12

MECHANICAL BEHAVIOR OF THIN LAMINATED PLATES

The definition of a laminate was given in Chapter 5.[1] In the same chapter the practical calculation methods for the laminate was also described. We propose here to justify these methods, meaning to study the behavior of the laminate when it is subjected to a combination of loadings. This study is necessary if one wants to have correct design with strains or stresses within their admissible values.[2]

12.1 LAMINATE WITH MIDPLANE SYMMETRY

12.1.1 Membrane Behavior

We consider in the following a laminate with **midplane symmetry**.[3] The total thickness of the laminate is denoted as h. It consists of n plies. Ply number k has a thickness denoted as e_k (see Figure 12.1). Plane x,y is the plane of symmetry.

12.1.1.1 Loadings

The laminate is subjected to loadings in its plane. The stress resultants are denoted as N_x, N_y, $T_{xy} = T_{yx}$. These are the membrane stress resultants.They are defined as:

- N_x: Stress resultant in the x direction over a unit width along the y direction.

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} (\sigma_x)_k \times e_k \qquad (12.1)$$

- N_y: Stress resultant along the y direction over a unit width along the x direction.

$$N_y = \int_{-h/2}^{h/2} \sigma_y \, dz = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} (\sigma_y)_k \times e_k \qquad (12.2)$$

1 See Section 5.2.
2 The problem of buckling of the laminates is not the scope of this chapter. See Appendix 2.
3 See Section 5.2.3.

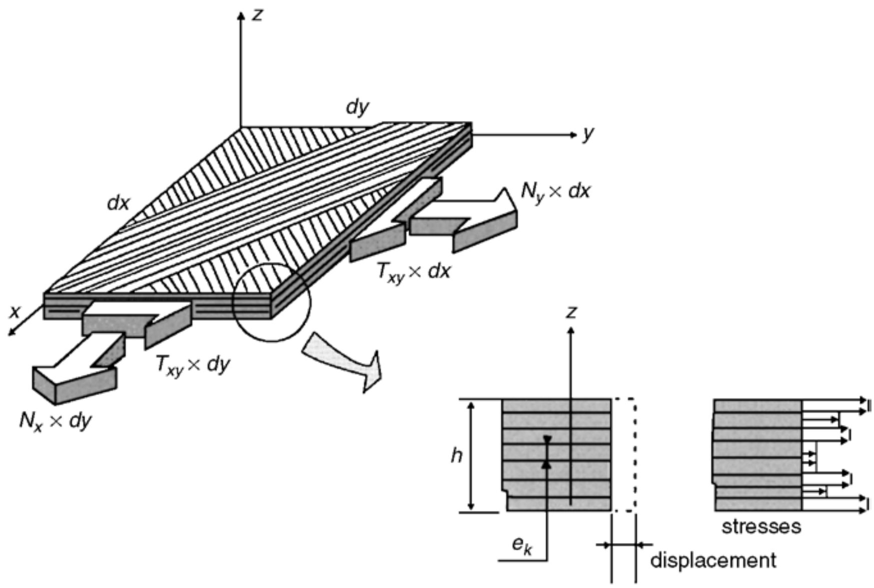

Figure 12.1 Definition of Laminate and Membrane Loading

- T_{xy} (or T_{yx}): Membrane shear stress resultant over a unit width along the y direction (or respectively along the x direction):

$$T_{xy} = \int_{-h/2}^{h/2} \tau_{xy}\, dz = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} (\tau_{xy})_k \times e_k \tag{12.3}$$

12.1.1.2 Displacement Field

The elastic displacement at each point of the laminate is assumed to be two dimensional, in the x,y plane of the laminate. It has the components: u_o , v_o. The nonzero strains can be written as:

$$\varepsilon_{ox} = \partial u_0 / \partial x$$
$$\varepsilon_{oy} = \partial v_0 / \partial y$$
$$\gamma_{oxy} = \partial u_0 / \partial y + \partial v_0 / \partial x$$

It was shown in the previous chapter (Equation 11.8) that one can express, in a given coordinate system, the stresses in a ply as functions of the strains. Then the stress resultant N_x defined in Equation 12.1 can be written as follows:

$$N_x = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \{\bar{E}_{11}^k \varepsilon_{ox} + \bar{E}_{12}^k \varepsilon_{oy} + \bar{E}_{13}^k \gamma_{oxy}\} e_k$$

then:

$$N_x = A_{11}\varepsilon_{ox} + A_{12}\varepsilon_{oy} + A_{13}\gamma_{oxy}$$

with:

$$A_{11} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{11}^{k} e_k; \quad A_{12} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{12}^{k} e_k; \quad A_{13} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{13}^{k} e_k$$

In an analogous manner, one obtains for the Equation 12.2:

$$N_y = A_{21}\varepsilon_{ox} + A_{22}\varepsilon_{oy} + A_{23}\gamma_{oxy}$$

with:

$$A_{2j} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{2j}^{k} e_k$$

and for the shear stress resultant T_{xy} one can write, starting from Equation 12.3:

$$T_{xy} = A_{31}\varepsilon_{ox} + A_{32}\varepsilon_{oy} + A_{33}\gamma_{oxy}$$

with:

$$A_{3j} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{3j}^{k} e_k$$

Therefore, it is possible to express the stress resultants in the following matrix form:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix}$$

with:

$$A_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^{k} e_k = A_{ji} \qquad (12.4)^{4}$$

Remarks:

- One observes from the above expressions that coefficients A_{ij} are independent of the stacking order of the plies.
- One can also see that the normal stress resultants N_x or N_y create angular distortions. This coupling will disappear if the laminate is **balanced**. This means that apart from the midplane symmetry, there are as many and identical plies that make with the *x* axis an angle $+\theta$ as those that make

[4] The developments of $\bar{E}_{ij}$ are given in Equation 11.8.

with the x axis an angle $-\theta$.[5] In effect, the coefficients $\bar{E}_{13}$ and $\bar{E}_{23}$ are antisymmetric in θ[6] and, therefore, cancel each other out for the pairs of plies at $\pm\theta$ when one calculates the terms A_{13} and A_{23}. The result is then:

$$A_{13} = A_{23} = 0$$

and the stress–strain relation for the laminate is reduced to

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} \tag{12.5}$$

- It is possible to substitute the stress resultants N_x, N_y, T_{xy} with the global average stresses (which are fictitious):

$$\begin{aligned} \sigma_{ox} &= N_x/h \\ \sigma_{oy} &= N_y/h \\ \tau_{oxy} &= T_{xy}/h \end{aligned} \tag{12.6}$$

