PART II

MECHANICAL BEHAVIOR OF LAMINATED MATERIALS

We have introduced in the previous part the anisotropic properties of composite materials from a qualitative point of view,[1] and we have indicated the characteristic elastic constants for the behavior of an anisotropic layer in its plane.

We have also mentioned the relations that allow one to predict the mechanical behavior of a fiber/matrix combination from the properties of the individual constituents.[2] In Chapter 5,[3] we have also given the elements necessary for the sizing of the laminates made of carbon/epoxy, Kevlar/epoxy, and glass/epoxy, in terms of strength and deformation.

This second part is dedicated to the justification and application of these properties and results. It requires a detailed study of the behavior of anisotropic composite layers and of the stacking that makes up the laminate. It is useful to note that the basics of the mechanics of continuous media—namely, the state of stress and strain at a point—already described in great details in many texts on mechanics of materials, are supposed to be known.

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1 See Section 3.1.
2 See Section 3.3.1.
3 See Section 5.2/5.3.

9

ANISOTROPIC ELASTIC MEDIA

9.1 REVIEW OF NOTATIONS

9.1.1 Continuum Mechanics

We consider the following classical notions and notations of the mechanics of continuous media:

- **State of stress at a point:** This is defined by a second order **tensor** with the symbol Â. The 3 by 3 matrix associated with this tensor is symmetric. In this matrix, there are six distinct terms, which are denoted as s_{ij}:

$$s_{11};\ s_{22};\ s_{33};\ s_{23};\ s_{13};\ s_{12}$$

- **State of strain at a point:** This is defined as a second order tensor e. The 3 by 3 matrix for this tensor is symmetric. It consists of six distinct terms denoted as e_{ij}:

$$e_{11};\ e_{22};\ e_{33};\ e_{23};\ e_{13};\ e_{12}$$

- **Linear elastic medium:** The strains are linear and homogeneous functions of the stresses. The corresponding relations are:

$$e_{ij} = j_{ijk\ell} ¥ s_{k\ell}$$ [1]

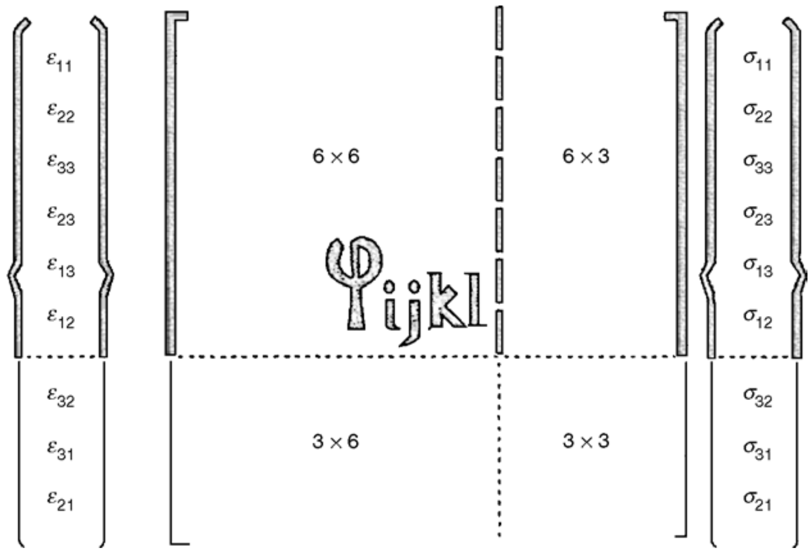
- **Homogeneous medium:** In this case, the matrix terms j_{ijkl} characterizing the elastic behavior of the medium are not point functions. They are the same at all points in the considered medium.

[1] For example:

$$e_{11} = j_{1111}\ s_{11} + j_{1112}\ s_{12} + j_{1113}\ s_{13} + j_{1121}\ s_{21} + j_{1122}\ s_{22} + j_{1123}\ s_{23} + j_{1131}\ s_{31} + j_{1132}\ s_{32} + j_{1133}\ s_{33}.$$

9.1.2 Number of Distinct $j_{ijk\ell}$ Terms

The above stress–strain relation can be written in matrix form as:



