy}}{\partial y} \tag{17.19}$$

- **Warping** $\eta_y\,(x, y, z)$:
  In an analogous manner, starting from the second equation of the local equilibrium:

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{yx}}{\partial x}$$

Then with Equations 17.15, 17.16, and 17.18:

$$G_{yz}^{k}\frac{\partial^2\eta_y}{\partial z^2} = -z\left(\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{12}}\right)\frac{\partial M_y}{\partial y}+z\left(\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}+\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}}\right)\frac{\partial M_x}{\partial y}\cdots$$

$$\cdots + z\frac{G_{xy}^{k}}{C_{33}}\frac{\partial M_{xy}}{\partial x}$$

Taking into account the equilibrium Equation 17.13, one can rewrite

$$G_{yz}^{k}\frac{\partial^2\eta_x}{\partial z^2} = -z\left(\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{12}}\right)\frac{\partial M_y}{\partial y}-z\left(\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}+\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}}\right)Q_y\cdots$$

$$\ldots + z\left(\frac{G_{xy}^{k}}{C_{33}}-\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}-\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}}\right)\frac{\partial M_{xy}}{\partial x} \qquad (17.20)$$

17.6.3.1 Particular Cases

Equations 17.19 and 17.20 simplify in the following particular cases:

- **Homogeneous orthotropic plate:**
  Then from relations 17.2, 17.8, and 17.9:

$$\bar{E}_{11}^{k} = \bar{E}_{11};\ \bar{E}_{12}^{k} = \bar{E}_{12};\ \bar{E}_{22}^{k} = \bar{E}_{22}$$

$$\frac{1}{\overline{\overline{EI}}_{11}} = \frac{C_{22}}{C_{11}C_{22}-C_{12}^2} = \frac{\bar{E}_{22}}{\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2}\times\frac{12}{h^3};$$

$$\frac{1}{\overline{\overline{EI}}_{22}} = \frac{\bar{E}_{11}}{\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2}\times\frac{12}{h^3}$$

$$\frac{1}{\overline{\overline{EI}}_{12}} = -\frac{C_{12}}{C_{11}C_{22}-C_{12}^2} = \frac{-\bar{E}_{12}}{\bar{E}_{11}\bar{E}_{22}-\bar{E}_{12}^2}\times\frac{12}{h^3};\quad \frac{1}{C_{33}} = \frac{1}{G_{xy}}\times\frac{12}{h^3}$$

Then Equation 17.19 and Equation 17.20 reduce to

$$\boxed{\begin{aligned} G_{xz}\frac{\partial^2\eta_x}{\partial z^2} &= -z\times\frac{12}{h^3}\times Q_x \\ G_{yz}\frac{\partial^2\eta_y}{\partial z^2} &= -z\times\frac{12}{h^3}\times Q_y \end{aligned}} \qquad (17.21)$$

- **Cylindrical bending about *x* or *y* axis** of a multilayered plate characterized by identical Poisson coefficient in the x,y plane of the plate as:

$$\forall k:\ \nu_{xy}^{k} = \nu_{xy};\ \nu_{yx}^{k} = \nu_{yx}$$

Then for any two plies k and m, one has (see Equations 17.2)[9]:

$$\frac{\bar{E}_{11}^{k}}{\bar{E}_{11}^{m}} = \frac{\bar{E}_{12}^{k}}{\bar{E}_{12}^{m}} = \frac{\bar{E}_{22}^{k}}{\bar{E}_{22}^{m}} = \alpha_{km}$$

then:

$$C_{ij} = \int_{-h/2}^{h/2} \bar{E}_{ij}^{k} z^2 dz = \sum_{k=1^{\text{st}}\,\text{ply}}^{n^{\text{th}}\,\text{ply}} \left\{ \bar{E}_{ij}^{k} \int_{z_{k-1}}^{z_k} z^2 dz \right\}$$

$$= \bar{E}_{ij}^{1} \int_{z_0}^{z_1} z^2 dz + \bar{E}_{ij}^{2} \int_{z_1}^{z_2} z^2 dz \cdots + \bar{E}_{ij}^{n} \int_{z_{n-1}}^{z_n} z^2\, dz$$