One then deduces from Equation 12.4 the average membrane stress–strain behavior of a laminate as:

$$\begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} = \frac{1}{h} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} \tag{12.7}$$

- One can also note that according to Equation 12.4 the terms of the matrix $\frac{1}{h}[A]$ above can be written as:

$$\frac{1}{h} \times A_{ij} = \sum_{k=1^{\text{st}}\,\text{ply}}^{n^{\text{th}}\,\text{ply}} \bar{E}_{ij}^{k} \times \frac{e_k}{h}$$

Then the ratios e_k/h can be rearranged to obtain the proportions of plies having the same orientation. In case where these proportions have already been fixed—and therefore their numerical values are known—it becomes possible to calculate the terms $\frac{1}{h}A_{ij}$ **without knowing the thickness**. For example, if the selected orientations are 0°, 90°, +45°, –45°, and by denoting $p^k(\%)$ as the percentages of the plies along the different orientations, one has:

$$\frac{1}{h} \times A_{ij} = \bar{E}_{ij}^{0°} \times p^{0°} + \bar{E}_{ij}^{90°} \times p^{90°} + \bar{E}_{ij}^{+45°} \times p^{+45°} + \bar{E}_{ij}^{-45°} \times p^{-45°} \tag{12.8}$$

5 See Figure 12.1 and figure in the Equation 11.8.
6 The expressions developed for $\bar{E}_{ij}$ are given in Equation 11.8.

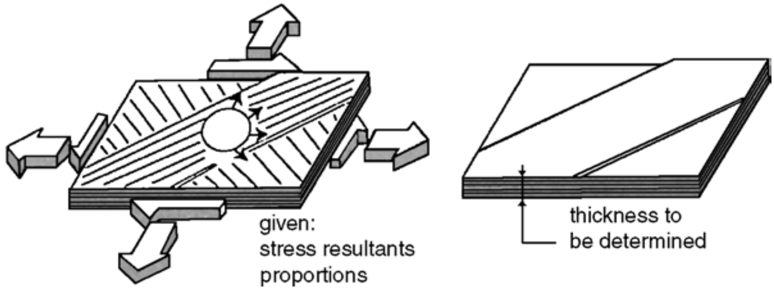

Figure 12.2 Practical Determination of a Laminate Subject to Membrane Loading

12.1.2 Apparent Moduli of the Laminate

Inversion of Equation 12.7 above allows one to obtain what can be called as **apparent moduli** and coupling coefficients associated with the membrane behavior in the plane *x,y*. These coefficients appear in the following relation:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} \qquad (12.9)$$

12.1.3 Consequence: Practical Determination of a Laminate Subject to Membrane Loading

Given:

- The stress resultants are given and denoted as: N_x, N_y, T_{xy}.
- Using the values of these stress resultants, one can estimate the ply proportions in the four orientations.[7] Assume in Figure 12.2 that the plies are identical (same material and same thickness).

The problem is to determine

- The apparent elastic moduli of the laminate and the associated coupling coefficients, in order to estimate strains under loading
- The minimum thickness for the laminate in order to avoid rupture of one of the plies in the laminate

[7] See Section 5.4.3.

12.1.3.1 Principle of Calculation

Apparent moduli of the laminate: The matrix $\frac{1}{h}[A]$ evaluated using Equation 12.8 can be inverted, and one obtains Equation 12.9 as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\nu_{yx}}{\bar{E}_y} & \frac{\eta_{xy}}{\bar{G}_{xy}} \\ -\frac{\nu_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\mu_{xy}}{\bar{G}_{xy}} \\ \frac{\eta_x}{\bar{E}_x} & \frac{\mu_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix}$$

We have already determined the apparent moduli and the coupling coefficients of the laminate.

Nonrupture of the laminate: Let σ_ℓ, σ_t, and $\tau_{\ell t}$ be the stresses in the orthotropic axes ℓ, t of one of the plies making up the laminate that is subjected to the loadings N_x, N_y, T_{xy}. Let h be the thickness of the laminate (unknown at the moment) so that the rupture limit of the ply using the Hill–Tsai failure criterion is just reached.

One then has for this ply[8]:

$$\frac{\sigma_\ell^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\sigma_t^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{\sigma_\ell \sigma_t}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{\tau_{\ell t}^2}{\underset{\text{rupture}}{\tau_{\ell t}^2}} = 1$$

Multiplying the two parts of this equation with the square of thickness h:

$$\frac{(\sigma_\ell h)^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(\sigma_t h)^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{(\sigma_\ell h)(\sigma_t h)}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(\tau_{\ell t} h)^2}{\underset{\text{rupture}}{\tau_{\ell t}^2}} = h^2 \tag{12.10}$$

To obtain the values $(\sigma_\ell h)$, $(\sigma_t h)$, $(\tau_{\ell t} h)$, one has to multiply with h the global stresses σ_{ox}, σ_{oy}, τ_{oxy} that are applied on the laminate, to become $(\sigma_{ox} h)$, $(\sigma_{oy} h)$, $(\tau_{oxy} h)$ which are the known stress resultants:

$$N_x = (\sigma_{ox} h); \quad N_y = (\sigma_{oy} h); \quad T_{xy} = (\tau_{oxy} h)$$

Then, for a ply, the calculation of the Hill–Tsai criterion can be done by substituting for the unknown global stresses the known stress resultants N_x, N_y, T_{xy}. This leads to the calculation of the thickness h so that the ply under consideration does not fracture.

In this way, each ply number k leads to a laminate thickness value denoted as h_k. The final thickness to be retained will the one with the highest value.

[8] For the Hill–Tsai failure criterion, see Section 5.2.3 and detailed explanation in Chapter 14.

