PART III

JUSTIFICATIONS, COMPOSITE BEAMS, AND THICK PLATES

We regroup in Part III elements that are less utilized than those in the previous parts. Nevertheless they are of fundamental interest for a better understanding of the principles for calculation of composite components. In the first two chapters, we focused on anisotropic properties and fracture strength of orthotropic materials, and then more particularly on transversely isotropic ones. The following two chapters allow us to consider that composite components in the form of beams can be "homogenized." This means that their study is analogous to the study of homogeneous beams that are common in the literature. Finally, the last chapter in this part describes with a similar procedure the behavior of thick composite plates subject to transverse loadings.

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ELASTIC COEFFICIENTS

The definition of a linear elastic anisotropic medium was given in Chapter 9. We have also given, without justification, the behavior relations characterizing the particular case of orthotropic materials. Now we propose to examine more closely the elastic constants which appear in stress–strain relations for these materials. In the case of transversely isotropic materials, we will study also the manner in which the constants evolve.

13.1 ELASTIC COEFFICIENTS IN AN ORTHOTROPIC MATERIAL

Recall: Consider the relation for elastic behavior written in Paragraph 9.1.1 in the form:

$$\varepsilon_{mn} = \varphi_{mnpq} \times \sigma_{pq}$$

Recall that the components φ_{mnpq} of a tensor expressed in the coordinate system 1,2,3 are written as Φ_{IJKL} in a coordinate system I,II,II using the relation:

$$\boxed{\Phi_{IJKL} = \cos_I^m \cos_J^n \cos_K^p \cos_L^q \varphi_{mnpq}} \tag{13.1}$$

in which:

$$\cos_I^m = \cos(\vec{m}, \vec{I})$$

By definition,[1] for mechanical behavior, an orthotropic medium has at any point two orthogonal planes of symmetry. Consider here two coordinate systems 1,2,3 and I,II,III, constructed on these planes and their intersection. One plane can be obtained from the other by a 180° rotation about the 3 axis as shown in Figure 13.1. One can deduce

$$[\cos_I^m] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

[1] See Section 9.2.

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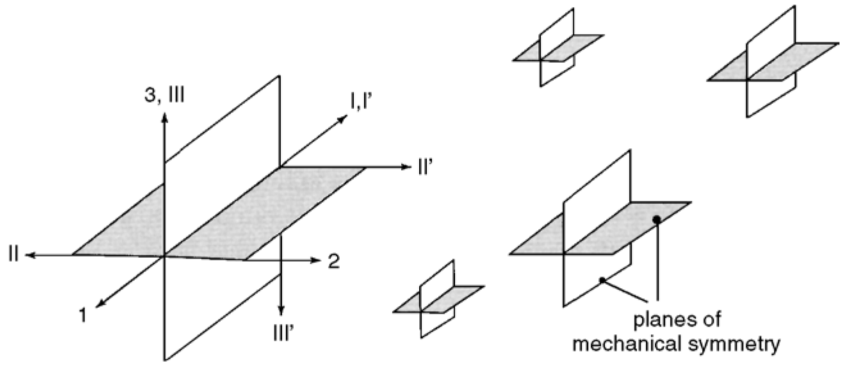

Figure 13.1 Orthotropic Medium

Application of Relation 13.1 above leads to

$$\Phi_{I\,I\,I\,I} = \varphi_{1111}; \quad \Phi_{I\,I\,II\,II} = \varphi_{1122}; \quad \Phi_{I\,I\,III\,III} = \varphi_{1133}$$
$$\Phi_{II\,II\,II\,II} = \varphi_{2222}; \quad \Phi_{II\,II\,III\,III} = \varphi_{2233}; \quad \Phi_{III\,III\,III\,III} = \varphi_{3333}$$
$$\Phi_{II\,III\,II\,III} = \varphi_{2323}; \quad \Phi_{I\,III\,I\,III} = \varphi_{1313}; \quad \Phi_{I\,II\,I\,II} = \varphi_{1212}$$

and:

$$\Phi_{I\,I\,II\,III} = -\varphi_{1123};$$

However, because the mechanical properties in the coordinates 1,2,3 and I,II,III are identical, one has

$$\Phi_{I\,I\,II\,III} = \varphi_{1123}$$

from this:

$$\Phi_{I\,I\,II\,III} = \varphi_{1123} = -\varphi_{1123} = 0$$

In an analogous manner:

