5

Failure, Analysis, and Design of Laminates

Chapter Objectives

- Understand the significance of stiffness, and hygrothermal and mechanical response of special cases of laminates.
- Establish the failure criteria for laminates based on failure of individual lamina in a laminate.
- Design laminated structures such as plates, thin pressure vessels, and drive shafts subjected to in-plane and hygrothermal loads.
- Introduce other mechanical design issues in laminated composites.

5.1 Introduction

The design of a laminated composite structure, such as a flat floor panel or a pressure vessel, starts with the building block of laminae, in which fiber and matrix are combined in a manufacturing process such as filament winding or prepregs. The material of the fiber and matrix, processing factors such as packing arrangements, and fiber volume fraction determine the stiffness, strength, and hygrothermal response of a single lamina. These properties can be found by using the properties of the individual constituents of the lamina or by experiments, as explained in Chapter 3. Then the laminate can have variations in material systems and in stacking sequence of plies to tailor a composite for a particular application.

In Chapter 4, we developed analysis to find the stresses and strains in a laminate under in-plane and hygrothermal loads. In this chapter, we will first use that analysis and failure theories studied in Chapter 2 to predict failure in a laminate. Then the fundamentals learned in Chapter 4 and the failure analysis discussed in this chapter will be used to design structures using laminated composites.

First, special cases of laminates that are important in the design of laminated structures will be introduced. Then the failure criterion analysis will be shown for a laminate. Eventually, we will be designing laminates mainly on the basis of optimizing for cost, weight, strength, and stiffness. Other mechanical design issues are briefly introduced at the end of the chapter.

5.2 Special Cases of Laminates

Based on angle, material, and thickness of plies, the symmetry or antisymmetry of a laminate may zero out some elements of the three stiffness matrices [A], [B], and [D]. These are important to study because they may result in reducing or zeroing out the coupling of forces and bending moments, normal and shear forces, or bending and twisting moments. This not only simplifies the mechanical analysis of composites, but also gives desired mechanical performance. For example, as already shown in Chapter 4, the analysis of a symmetric laminate is simplified due to the zero coupling matrix [B]. Mechanically, symmetric laminates result in no warpage in a flat panel due to temperature changes in processing.

5.2.1 Symmetric Laminates

A laminate is called symmetric if the material, angle, and thickness of plies are the same above and below the midplane. An example of symmetric laminates is $[0/30/60]_s$:

0
30
60
30
0

For symmetric laminates from the definition of [B] matrix, it can be proved that [B] = 0. Thus, Equation (4.29) can be decoupled to give

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \tag{5.1a}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}. \tag{5.1b}$$

This shows that the force and moment terms are uncoupled. Thus, if a laminate is subjected only to forces, it will have zero midplane curvatures. Similarly, if it is subjected only to moments, it will have zero midplane strains.

The uncoupling between extension and bending in symmetric laminates makes analyzing such laminates simpler. It also prevents a laminate from twisting due to thermal loads, such as cooling down from processing temperatures and temperature fluctuations during use such as in a space shuttle, etc.

5.2.2 Cross-Ply Laminates

A laminate is called a cross-ply laminate (also called laminates with specially orthotropic layers) if only 0 and 90° plies were used to make a laminate. An example of a cross ply laminate is a $[0/90_2/0/90]$ laminate:

0
90
90
0
90

For cross-ply laminates, $A_{16} = 0$, $A_{26} = 0$, $B_{16} = 0$, $B_{26} = 0$, $D_{16} = 0$, and $D_{26} = 0$; thus, Equation (4.29) can be written as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}. \tag{5.2}$$

In these cases, uncoupling occurs between the normal and shear forces, as well as between the bending and twisting moments. If a cross-ply laminate is also symmetric, then in addition to the preceding uncoupling, the coupling matrix $[B] = 0$ and no coupling takes place between the force and moment terms.

5.2.3 Angle Ply Laminates

A laminate is called an angle ply laminate if it has plies of the same material and thickness and only oriented at +θ and –θ directions. An example of an angle ply laminate is [–40/40/–40/40]:

–40
40
–40
40

If a laminate has an even number of plies, then $A_{16} = A_{26} = 0$. However, if the number of plies is odd and it consists of alternating +θ and –θ plies, then it is symmetric, giving $[B] = 0$, and A_{16}, A_{26}, D_{16}, and D_{26} also become small as the number of layers increases for the same laminate thickness. This behavior is similar to the symmetric cross-ply laminates. However, these angle ply laminates have higher shear stiffness and shear strength properties than cross-ply laminates.

5.2.4 Antisymmetric Laminates

A laminate is called antisymmetric if the material and thickness of the plies are the same above and below the midplane, but the ply orientations at the same distance above and below the midplane are negative of each other. An example of an antisymmetric laminate is:

45
60
–60
–45

From Equation (4.28a) and Equation (4.28c), the coupling terms of the extensional stiffness matrix, $A_{16} = A_{26} = 0$, and the coupling terms of the bending stiffness matrix, $D_{16} = D_{26} = 0$:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{22} & B_{16} & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}. \tag{5.3}$$

5.2.5 Balanced Laminate

A laminate is balanced if layers at angles other than 0 and 90° occur only as plus and minus pairs of +θ and –θ. The plus and minus pairs do not need to be adjacent to each other, but the thickness and material of the plus and minus pairs need to be the same. Here, the terms $A_{16} = A_{26} = 0$. An example of a balanced laminate is [30/40/–30/30/–30/–40]:

30
40
–30
30
–30
–40

From Equation (4.28a),

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{26} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}. \tag{5.4}$$

5.2.6 Quasi-Isotropic Laminates

For a plate of isotropic material with Young's modulus, E, Poisson's ratio, ν, and thickness, h, the three stiffness matrices are

$$[A] = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 \\ 0 & 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix} h\,, \tag{5.5}$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.6}$$

$$[D]=\begin{bmatrix} \frac{E}{12(1-\nu^2)} & \frac{\nu E}{12(1-\nu^2)} & 0 \\ \frac{\nu E}{12(1-\nu^2)} & \frac{E}{12(1-\nu^2)} & 0 \\ 0 & 0 & \frac{E}{24(1+\nu)} \end{bmatrix} h^3 . \tag{5.7}$$

A laminate is called quasi-isotropic if its extensional stiffness matrix [A] behaves like that of an isotropic material. This implies not only that $A_{11} = A_{22}$, $A_{16} = A_{26} = 0$, and $A_{66} = \frac{A_{11}-A_{12}}{2}$, but also that these stiffnesses are independent of the angle of rotation of the laminate. The reason for calling such a laminate quasi-isotropic and not isotropic is that the other stiffness matrices, [B] and [D], may not behave like isotropic materials. Examples of quasi-isotropic laminates include [0/±60], $[0/\pm45/90]_s$, and [0/36/72/–36/–72].