$$C_{ij} = \bar{E}_{ij}^{1} \left\{ \int_{z_0}^{z_1} z^2 dz + \alpha_{12} \int_{z_1}^{z_2} z^2 dz \cdots + \alpha_{n-1,\,n} \int_{z_{n-1}}^{z_n} z^2 dz \right\} = \bar{E}_{ij}^{1} \times \frac{\alpha h^3}{12}$$

where α is a nondimensional coefficient. One then has

$$\frac{1}{\overline{\overline{EI}}_{11}} = \frac{C_{22}}{C_{11}C_{22} - C_{12}^2} = \frac{\overline{EI}_{22}^{1}}{\bar{E}_{11}^{1}\,\bar{E}_{22}^{1} - (\bar{E}_{12}^{1})^2} \times \frac{12}{\alpha h^3};$$

$$\frac{1}{\overline{\overline{EI}}_{22}} = \frac{\overline{EI}_{11}^{1}}{\bar{E}_{11}^{1}\,\bar{E}_{22}^{1} - (\bar{E}_{12}^{1})^2} \times \frac{12}{\alpha h^3}$$

$$\frac{1}{\overline{\overline{EI}}_{12}} = \frac{\overline{EI}_{12}^{1}}{\bar{E}_{11}^{1}\,\bar{E}_{22}^{1} - (\bar{E}_{12}^{1})^2} \times \frac{12}{\alpha h^3}$$

In [17.19], we have the simplification :

$$\frac{\bar{E}_{11}^{k}}{\overline{\overline{EI}}_{12}} + \frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{22}} = \frac{-\bar{E}_{11}^{k}\,\bar{E}_{12}^{1} + \bar{E}_{12}^{k}\,\bar{E}_{11}^{1}}{\bar{E}_{11}^{1}\,\bar{E}_{22}^{1} - (\bar{E}_{12}^{1})^2} \times \frac{12}{\alpha h^3}$$

$$= \frac{\alpha_{k1}(-\bar{E}_{11}^{1}\,\bar{E}_{12}^{1} + \bar{E}_{12}^{1}\,\bar{E}_{11}^{1})}{\bar{E}_{11}^{1}\,\bar{E}_{22}^{1} - (\bar{E}_{12}^{1})^2} \times \frac{12}{\alpha h^3} = 0$$

as well as an analogous simplification in Equation 17.20:

$$\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{12}} = 0$$

Equations 17.19 and 17.20 simplify as follows[10]:

---

[9] Recall the relation $\nu_{yx}E_x = \nu_{xy}E_y$ (see Equation 9.4).

[10] In the first case in Equation 17.22, $M_{xy} = Q_y = 0$ and Equation 17.20 disappears. In the second case, $M_{xy} = Q_y = 0$ and Equation 17.19 disappears.

1. cylindrical bending about y axis

$$G_{xz}^{k}\frac{\partial^2\eta_x}{\partial z^2} = -z\left(\frac{\bar{E}_{11}^{k}}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right)Q_x$$

2. cylindrical bending about x axis

$$G_{yz}^{k}\frac{\partial^2\eta_y}{\partial z^2} = -z\left(\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}} + \frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right)Q_y \tag{17.22}$$

- **The case of a multilayer plate** such that for any two plies k and m one has in the plane of the plate[11]:

$$\frac{\bar{E}_{ij}^{k}}{\bar{E}_{ij}^{m}} = \alpha_{km} \qquad \forall i \quad \text{and} \quad j = 1, 2, 3$$

Then Equations 17.19 and 17.20 reduce to

$$G_{xz}^{k}\frac{\partial^2\eta_x}{\partial z^2} = -z\left(\frac{\bar{E}_{11}^{k}}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right)Q_x$$

$$G_{yz}^{k}\frac{\partial^2\eta_y}{\partial z^2} = -z\left(\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}} + \frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right)Q_y \tag{17.23}$$

The preceding particular cases constitute a severe restriction among the variety of practical laminations. Nevertheless we will conserve in the following the simplified forms of Equations 17.21, 17.22, and 17.23 because they well show the direct connection between the warpings η_x and η_y and the transverse shear forces Q_x and Q_y, respectively.