$$\varphi_{II\,II\,II\,III} = 0; \quad \Phi_{III\,III\,II\,III} = 0$$
$$\Phi_{I\,I\,I\,III} = 0; \quad \Phi_{II\,II\,I\,III} = 0; \quad \Phi_{III\,III\,I\,III} = 0$$
$$\Phi_{II\,III\,I\,II} = 0; \quad \Phi_{I\,III\,I\,II} = 0$$

finally:

$$\Phi_{I\,I\,I\,II} = \varphi_{1112}; \quad \Phi_{II\,II\,I\,II} = \varphi_{2212}; \quad \Phi_{III\,III\,I\,II} = \varphi_{3312}$$
$$\Phi_{II\,III\,I\,III} = \varphi_{2313}$$

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Until now, we have taken into account the symmetry with respect to plane 1,3. Consider now the coordinates 1,2,3 and I', II', III' (see Figure 13.1), which can be obtained from each other by a 180° rotation about the 2 axis (symmetry with respect to plane 1,2). One has

$$[\cos_I^m] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The same procedure as above will lead to

$$\Phi_{I'I'I'II'} = -\varphi_{1112} = \varphi_{1112} = 0; \quad \Phi_{II'II'I'II'} = -\varphi_{2212} = \varphi_{2212} = 0$$
$$\Phi_{III'III'I'II'} = -\varphi_{3312} = \varphi_{3312} = 0; \quad \Phi_{II'III'I'III'} = -\varphi_{2313} = \varphi_{2313} = 0$$

Considering the symmetry of the coefficients φ_{mnpq} indicated in Relation 9.1,[2] we have written here the only nonzero terms. For the mechanical behavior, one obtains by simplification of Equation 9.2:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \varphi_{1111} & \varphi_{1122} & \varphi_{1133} & 0 & 0 & 0 \\ \varphi_{2211} & \varphi_{2222} & \varphi_{2233} & 0 & 0 & 0 \\ \varphi_{3311} & \varphi_{3322} & \varphi_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\varphi_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\varphi_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\varphi_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (13.2)$$

There remain then only **nine** distinct elastic coefficients, which can be written in the form of Young's moduli and Poisson ratios as:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \qquad (13.3)$$

[2] Recall the symmetry relations: $\varphi_{ijkl} = \varphi_{ijlk}$; $\varphi_{ijkl} = \varphi_{jikl}$; $\varphi_{ijkl} = \varphi_{klij}$.

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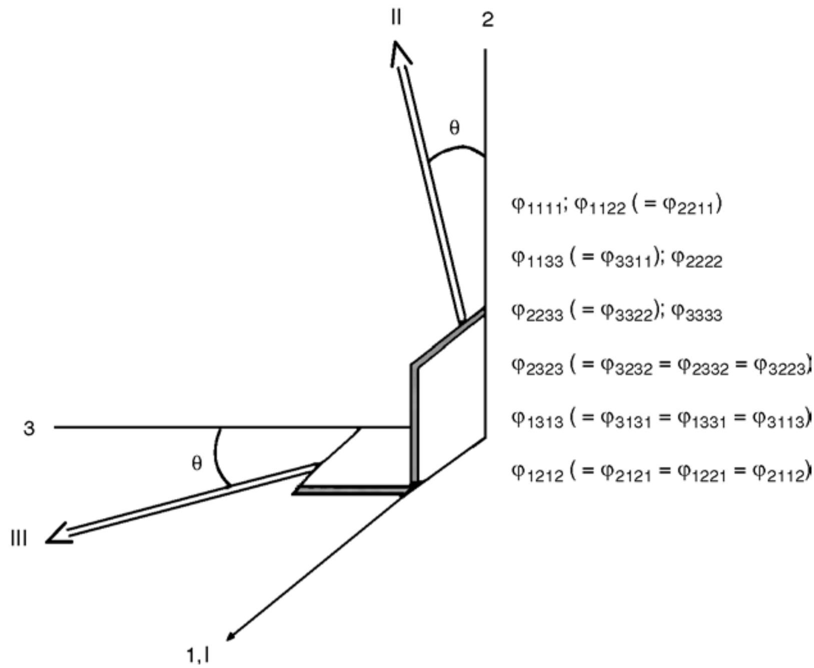