- Due to the symmetry of the stresses ($s_{k\ell} = s_{\ell k}$), the corresponding coefficients are the same, i.e., $j_{ijk\ell} = j_{ij\ell k}$.
- Due to the symmetry of the strains ($e_{ij} = e_{ji}$), the corresponding coefficients are the same, i.e., $j_{ijk\ell} = j_{jik\ell}$. In other words, the knowledge of only the coefficients of the 6 ¥ 6 matrix written above is required.
- In addition, application of the theorem of virtual work on the stresses shows that the coefficients $j_{ijk\ell}$ are symmetric, meaning: $j_{ijk\ell} = j_{k\ell ij}$.[2]

Therefore, the 6 ¥ 6 matrix mentioned previously is symmetric. The number of distinct coefficients is:

$$\frac{6(6+1)}{2} = 21 \text{ coefficients}$$

[2] Consider two simple stress states:

- **State No. 1:** One single stress, $(s_{k\ell})_1$, which causes the strain:

$$(e_{ij})_1 = j_{ijk\ell}\,(s_{k\ell})_1$$

- **State No. 2:** One single stress, $(s_{pq})_2$, which causes the strain:

$$(e_{mn})_2 = j_{mnpq}\,(s_{pq})_2$$

One can write that the work of the stress in state No. 1 on the strain in state No. 2 is equal to the work of the stress in state No. 2 on the strain in state No. 1, as:

$$(s_{k\ell})_1 \text{¥} (e_{k\ell})_2 = (s_{pq})_2 \text{¥} (e_{pq})_1$$

which means: $(s_{k\ell})_1 \text{¥} j_{k\ell pq} \text{¥} (s_{pq})_2 = (s_{pq})_2 \text{¥} j_{pqk\ell} \text{¥} (s_{k\ell})_1$

from which one has: $j_{k\ell pq} = j_{pqk\ell}$

- In summary:

$$\boxed{\begin{array}{l}\text{stress reciprocity: } \varphi_{ijk\ell} = \varphi_{ij\ell k}\\ \text{strain definition: } \varphi_{ijk\ell} = \varphi_{jik\ell}\\ \text{symmetry: } \varphi_{ijk\ell} = \varphi_{k\ell ij}\\ \text{There remain 21 distinct coefficients } \varphi_{ijk\ell}\end{array}} \qquad (9.1)$$

The previous stress–strain relation can then be written as:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \varphi_{1111} & \varphi_{1122} & \varphi_{1133} & 2\varphi_{1123} & 2\varphi_{1113} & 2\varphi_{1112} \\ \varphi_{2211} & \varphi_{2222} & \varphi_{2233} & 2\varphi_{2223} & 2\varphi_{2213} & 2\varphi_{2212} \\ \varphi_{3311} & \varphi_{3322} & \varphi_{3333} & 2\varphi_{3323} & 2\varphi_{3313} & 2\varphi_{3312} \\ \varphi_{2311} & \varphi_{2322} & \varphi_{2333} & 2\varphi_{2323} & 2\varphi_{2313} & 2\varphi_{2312} \\ \varphi_{1311} & \varphi_{1322} & \varphi_{1333} & 2\varphi_{1323} & 2\varphi_{1313} & 2\varphi_{1312} \\ \varphi_{1211} & \varphi_{1222} & \varphi_{1233} & 2\varphi_{1223} & 2\varphi_{1213} & 2\varphi_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

The matrix above does not have the general symmetry as in the general form (9×9) presented previously. (Note the coefficients 2 in this matrix). One can get around this inconvenience by doubling the terms ε_{23}, ε_{13}, ε_{12}, introducing the shear strains:

$$\gamma_{23} = 2\varepsilon_{23}; \quad \gamma_{13} = 2\varepsilon_{13}; \quad \gamma_{12} = 2\varepsilon_{12}$$

from which the stress–strain behavior can then be written in a symmetric form as:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \begin{Bmatrix} \\ \\ \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \varphi_{1111} & \varphi_{1122} & \varphi_{1133} & 2\varphi_{1123} & 2\varphi_{1113} & 2\varphi_{1112} \\ \varphi_{2211} & \varphi_{2222} & \varphi_{2233} & 2\varphi_{2223} & 2\varphi_{2213} & 2\varphi_{2212} \\ \varphi_{3311} & \varphi_{3322} & \varphi_{3333} & 2\varphi_{3323} & 2\varphi_{3313} & 2\varphi_{3312} \\ 2\varphi_{2311} & 2\varphi_{2322} & 2\varphi_{2333} & 4\varphi_{2323} & 4\varphi_{2313} & 4\varphi_{2312} \\ 2\varphi_{1311} & 2\varphi_{1322} & 2\varphi_{1333} & 4\varphi_{1323} & 4\varphi_{1313} & 4\varphi_{1312} \\ 2\varphi_{1211} & 2\varphi_{1222} & 2\varphi_{1233} & 4\varphi_{1223} & 4\varphi_{1213} & 4\varphi_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \qquad (9.2)$$

9.2 ORTHOTROPIC MATERIALS

Definition: An orthotropic material is a homogeneous linear elastic material having two planes of symmetry at every point in terms of mechanical properties, these two planes being perpendicular to each other.