17.6.3.2 Consequences

Setting η_x and η_y in the forms:

$$\left.\begin{aligned} \eta_x(x, y, z) &= \frac{Q_x}{\langle hG_{xz}\rangle} \times g(z) \\ \eta_y(x, y, z) &= \frac{Q_y}{\langle hG_{yz}\rangle} \times p(z) \end{aligned}\right\} \tag{17.24}$$

The constitutive Equations 17.10 and 17.11 are written as:

- $Q_x = \langle hG_{xz}\rangle\left(\frac{\partial w_0}{\partial x} + \theta_y\right) + \frac{Q_x}{\langle hG_{xz}\rangle}\int_{-h/2}^{h/2} G_{xz}\frac{dg}{dz}dz$

[11] Such a limiting case is rare in practice, because it imposes in particular: $\frac{E_{33}^{k}}{E_{33}^{m}} = \frac{G_{xy}^{k}}{G_{xy}^{m}} = \alpha_{km}$.

then by setting :

$$k_x = \left(1 - \frac{1}{\langle hG_{xz}\rangle}\int_{-h/2}^{h/2} G_{xz}\frac{dg}{dz}dz\right)$$
$$Q_x = \frac{\langle hG_{xz}\rangle}{k_x}\left(\frac{\partial w_0}{\partial x} + \theta_y\right) \tag{17.25}$$

- $Q_y = \langle hG_{yz}\rangle\left(\frac{\partial w_0}{\partial y} - \theta_x\right) + \frac{Q_y}{\langle hG_{yz}\rangle}\int_{-h/2}^{h/2} G_{yz}\frac{dp}{dz}dz$

  then by setting :

$$k_y = \left(1 - \frac{1}{\langle hG_{yz}\rangle}\int_{-h/2}^{h/2} G_{yz}\frac{dp}{dz}dz\right)$$
$$Q_y = \frac{\langle hG_{yz}\rangle}{k_y}\left(\frac{\partial w_0}{\partial y} - \theta_x\right) \tag{17.26}$$

There appear two transverse shear coefficients k_x and k_y which require the knowledge of the functions $g(z)$ and $p(z)$ for their calculations.

17.6.4 Warping Functions

- **Boundary conditions:** We have assumed that the upper and lower faces of the plate were free of any shear. Then the transverse shear in Equations 17.17 and 17.18 leads to
- $\left(\frac{\partial w_0}{\partial y} + \theta_y\right) + \frac{Q_x}{\langle hG_{xz}\rangle}\frac{dg}{dz} = 0$ for $z = \pm h/2$

  then with Equation 17.25:

$$k_x + \frac{dg}{dz} = 0 \quad \text{for} \quad z = \pm h/2$$

- $\left(\frac{\partial w_0}{\partial y} - \theta_y\right) - \frac{Q_y}{\langle hG_{yz}\rangle}\frac{dp}{dz} = 0$ for $z = \pm h/2$

  then with [17.26]:

$$k_y + \frac{dp}{dz} = 0 \quad \text{for} \quad z = \pm h/2$$

- **Continuity at the interfaces:** The continuity of the transverse shear at the interfaces between layers results from the assumed perfect bonding between two plies (see Paragraph 15.1.2). One then has at the interface between two consecutive plies k and $k + 1$:

$$\tau_{xz}^{k} = \tau_{xz}^{k+1}; \ \tau_{yz}^{k} = \tau_{yz}^{k+1}$$

then with Equations 17.17, 17.18 and Equations 17.25, 17.26:

$$G_{xz}^{k}\left(k_x+\frac{dg_k}{dz}\right) = G_{xz}^{k+1}\left(k_x+\frac{dg_{k+1}}{dz}\right)$$

$$G_{yz}^{k}\left(k_y+\frac{dp_k}{dz}\right) = G_{yz}^{k+1}\left(k_y+\frac{dp_{k+1}}{dz}\right)$$