Example 5.1

A [0/±60] graphite/epoxy laminate is quasi-isotropic. Find the three stiffness matrices [A], [B], and [D] and show that

1. $A_{11} = A_{22}$; $A_{16} = A_{26} = 0$; $A_{66} = \frac{A_{11}-A_{12}}{2}$.
2. [B] $\neq 0$, unlike isotropic materials.
3. [D] matrix is unlike isotropic materials.

Use properties of unidirectional graphite/epoxy lamina from Table 2.1. Each lamina has a thickness of 5 mm.

Solution

From Example 2.6, the reduced stiffness matrix [Q] for the 0° graphite/epoxy lamina is

$$[Q]=\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\ Pa .$$

From Equation (2.104), the transformed reduced stiffness matrices for the three plies are

$$[\bar{Q}]_0 = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)\ Pa\ ,$$

$$[\bar{Q}]_{60} = \begin{bmatrix} 23.65 & 32.46 & 20.05 \\ 32.46 & 109.4 & 54.19 \\ 20.05 & 54.19 & 36.74 \end{bmatrix} (10^9)\ Pa\ ,$$

$$[\bar{Q}]_{-60} = \begin{bmatrix} 23.65 & 32.46 & -20.05 \\ 32.46 & 109.4 & -54.19 \\ -20.05 & -54.19 & 36.74 \end{bmatrix} (10^9)\ Pa.$$

The total thickness of the laminate is h = (0.005)(3) = 0.015 m.

The midplane is 0.0075 m from the top and bottom of the laminate. Thus, using Equation (4.20),

$$h_0 = -0.0075 \text{ m}$$

$$h_1 = -0.0025 \text{ m}$$

$$h_2 = 0.0025 \text{ m}$$

$$h_3 = 0.0075 \text{ m}$$

Using Equation (4.28a) to Equation (4.28c), one can now calculate the stiffness matrices [A], [B], and [D], respectively, as shown in Example 4.2:

$$[A] = \begin{bmatrix} 1.146 & 0.3391 & 0 \\ 0.3391 & 1.146 & 0 \\ 0 & 0 & 0.4032 \end{bmatrix} (10^9)\ Pa\text{-}m\ ,$$

$$[B] = \begin{bmatrix} -3.954 & 0.7391 & -0.5013 \\ 0.7391 & 2.476 & -1.355 \\ -0.5013 & -1.355 & 0.7391 \end{bmatrix} (10^6)\ Pa\text{-}m^2\ ,$$

$$[D] = \begin{bmatrix} 28.07 & 5.126 & -2.507 \\ 5.126 & 17.35 & -6.774 \\ -2.507 & -6.774 & 6.328 \end{bmatrix} (10^3)\ Pa\text{-}m^3 .$$

1. From the extensional stiffness matrix [A],

$$A_{11} = A_{22} = 1.146 \times 10^9\ Pa\text{-}m$$

$$A_{16} = A_{26} = 0$$

$$\frac{A_{11} - A_{12}}{2} = \frac{1.146 - 0.3391}{2} \times 10^9$$

$$= 0.4032 \times 10^9\ Pa\text{-}m$$

$$= A_{66}.$$

This behavior is similar to that of an isotropic material. However, a quasi-isotropic laminate should give the same [A] matrix, if a constant angle is added to each of the layers of the laminate. For example, adding 30° to each ply angle of the [0/±60] laminate gives a [30/90/–30] laminate, which has the same [A] matrix as the [0/±60] laminate.

2. Unlike isotropic materials, the coupling stiffness matrix [B] of the [0/±60] laminate is nonzero.
3. In an isotropic material,

$$D_{11} = D_{22} ,$$

$$D_{16} = D_{26} = 0 ,$$

and

$$D_{66} = \frac{D_{11} - D_{12}}{2} .$$

In this example, unlike isotropic materials, $D_{11} \neq D_{22}$ because

$$D_{11} = 28.07 \times 10^3\ Pa\text{-}m^3$$

$$D_{22} = 17.35 \times 10^3 \; Pa\text{-}m^3$$

$$D_{16} \neq 0, D_{26} \neq 0 \text{ as}$$

$$D_{16} = -2.507 \times 10^3 \; Pa\text{-}m^3$$

$$D_{26} = -6.774 \times 10^3 \; Pa\text{-}m^3$$

$$\frac{D_{11} - D_{12}}{2} \neq D_{66}$$

because

$$\frac{D_{11} - D_{12}}{2} = \frac{28.07 \times 10^3 - 5.126 \times 10^3}{2}$$

$$= 11.47 \times 10^3 \; Pa\text{-}m^3$$

$$D_{66} = 6.328 \times 10^3 \; Pa\text{-}m^3.$$

One can make a quasi-isotropic laminate by having a laminate with N ($N \geq 3$) lamina of the same material and thickness, where each lamina is oriented at an angle of $180°/N$ between each other. For example, a three-ply laminate will require the laminae to be oriented at $180°/3 = 60°$ to each other. Thus, [0/60/–60], [30/90/–30], and [45/–75/–15] are all quasi-isotropic laminates. One can make the preceding combinations symmetric or repeated to give quasi-isotropic laminates, such as $[0/\pm 60]_s$, $[0/\pm 60]_s$, and $[0/\pm 60]_{2s}$ laminates. The symmetry of the laminates zeros out the coupling matrix [B] and makes its behavior closer (not same) to that of an isotropic material.

Example 5.2

Show that the extensional stiffness matrix for a general N-ply quasi-isotropic laminate is given by

$$[A] = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & \dfrac{U_1 - U_4}{2} \end{bmatrix} h \,. \tag{5.8}$$

where U_1 and U_4 are the stiffness invariants given by Equation (2.132) and h is the thickness of the laminate. Also, find the in-plane engineering stiffness constants of the laminate.

Solution

From Equation (2.131a), for a general angle ply with angle θ,

$$\bar{Q}_{11} = U_1 + U_2 \text{ Cos}2\theta + U_3 \text{ Cos}4\theta. \tag{5.9}$$

For the k^{th} ply of the quasi-isotropic laminate with an angle θ_k,

$$(\bar{Q}_{11})_k = U_1 + U_2 \text{ Cos}2\theta_k + U_3 \text{ Cos}4\theta_k, \tag{5.10}$$

where

$$\theta_1 = \frac{\pi}{N}, \theta_2 = \frac{2\pi}{N}, \ldots, \theta_k = \frac{k\pi}{N}, \ldots, \theta_{N-1} = \frac{(N-1)\pi}{N}, \theta_N = \pi.$$

From Equation (4.28a),

$$A_{11} = \sum_{k=1}^{N} t_k (\bar{Q}_{11})_k \ , \tag{5.11}$$

where t_k = thickness of k^{th} lamina.