Figure 13.2 Transversely Isotropic Material

13.2 ELASTIC COEFFICIENTS FOR A TRANSVERSELY ISOTROPIC MATERIAL

Recall: By definition,[3] a transversely isotropic material (Figure 13.2) is such that any plane including a preferred axis is a plane of mechanical symmetry. We have already noted that this is a particular case of orthotropic materials. Therefore, the only nonzero elastic constants are shown in Figure 13.2.[4]

The preferred direction is axis 1 in Figure 13.2. Considering that the coordinates 1,2,3 and I,II,III can be obtained from each other by a rotation of θ. One then has

$$[\cos_I^m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \quad \text{with} \quad \begin{aligned} c &= \cos\theta \\ s &= \sin\theta \end{aligned}$$

From the definition of material, the matrix of elastic coefficients has to remain invariant in this rotation. The Relation 13.1 allows one to write

- $\Phi_{\mathrm{I\,I\,I\,I}} = \varphi_{1111}$
- $\Phi_{\mathrm{I\,I\,II\,II}} = \varphi_{1122}c^2 + \varphi_{1133}s^2 = \varphi_{1122}$

[3] See Section 9.3.

[4] Using the symmetries in Equation 9.1 and also using the discussion of the previous paragraph.

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then:

$$\varphi_{1122}(c^2-1)+\varphi_{1133}s^2 = 0$$

$$\boxed{\varphi_{1122} = \varphi_{1133}}$$

- $$\begin{aligned}\Phi_{\text{II II II II}} = {} & \varphi_{2222}c^4+\varphi_{2233}s^2c^2+\varphi_{2323}s^2c^2+\varphi_{2332}s^2c^2\cdots \\ & \cdots+\varphi_{3223}s^2c^2+\varphi_{3232}s^2c^2+\varphi_{3322}s^2c^2+\varphi_{3333}s^4\end{aligned}$$

and:

$$\Phi_{\text{II II II II}} = \varphi_{2222}$$

Then, taking into account the symmetries, we obtain the relation:

$$\varphi_{2222}(c^4-1)+\varphi_{3333}s^4+2s^2c^2(\varphi_{2233}+2\varphi_{2323}) = 0 \qquad \textbf{(a)}$$

- $$\begin{aligned}\Phi_{\text{III III III III}} = {} & \varphi_{2222}s^4+\varphi_{2233}s^2c^2+\varphi_{2323}s^2c^2+\varphi_{2332}s^2c^2\cdots \\ & \cdots+\varphi_{3232}s^2c^2+\varphi_{3232}s^2c^2+\varphi_{3322}s^2c^2+\varphi_{3333}s^4\end{aligned}$$

and:

$$\Phi_{\text{III III III III}} = \varphi_{3333}$$

then taking in account symmetry, we have:

$$\varphi_{2222}s^4+\varphi_{3333}(c^4-1)+2s^2c^2(\varphi_{2233}+2\varphi_{2323}) = 0 \qquad \textbf{(b)}$$

Examining member by member the difference of relations shown in (a) and (b) above, one obtains:

$$\boxed{\varphi_{2222} = \varphi_{3333}}$$

Replacing in (a):

$$\varphi_{2222}(c^4+s^4-1)+2s^2c^2(\varphi_{2233}+2\varphi_{2323}) = 0$$
$$-2s^2c^2\varphi_{2222}+2s^2c^2(\varphi_{2233}+2\varphi_{2323}) = 0$$

$$\boxed{2\varphi_{2323} = \varphi_{2222}-\varphi_{2233}}$$

- $$\Phi_{\text{I III I III}} = \varphi_{1212}s^2+\varphi_{1313}c^2 = \varphi_{1313}$$

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then

$$\varphi_{1212}s^2 + \varphi_{1313}(c^2 - 1) = 0$$

$$\boxed{\varphi_{1212} = \varphi_{1313}}$$

We have written four relations for the nine coefficients; there remain five distinct elastic coefficients. Equation 13.2 is reduced to

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \varphi_{1111} & \varphi_{1122} & \varphi_{1122} & 0 & 0 & 0 \\ \varphi_{2211} & \varphi_{2222} & \varphi_{2233} & 0 & 0 & 0 \\ \varphi_{2211} & \varphi_{3322} & \varphi_{2222} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(\varphi_{2222} - \varphi_{2233}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\varphi_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\varphi_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \quad (13.4)$$

or in the form of Young's moduli and Poisson ratios:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{21}}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu}{E_2} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{-\nu}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \quad (13.5)$$

13.2.1 Rotation about an Orthotropic Transverse Axis

13.2.1.1 Problem

How can one transform the elastic coefficients of the previous constitutive equation when writing them in coordinate axes x,y,z other than the orthotropic axes ℓ,t,z?[5] The coordinate axes x,y,z are obtained from the orthotropic axes by a rotation θ about the z axis, as shown in Figure 13.3.