- **Formulation of the warping functions:** Let us substitute to $g(z)$ and $p(z)$ the functions $g_0(z)$ and $p_0(z)$ such that:

$$g_0(z) = g(z)+z\times k_x;\; p_0(z) = p(z)+z\times k_y$$

$g_0(z)$ and $p_0(z)$ are called the **warping functions**. Then the boundary conditions and the interface conditions simplify, and Equations 17.23 allow one to formulate the problems that permit a simple calculation of warping functions $g_0(z)$ and $p_0(z)$. One obtains

$$\begin{cases} \dfrac{d^2g_0}{dz^2} = -z\times\dfrac{\langle hG_{xz}\rangle}{G_{xz}^k}\left(\dfrac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}}+\dfrac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right) \\ \dfrac{dg_0}{dz} = 0 \quad \text{for} \quad z = \pm h/2 \\ G_{xz}^k\dfrac{dg_{0k}}{dz} = G_{xz}^{k+1}\dfrac{dg_{0k+1}}{dz} \quad \text{for} \quad z = z_k \end{cases} \tag{17.27}$$

$$\begin{cases} \dfrac{d^2p_0}{dz^2} = -z\times\dfrac{\langle hG_{yz}\rangle}{G_{yz}^k}\left(\dfrac{\bar{E}_{22}^k}{\overline{\overline{EI}}_{22}}+\dfrac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right) \\ \dfrac{dp_0}{dz} = 0 \quad \text{for} \quad z = \pm h/2 \\ G_{yz}^k\dfrac{dp_{0k}}{dz} = G_{yz}^{k+1}\dfrac{dp_{0k+1}}{dz} \quad \text{for} \quad z = z_k \end{cases} \tag{17.28}$$

The antisymmetric functions g_0 and p_0 are then defined in a unique manner.

17.6.5 Consequences

- **Form of the transverse shear stresses:** Equations 17.17 and 17.18 then take the simple forms:

$$\tau_{xz} = Q_x\times\frac{G_{xz}^k}{\langle hG_{xz}\rangle}\frac{dg_0}{dz};\quad \tau_{yz} = Q_y\times\frac{G_{yz}^k}{\langle hG_{yz}\rangle}\frac{dp_0}{dz} \tag{17.29}$$

- **Transverse shear coefficients:** One obtains these coefficients from the Equation 17.5:
  - $\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\eta_x z\,dz = 0$

    here is :

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\times\frac{Q_x}{\langle hG_{xz}\rangle}(g_0-k_x z)z\,dz = 0$$

noting that :

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)z^2dz = \frac{C_{11}}{\overline{\overline{EI}}_{11}}+\frac{C_{12}}{\overline{\overline{EI}}_{12}} = \frac{C_{11}C_{12}-C_{12}^2}{C_{11}C_{12}-C_{12}^2} = 1.$$

One obtains :

$$k_x = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)g_0 z\,dz \qquad (17.30)$$

  - $\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\eta_y z dz = 0$

    here is :

$$\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)\times\frac{Q_y}{\langle hG_{yz}\rangle}(p_0-k_y z)z\,dz = 0$$

leading to:

$$k_y = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)p_0 z\,dz \qquad (17.31)$$

In summary, in the absence of body forces (inertia forces, example), the bending behavior uncoupled from the membrane behavior of a thick laminated plate can be simplified in a few particular cases noted below. The characteristic relations are summarized in the following table.

Bending Behavior (no in-plane stress resultants)

	homogeneous orthotropic plate/orthotropic axes : x, y, z
or	Laminated plate/midplane symmetry/orthotropic axes of plies: x,y,z/same Poisson ratios ν_{xy} and ν_{yx} for all plies/cylindrical bending about x or y axis.
or	Laminated plate/midplane symmetry/orthotropic axes of plies: x,y,z/elastic constants are proportional from one ply to another
●	**Equilibrium relation:** $\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p_z = 0;\ \frac{\partial M_y}{\partial x}-\frac{\partial M_{xy}}{\partial y}-Q_x = 0;\ \frac{\partial M_{xy}}{\partial x}+\frac{\partial M_x}{\partial y}+Q_y = 0$

- **Constitutive relations:**

$$\begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{\langle hG_{xz}\rangle}{k_x} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{\langle hG_{yz}\rangle}{k_y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\theta_y}{\partial x} \\ -\dfrac{\partial\theta_x}{\partial y} \\ \dfrac{\partial\theta_y}{\partial y} - \dfrac{\partial\theta_x}{\partial x} \\ \dfrac{\partial w_0}{\partial x} + \theta_y \\ \dfrac{\partial w_0}{\partial y} - \theta_x \end{Bmatrix}$$

with

$$[C]^{-1} = \left[\frac{1}{\overline{\overline{EI}}}\right]$$

- **Stresses**
  - ■ stresses within the ply :