12.1.3.2 Calculation Procedure

1. **Complete calculation:** The ply proportions are given, the matrix $\frac{1}{h}[A]$ of the Equation 12.7 is known, and then—after inversion—we obtain the elastic moduli of the laminate (Equation 12.9).[9] Multiplying 12.9 with the thickness h (unknown) of the laminate:

$$\begin{Bmatrix} h\varepsilon_{ox} \\ h\varepsilon_{oy} \\ h\gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix}$$

Then introducing a multiplication factor of h for the stresses in the ply—or the group of plies—corresponding to the orientation k (see Equation 11.8):

$$\underset{\text{ply } n^\circ k}{\begin{Bmatrix} h\sigma_x \\ h\sigma_y \\ h\tau_{xy} \end{Bmatrix}} = \underset{\text{ply } n^\circ k}{\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}} \underset{\text{laminate}}{\begin{Bmatrix} h\varepsilon_{ox} \\ h\varepsilon_{oy} \\ h\gamma_{oxy} \end{Bmatrix}}$$

and in the orthotropic coordinates of the ply (see Equation 11.4):

$$\underset{\text{ply } n^\circ k}{\begin{Bmatrix} h\sigma_\ell \\ h\sigma_t \\ h\tau_{\ell t} \end{Bmatrix}} = \underset{\text{ply } n^\circ k}{\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ sc & -sc & (c^2 - s^2) \end{bmatrix}} \underset{\text{ply } n^\circ k}{\begin{Bmatrix} h\sigma_x \\ h\sigma_y \\ h\tau_{xy} \end{Bmatrix}} \qquad c = \cos\theta;\ s = \sin\theta$$

Saturation of the Hill–Tsai criterion leads then to Equation 12.10 where the above known stress resultants values appear in the numerator as:

$$\frac{(h\sigma_\ell)^2}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(h\sigma_t)^2}{\underset{\text{rupture}}{\sigma_t^2}} - \frac{(h\sigma_\ell)(h\sigma_t)}{\underset{\text{rupture}}{\sigma_\ell^2}} + \frac{(h\tau_{\ell t}^2)}{\underset{\text{rupture}}{\tau_{\ell t}^2}} = h^2 \times 1$$

After having written an analogous expression for each orientation k of the plies, one retains for the final value of the laminate thickness, the maximum value found for h.

[9] One can read directly these moduli in Tables 5.1 to 5.15 of Section 5.4.2 for balanced laminates of carbon, Kevlar, and glass/epoxy with V_f = 60% fiber volume fraction.

(2) **Simplified calculation:** One can write more rapidly the Equation 12.10 if one knows at the beginning for each orientation the stresses due to a global uniaxial state of unit stress applied on the laminate: first $\sigma'_{ox} = 1$ (for example, 1 MPa), then $\sigma''_{oy} = 1$ MPa, then $\tau'''_{oxy} = 1$ MPa.

■ Assume first that the state of stress is given as:

$$\left|\begin{array}{l} \sigma'_{ox} = 1(\text{MPa}) \\ \sigma'_{ox} = 0 \\ \tau'_{oxy} = 0 \end{array}\right.$$

Inverting the Equation 12.9 leads to

$$\begin{Bmatrix} \varepsilon'_{ox} \\ \varepsilon'_{oy} \\ \gamma'_{oxy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\bar{E}_x} & -\dfrac{\bar{\nu}_{yx}}{\bar{E}_y} & \dfrac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\dfrac{\bar{\nu}_{xy}}{\bar{E}_x} & \dfrac{1}{\bar{E}_y} & \dfrac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \dfrac{\bar{\eta}_x}{\bar{E}_x} & \dfrac{\bar{\mu}_y}{\bar{E}_y} & \dfrac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} 1 \text{ MPa} \\ 0 \\ 0 \end{Bmatrix}$$

which can be considered as "**unitary strains**" of the laminate. These allow the calculation of the stresses in each ply by means of Equations 11.8 and then 11.4, successively, as:

$$\underset{\text{ply } n^\circ k}{\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau'_{xy} \end{Bmatrix}} = \underset{\text{ply } n^\circ k}{\begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}} \underset{\text{laminate}}{\begin{Bmatrix} \varepsilon'_{ox} \\ \varepsilon'_{oy} \\ \gamma'_{oxy} \end{Bmatrix}}$$

and in the orthotropic coordinates of the ply (Equation 11.4):

$$\underset{\text{ply } n^\circ k}{\begin{Bmatrix} \sigma'_\ell \\ \sigma'_t \\ \tau'_{\ell t} \end{Bmatrix}} = \underset{\text{ply } n^\circ k}{\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ sc & -sc & (c^2 - s^2) \end{bmatrix}} \underset{\text{ply } n^\circ k}{\begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau'_{xy} \end{Bmatrix}} \qquad \begin{array}{l} c = \cos\theta \\ s = \sin\theta \end{array}$$

■ Consider then the state of stresses:

$$\left|\begin{array}{l} \sigma''_{ox} = 0 \\ \sigma''_{oy} = 1 \text{ (MPa)} \\ \tau''_{oxy} = 0 \end{array}\right.$$

Following the same procedure, one can calculate σ''_ℓ, σ''_t, and $\tau''_{\ell t}$ in the orthotropic axes of each ply for a global stress on the laminate that is reduced to $\sigma''_{oy} = 1$ MPa.

■ Finally consider the state of stresses:

$$\left| \begin{array}{l} \sigma_{ox}''' = 0 \\ \sigma_{oy}''' = 0 \\ \tau_{oxy}''' = 1 \ \text{(MPa)} \end{array} \right.$$

Following the same procedure, one obtains σ_ℓ'', σ_t'', and $\tau_{\ell t}''$ in the orthotropic axes of each ply for a global stress applied on the laminate, and that is reduced to $\tau_{oxy}'' = 1$ MPa.[10]

It is then easy to determine by simple rule of proportion (or multiplication)[11] the quantities $(\sigma_\ell h)$, $(\sigma_t h)$, and $(\tau_{\ell t} h)$ in each ply, corresponding to loadings that are no longer unitary, but equal successively to

$$N_x = (\sigma_{ox} h)$$

then:

$$N_y = (\sigma_{oy} h)$$

then:

$$T_{xy} = (\tau_{oxy} h)$$

Subsequently, the **principle of superposition** allows one to determine $(\sigma_\ell h)_{\text{total}}$ $(\sigma_t h)_{\text{total}}$ and $(\tau_{\ell t} h)_{\text{total}}$ when one applies **simultaneously** N_x, N_y, and T_{xy}. From these it is possible to write the modified Hill–Tsai expression in the form of Equation 12.10, which will provide the thickness for the laminate needed to avoid the fracture of the ply under consideration.

If h_k is the laminate thickness obtained from the ply number k, after having gone over all the plies, one will retain for the final thickness h the thickness of highest value found as:

$$h = \sup \{h_k\}$$[12]

Remark: The principle of calculation is conserved when the plies have different thicknesses with any orientations. It then becomes indispensable to program the procedure, or to use existing computer programs. Then one can propose a complete composition for the laminate and verify that the solution is satisfactory regarding the criterion mentioned previously (deformation and fracture). This is

[10] This calculation can be easily programmed on a computer: *cf.* Application 18.2.2 "Program for Calculation of a Laminate." One will find in Appendix 1 at the end of the book the values σ_ℓ, σ_t, $\tau_{\ell t}$ obtained for the particular case of a carbon/epoxy laminate with ply orientations of 0°, 90°, +45°, −45°. These values are given in Plates 1 to 12.