Because the thickness of the laminate is h and all laminae are of the same thickness,

$$t_k = \frac{h}{N}, \quad k = 1, 2, \ldots\ldots\ldots\ldots, N, \tag{5.12}$$

and, substituting Equation (5.10) in Equation (5.11),

$$\begin{aligned} A_{11} &= \frac{h}{N} \sum_{k=1}^{N} (U_1 + U_2 \text{ Cos } 2\theta_k + U_3 \text{ Cos } 4\theta_k) \\ &= hU_1 + U_2 \frac{h}{N} \sum_{k=1}^{N} \text{Cos } 2\theta_k + U_3 \frac{h}{N} \sum_{k=1}^{N} \text{Cos } 4\theta_k . \end{aligned} \tag{5.13}$$

Using the following identity,[1]

$$\sum_{k=1}^{N} \text{Cos } kx \equiv \frac{\text{Sin}\left(N+\frac{1}{2}\right)x}{2 \text{ Sin}\left(\frac{x}{2}\right)} - \frac{1}{2} . \tag{5.14}$$

Then,

$$\sum_{k=1}^{N} \text{Cos } 2\theta_k = 0 \ \ for \ \ N \geq 1 \tag{5.15a}$$

$$\sum_{k=1}^{N} \text{Cos } 4\theta_k = 0 \ \ for \ \ N \geq 3. \tag{5.15b}$$

Thus,

$$A_{11} = U_1 h \ . \tag{5.16a}$$

Similarly, it can be shown that

$$A_{12} = U_4 h \ , \tag{5.16b}$$

$$A_{22} = U_1 h \ , \tag{5.16c}$$

$$A_{66} = \left(\frac{U_1 - U_4}{2}\right) h \ . \tag{5.16d}$$

Therefore,

$$[A] = \begin{bmatrix} U_1 & U_4 & 0 \\ U_4 & U_1 & 0 \\ 0 & 0 & \frac{U_1 - U_4}{2} \end{bmatrix} h. \tag{5.17}$$

Because Equation (5.15b) is valid only for $N \geq 3$, this proves that one needs at least three plies to make a quasi-isotropic laminate.

For a symmetric quasi-isotropic laminate, the extensional compliance matrix is given by

$$[A^*] = \frac{1}{h}\begin{bmatrix} \frac{U_1}{U_1^2 - U_4^2} & -\frac{U_4}{U_1^2 - U_4^2} & 0 \\ -\frac{U_4}{U_1^2 - U_4^2} & \frac{U_1}{U_1^2 - U_4^2} & 0 \\ 0 & 0 & \frac{2}{U_1 - U_4} \end{bmatrix}. \tag{5.18}$$

From the definitions of engineering constants given in Equations (4.35), (4.37), (4.39), (4.42), and (4.45), and using Equation (5.18), the elastic moduli of the laminate are independent of the angle of the lamina and are given by

$$E_x = E_y = E_{iso} = \frac{1}{A_{11}^* h} = \frac{U_1^2 - U_4^2}{U_1}, \tag{5.19a}$$

$$G_{xy} = G_{iso} = \frac{1}{A_{66}^* h} = \frac{U_1 - U_4}{2}, \tag{5.19b}$$

$$\nu_{xy} = \nu_{yx} = \nu_{iso} = -\frac{A_{12}^*}{A_{22}^*} = \frac{U_4}{U_1}. \tag{5.19c}$$

5.3 Failure Criterion for a Laminate

A laminate will fail under increasing mechanical and thermal loads. The laminate failure, however, may not be catastrophic. It is possible that some layer fails first and that the composite continues to take more loads until all the plies fail. Failed plies may still contribute to the stiffness and strength of the laminate. The degradation of the stiffness and strength properties of each failed lamina depends on the philosophy followed by the user.

- When a ply fails, it may have cracks parallel to the fibers. This ply is still capable of taking load parallel to the fibers. Here, the cracked ply can be replaced by a hypothetical ply that has no transverse stiffness, transverse tensile strength, and shear strength. The longitudinal modulus and strength remain unchanged.

- When a ply fails, fully discount the ply and replace the ply of near zero stiffness and strength. Near zero values avoid singularities in stiffness and compliance matrices.

The procedure for finding the successive loads between first ply failure and last ply failure given next follows the fully discounted method:

1. Given the mechanical loads, apply loads in the same ratio as the applied loads. However, apply the actual temperature change and moisture content.
2. Use laminate analysis to find the midplane strains and curvatures.
3. Find the local stresses and strains in each ply under the assumed load.
4. Use the ply-by-ply stresses and strains in ply failure theories discussed in Section 2.8 to find the strength ratio. Multiplying the strength ratio to the applied load gives the load level of the failure of the first ply. This load is called the *first ply failure* load.
5. Degrade fully the stiffness of damaged ply or plies. Apply the actual load level of previous failure.
6. Go to step 2 to find the strength ratio in the undamaged plies:
   - If the strength ratio is more than one, multiply the strength ratio to the applied load to give the load level of the next ply failure and go to step 2.
   - If the strength ratio is less than one, degrade the stiffness and strength properties of all the damaged plies and go to step 5.
7. Repeat the preceding steps until all the plies in the laminate have failed. The load at which all the plies in the laminate have failed is called the *last ply failure*.

The procedure for partial discounting of fibers is more complicated. The noninteractive maximum stress and maximum strain failure criteria are used to find the mode of failure. Based on the mode of failure, the appropriate elastic moduli and strengths are partially or fully discounted.

Example 5.3

Find the ply-by-ply failure loads for a $[0/\overline{90}]_s$ graphite/epoxy laminate. Assume the thickness of each ply is 5 mm and use properties of unidirectional graphite/epoxy lamina from Table 2.1. The only load applied is a tensile normal load in the x-direction — that is, the direction parallel to the fibers in the 0° ply.

Solution

Because the laminate is symmetric and the load applied is a normal load, only the extensional stiffness matrix is required. From Example 4.4, the extensional compliance matrix is

$$[A^*] = \begin{bmatrix} 5.353\times10^{-10} & -2.297\times10^{-11} & 0 \\ -2.297\times10^{-11} & 9.886\times10^{-10} & 0 \\ 0 & 0 & 9.298\times10^{-9} \end{bmatrix} \frac{1}{Pa\text{-}m},$$

which, from Equation (5.1a), gives the midplane strains for symmetric laminates subjected to $N_x = 1$ N/m as

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}.$$

The midplane curvatures are zero because the laminate is symmetric and no bending and no twisting loads are applied.

The global strains in the top 0° ply at the top surface can be found as follows using Equation (4.16),

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix} + (0.0075)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}.$$

Using Equation (2.103), one can find the global stresses at the top surface of the top 0° ply as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{0°,top} = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 5.353\times10^{-10} \\ -2.297\times10^{-11} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9.726\times10^{1} \\ 1.313 \\ 0 \end{bmatrix} Pa\ .$$

Using the transformation Equation (2.94), the local stresses at the top surface of the top 0° ply are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_{0^\circ,\ top} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9.726\times10^{1} \\ 1.313\times10^{0} \\ 0 \end{bmatrix} .$$

$$= \begin{bmatrix} 9.726\times10^{1} \\ 1.313 \\ 0 \end{bmatrix} Pa$$

All the local stresses and strains in the laminate are summarized in Table 5.1 and Table 5.2.