Recall Relation 13.1, which allows the calculation of components Φ_{IJKL} in the coordinate axes x,y,z as functions of the components φ_{mnpq} in the coordinate axes ℓ,t,z to be:

$$\boxed{\begin{array}{l} \Phi_{IJKL} = \cos_I^m \cos_J^n \cos_K^p \cos_L^q \times \varphi_{mnpq} \\ \text{(axes } x,\ y,\ z) \qquad\qquad \text{(axes } \ell,\ t,\ z) \end{array}}$$

[5] The orthotropic axes 1,2,3 of Equation 13.5 are therefore denoted as ℓ,t,z.

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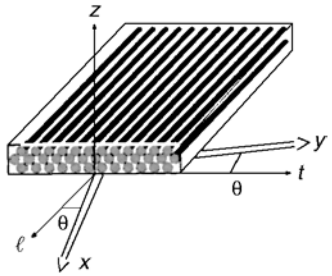

Figure 13.3 Rotation above an Orthotropic Transverse Axis

with (see Figure 13.3):

$$[\cos_I^m] = \begin{bmatrix} \cos(\ell, x) & \cos(\ell, y) & \cos(\ell, z) \\ \cos(t, x) & \cos(t, y) & \cos(t, z) \\ \cos(z, x) & \cos(z, y) & \cos(z, z) \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Noting that the only nonzero coefficients φ_{mnpq} are those that appear in Equation 13.4, one obtains:

- $\Phi_{\mathrm{IIII}} = c^4\varphi_{1111} + c^2s^2\varphi_{1122} + c^2s^2\varphi_{1212} + c^2s^2\varphi_{1221}\cdots$
  $\cdots + c^2s^2\varphi_{2112} + c^2s^2\varphi_{2121} + c^2s^2\varphi_{2211} + s^4\varphi_{2222}$
- $\Phi_{\mathrm{IIII}} = c^4\varphi_{1111} + s^4\varphi_{2222} + 2c^2s^2(\varphi_{1122} + 2\varphi_{1212})$

Expressing this coefficient as a function of the "technical" constants which appear in Equation 13.5, one obtains:

$$\Phi_{\mathrm{IIII}} = \frac{c^4}{E_\ell} + \frac{s^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)$$

- $\Phi_{\mathrm{IIIIII}} = c^2s^2\varphi_{1111} + c^4\varphi_{1122} - c^2s^2\varphi_{1212} - c^2s^2\varphi_{1221}\cdots$
  $\cdots - s^2c^2\varphi_{2112} - s^2c^2\varphi_{2121} + s^4\varphi_{2211} + s^2c^2\varphi_{2222}$
- $\Phi_{\mathrm{IIIIII}} = (c^4 + s^4)\varphi_{1122} + c^2s^2(\varphi_{1111} + \varphi_{2222} - 4c^2s^2\varphi_{1212})$

or in the "technical" form:

$$\Phi_{\mathrm{IIIIII}} = -\frac{\nu_{t\ell}}{E_t}(c^4 + s^4) + c^2s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} - \frac{1}{G_{\ell t}}\right)$$

- $\Phi_{\mathrm{II\,III\,III}} = c^2\varphi_{1133} + s^2\varphi_{2233}$ and as $\varphi_{1133} = \varphi_{1122}$[6]
  $\Phi_{\mathrm{II\,III\,III}} = c^2\varphi_{1122} + s^2\varphi_{2233}$

[6] Because this is a transversely isotropic material; see Equations 9.2 and 13.4.