[11] For example, one has the following:

$$\sigma_{ox}' = 1 \ \text{MPa} \rightarrow \sigma_\ell', \ \sigma_t', \ \tau_{\ell t}'$$
$$\sigma_{ox}(\text{MPa}) \ \rightarrow \sigma_\ell, \ \sigma_t, \ \tau_{\ell t}$$

then: $$\frac{\sigma_{ox}}{\sigma_{ox}'} = \frac{\sigma_\ell}{\sigma_\ell'} \Rightarrow \sigma_\ell = \sigma_\ell' \times \frac{\sigma_{ox}}{1}, \text{ and } h\sigma_\ell = \frac{\sigma_\ell'}{1} \times N_x$$

[12] This method to determine the thickness is illustrated by an example: See Application 18.1.6.

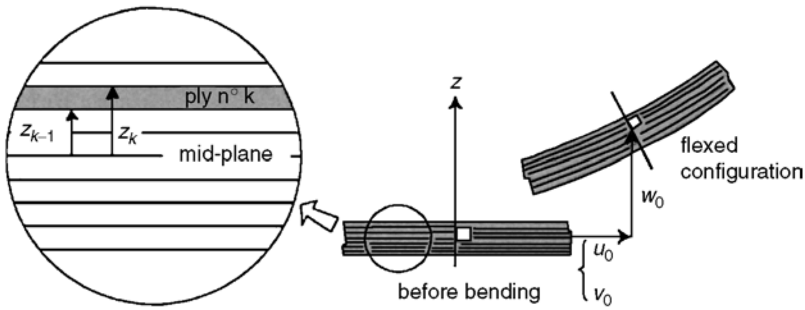

Figure 12.3 Bending of the Laminate

facilitated by using the user friendly aspect of the program, allowing rapid return of the solution.

12.1.4 Flexure Behavior

In the previous paragraph, we have limited discussion to loadings consisting of N_x, N_y, and T_{xy} applying in the midplane of the laminate. We will now examine the cases that can cause deformation outside of the plane of the laminate. The laminate considered is—as before—supposed to have **midplane symmetry**.

12.1.4.1 Displacement Fields

- **Hypothesis:** Assume that a line perpendicular to the midplane of laminate before deformation (see Figure 12.3) remains perpendicular to the mid-plane surface after deformation.
- **Consequence:** If one denotes as before u_o and v_o the components of the displacement in the midplane and w_o as the displacement out of the plane (see Figure 12.3), the displacement of any point at a position z in the laminate (in the nondeformed configuration) can be written as

$$\begin{cases} u = u_o - z\dfrac{\partial w_0}{\partial x} \\ v = v_o - z\dfrac{\partial w_0}{\partial y} \\ w = w_o \end{cases} \tag{12.11}$$

One can then deduce the nonzero strains:

$$\begin{cases} \varepsilon_x = \varepsilon_{ox} - z\dfrac{\partial^2 w_0}{\partial x^2} \\ \varepsilon_y = \varepsilon_{oy} - z\dfrac{\partial^2 w_0}{\partial y^2} \\ \gamma_{xy} = \gamma_{oxy} - z \times 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{cases} \tag{12.12}$$

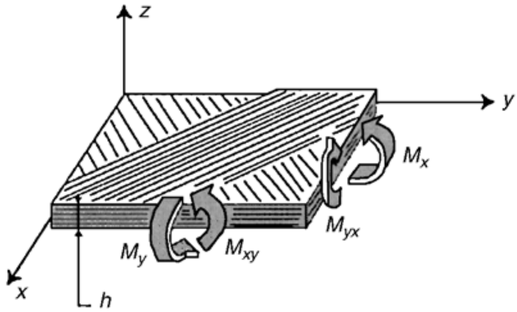

Figure 12.4 Moment Resultants

12.1.4.2 Loadings

In addition to the membrane stress resultants N_x, N_y, T_{xy} in the previous paragraphs, one can add the **moment resultants** along the x and y directions (see Figure 12.4).

As in the case of the membrane stress resultants, the moment resultants serve to synthesize the cohesive forces that appear by sectioning, following classical method that is common for all structures (beams, plates, etc.). One can interpret these as the unit moments of the cohesive forces.[13] They are written as:

- M_y: Moment resultant along the y axis, due to the stresses σ_x, over a unit width along the y direction.

$$M_y = \int_{-h/2}^{h/2} \sigma_x z \, dz \tag{12.13}$$

- M_x: Moment resultant along the x direction, due to the stress σ_y, over a unit width along the x direction.

$$M_x = -\int_{-h/2}^{h/2} \sigma_y z \, dz \tag{12.14}$$

- M_{xy}: (or $-M_{yx}$): Twisting moment along the x axis (or y axis), due to the shear stress τ_{xy} over a unit width along the y direction (or x direction):

$$M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z \, dz \tag{12.15}$$

[13] The expression of M_y can be written in integral form *as*:

$$M_y = \left[\int_{-h/2}^{h/2} z\vec{z} \wedge \sigma_x \vec{x} dz\right] . \vec{y} = \int_{-h/2}^{h/2} \sigma_x z \, dz$$

also:

$$M_x = \left[\int_{-h/2}^{h/2} z\vec{z} \wedge \sigma_y \vec{y} dz\right] . \vec{x} = -\int_{-h/2}^{h/2} \sigma_y z \, dz$$

Finally:

$$M_{xy} = \left[\int_{-h/2}^{h/2} z\vec{z} \wedge \tau_{xy} \vec{y} dz\right] . \vec{x} = -\int_{-h/2}^{h/2} \tau_{xy} z \, dz$$