TABLE 5.1

Local Stresses (Pa) in Example 5.3

Ply no.	Position	σ_1	σ_2	τ_{12}
1 (0°)	Top	9.726×10^{1}	1.313×10^{0}	0.0
	Middle	9.726×10^{1}	1.313×10^{0}	0.0
	Bottom	9.726×10^{1}	1.313×10^{0}	0.0
2 (90°)	Top	-2.626×10^{0}	5.472×10^{0}	0.0
	Middle	-2.626×10^{0}	5.472×10^{0}	0.0
	Bottom	-2.626×10^{0}	5.472×10^{0}	0.0
3 (0°)	Top	9.726×10^{1}	1.313×10^{0}	0.0
	Middle	9.726×10^{1}	1.313×10^{0}	0.0
	Bottom	9.726×10^{1}	1.313×10^{0}	0.0

TABLE 5.2

Local Strains in Example 5.3

Ply no.	Position	ε_1	ε_2	τ_{12}
1 (0°)	Top	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Middle	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Bottom	5.353×10^{-10}	-2.297×10^{-11}	0.0
2 (90°)	Top	-2.297×10^{-11}	5.353×10^{-10}	0.0
	Middle	-2.297×10^{-11}	5.353×10^{-10}	0.0
	Bottom	-2.297×10^{-11}	5.353×10^{-10}	0.0
3 (0°)	Top	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Middle	5.353×10^{-10}	-2.297×10^{-11}	0.0
	Bottom	5.353×10^{-10}	-2.297×10^{-11}	0.0

The Tsai–Wu failure theory applied to the top surface of the top 0° ply is applied as follows. The local stresses are

$\sigma_1 = 9.726 \times 10^1$ Pa

$\sigma_2 = 1.313$ Pa

$\tau_{12} = 0$

Using the parameters H_1, H_2, H_6, H_{11}, H_{22}, H_{66}, and H_{12} from Example 2.19, the Tsai–Wu failure theory Equation (2.152) gives the strength ratio as

$$(0)\,(9.726 \times 10^1)\,SR + (2.093 \times 10^{-8})\,(1.313)\,SR + (0 \times 0) +$$
$$(4.4444 \times 10^{-19})\,(9.726 \times 10^1)^2 (SR)^2 + (1.0162 \times 10^{-16})\,(1.313)^2 (SR)^2$$
$$+ (2.1626 \times 10^{-16})\,(0)^2 + 2(-3.360 \times 10^{-18})\,(9.726 \times 10^1)\,(1.313)(SR)^2 = 1$$
$$SR = 1.339 \times 10^7.$$

The maximum strain failure theory can also be applied to the top surface of the top 0° ply as follows. The local strains are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 5.353 \times 10^{-10} \\ -2.297 \times 10^{-11} \\ 0.000 \end{bmatrix}.$$

Then, according to maximum strain failure theory (Equation 2.143), the strength ratio is given by

$$SR = \min\{[(1500 \times 10^6)/(181 \times 10^9)]/(5.353 \times 10^{-10}),$$
$$[(246 \times 10^6)/(10.3 \times 10^9)]/(2.297 \times 10^{-11})\} = 1.548 \times 10^7.$$

The strength ratios for all the plies in the laminate are summarized in Table 5.3 using the maximum strain and Tsai–Wu failure theories. The symbols in

TABLE 5.3

Strength Ratios in Example 5.3

Ply no.	Position	Maximum strain	Tsai–Wu
1 (0°)	Top	1.548×10^7 (1T)	1.339×10^7
	Middle	1.548×10^7 (1T)	1.339×10^7
	Bottom	1.548×10^7 (1T)	1.339×10^7
2 (90°)	Top	7.254×10^6 (2T)	7.277×10^6
	Middle	7.254×10^6 (2T)	7.277×10^6
	Bottom	7.254×10^6 (2T)	7.277×10^6
3 (0°)	Top	1.548×10^7 (1T)	1.339×10^7
	Middle	1.548×10^7 (1T)	1.339×10^7
	Bottom	1.548×10^7 (1T)	1.339×10^7

the parentheses in the maximum strain failure theory column denote the mode of failure and are explained at the bottom of Table 2.3.

From Table 5.3 and using the Tsai–Wu theory, the minimum strength ratio is found for the 90° ply. This strength ratio gives the maximum value of the allowable normal load as

$$N_x = 7.277 \times 10^6 \frac{N}{m}$$

and the maximum value of the allowable normal stress as

$$\frac{N_x}{h} = \frac{7.277 \times 10^6}{0.015},$$

$$= 0.4851 \times 10^9 \ Pa$$

where h = thickness of the laminate.

The normal strain in the x-direction at this load is

$$(\varepsilon_x^0)_{\text{first ply failure}} (5.353 \times 10^{-10})(7.277 \times 10^6)$$

$$= 3.895 \times 10^{-3}.$$

Now, degrading the 90° ply completely involves assuming zero stiffnesses and strengths of the 90° lamina. Complete degradation of a ply does not allow further failure of that ply. For the undamaged plies, the $[0/90]_s$ laminate has two reduced stiffness matrices as

$$[Q] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} GPa$$

and, for the damaged ply,

$$[Q] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} GPa .$$

Using Equation (4.28a), the extensional stiffness matrix

$$A_{ij} = \sum_{k=1}^{3} [\bar{Q}_{ij}]_k (h_k - h_{k-1})$$

$$[A] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)(0.005)$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (10^9)(0.005)$$

$$+ \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9)(0.005)$$

$$[A] = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^7)\ Pa\text{-}m.$$

Inverting the new extensional stiffness matrix [A], the new extensional compliance matrix is

$$[A^*] = \begin{bmatrix} 5.525 \times 10^{-10} & -1.547 \times 10^{-10} & 0 \\ -1.547 \times 10^{-10} & 9.709 \times 10^{-9} & 0 \\ 0 & 0 & 1.395 \times 10^{-8} \end{bmatrix} \frac{1}{Pa\text{-}m},$$

which gives midplane strains subjected to N_x = 1 N/m by Equation (5.1a) as

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 5.525 \times 10^{-10} & -1.547 \times 10^{-10} & 0 \\ -1.547 \times 10^{-10} & 9.709 \times 10^{-9} & 0 \\ 0 & 0 & 1.395 \times 10^{-8} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

TABLE 5.4

Local Stresses after First Ply Failure in Example 5.3

Ply no.	Position	σ_1	σ_2	τ_{12}
1 (0°)	Top	1.0000×10^2	0.0	0.0
	Middle	1.0000×10^2	0.0	0.0
	Bottom	1.0000×10^2	0.0	0.0
2 (90°)	Top	—	—	—
	Middle	—	—	—
	Bottom	—	—	—
3 (0°)	Top	1.0000×10^2	0.0	0.0
	Middle	1.0000×10^2	0.0	0.0
	Bottom	1.0000×10^2	0.0	0.0

TABLE 5.5

Local Strains after First Ply Failure in Example 5.3

Ply no.	Position	ε_1	ε_2	γ_{12}
1 (0°)	Top	5.25×10^{-10}	-1.547×10^{-10}	0.0
	Middle	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Bottom	5.525×10^{-10}	-1.547×10^{-10}	0.0
2 (90°)	Top	—	—	—
	Middle	—	—	—
	Bottom	—	—	—
3 (0°)	Top	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Middle	5.525×10^{-10}	-1.547×10^{-10}	0.0
	Bottom	5.525×10^{-10}	-1.547×10^{-10}	0.0

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} 5.525 \times 10^{-10} \\ -1.547 \times 10^{-10} \\ 0 \end{bmatrix}.$$

These strains are close to those obtained before the ply failure only because the 90° ply takes a small percentage of the load out of the normal load in the x-direction.