    σ_x: cf. [17.14]; σ_y: cf. [17.15]; τ_{xy}: cf. [17.16]

  - ■ transverse shear stresses

$$\tau_{xz} = Q_x \frac{G_{xz}^k}{\langle hG_{xz}\rangle}\frac{dg_0}{dz}; \qquad \tau_{yz} = Q_y \frac{G_{yz}^k}{\langle hG_{yz}\rangle}\frac{dh_0}{dz}$$

- **Warping functions**
  - ■ $g_0\,(z)$ is the solution of the problem:

$$\begin{cases} \dfrac{d^2 g_0}{dz^2} = -z\dfrac{\langle hG_{xz}\rangle}{G_{xz}^k}\left(\dfrac{\bar{E}_{11}^k}{\overline{\overline{EI}}_{11}} + \dfrac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right) \\ \dfrac{dg_0}{dz} = 0 \quad \text{for} \quad z = \pm h/2 \\ G_{xz}^k \dfrac{dg_{0k}}{dz} = G_{xz}^{k+1}\dfrac{dg_{0k+1}}{dz} \quad \text{for} \quad z = z_k \end{cases}$$

  - ■ $p_0\,(z)$ is the solution of the problem:

$$\begin{cases} \dfrac{d^2 p_0}{dz^2} = -z\dfrac{\langle hG_{yz}\rangle}{G_{yz}^k}\left(\dfrac{\bar{E}_{22}^k}{\overline{\overline{EI}}_{22}} + \dfrac{\bar{E}_{12}^k}{\overline{\overline{EI}}_{12}}\right) \\ \dfrac{dp_0}{dz} = 0 \quad \text{for} \quad z = \pm h/2 \\ G_{yz}^k \dfrac{dp_{0k}}{dz} = G_{yz}^{k+1}\dfrac{dp_{0k+1}}{dz} \quad \text{for} \quad z = z_k \end{cases}$$

- **Transverse shear coefficients k_x and k_y:**
  - ■ They are given by the formula:

$$k_x = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right) g_0 z\, dz; \quad k_y = \int_{-h/2}^{h/2}\left(\frac{\bar{E}_{22}}{\overline{\overline{EI}}_{22}} + \frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right) p_0 z\, dz$$

(17.32)

17.6.6 Interpretation in Terms of Energy

We will limit ourselves to the surface energy density due to transverse shear stresses as:

$$W_\tau = \frac{1}{2}\int_{-h/2}^{h/2}(\tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})dz = \frac{1}{2}\int_{-h/2}^{h/2}\left(\frac{\tau_{xz}^2}{G_{xz}} + \frac{\tau_{yz}^2}{G_{yz}}\right)dz$$

Substituting Equation 17.29, one obtains

$$W_\tau = \frac{1}{2}\int_{-h/2}^{h/2} Q_x^2 \frac{G_{xz}}{\langle hG_{xz}\rangle^2}\left(\frac{dg_0}{dz}\right)^2 dz + \frac{1}{2}\int_{-h/2}^{h/2} Q_y^2 \frac{G_{yz}}{\langle hG_{yz}\rangle^2}\left(\frac{dp_0}{dz}\right)^2 dz$$

The first integral can be rewritten as:

$$\frac{1}{2}\frac{Q_x^2}{\langle hG_{xz}\rangle^2}\int_{-h/2}^{h/2} G_{xz}\left[\frac{d}{dz}\left(g_0\frac{dg_0}{dz}\right) - g_0\frac{d^2 g_0}{dz^2}\right]dz$$

or, taking into account Equation 17.27:

$$\frac{1}{2}\frac{Q_x^2}{\langle hG_{xz}\rangle^2}\left\{G_{xz}\left[g_0\frac{dg_0}{dz}\right]_{-h/2}^{h/2} + \langle hG_{xz}\rangle\int_{-h/2}^{h/2}\left(\frac{\bar{E}_{11}}{\overline{\overline{EI}}_{11}} + \frac{\bar{E}_{12}}{\overline{\overline{EI}}_{12}}\right)g_0 z\, dz\right\}$$