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or in the "technical" form:

$$\Phi_{\mathrm{I\,I\,III\,III}} = -\left(c^2\frac{\nu_{t\ell}}{E_t} + s^2\frac{\nu}{E_t}\right)$$

- $\Phi_{\mathrm{I\,I\,II\,III}} = 0$
- $\Phi_{\mathrm{I\,I\,I\,III}} = 0$
- $\Phi_{\mathrm{I\,I\,I\,II}} = -c^3 s\varphi_{1\,1\,1\,1} + c^3 s\varphi_{1\,1\,2\,2} + c^3 s\varphi_{1\,2\,1\,2} - cs^3\varphi_{1\,2\,2\,1} + sc^3\varphi_{2\,1\,1\,2}\ldots$

$$\cdots - s^3 c\varphi_{2\,1\,2\,1} - s^3 c\varphi_{2\,2\,1\,1} + s^3 c\varphi_{2\,2\,2\,2}$$

$$\Phi_{\mathrm{I\,I\,I\,II}} = -sc\{c^2\varphi_{1\,1\,1\,1} - s^2\varphi_{2\,2\,2\,2} - (c^2 - s^2)(\varphi_{1\,1\,2\,2} + 2\varphi_{1\,2\,1\,2})\}$$

or in the "technical" form:

$$\Phi_{\mathrm{I\,I\,I\,II}} = -cs\left\{\frac{c^2}{E_\ell} - \frac{s^2}{E_t} + (c^2 - s^2)\left(\frac{\nu_{t\ell}}{E_t} - \frac{1}{2G_{\ell t}}\right)\right\}$$

- $\Phi_{\mathrm{II\,II\,II\,II}} = s^4\varphi_{1\,1\,1\,1} + s^2c^2\varphi_{1\,1\,2\,2} + s^2c^2\varphi_{1\,2\,1\,2} + s^2c^2\varphi_{1\,2\,2\,1}\ldots$

$$\ldots + s^2c^2\varphi_{2\,1\,1\,2} + s^2c^2\varphi_{2\,1\,2\,1} + s^2c^2\varphi_{2\,2\,1\,1} + c^4\varphi_{2\,2\,2\,2}$$

$$\Phi_{\mathrm{II\,II\,II\,II}} = s^4\varphi_{1\,1\,1\,1} + c^4\varphi_{2\,2\,2\,2} + s^2c^2(4\varphi_{1\,2\,1\,2} + 2\varphi_{1\,1\,2\,2})$$

or in "technical" form:

$$\Phi_{\mathrm{II\,II\,II\,II}} = \frac{s^4}{E_\ell} + \frac{c^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)$$

- $\Phi_{\mathrm{II\,II\,III\,III}} = s^2\varphi_{1\,1\,3\,3} + c^2\varphi_{2\,2\,3\,3}$ and as $\varphi_{1\,1\,3\,3} = \varphi_{1\,1\,2\,2}$[7]

$$\Phi_{\mathrm{II\,II\,III\,III}} = s^2\varphi_{1\,1\,2\,2} + c^2\varphi_{2\,2\,3\,3}$$

or in "technical" form:

$$\Phi_{\mathrm{II\,II\,III\,III}} = -\left(s^2\frac{\nu_{t\ell}}{E_t} + c^2\frac{\nu}{E_t}\right)$$

- $\Phi_{\mathrm{II\,II\,II\,III}} = 0$
- $\Phi_{\mathrm{II\,II\,I\,III}} = 0$
- $\Phi_{\mathrm{II\,II\,I\,II}} = -s^3 c\varphi_{1\,1\,1\,1} + s^3 c\varphi_{1\,1\,2\,2} - sc^3\varphi_{1\,2\,1\,2} + s^3 c\varphi_{1\,2\,2\,1}\ldots$

$$\cdots(-sc^3\varphi_{2\,1\,1\,2} + s^3 c\varphi_{2\,1\,2\,1} - sc^3\varphi_{2\,2\,1\,1} + c^3 s\varphi_{2\,2\,2\,2})$$

$$\Phi_{\mathrm{II\,II\,I\,II}} = -sc\{s^2\varphi_{1\,1\,1\,1} + c^2\varphi_{2\,2\,2\,2} + (c^2 - s^2)(\varphi_{1\,1\,2\,2} + 2\varphi_{1\,2\,1\,2})\}$$

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7 See Equations 9.2 and 13.4.