The local stresses in each layer are found using earlier techniques given in this example and are shown in Table 5.4. The strength ratios in each layer are also found using methods given in this example and are shown in Table 5.5.

From Table 5.6 and using Tsai–Wu failure theory, the minimum strength ratio is found in both the 0° plies. This strength ratio gives the maximum value of the normal load as

$$N_x = 1.5 \times 10^7 \frac{N}{m}$$

TABLE 5.6

Strength Ratios after First Ply Failure in Example 5.3

Ply no.	Position	Max strain	Tsai–Wu
1 (0°)	Top	1.5000×10^7 (1T)	1.5000×10^7
	Middle	1.5000×10^7(1T)	1.5000×10^7
	Bottom	1.5000×10^7(1T)	1.5000×10^7
2 (90°)	Top	—	—
	Middle	—	—
	Bottom	—	—
3 (0°)	Top	1.5000×10^7(1T)	1.5000×10^7
	Middle	1.5000×10^7(1T)	1.5000×10^7
	Bottom	1.5000×10^7(1T)	1.5000×10^7

and the maximum value of the allowable normal stress as

$$\frac{N_x}{h} = \frac{1.5 \times 10^7}{0.015},$$
$$= 1.0 \times 10^9 \ Pa$$

where h is the thickness of the laminate.

The normal strain in the x-direction at this load is

$$(\varepsilon_x^o)_{\text{last ply failure}} = (5.525 \times 10^{-10})(1.5 \times 10^7)$$
$$= 8.288 \times 10^{-3}.$$

The preceding load is also the last ply failure (LPF) because none of the layers is left undamaged. Plotting the stress vs. strain curve for the laminate until last ply failure shows that the curve will consist of two linear curves, each ending at each ply failure. The slope of the two lines will be the Young's modulus in x direction for the undamaged laminate and for the FPF laminate — that is, using Equation (4.35),

$$E_x = \frac{1}{(0.015)(5.353 \times 10^{-10})},$$
$$= 124.5 \ GPa$$

until first ply failure, and

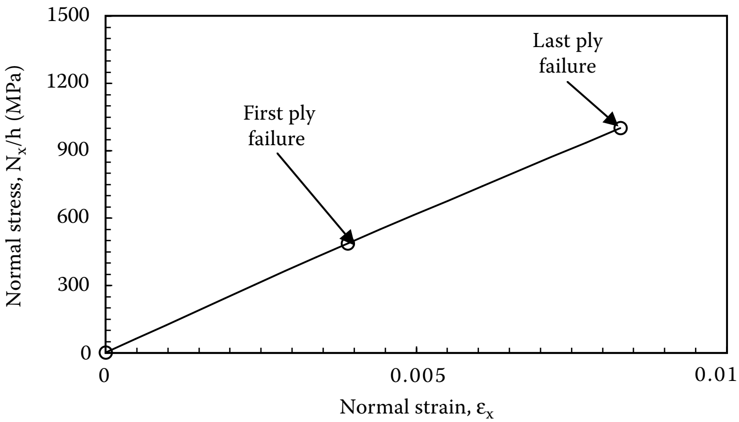

FIGURE 5.1
Stress–strain curve showing ply-by-ply failure of a laminated composite.

$$E_x = \frac{(N_x / h)_{\text{last ply failure}} - (N_x / h)_{\text{first play failure}}}{(\varepsilon_x^o)_{\text{last play failure}} - (\varepsilon_x^o)_{\text{first play failure}}}$$

$$= \frac{0.1 \times 10^{10} - 0.4851 \times 10^9}{8.288 \times 10^{-3} - 3.895 \times 10^{-3}},$$

$$= 117.2\ GPa$$

after first ply failure and until last ply failure (Figure 5.1).

Example 5.4

Repeat Example 5.3 for the first ply failure and use Tsai–Wu failure theory now with an additional thermal load: a temperature change of –75°C.

Solution

The laminate is symmetric and the load applied is a normal load and a temperature change. Thus, only the extensional stiffness matrix is needed. From Example 5.3,

$$[A^*] = \begin{bmatrix} 5.353 \times 10^{-10} & -2.297 \times 10^{-11} & 0 \\ -2.297 \times 10^{-11} & 9.886 \times 10^{-10} & 0 \\ 0 & 0 & 9.298 \times 10^{-9} \end{bmatrix} \frac{1}{Pa\text{-}m}.$$

Corresponding to a temperature change of –75°C, the mechanical stresses can be found as follows. The fictitious thermal forces given by Equation (4.64) are

$$\begin{bmatrix} N_x^T \\ N_y^T \\ N_{xy}^T \end{bmatrix} = (-75)\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.200\times 10^{-7} \\ 0.225\times 10^{-4} \\ 0 \end{bmatrix}[-0.0025-(-0.0075)]$$

$$+(-75)\begin{bmatrix} 10.35 & 2.897 & 0 \\ 2.897 & 181.8 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.225\times 10^{-4} \\ 0.200\times 10^{-7} \\ 0 \end{bmatrix}[0.0025-(-0.0025)]$$

$$+(-75)\begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix}(10^9)\begin{bmatrix} 0.200\times 10^{-7} \\ 0.225\times 10^{-4} \\ 0 \end{bmatrix}[0.0075-(-0.0025)]$$

$$= \begin{bmatrix} -1.389\times 10^{5} \\ -2.004\times 10^{5} \\ 0 \end{bmatrix} Pa\text{-}m\ .$$

Because the laminate is symmetric, the fictitious thermal moments are zero. This also then gives only midplane strains in the laminate without any plate curvatures. The midplane strain due to the thermal load is given by

$$\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} 5.353\times 10^{-10} & -2.297\times 10^{-11} & 0 \\ -2.297\times 10^{-11} & 9.886\times 10^{-10} & 0 \\ 0 & 0 & 9.298\times 10^{-9} \end{bmatrix}\begin{bmatrix} -1.389\times 10^{5} \\ -2.004\times 10^{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.6977\times 10^{-4} \\ -0.1950\times 10^{-3} \\ 0 \end{bmatrix}.$$

The laminate is symmetric and no bending or torsional moments are applied; therefore, the global strains in the laminate are the same as the midplane strains. The free expansional thermal strains in the top 0° ply are