where one recognizes Equation 17.30 of the transverse shear coefficient k_x, The first integral is reduced to

$$\frac{1}{2}k_x\frac{Q_x^2}{\langle hG_{xz}\rangle}$$

Following a similar approach for the second integral and taking into account Equations 17.28 and 17.31 for the transverse shear coefficient k_y, the surface energy due to transverse shear takes the form:

$$W_\tau = \frac{1}{2}k_x\frac{Q_x^2}{\langle hG_{xz}\rangle} + \frac{1}{2}k_y\frac{Q_y^2}{\langle hG_{yz}\rangle}$$

17.7 EXAMPLES

Examples for plates in bending are shown in details in Part Four of this book, in Chapter 18, “Applications.” We give here a few useful elements to treat these examples.

17.7.1 Homogeneous Orthotropic Plate

- **Warping functions:** Equation 17.27 becomes

$$\bar{E}^k_{11} = \bar{E}_{11};\ \bar{E}^k_{12} = \bar{E}_{12};\quad \bar{E}^k_{22} = \bar{E}_{22}\ ;\ G^k_{xz} = G_{xz}$$

$$\frac{d^2 g_0}{dz^2} = -zh\left(\frac{\bar{E}_{11}\bar{E}_{22}}{\bar{E}_{11}\bar{E}_{22} - \bar{E}^2_{12}}\frac{12}{h^3} - \frac{\bar{E}^2_{12}}{\bar{E}_{11}\bar{E}_{22} - \bar{E}^2_{12}}\frac{12}{h^3}\right) = -z \times \frac{12}{h^2}$$

$$\frac{dg_0}{dz} = 0 \text{ for } z = \pm h/2$$

then[12] $$\frac{dg_0}{dz} = \frac{3}{2}\left(1 - 4\frac{z^2}{h^2}\right);\quad g_0 = \frac{3}{2}z\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)$$

- **Transverse shear stresses and shear coefficients:** One deduces from Equation 17.32:

$$\tau_{xz} = \frac{Q_x}{h} \times \frac{3}{2}\left(1 - 4\frac{z^2}{h^2}\right) \tag{17.33}$$

$$k_x = \frac{12}{h^3}\int_{-h/2}^{h/2}\frac{3}{2}\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right)z^2 dz$$

$$k_x = \frac{6}{5} \tag{17.34}$$

In an analogous manner starting from Equation 17.28:

$$p_0(z) = g_0(z)$$

then:

$$\tau_{yz} = \frac{Q_y}{h} \times \frac{3}{2}\left(1 - 4\frac{z^2}{h^2}\right) \tag{17.35}$$

$$k_y = \frac{6}{5} \tag{17.36}$$

Remark: In Application 18.3.7 (Chapter 18), one treats the case of a thick homogeneous orthotropic plate in cylindrical bending about the y axis. The plate supports a uniformly distributed load. One can consider there the strong influence of transverse shear in bending. Two characteristics of the plate then apply directly on the deflection:

[12] g_0 is, as g, antisymmetric in z (see Equation 17.4).

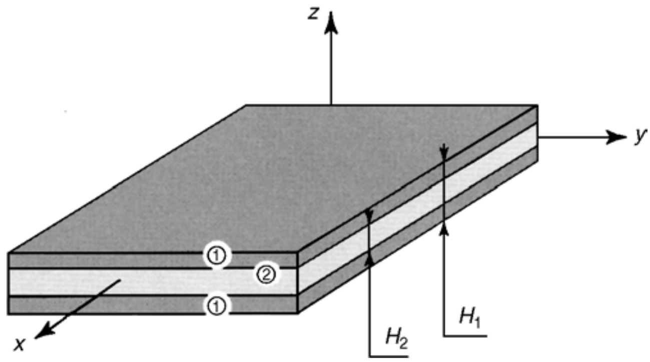

Figure 17.3 Sandwich Plate

- The relative thickness h/a, where a is the length of the bent side of the plate
- The ratio E_x/G_{xz} (for certain combinations of fiber/matrix, this ratio becomes large compared with unity; for example, for unidirectional)