Taking Equation 11.8 into consideration, which allows one to express, in a certain coordinate system, the stresses in a ply as functions of strains, the moment resultant M_y (Equation 12.13) can be written as:

$$M_y = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \left\{ \int_{z_{k-1}}^{z_k} (\bar{E}_{11}^k \;\; \varepsilon_x + \bar{E}_{12}^k \;\; \varepsilon_y + \bar{E}_{13}^k \;\; \gamma_{xy}) z \; dz \right\}$$

which, when using Equation 12.12 becomes

$$M_y = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \left\{ \int_{z_{k-1}}^{z_k} \left\{ \bar{E}_{11}^k \left(z\varepsilon_{ox} - z^2 \frac{\partial^2 w_o}{\partial x^2} \right) + \bar{E}_{12}^k \left(z\varepsilon_{oy} - z^2 \frac{\partial^2 w_o}{\partial y^2} \right) \cdots \right.\right.$$

$$\left.\left. \cdots + \bar{E}_{13}^k \left(z\gamma_{oxy} - z^2 2 \frac{\partial^2 w_o}{\partial x \partial y} \right) \right\} dz \right\}$$

Due to midplane symmetry, every integral of the form:

$$\int_{z_{k-1}}^{z_k} \bar{E}_{1j} z \; dz$$

in the above expression is accompanied by an integral of the form:

$$\int_{-z_k}^{-z_{k-1}} \bar{E}_{1j} z \; dz$$

that is opposite in sign. Integrals of this type disappear and there remains

$$M_y = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} - \left\{ \bar{E}_{11}^k \frac{(z_k^3 - z_{k-1}^3)}{3} \frac{\partial^2 w_o}{\partial x^2} + \bar{E}_{12}^k \frac{(z_k^3 - z_{k-1}^3)}{3} \frac{\partial^2 w_o}{\partial y^2} \cdots \right.$$

$$\left. \cdots + \bar{E}_{13}^k \frac{(z_k^3 - z_{k-1}^3)}{3} 2 \frac{\partial^2 w_o}{\partial x \partial y} \right\}$$

which can be written as:

$$M_y = -C_{11} \frac{\partial^2 w_o}{\partial x^2} - C_{12} \frac{\partial^2 w_o}{\partial y^2} - C_{13} 2 \frac{\partial^2 w_o}{\partial x \partial y}$$

with

$$C_{1j} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{1j}^k \frac{(z_k^3 - z_{k-1}^3)}{3}$$

Proceeding in an analogous manner with M_x and M_{xy} (Equations 12.14 and 12.15), one obtains the following matrix form:

$$\begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w_o}{\partial x^2} \\ -\dfrac{\partial^2 w_o}{\partial y^2} \\ -2\dfrac{\partial^2 w_o}{\partial x \partial y} \end{Bmatrix} \tag{12.16}$$

with :

$$C_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^k \frac{(z_k^3 - z_{k-1}^3)}{3}$$

Remarks:

- One can observe that in Equation 12.16 the coefficients C_{ij} **depend** on the stacking sequence of the plies.
- Does a laminated plate bend under membrane loadings? Using the displacement field due to flexure to express, for example, the stress resultant N_x (Equation 12.11), one has

$$N_x = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \left\{ \int_{z_{k-1}}^{z_k} \left\{ \bar{E}_{11}^k \left(\varepsilon_{ox} - z\frac{\partial^2 w_o}{\partial x^2} \right) + \bar{E}_{12}^k \left(\varepsilon_{oy} - z\frac{\partial^2 w_o}{\partial y^2} \right) \cdots \right.\right.$$

$$\left.\left. \cdots + \bar{E}_{13}^k \left(\gamma_{oxy} - z \times 2\frac{\partial^2 w_o}{\partial x \partial y} \right) \right\} dz \right\}$$

Making use of the remark mentioned above, the midplane symmetry causes the disappearance of integrals of the type:

$$\int_{z_{k-1}}^{z_k} \bar{E}_{1j}^k z \, dz$$

As a consequence, one finds again the Equation 12.4 as:

$$N_1 = A_{11}\varepsilon_{ox} + A_{12}\varepsilon_{oy} + A_{13}\gamma_{oxy}$$

As a result of the midplane symmetry, the membrane behavior is independent of the flexural behavior.

- Even in the case of balanced laminate (as many plies oriented at angle θ as the number of plies oriented at an angle $-\theta$), terms C_{13} and C_{23} in Equation 12.16 are not zero. This modifies the deformed bending configuration compared with the isotropic case (see Figure 12.5).

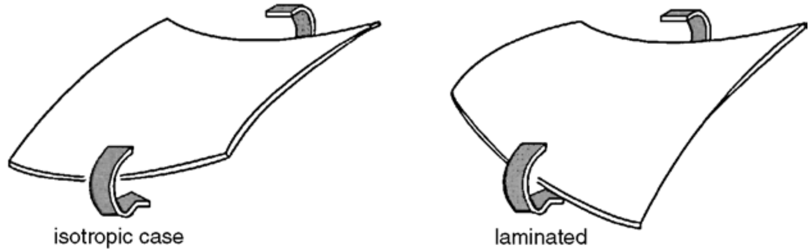

Figure 12.5 Bending Configurations of a Plate

- The terms C_{13} and C_{23} disappear only in the following cases:
  - The plies are oriented uniquely in the 0° and 90°. Then the product $\cos\theta \times \sin\theta$ is zero and[14]:

$$\bar{E}_{13}^{k} = \bar{E}_{23}^{k} = 0 \quad \forall k$$

  - The laminate [0/90/45/–45] is constituted mainly with balanced fabric layers (in each fabric layer, the fibers along the warp and fill directions are, by first approximation[15] at the same z location), or mats layers.
- The stresses in the different plies are obtained from the Equations 11.8. For example, in the ply number k, one has:

$$\sigma_x = \bar{E}_{11}^{k}\ \varepsilon_x + \bar{E}_{12}^{k}\ \varepsilon_y + \bar{E}_{13}^{k}\ \gamma_{xy}$$

and taking into consideration Equations 12.12 for the strains:

$$\sigma_x = [\bar{E}_{11}^{k}\ \varepsilon_{ox} + \bar{E}_{12}^{k}\ \varepsilon_{oy} + \bar{E}_{13}^{k}\ \gamma_{oxy}] - z\left[\bar{E}_{11}^{k}\frac{\partial^2 w_o}{\partial x^2}\cdots\right.$$

$$\left.\cdots + \bar{E}_{12}^{k}\frac{\partial^2 w_o}{\partial y^2} + \bar{E}_{13}^{k} \times 2\frac{\partial^2 w_o}{\partial x \partial y}\right]$$

one can resume by:

$$\sigma_x = \sigma_{x_{\text{membrane}}} + \sigma_{x_{\text{flexure}}}$$

Along the thickness of the laminate, the stress σ_x can be considered as the sum of two parts: a constant part and a linearly varying part, as seen in Figure 12.6. One can also observe analogous forms for the stresses σ_y and τ_{xy}.