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}_{0^\circ} \Delta T$$

$$= \begin{bmatrix} 0.200 \times 10^{-7} \\ 0.225 \times 10^{-4} \\ 0 \end{bmatrix} (-75)$$

$$= \begin{bmatrix} -0.1500 \times 10^{-5} \\ -0.16875 \times 10^{-2} \\ 0 \end{bmatrix}.$$

From Equation (4.70), the global mechanical strain at the top surface of the top 0° ply is

$$\begin{bmatrix} -0.6977 \times 10^{-4} \\ -0.1950 \times 10^{-3} \\ 0 \end{bmatrix} - \begin{bmatrix} -0.1500 \times 10^{-5} \\ -0.16875 \times 10^{-2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.6827 \times 10^{-4} \\ 0.14925 \times 10^{-2} \\ 0 \end{bmatrix}$$

From Equation (2.103), the global mechanical stresses at the top of the top 0° ply are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} (10^9) \begin{bmatrix} -0.6827 \times 10^{-4} \\ 0.14925 \times 10^{-2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -8.088 \times 10^{6} \\ 1.524 \times 10^{7} \\ 0 \end{bmatrix} Pa$$

Now, if the mechanical loads were given, the resulting mechanical stresses could then be added to the previous stresses due to the temperature difference.

Then, the failure criteria could be used to find out whether the ply has failed. However, we are asked to find out the mechanical load that could be applied in the presence of the temperature difference. This can be done as follows.

The stress at the top of the 0° ply, per Example 5.3 for a unit load $N_x = 1$ N/m, is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 9.726 \times 10^1 \\ 1.313 \times 10^0 \\ 0.0 \end{bmatrix} Pa.$$

If the unknown load is N_x, then the overall stress at the top surface of the top 0° ply is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} -8.088 \times 10^6 + 9.726 \times 10^1 N_x \\ 1.524 \times 10^7 + 1.313 \times 10^0 N_x \\ 0 \end{bmatrix} Pa.$$

Now, the failure theories can be applied to find the value of N_x. Using transformation equation (2.94), the local stresses at the top surface of the top 0° ply are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} -8.088 \times 10^6 + 9.726 \times 10^1 N_x \\ 1.524 \times 10^7 + 1.313 \times 10^0 N_x \\ 0 \end{bmatrix} Pa.$$

Using the parameters H_1, H_2, H_6 H_{11}, H_{22}, H_{66} and H_{12} from Example 2.19, the Tsai–Wu failure criterion (Equation 2.146) is

$$(0)\,[-8.088 \times 10^6 + 9.726 \times 10^1\, N_x] + (2.093 \times 10^{-8})[1.524 \times 10^7 + 1.313$$

$$\times\, 10^0\, N_x] + (0)(0) + 4.4444 \times 10^{-19}\, [-8.088 \times 10^6 + 9.726 \times 10^1\, N_x]^2$$

$$+\, 1.0162 \times 10^{-16}\, [1.524 \times 10^7 + 1.313 \times 10^0\, N_x]^2 + 2.1626 \times 10^{-16}\, [0]^2$$

$$+\, 2[-3.360 \times 10^{-18}]\, [-8.088 \times 10^6 + 9.726 \times 10^1\, N_x]\, [1.524 \times 10^7 + 1.313$$

$$\times\, 10^0\, N_x] < 1.$$

TABLE 5.7

Strength Ratios of Example 5.4

Ply no.	Position	Tsai–Wu
1 (0°)	Top	1.100×10^7
	Middle	1.100×10^7
	Bottom	1.100×10^7
2 (90°)	Top	4.279×10^6
	Middle	4.279×10^6
	Bottom	4.279×10^6
3 (0°)	Top	1.100×10^7
	Middle	1.100×10^7
	Bottom	1.100×10^7

As can be seen, this results in a quadratic polynomial in the left-hand side of the strength criteria — that is,

$$3.521 \times 10^{-15} N_x^2 + 2.096 \times 10^{-8} N_x - 0.6566 = 0 .$$

This gives two roots for which the inequality is satisfied for $N_x < 1.100 \times 10^7$ and $N_x > -1.695 \times 10^7$.

Because the load N_x is tensile, $N_x = 1.100 \times 10^7$ is the valid solution. Similarly, the values of strength ratios for all the plies in the laminate are found and summarized in Table 5.7.

Using the lowest value of strength ratio of 4.279×10^6 gives $N_x = 4.279 \times 10^6$ N/m as the load at which the first ply failure would take place. Compare this with the value of $N_x = 7.277 \times 10^6$ in Example 5.3, in which no temperature change was applied.

5.4 Design of a Laminated Composite

Because we have developed the laminated plate theory for composites subjected to in-plane mechanical loads, temperature, and moisture, the designs in this chapter are also limited to such loads and simple shapes. Factors not covered in this section include stability; out-of plane loads; and fracture, impact, and fatigue resistance; interlaminar strength; damping characteristics; vibration control; and complex shapes. These factors are introduced briefly in Section 5.5.

Design of laminated composites includes constraints on optimizing and constraining factors such as

- Cost
- Mass as related to aerospace and automotive industry to reduce energy cost

- Stiffness (to limit deformations) as related to aircraft skins to avoid buckling
- Thermal and moisture expansion coefficients as related to space antennas to maintain dimensional stability

These factors are similar to those used with designing with monolithic materials; thus, the main issue with designing with composites as opposed to monolithic materials involves understanding the orthotropic nature of composite plies.

The possibility of different fiber-matrix systems combined with the variables such as fiber volume fraction first dictate the properties of a lamina. Then, laminae can be placed at angles and at particular distances from the midplane in the laminate. The material systems and the stacking sequence then determine the stresses and strains in the laminate. The failure of the composite may be based on the first ply failure (FPF) or the last ply failure (LPF). Although one may think that all plies failing at the same time is an ideal laminate, others may argue that differences between the two give time for detection and repair or replacement of the part.

Laminate selection is a computationally intensive and repetitive task due to the many possibilities of fiber-matrix combinations, material systems, and stacking sequence. Computer programs have made these calculations easy and the reader is directed to use the PROMAL[2] program included in this book or any other equivalent program of choice to fully appreciate designing with composites. A more scientific approach to optimization of laminated composites is out of scope of this book, and the reader is referred to Gurdal et al.[3]

Example 5.5

1. An electronic device uses an aluminum plate of cross-section 4 in. × 4 in. to take a pure bending moment of 13,000 lb-in. The factor of safety is 2. Using the properties of aluminum given in Table 3.4, find the thickness of the plate.
2. The designer wants at least to halve the thickness of the plate to make room for additional hardware on the electronic device. The choices include unidirectional laminates of graphite/epoxy, glass/epoxy, or their combination (hybrid laminates). The ply thickness is 0.125 mm (0.0049213 in.). Design a plate with the lowest cost if the manufacturing cost per ply of graphite/epoxy and glass/epoxy is ten and four units, respectively. Use the properties of unidirectional graphite/epoxy and glass/epoxy laminae from Table 2.2.
3. Did your choice of the laminate composite design decrease the mass? If so, by how much?