Then one can show that[3] the number of independent elastic constants is nine. The constitutive relation expressed in the so-called "**orthotropic**" axes, defined by three axes constructed on the two orthogonal planes and their intersection line, can be written in the following form, called the **engineering notation** because it utilizes the elastic modulus and Poisson ratios:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{9.3}$$

where

E_1, E_2, E_3 are the longitudinal elastic moduli.
G_{23}, G_{13}, G_{12} are the shear moduli.
ν_{12}, ν_{13}, ν_{23}, ν_{21}, ν_{31}, ν_{32} are the Poisson ratios.

In addition, the symmetry of the stress–strain matrix above leads to the following relations:

$$\boxed{\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}; \quad \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}} \tag{9.4}$$

9.3 TRANSVERSELY ISOTROPIC MATERIALS

Definition: A transversely isotropic material is a homogeneous linear elastic material such that any plane including a preferred axis, is a plane of mechanical symmetry.

One can show that[4] the constitutive relation has five independent elastic constants. For the fiber/matrix composite shown in Figure 9.1 the preferred axis is ℓ. The fibers are distributed uniformly in the direction along ℓ. All directions perpendicular to the fibers characterize the transverse direction t.

[3] Proof is shown in Section 13.1.
[4] Proof is shown in Section 13.2.

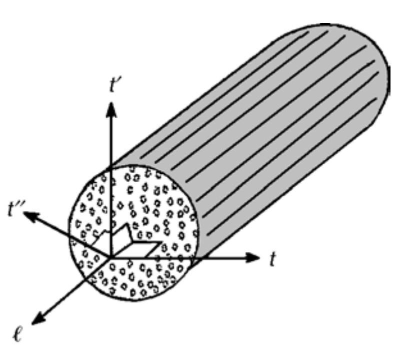


Figure 9.1 Transversely Isotropic Unidirectional

The engineering stress–strain relation has the form:

$$
\begin{Bmatrix} \varepsilon_{\ell\ell} \\ \varepsilon_{tt} \\ \varepsilon_{t't'} \\ \gamma_{tt'} \\ \gamma_{\ell t'} \\ \gamma_{\ell t} \end{Bmatrix} =
\begin{bmatrix}
\frac{1}{E_\ell} & -\frac{\nu_{t\ell}}{E_t} & -\frac{\nu_{t\ell}}{E_t} & 0 & 0 & 0 \\
-\frac{\nu_{\ell t}}{E_\ell} & \frac{1}{E_t} & -\frac{\nu_t}{E_t} & 0 & 0 & 0 \\
-\frac{\nu_{\ell t}}{E_\ell} & -\frac{\nu_t}{E_t} & \frac{1}{E_t} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2(1+\nu_t)}{E_t} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{\ell t}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G_{\ell t}}
\end{bmatrix}
\begin{Bmatrix} \sigma_{\ell\ell} \\ \sigma_{tt} \\ \sigma_{t't'} \\ \tau_{tt'} \\ \tau_{\ell t'} \\ \tau_{\ell t} \end{Bmatrix} \tag{9.5}
$$

Remarks: The independent elastic constants are

- Young modulus along the ℓ direction: E_ℓ.
- Young modulus along any transverse direction t: E_t.
- Shear modulus in the plane ℓ, t: $G_{\ell t}$.
- Poisson coefficients: $\nu_{\ell t}$ and ν_t.

The symmetry of the coefficients of the constitutive relation leads to

$$\boxed{\frac{\nu_{\ell t}}{E_\ell} = \frac{\nu_{t\ell}}{E_t}}$$

One may also note that the shear modulus in the plane t, t' can be written as:

$$\frac{E_t}{2(1+\nu_t)}$$

This equation is classic for isotropic materials.