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or in "technical" form:

$$\Phi_{\text{II II I II}} = -cs\left\{\frac{s^2}{E_\ell} - \frac{c^2}{E_t} - (c^2 - s^2)\left(\frac{\nu_{t\ell}}{E_t} - \frac{1}{2G_{\ell t}}\right)\right\}$$

- $\Phi_{\text{III III III III}} = \varphi_{3333}$

in "technical" form:

$$\Phi_{\text{III III III III}} = \frac{1}{E_t}$$

- $\Phi_{\text{III III II III}} = 0$
- $\Phi_{\text{III III I III}} = 0$
- $\Phi_{\text{III III I II}} = -sc\varphi_{3311} + sc\varphi_{3322}$ and as $\varphi_{3311} = \varphi_{1122}$ [8]
  $\Phi_{\text{III III I II}} = -sc\varphi_{1122} + sc\varphi_{2233}$

in "technical" form:

$$\Phi_{\text{III III I II}} = -sc\left(\frac{\nu - \nu_{t\ell}}{E_t}\right)$$

- $\Phi_{\text{II III II III}} = s^2\varphi_{1313} + c^2\varphi_{2323}$

we know[8] that for a transversely isotropic material, one has:

$$\varphi_{1313} = \varphi_{1212} \quad \text{and} \quad 2\varphi_{2323} = \varphi_{2222} - \varphi_{2233}$$

then:

$$\Phi_{\text{II III II III}} = s^2\varphi_{1212} + c^2\left(\frac{\varphi_{2222} - \varphi_{2233}}{2}\right)$$

in "technical" form:

$$\Phi_{\text{II III II III}} = \frac{s^2}{4G_{\ell t}} + \frac{c^2(1+\nu)}{2E_t}$$

- $\Phi_{\text{II III I III}} = -sc\varphi_{1313} + sc\varphi_{2323}$ is still:[8]
  $\Phi_{\text{II III I III}} = -sc\left(\varphi_{2121} - \frac{1}{2}(\varphi_{2222} - \varphi_{2233})\right)$

[8] See Equations 9.2 and 13.4.

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or in "technical" form:

$$\Phi_{\mathrm{II\,III\,I\,III}} = -sc\left(\frac{1}{4G_{\ell t}} - \frac{(1+\nu)}{2E_t}\right)$$

- $\Phi_{\mathrm{III\,II\,I\,II}} = 0$
- $\Phi_{\mathrm{I\,III\,I\,III}} = c^2\varphi_{1\,3\,1\,3} + s^2\varphi_{2\,3\,2\,3}$ is still:

$$\Phi_{\mathrm{I\,III\,I\,III}} = c^2\varphi_{1\,2\,1\,2} + s^2\frac{(\varphi_{2\,2\,2\,2} - \varphi_{2\,2\,3\,3})}{2}$$

or in "technical" form:

$$\Phi_{\mathrm{I\,III\,I\,III}} = \frac{c^2}{4G_{\ell t}} + s^2\frac{(1+\nu)}{2E_t}$$

- $\Phi_{\mathrm{I\,III\,I\,II}} = 0$
- $\Phi_{\mathrm{I\,II\,I\,II}} = s^2c^2\varphi_{1\,1\,1\,1} - s^2c^2\varphi_{1\,1\,2\,2} + c^4\varphi_{1\,2\,1\,2} - s^2c^2\varphi_{1\,2\,2\,1}\ldots$
  $\cdots - s^2c^2\varphi_{2\,1\,1\,2} + s^4\varphi_{2\,1\,2\,1} - s^2c^2\varphi_{2\,2\,1\,1} + s^2c^2\varphi_{2\,2\,2\,2}$

$$\Phi_{\mathrm{I\,II\,I\,II}} = s^2c^2(\varphi_{1\,1\,1\,1} + \varphi_{2\,2\,2\,2} - 2\varphi_{1\,1\,2\,2}) + (c^2 - s^2)^2\varphi_{1\,2\,1\,2}$$

or in "technical" form:

$$\Phi_{\mathrm{I\,II\,I\,II}} = s^2c^2\left\{\frac{1}{E_\ell} + \frac{1}{E_t} + 2\frac{\nu_{t\ell}}{E_t}\right\} + (c^2 - s^2)^2\frac{1}{4G_{\ell t}}$$