Solution

1. The maximum normal stress in a plate under bending is given by

$$\sigma = \pm \frac{M\frac{t}{2}}{I}, \tag{5.20}$$

where

M = bending moment (lb-in.)
t = thickness of plate (in.)
I = second moment of area (in.^{4})

For a rectangular cross-section, the second moment of area is

$$I = \frac{bt^3}{12}, \tag{5.21}$$

where b = width of plate (in.).

Using the given factor of safety, $F_s = 2$, and given $b = 4$ in., the thickness of the plate using the maximum stress criterion is

$$t = \sqrt{\frac{6MF_s}{b\sigma_{ult}}}, \tag{5.22}$$

where σ_{ult} = 40.02 Ksi from Table 3.4

$$t = \sqrt{\frac{6(13000)2}{4(40.02)10^3}}$$

$$= 0.9872 \ in.$$

2. Now the designer wants to replace the 0.9872 in. thick aluminum plate by a plate of maximum thickness of 0.4936 in. (half that of aluminum) made of laminated composites. The bending moment per unit width is

$$M_{xx} = \frac{13,000}{4}$$

$$= 3,250 \ lb\text{-}in./in.$$

Using the factor of safety of two, the plate is designed to take a bending moment per unit width of

$$M_{xx} = 3{,}250 \times 2$$

$$= 6{,}500 \ lb\text{-}in./in.$$

The simplest choices are to replace the aluminum plate by an all graphite/epoxy laminate or an all glass/epoxy laminate. Using the procedure described in Example 5.3 or using the PROMAL[2] program, the strength ratio for using a single 0° ply for the previous load for glass/epoxy ply is

$$SR = 5.494 \times 10^{-5}.$$

The bending moment per unit width is inversely proportional to the square of the thickness of the plate, so the minimum number of plies required would be

$$N_{Gl/Ep} = \sqrt{\frac{1}{5.494 \times 10^{-5}}}$$

$$= 135 \text{ plies.}$$

This gives the thickness of the all-glass/epoxy laminate as

$$t_{Gl/Ep} = 135 \times 0.0049213 \text{ in.}$$

$$= 0.6643 \text{ in.}$$

The thickness of an all-glass/epoxy laminate is more than 0.4935 in. and is thus not acceptable.

Similarly, for an all graphite/epoxy laminate made of only 0° plies, the minimum number of plies required is

$$N_{Gr/Ep} = 87 \text{ plies.}$$

This gives the thickness of the plate as

$$t_{Gr/Ep} = 87 \times 0.0049213$$

$$= 0.4282 \text{ in.}$$

The thickness of an all-graphite/epoxy laminate is less than 0.4936 in. and is acceptable.

Even if an all-graphite/epoxy laminate is acceptable, because graphite/epoxy is 2.5 times more costly than glass/epoxy, one would suggest the use of a hybrid laminate. The question that arises now concerns the sequence in which the unidirectional plies should be stacked. In a plate under a bending moment, the magnitude of ply stresses is maximum on the top and bottom face. Because the longitudinal tensile and compressive strengths are larger in the graphite/epoxy lamina than in a glass/epoxy lamina, one would put the former as the facing material and the latter in the core.

The maximum number of plies allowed in the hybrid laminate is

$$N_{\max} = \frac{\textit{Maximum Allowable Thickness}}{\textit{Thickness of each ply}}$$

$$= \frac{0.4936}{0.0049213}$$

$$= 100 \text{ plies.}$$

Several combinations of 100-ply symmetric hybrid laminates of the form $[0_n^{Gr}/0_m^{Gl}/0_n^{Gr}]$ are now subjected to the applied bending moment. Minimum strength ratios in each laminate stacking sequence are found. Only if the strength ratios are greater than one — that is, the laminate is safe — is the cost of the stacking sequence determined. A summary of these results is given in Table 5.8.

From Table 5.8, an acceptable hybrid laminate with the lowest cost is case VI, $[0_{16}^{Gr}/0_{68}^{Gl}/0_{16}^{Gr}]$.

TABLE 5.8

Cost of Various Glass/Epoxy–Graphite/Epoxy Hybrid Laminates

	Number of plies			
Case	**Glass/epoxy (m)**	**Graphite/epoxy (2n)**	**Minimum SR**	**Cost**
I	0	87	1.023	870
II	20	80	1.342	880
III	60	40	1.127	640
IV	80	20	0.8032	—
V	70	30	0.9836	—
VI	68	32	1.014	592
VII	66	34	1.043	604

3. The volume of the aluminum plate is

$$V_{Al} = 4 \times 4 \times 0.9871$$

$$= 15.7936 \text{ in.}^3$$

The mass of the aluminum plate is (specific gravity = 2.7 from Table 3.2),

$$M_{Al} = V_{Al}\, \rho_{Al}$$

$$= 15.793 \times [(2.7)\ (3.6127 \times 10^{-2})]$$

$$= 1.540 \text{ lbm.}$$

The volume of the glass/epoxy in the hybrid laminate is

$$V_{Gl/Ep} = 4 \times 4 \times 0.0049213 \times 68$$

$$= 5.354 \text{ in.}^3$$

The volume of graphite/epoxy in the hybrid laminate is

$$V_{Gr/Ep} = 4 \times 4 \times 0.0049213 \times 32$$

$$= 2.520 \text{ in.}^3$$

Using the specific gravities of glass, graphite, and epoxy given in Table 3.1 and Table 3.2 and considering that the density of water is 3.6127×10^{-2} lbm/in.^{3}:

$$\rho_{Gl} = 2.5 \times (3.6127 \times 10^{-2}) = 0.9032 \times 10^{-1} \text{ lbm/in.}^3$$

$$\rho_{Gr} = 1.8 \times (3.6127 \times 10^{-2}) = 0.6503 \times 10^{-1} \text{ lbm/in.}^3$$

$$\rho_{Ep} = 1.2 \times (3.6127 \times 10^{-2}) = 0.4335 \times 10^{-1} \text{ lbm/in.}^3$$

The fiber volume fraction is given in Table 2.1 and, substituting in Equation (3.8), the density of glass/epoxy and graphite/epoxy laminae is

$$\rho_{Gl/Ep} = (0.9032 \times 10^{-1})\ (0.45) + (0.4335 \times 10^{-1})\ (0.55)$$

$$= 0.6449 \times 10^{-1} \text{ lbm/in.}^3$$

$$\rho_{Gr/Ep} = (0.6503 \times 10^{-1})\,(0.70) + (0.4335 \times 10^{-1})\,(0.30)$$

$$= 0.5853 \times 10^{-1} \text{ lbm/in.}^3$$

The mass of the hybrid laminate then is

$$M_h = (5.354)\,(0.6449 \times 10^{-1}) + (2.520)(0.5853 \times 10^{-1})$$

$$= 0.4928 \text{ lbm.}$$

The percentage savings using the composite laminate over aluminum is

$$= \frac{1.540 - 0.4928}{1.540} \times 100$$

$$= 68\%.$$

This example dictated the use of unidirectional laminates. How will the design change if multiple loads are present? Examples of multiple loads include a leaf spring subjected to bending moment as well as torsion or a thin pressure vessel subjected to an internal pressure to yield a biaxial state of stress. In such cases, one may have a choice not only of material systems and their combination, but also of orientation of plies. Combinations of angle plies can be infinite, so attention may be focused on angle plies of 0°, 90°, 45°, and –45° and their combinations. This reduces the possibilities to a finite number for a limited number of material systems; however, but the number of combinations can still be quite large to handle.