[14] See Equations 11.8

[15] See Section 5.2.3.5, Particular Cases of Balanced Fabrics.

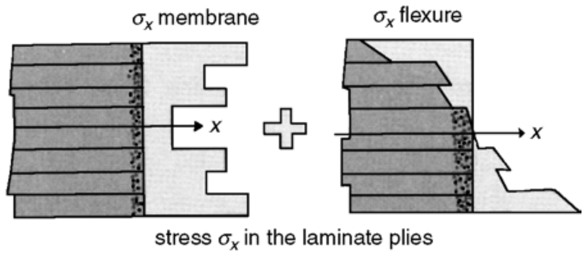

Figure 12.6 Total Normal Stress in a Laminate

12.1.5 Consequence: Practical Determination for a Laminate Subject to Flexure

Given:

- The moment resultants are known.
- Using these resultants, one is led to estimate proportions of plies along the four orientations (or more, eventually)[16] and to predict the stacking sequence.

Principle for the calculation:

- **Nonrupture of laminate:** Following a procedure analogous to that described in Section 12.1.3, it is possible to calculate the stresses σ_ℓ, σ_t, $\tau_{\ell t}$ along the orthotropic axes of each of the plies. This allows the control of their integrity using the Hill–Tsai failure criterion. This requires the use of a computer program which can allow the adjustment of the composition of the laminate.
- **Flexure deformation:** The determination of the deformed configuration of the laminate under flexure poses the same problem as with the isotropic plates: outside of a few cases of academic interest, it is necessary to use a computer program based on the finite element method.[17]

12.1.6 Simplified Calculation for Flexure

It is possible, for a first estimate, to perform simplified calculations by considering that the moment M_y is related uniquely to the curvature $\frac{\partial^2 w_o}{\partial x^2}$ and the moment M_x to the curvature $\frac{\partial^2 w_o}{\partial y^2}$. One then can determine experimentally:

1. **The apparent failure stresses in flexure**

An experiment on a sample can provide the value for the moment at failure, denoted by M_{rupture} on Figure 12.7 (per unit width of the sample). Analogy with

[16] See Section 5.2.
[17] These elements are constituted on the basis presented above and can include the effects which were not taken into account previously: in particular, the transverse shear stresses in flexure due to the transverse shear stress resultants (consult this subject in Chapter 17).

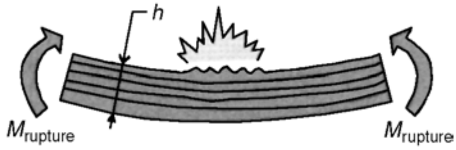

Figure 12.7 Bending Failure

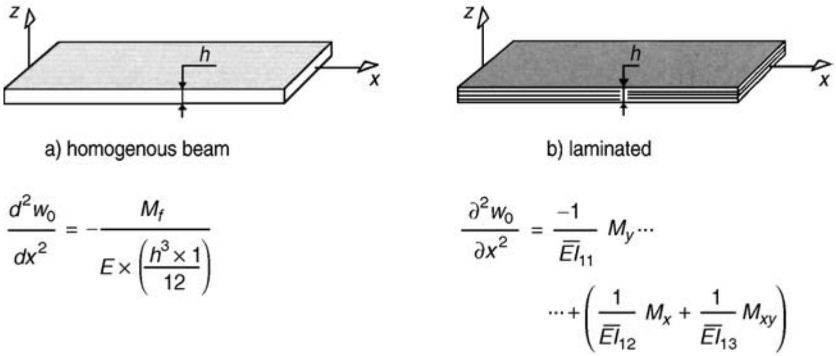

Figure 12.8 Homogeneous and Laminated Beams

the flexure of beams leads to:

$$\|\sigma_{\text{rupture}}\| = \frac{M_{\text{rupture}} \times h/2}{h^3/12} \text{ then: } \|\sigma_{\text{rupture}}\| = M_{\text{rupture}} \times \frac{6}{h^2}$$

2. Apparent flexure moduli

These are obtained starting from the comparison of relations between the "composite" and "homogeneous." One recalls on Figure 12.8(a) the relation between the moment and curvature for a homogeneous beam with unit width, obtained by integration of the local behavior[18]:

$$\varepsilon_x = \frac{\sigma_x}{E} \rightarrow \frac{h^3 \times 1}{12} \times \frac{d^2 w_o}{dx^2} = -\frac{M_f}{E}$$

If one notes that Equation 12.16, recalled as:

$$\begin{Bmatrix} M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = [C] \begin{Bmatrix} -\partial^2 w_o / \partial x^2 \\ -\partial^2 w_o / \partial y^2 \\ -2 \times \partial^2 w_o / \partial x \, \partial y \end{Bmatrix}$$

[18] Recall that $\varepsilon_x = \frac{\partial u}{\partial x}$ with $u = -z\frac{dw_o}{dx}$; then $z^2 \frac{d^2 w_o}{dx^2} = -z\frac{\sigma_x}{E}$ which can be integrated into the thickness.

can be inverted, and noting:

$$[C]^{-1} = \begin{bmatrix} 1/\overline{EI}_{11} & 1/\overline{EI}_{12} & 1/\overline{EI}_{13} \\ 1/\overline{EI}_{21} & 1/\overline{EI}_{22} & 1/\overline{EI}_{23} \\ 1/\overline{EI}_{31} & 1/\overline{EI}_{32} & 1/\overline{EI}_{33} \end{bmatrix}$$

one obtains:

$$\frac{\partial^2 w_o}{\partial x^2} = \frac{-1}{\overline{EI}_{11}} \times M_y + \frac{1}{\overline{EI}_{12}} \times M_x + \frac{1}{\overline{EI}_{13}} \times M_{xy}$$

The identification of the behavior noted in Figure 12.8(a), on the one hand, with only the first term of the moment M_y in the equation in Figure 12.8(b), on the other hand, that is:

$$\overline{EI}_{11} \equiv E\,\frac{h^3 \times 1}{12}$$

leads to an approximate equation of an equivalent modulus E that one can interpret as the **flexure modulus** along the x direction of the homogeneous material:

$$\underset{(\text{along } x)}{E_{\text{flexure}}} = \frac{12}{h^3} \times \overline{EI}_{11}$$