17.7.2 Sandwich Plate

The plate consists of two orthotropic materials:

Material (1) for the skins
Material (2) for the core (see Figure 17.3)

Assuming the proportionality of elastic coefficients for the two materials leads to (see Section 17.6.3):

then

$$C_{ij} = \int_{-h/2}^{h/2} \bar{E}_{ij} z^2 dz = \bar{E}_{ij}^{①} \int_{-H_1/2}^{-H_2/2} z^2 dz + \bar{E}_{ij}^{②} \int_{-H_2/2}^{H_2/2} z^2 dz + \bar{E}_{ij}^{①} \int_{H_2/2}^{H_1/2} z^2 dz$$

$$C_{ij} = \bar{E}_{ij}^{①} \left(\frac{H_1^3 - H_2^3}{12} \right) + \bar{E}_{ij}^{②} \frac{H_2^3}{12}$$

$$C_{ij} = \bar{E}_{ij}^{①} \times \frac{\alpha H_1^3}{12} \text{ with } \frac{\alpha H_1^3}{12} = \frac{H_1^3 - H_2^3}{12} + \alpha_{12} \frac{H_2^3}{12}$$

one deduces from there:

$$\frac{1}{\overline{\overline{EI}}_{11}} = \frac{C_{11}}{C_{11}C_{22} - C_{12}^2} = \frac{\bar{E}_{11}^{①}}{\bar{E}_{11}^{①}\bar{E}_{22}^{①} - (\bar{E}_{12}^{①})^2} \times \frac{12}{\alpha H_1^3}$$

$$\frac{1}{\overline{\overline{EI}}_{12}} = \frac{-C_{12}}{C_{11}C_{22} - C_{12}^2} = \frac{-\bar{E}_{12}^{①}}{\bar{E}_{11}^{①}\bar{E}_{22}^{①} - (\bar{E}_{12}^{①})^2} \times \frac{12}{\alpha H_1^3}$$

17.7.2.1 Warping Functions

- From the above one can write in Equation 17.27[13]:

$$\left(\frac{\bar{E}_{11}^{k}}{\overline{\overline{EI}}_{11}}+\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right) = \frac{E_x^k}{E_x^{①}}\times\frac{12}{\alpha H_1^3} = \frac{E_x^k}{E_x^{①}\frac{\left(H_1^3-H_2^3\right)}{12}+E_x^{②}\frac{H_2^3}{12}}$$

In addition :

$$\langle hG_{xz}\rangle = G_{xz}^{①}(H_1-H_2)+G_{xz}^{①}H_2$$

Equation 17.27 then can be written as:

$$\begin{cases}\dfrac{d^2g_0}{dz^2} = -z\times\dfrac{E_x^k}{G_{xz}^k}\times 12\dfrac{G_{xz}^{①}(H_1-H_2)+G_{xz}^{②}H_2}{E_x^{①}(H_1^3-H_2^3)+E_x^{②}\ H_2^3}\\ \dfrac{dg_0}{dz} = 0 \text{ for } z = \pm H_1/2\\ G_{xz}\dfrac{dg_0}{dz} \text{ continuous for } z = \pm H_2/2\end{cases}$$

- In Equation 17.28, one obtains an analogous formulation. In effect, one can write

$$\left(\frac{\bar{E}_{22}^{k}}{\overline{\overline{EI}}_{22}}+\frac{\bar{E}_{12}^{k}}{\overline{\overline{EI}}_{12}}\right) = \frac{E_y^k}{E_y^{①}}\times\frac{12}{\alpha h^3} = \frac{E_y^k}{E_y^{①}\frac{\left(H_1^3-H_2^3\right)}{12}+E_y^{②}\frac{H_2^3}{12}}$$

The problem [17.28] is then written as:

$$\begin{cases}\dfrac{d^2p_0}{dz^2} = -z\times\dfrac{E_y^k}{G_{yz}^k}\times 12\dfrac{G_{yz}^{①}(H_1-H_2)+G_{yz}^{②}H_2}{E_y^{①}(H_1^3-H_2^3)+E_y^{②}H_2^3}\\ \dfrac{dp_0}{dz} = 0 \qquad z = \pm H_1/2\\ G_{yz}\dfrac{dp_0}{dz} \text{ continuous for } z = \pm H_2/2\end{cases}$$

- **Remark:** These problems are identical to that which allows the calculation of the warping function for the bending of a sandwich beam, and one can consider it in Chapter 18, application 18.3.5. One can then carry out the same steps of calculation. The results obtained are shown below.