All the nonzero coefficients Φ_{IJKL} found above allow one to write the constitutive relation in the form[9]:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \Phi_{\mathrm{I\,I\,I\,I}} & \Phi_{\mathrm{I\,I\,II\,II}} & \Phi_{\mathrm{I\,I\,III\,III}} & 0 & 0 & 2\Phi_{\mathrm{I\,I\,I\,II}} \\ \Phi_{\mathrm{II\,II\,I\,I}} & \Phi_{\mathrm{II\,II\,II\,II}} & \Phi_{\mathrm{II\,II\,III\,III}} & 0 & 0 & 2\Phi_{\mathrm{II\,II\,I\,II}} \\ \Phi_{\mathrm{III\,III\,I\,I}} & \Phi_{\mathrm{III\,III\,II\,II}} & \Phi_{\mathrm{III\,III\,III\,III}} & 0 & 0 & 2\Phi_{\mathrm{III\,III\,I\,II}} \\ 0 & 0 & 0 & 4\Phi_{\mathrm{II\,III\,II\,III}} & 4\Phi_{\mathrm{II\,III\,I\,III}} & 0 \\ 0 & 0 & 0 & 4\Phi_{\mathrm{I\,III\,II\,III}} & 4\Phi_{\mathrm{I\,III\,I\,III}} & 0 \\ 2\Phi_{\mathrm{I\,II\,I\,I}} & 2\Phi_{\mathrm{I\,II\,II\,II}} & 2\Phi_{\mathrm{I\,II\,III\,III}} & 0 & 0 & 4\Phi_{\mathrm{I\,II\,I\,II}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{13.6}$$

[9] This is deduced from the general Equation 9.2.

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13.2.1.2 Technical Form

In analogy with the technical form of Equation 13.5, which was written in orthotropic axes, one can write the constitutive equation in terms of equivalent moduli and Poisson coefficients, as:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{-\nu_{zx}}{E_z} & 0 & 0 & \frac{\eta_{xy}}{G_{xy}} \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{-\nu_{zy}}{E_z} & 0 & 0 & \frac{\mu_{xy}}{G_{xy}} \\ \frac{-\nu_{xz}}{E_x} & \frac{-\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & \frac{\zeta_{xy}}{G_{xy}} \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & \frac{\xi_{xz}}{G_{xz}} & 0 \\ 0 & 0 & 0 & \frac{\xi_{yz}}{G_{yz}} & \frac{1}{G_{xz}} & 0 \\ \frac{\eta_x}{E_x} & \frac{\mu_y}{E_y} & \frac{\zeta_z}{E_z} & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{13.7}$$

In this equation, there are coupling terms characterized by the coefficients η_{xy}, μ_{xy}, ζ_{xy}, and ξ_{xy}, which are not similar to the Poisson coefficients.

The values of elastic constants in Relation 13.7 are deduced immediately from the technical forms obtained above for the coefficients Φ_{IJKL}. These constants are detailed below. One obtains subsequently the elastic modulus and Poisson coefficients in the x,y,z coordinates.

$\frac{1}{E_x} = \frac{c^4}{E_\ell} + \frac{s^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)$	➡	$E_x(\theta) = \dfrac{1}{\frac{c^4}{E_\ell} + \frac{s^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - \frac{2\nu_{t\ell}}{E_t}\right)}$
$\frac{1}{E_y} = \frac{s^4}{E_\ell} + \frac{c^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - 2\frac{\nu_{t\ell}}{E_t}\right)$	➡	$E_y(\theta) = \dfrac{1}{\frac{s^4}{E_\ell} + \frac{c^4}{E_t} + s^2c^2\left(\frac{1}{G_{\ell t}} - \frac{2\nu_{t\ell}}{E_t}\right)}$
$\frac{1}{E_z} = \frac{1}{E_t}$	➡	$E_z(\theta) = E_t(\forall\theta)$
$-\frac{\nu_{yx}}{E_y} = -\frac{\nu_{t\ell}}{E_t}(c^4 + s^4)\cdots$ $\cdots + c^2s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} - \frac{1}{G_{\ell t}}\right)$	➡	$\frac{\nu_{yx}}{E_y}(\theta) = \frac{\nu_{t\ell}}{E_t}(c^4 + s^4)\cdots$ $\cdots - c^2s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} - \frac{1}{G_{\ell t}}\right)$
$-\frac{\nu_{zx}}{E_z} = -\left(c^2\frac{\nu_{t\ell}}{E_t} + s^2\frac{\nu}{E_t}\right)$	➡	$\nu_{zx}(\theta) = c^2\nu_{t\ell} + s^2\nu$
$-\frac{\nu_{zy}}{E_z} = -\left(s^2\frac{\nu_{t\ell}}{E_t} + c^2\frac{\nu}{E_t}\right)$	➡	$\nu_{zy}(\theta) = s^2\nu_{t\ell} + c^2\nu$
$\frac{1}{G_{yz}} = c^2\frac{2(1+\nu)}{E_t} + \frac{s^2}{G_{\ell t}}$	➡	$G_{yz}(\theta) = \dfrac{1}{c^2\frac{2(1+\nu)}{E_t} + \frac{s^2}{G_{\ell t}}}$