Note: When the plies of the laminate are oriented uniquely along the 0° and 90° directions, or when the laminate [0/90/45/–45] is constituted uniquely of balanced fabrics and of mats, excluding the unidirectional layers, then one has in the matrix [C]:

$$C_{13} = C_{23} = 0$$

then:

$$\overline{EI}_{11} = C_{11} - \frac{C_{12}^2}{C_{22}}$$

12.1.7 Case of Thermomechanical Loading

12.1.7.1 Membrane Behavior

When one considers variation in temperature, which is assumed to be **identical** in all plies of the laminate, the stresses are given by the modified Equations 11.10. Following the procedure of Section 12.2, with the same hypotheses and notations, the stress resultant N_x (Equation 12.1) becomes

$$N_x = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \{\bar{E}_{11}^k\,\varepsilon_{ox} + \bar{E}_{12}^k\,\varepsilon_{oy} + \bar{E}_{13}^k\,\gamma_{oxy}\}e_k - \Delta T \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_1^k \times e_k$$

then:

$$N_x = A_{11}\varepsilon_{ox} + A_{12}\varepsilon_{oy} + A_{13}\gamma_{oxy} - \Delta T\langle \alpha E h\rangle_x$$

with:

$$A_{1j} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{1j}^{k}\ e_k\ ; \langle \alpha E h \rangle_x = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_1^k\ e_k$$

Following the same procedure for N_y and T_{xy}, the stress resultants are expressed as:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} - \Delta T \begin{Bmatrix} \langle \alpha E h \rangle_x \\ \langle \alpha E h \rangle_y \\ \langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

with :

$$A_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^{k} \times e_k = A_{ji} \qquad \text{cf. [11.8]}$$

$$\left.\begin{aligned} \langle \alpha E h \rangle_x &= \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_1^k \times e_k \\ \langle \alpha E h \rangle_y &= \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_2^k \times e_k \\ \langle \alpha E h \rangle_{xy} &= \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_3^k \times e_k \end{aligned}\right\} \qquad \text{cf. [11.10]} \tag{12.17}$$

Inversion of the above relation allows one to show the apparent moduli of the laminate (see Paragraph 12.1.2) and thermal membrane strains:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T [A]^{-1} \begin{Bmatrix} \langle \alpha E h \rangle_x \\ \langle \alpha E h \rangle_y \\ \langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

or with Equation 12.9:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\bar{\bar{E}}_x} & -\dfrac{\bar{\nu}_{yx}}{\bar{\bar{E}}_y} & \dfrac{\bar{\eta}_{xy}}{\bar{\bar{G}}_{xy}} \\ -\dfrac{\bar{\nu}_{xy}}{\bar{\bar{E}}_x} & \dfrac{1}{\bar{\bar{E}}_y} & \dfrac{\bar{\mu}_{xy}}{\bar{\bar{G}}_{xy}} \\ \dfrac{\bar{\eta}_x}{\bar{\bar{E}}_x} & \dfrac{\bar{\mu}_y}{\bar{\bar{E}}_y} & \dfrac{1}{\bar{\bar{G}}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T [A]^{-1} \begin{Bmatrix} \langle \alpha E h \rangle_x \\ \langle \alpha E h \rangle_y \\ \langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

which can be rewritten as:

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = h[A]^{-1} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \times h[A]^{-1} \begin{Bmatrix} \frac{1}{h}\langle \alpha E h \rangle_x \\ \frac{1}{h}\langle \alpha E h \rangle_y \\ \frac{1}{h}\langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

Remarks:

- Evaluation of terms $(1/h)\langle\alpha Eh\rangle_x$, $(1/h)\langle\alpha Eh\rangle_y$, and $(1/h)\langle\alpha Eh\rangle_{xy}$ only requires the knowledge of the proportions of plies along the different orientations and not their thicknesses.[19]
- The matrix $h[A]^{-1}$, already mentioned in Section 12.1.2, contains the global moduli of the laminate. One can then write (see Equation 12.9):

$$\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \begin{bmatrix} \frac{1}{\bar{E}_x} & \frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \frac{1}{h}\langle \alpha E h \rangle_x \\ \frac{1}{h}\langle \alpha E h \rangle_y \\ \frac{1}{h}\langle \alpha E h \rangle_{xy} \end{Bmatrix}$$

The last term of the above equation allows one to show the global expansion coefficients of the laminate, which are denoted as α_{ox}, α_{oy}, and α_{oxy}, as below:

$$\begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \frac{1}{h}\langle \alpha E h \rangle_x \\ \frac{1}{h}\langle \alpha E h \rangle_y \\ \frac{1}{h}\langle \alpha E h \rangle_{xy} \end{Bmatrix} \tag{12.18}$$

In summary, the membrane thermomechanical behavior of a laminate with midplane symmetry can be written as:

$$\boxed{\begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\bar{E}_x} & -\frac{\bar{\nu}_{yx}}{\bar{E}_y} & \frac{\bar{\eta}_{xy}}{\bar{G}_{xy}} \\ -\frac{\bar{\nu}_{xy}}{\bar{E}_x} & \frac{1}{\bar{E}_y} & \frac{\bar{\mu}_{xy}}{\bar{G}_{xy}} \\ \frac{\bar{\eta}_x}{\bar{E}_x} & \frac{\bar{\mu}_y}{\bar{E}_y} & \frac{1}{\bar{G}_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{ox} \\ \sigma_{oy} \\ \tau_{oxy} \end{Bmatrix} + \Delta T \begin{Bmatrix} \alpha_{ox} \\ \alpha_{oy} \\ \alpha_{oxy} \end{Bmatrix}} \tag{12.19}$$

[19] See Application18.2 "Residual Thermal Stresses due to Curing of the Laminate."

This is an equation in which α_{ox}, α_{oy}, and α_{oxy} are given by Equations 12.17 and 12.18.[20]

12.1.7.2 Flexure Behavior

Following the procedure in Section 12.14 with the same notations, the moment resultant M_y (Equation 12.13) becomes, using the modified Equations 11.10:

$$M_y = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \left\{ \int_{z_{k-1}}^{z_k} (\bar{E}_{11}\ \varepsilon_x + \bar{E}_{12}^k\ \varepsilon_y + \bar{E}_{13}^k\ \gamma_{xy}) z\ dz \right\}$$

$$-\Delta T \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \left(\int_{z_{k-1}}^{z_k} \overline{\alpha E_1^k} \times z\, dz \right)$$

The plate is assumed to have midplane symmetry, then each integral of the form $\int_{z_{k-1}}^{z_k} \overline{\alpha E_1} z\, dz$ is associated with another integral such that $\int_{-z_k}^{-z_{k-1}} \overline{\alpha E_1} z\, dz$ is equal and opposite in sign. There remains the following expression, with the notations of Section 12.1.4:

$$M_y = -C_{11}\frac{\partial^2 w_0}{\partial x^2} - C_{12}\frac{\partial^2 w_0}{\partial y^2} - C_{13} \times 2\frac{\partial^2 w_0}{\partial x \partial y}$$

Due to the midplane symmetry, the behavior in flexure 12.16 is not modified when the laminate is subjected to thermomechanical loading.