[13] See Equations 17.2.

17.7.2.2 Transverse Shear Stresses

- Stress τ_{xz}:

$$-\frac{H_2}{2} \le z \le \frac{H_2}{2}; \quad \tau_{xz} = Q_x \times 6 \times \frac{E_x^{②}\left(\frac{H_2^2}{4} - z^2\right) + E_x^{①}\left(\frac{H_1^2}{4} - \frac{H_2^2}{4}\right)}{E_x^{①}(H_1^3 - H_2^3) + E_x^{②} H_2^3}$$

$$\frac{H_2}{2} \le z \le \frac{H_1}{2}: \quad \tau_{xz} = Q_x \times 6 \times \frac{E_x^{①}\left(\frac{H_1^2}{4} - z^2\right)}{E_x^{①}(H_1^3 - H_2^3) + E_x^{②} H_2^3} \tag{17.37}$$

- Stress τ_{yz}:

$$-\frac{H_2}{2} \le z \le \frac{H_2}{2}; \quad \tau_{yz} = Q_y \times 6 \times \frac{E_y^{②}\left(\frac{H_2^2}{4} - z^2\right) + E_y^{①}\left(\frac{H_1^2}{4} - \frac{H_2^2}{4}\right)}{E_y^{①}(H_1^3 - H_2^3) + E_y^{②} H_2^3}$$

$$\frac{H_2}{2} \le z \le \frac{H_1}{2}; \quad \tau_{yz} = Q_y \times 6 \times \frac{E_y^{①}\left(\frac{H_1^2}{4} - z^2\right)}{E_y^{①}(H_1^3 - H_2^3) + E_y^{②} H_2^3} \tag{17.38}$$

17.7.2.3 Transverse Shear Coefficients

$$k_x = \frac{a_x}{8[E_x^{①}(H_1^3 - H_2^3) + E_x^{②} H_2^3]}\left\{\frac{E_x^{②}}{G_{xz}^{②}} H_2^3\left[E_x^{①} H_1^2 + \left(\frac{4}{5}E_x^{②} - E_x^{①}\right)H_2^2\right]\ldots\right.$$

$$\left.\cdots + \frac{(E_x^{①})^2}{G_{xz}^{①}}\left(\frac{4}{5}H_1^5 + \frac{H_2^5}{5} - H_1^2 H_2^3\right)\right\} + \frac{3 b_x E_x^{①}(H_1^2 - H_2^2)}{E_x^{①}(H_1^3 - H_2^3) + E_x^{②} H_2^3}$$

$$\text{with} \quad a_x = 12 \times \frac{G_{xz}^{①}(H_1 - H_2) + G_{xz}^{②} H_2}{E_x^{①}(H_1^3 - H_2^3) + E_x^{②} H_2^3}$$

$$b_x = \frac{a_x}{16} H_2 \frac{E_x^{①}}{G_{xz}^{①}}\left\{\frac{H_2^2}{3} + H_1^2\left(\frac{G_{xz}^{①}}{G_{xz}^{②}} - 1\right) - H_2^2 \frac{G_{xz}^{①}}{G_{xz}^{②}}\left(1 - \frac{2}{3}\frac{E_x^{②}}{E_x^{①}}\right)\right\} \tag{17.39}$$

k_y is given by an expression formally identical to that in which the index x is replaced by y.

In Application 18.3.8 we treat the case of a rectangular sandwich plate in cylindrical bending, clamped on one side and subjected to a uniform linear force on another. The plate is free on the other sides. One shows the influence of transverse shear on the deflection. This influence increases when:

- The mechanical characteristics (moduli) of the core are weaker than those of the skins.
- The relative thickness of the core is important (thin skins).
- The relative thickness of the plate is large (thick plate).