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$\dfrac{1}{G_{xz}} = s^2\dfrac{2(1+\nu)}{E_t} + \dfrac{c^2}{G_{\ell t}}$	$\Rightarrow G_{xz}(\theta) = \dfrac{1}{s^2\frac{2(1+\nu)}{E_t} + \frac{c^2}{G_{\ell t}}}$
$\dfrac{1}{G_{xy}} = 4c^2s^2\left(\dfrac{1}{E_\ell} + \dfrac{1}{E_t} + 2\dfrac{\nu_{t\ell}}{E_t}\right) + \dfrac{(c^2-s^2)^2}{G_{\ell t}}$	$\Rightarrow G_{xy}(\theta) = \dfrac{1}{4c^2s^2\left(\frac{1}{E_\ell} + \frac{1}{E_t} + 2\frac{\nu_{t\ell}}{E_t}\right) + \frac{(c^2-s^2)^2}{G_{\ell t}}}$
$\dfrac{\eta_{xy}}{G_{xy}} = -2cs\left\{\dfrac{c^2}{E_\ell} - \dfrac{s^2}{E_t} \cdots \right.$ $\left. \cdots + (c^2-s^2)\left(\dfrac{\nu_{t\ell}}{E_t} - \dfrac{1}{2G_{\ell t}}\right)\right\}$;	$\dfrac{\mu_{xy}}{G_{xy}} = -2cs\left\{\dfrac{s^2}{E_\ell} - \dfrac{c^2}{E_t} \cdots \right.$ $\left. \cdots - (c^2-s^2)\left(\dfrac{\nu_{t\ell}}{E_t} - \dfrac{1}{2G_{\ell t}}\right)\right\}$
$\dfrac{\zeta_{xy}}{G_{xy}} = -2cs\dfrac{(\nu - \nu_{t\ell})}{E_t}$;	$\dfrac{\zeta_{xz}}{G_{xz}} = -cs\left(\dfrac{1}{G_{\ell t}} - \dfrac{2(1+\nu)}{E_t}\right)$

(13.8)

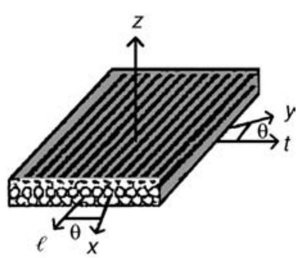
13.3 CASE OF A PLY

One can observe from Equation 13.7 that the stress–strain relations in the plane x,y appear decoupled in the case when $\sigma_{zz} = 0$. We suppose that this applies for the plies making a thin laminate. Each ply will be characterized in its plane by the following relations which are extracted from relations 13.5[10] and 13.7:

- In the orthotropic axes ℓ,t:

$$\begin{Bmatrix} \varepsilon_\ell \\ \varepsilon_t \\ \gamma_{\ell t} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_\ell} & \frac{-\nu_{t\ell}}{E_t} & 0 \\ \frac{-\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & 0 \\ 0 & 0 & \frac{1}{G_{\ell t}} \end{bmatrix} \begin{Bmatrix} \sigma_\ell \\ \sigma_t \\ \tau_{\ell t} \end{Bmatrix} \qquad (13.9)$$

- In the x,y axes, making an angle θ with the orthotropic axes:



[10] The orthotropic axes of Equation 13.5 are denoted as l,t,z for a ply (see Section 3.3.1).

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$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{xy}}{G_{xy}} \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\mu_{xy}}{G_{xy}} \\ \frac{\eta_x}{E_x} & \frac{\mu_y}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} \tag{13.10}$$

For the constants, the values were shown in detail in 13.8.

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