Remark:

In the preceding discussion, it is assumed that the temperature field is uniform across the thickness of the laminate.

12.2 LAMINATE WITHOUT MIDPLANE SYMMETRY

12.2.1 Coupled Membrane–Flexure Behavior

If one considers again the calculations of Section 12.1.4 without midplane symmetry, one can see the presence of new integrals as:

$$\int_{z_{k-1}}^{z_k} \bar{E}_{ij}^k z\ dz = \bar{E}_{ij}^k \left(\frac{z_k^2 - z_{k-1}^2}{2} \right)$$

for the ply k. When the summation over all plies is taken, these integrals lead to nonzero terms with the form:

$$B_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^k \left(\frac{z_k^2 - z_{k-1}^2}{2} \right)$$

[20] One indicates in Tables 5.4, 5.9, and 5.14 of Section 5.4 the values of expansion coefficients of the laminates made of carbon, Kevlar, and glass/epoxy with $V_f = 60\%$ fiber volume fraction.

Then one has for the development of M_y (see Section 12.1.4):

$$M_y = -C_{11}\frac{\partial^2 w_0}{\partial x^2} - C_{12}\frac{\partial^2 w_0}{\partial y^2} - C_{13}\times 2\frac{\partial^2 w_0}{\partial x\partial y} + B_{11}\varepsilon_{ox} + B_{12}\varepsilon_{oy} + B_{13}\gamma_{oxy}$$

In this expression appears the coupling between bending and membrane behavior.

In a similar manner, the stress resultant N_x which was developed in Section 12.1.4 is rewritten as:

$$N_x = A_{11}\varepsilon_{ox} + A_{12}\varepsilon_{oy} + A_{13}\gamma_{oxy} - B_{11}\frac{\partial^2 w_0}{\partial x^2} - B_{12}\frac{\partial^2 w_0}{\partial y^2} - B_{13}\times 2\frac{\partial^2 w_0}{\partial x\partial y}$$

where one can find the coupling as mentioned previously.

Developing along the same manner the resultants M_x, M_{xy}, N_y, and T_{xy}, one can regroup the obtained relations. Therefore, the global relation for the behavior can be written as:

$$\left\{\begin{array}{c} N_x \\ N_y \\ T_{xy} \\ \hline M_y \\ -M_x \\ -M_{xy} \end{array}\right\} = \left[\begin{array}{c|c} A & B \\ \hline B & C \end{array}\right] \left\{\begin{array}{c} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ \hline -\partial^2 w_0/\partial x^2 \\ -\partial^2 w_0/\partial y^2 \\ -2\partial^2 w_0/\partial x\partial y \end{array}\right\} \tag{12.20}$$

with :

$$A_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^k e_k;\; B_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^k\left(\frac{z_k^2 - z_{k-1}^2}{2}\right)$$

$$C_{ij} = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \bar{E}_{ij}^k\left(\frac{z_k^3 - z_{k-1}^3}{3}\right)$$

12.2.2 Case of Thermomechanical Loading

Using the expression developed for moment resultant M_y as shown in Section 12.1.7.2, one can find the following form of integrals for each ply k:

$$\int_{z_{k-1}}^{z_k} \overline{\alpha E}_1^k \times z\,dz = \overline{\alpha E}_1^k\left(\frac{z_k^2 - z_{k-1}^2}{2}\right)$$

after summing over all plies of the laminate, it appears a nonzero term of the form:

$$\langle \alpha E h^2\rangle_x = \sum_{k=1^{\text{st}}\text{ply}}^{n^{\text{th}}\text{ply}} \overline{\alpha E}_1^k\left(\frac{z_k^2 - z_{k-1}^2}{2}\right)$$

A similar development for other resultants lead to the following relation for thermomechanical behavior:

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \\ \hdashline M_y \\ -M_x \\ -M_{xy} \end{Bmatrix} = \left[\begin{array}{c:c} A & B \\ \hdashline B & C \end{array} \right] \begin{Bmatrix} \varepsilon_{ox} \\ \varepsilon_{oy} \\ \gamma_{oxy} \\ \hdashline -\partial^2 w_0/\partial x^2 \\ -\partial^2 w_0/\partial y^2 \\ -2\partial^2 w_0/\partial x \partial y \end{Bmatrix} - \Delta T \begin{Bmatrix} \langle \alpha E h \rangle_x \\ \langle \alpha E h \rangle_y \\ \langle \alpha E h \rangle_{xy} \\ \hdashline \langle \alpha E h^2 \rangle_x \\ \langle \alpha E h^2 \rangle_y \\ \langle \alpha E h^2 \rangle_{xy} \end{Bmatrix}$$

with: $$A_{ij} = \sum_k \bar{E}_{ij}^k \, e_k; \quad B_{ij} = \sum_k \bar{E}_{ij}^k \left(\frac{z_k^2 - z_{k-1}^2}{2} \right) \qquad (12.21)$$

$$C_{ij} = \sum_k \bar{E}_{ij}^k \left(\frac{z_k^3 - z_{k-1}^3}{3} \right)$$

$$\langle \alpha E h \rangle_x = \sum_k \overline{\alpha E}_1^k \, e_k; \quad \langle \alpha E h \rangle_y = \sum_k \overline{\alpha E}_2^k \, e_k; \quad \langle \alpha E h \rangle_{xy} = \sum_k \overline{\alpha E}_3^k \, e_k$$

$$\langle \alpha E h^2 \rangle_x = \sum_k \overline{\alpha E}_1^k \frac{(z_k^2 - z_{k-1}^2)}{2}; \quad \langle \alpha E h^2 \rangle_y = \sum_k \overline{\alpha E}_2^k \frac{(z_k^2 - z_{k-1}^2)}{2};$$

$$\langle \alpha E h^2 \rangle_{xy} = \sum_k \overline{\alpha E}_3^k \frac{(z_k^2 - z_{k-1}^2)}{2}$$