Example 5.6

An electronic device uses an aluminum plate of 1-in. thickness and a top cross–sectional area of 4 in. × 4 in. to take a pure bending moment. The designer wants to replace the aluminum plate with graphite/epoxy unidirectional laminate. The ply thickness of graphite/epoxy is 0.125 mm (0.0049213 in.).

1. Use the properties of aluminum and unidirectional graphite/epoxy as given in Table 3.4 and Table 2.2, respectively, to design a plate of graphite/epoxy with the same bending stiffness in the needed direction of load as that of the aluminum beam.
2. Does the laminate design decrease the mass? If so, by how much?

Solution

1. The bending stiffness, E_b, of the aluminum plate is given by:

$$E_b = EI \tag{5.23}$$

$$= E\left(\frac{1}{12}bh^3\right),$$

where

E = Young's modulus of aluminum
b = width of beam
h = thickness of beam

$$E_b = 10.3 \times 10^6\left(\frac{1}{2}(4)(1)^3\right)$$

$$= 3.433 \times 10^6 \text{ lb–in.}^2$$

To find the thickness of a graphite/epoxy laminate with unidirectional plies and the same flexural rigidity, let us look at the bending stiffness of a laminate of thickness, h:

$$E_b = E_x I$$

$$= E_x \frac{1}{12}bh^3 ,$$

where E_x = Young's modulus in direction of fibers.

Because $E_x = E_1 = 26.25$ Msi for a 0° ply from Table 2.2,

$$3.433 \times 10^6 = 26.25 \times 10^6\left(\frac{1}{2}\ 4h^3\right),$$

giving

$$h = 0.732 \text{ in.}$$

Thus, a 1-in. thick aluminum beam can be replaced with a graphite/epoxy laminate of 0.732 in. thickness. Note that, although the Young's modulus of graphite /epoxy is approximately 2.5 times that of aluminum, the thickness of aluminum plate is approximately only

1.4 times that of the graphite /epoxy of laminate because the bending stiffness of a beam is proportional to the cube of the thickness. Thus, the lightest beam for such bending would be influenced by the cube root of the Young's moduli. From the thickness of 0.732 in. of the laminate and a thickness of 0.0049312 in. of the lamina, the number of 0° graphite/epoxy plies needed is

$$n = \frac{0.732}{0.0049213} = 149 \ .$$

The resulting graphite/epoxy laminate then is $[0_{149}]$.

2. The volume of the aluminum plate V_{Al} is

$$V_{Al} = 4 \times 4 \times 1.0$$

$$= 16 \text{ in.}^3$$

The mass of the aluminum plate is (specific gravity = 2.7 from Table 3.2; density of water is 3.6127×10^{-2} lbm/in.^{3}):

$$M_{Al} = V_{Al}\,\rho_{Al}$$

$$= 16 \times (2.7 \times 3.6127 \times 10^{-2})$$

$$= 1.561 \text{ lbm.}$$

The volume of a $[0_{149}]$ graphite/epoxy laminate is

$$V_{Gr/Ep} = 4 \times 4 \times 0.0049213 \times 149$$

$$= 11.73 \text{ in.}^3$$

The density of a graphite/epoxy from Example 5.5 is

$$\rho_{Gr/Ep} = 0.5853 \times 10^{-1} \frac{lbm}{in^3} \ .$$

The mass of the graphite/epoxy laminate beam is

$$M_{Gr/Ep} = (0.5853 \times 10^{-1})\,(11.73)$$

$$= 0.6866 \text{ lbm.}$$

Therefore, the percentage saving in using graphite/epoxy composite laminate over aluminum is

$$\frac{M_{Al} - M_{Gr/Ep}}{M_{Al}}$$

$$= \frac{1.561 - 0.6866}{1.561} \times 100$$

$$= 56\%.$$

Example 5.7

A 6-ft-long cylindrical pressure vessel (Figure 5.2) with an inner diameter of 35 in. is subjected to an internal gauge pressure of 150 psi. The vessel operates at room temperature and curing residual stresses are neglected. The cost of a graphite/epoxy lamina is 250 units/lbm and cost of a glass/epoxy lamina is 50 units/lbm. The following are other specifications of the design:

1. Only 0°, +45°, –45°, +60°, –60°, and 90° plies can be used.
2. Only symmetric laminates can be used.
3. Only graphite/epoxy and glass/epoxy laminae, as given in Table 2.2, are available, but hybrid laminates made of these two laminae are allowed. The thickness of each lamina is 0.005 in.*

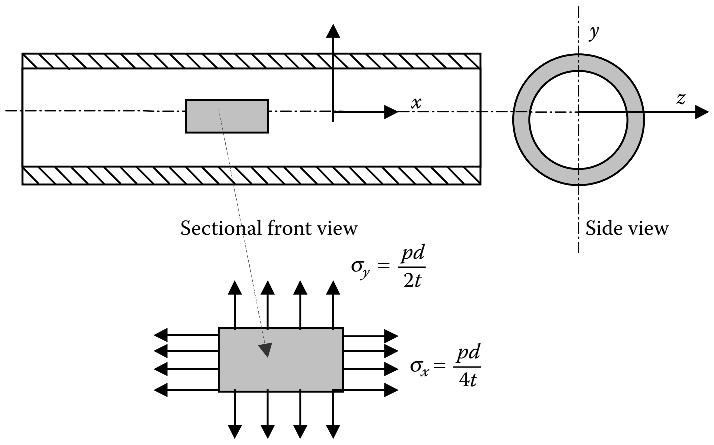

FIGURE 5.2
Fiber composite pressure vessel.

* Note that the thickness of each lamina is given as 0.005 in., and is not 0.125 mm (0.0004921 in.), as given in the material database of the PROMAL program. Material properties for two new materials need to be entered in the database.

4. Calculate specific gravities of the laminae using Table 3.3 and Table 3.4 and fiber volume fractions given in Table 2.2.
5. Neglect the end effects and the mass and cost of ends of the pressure vessel in your design.
6. Use Tsai–Wu failure criterion for calculating strength ratios.
7. Use a factor of safety of 1.95.

Design for ply orientation, stacking sequence, number of plies, and ply material and give separate designs (laminate code, including materials) based on each of the following design criteria:

1. Minimum mass
2. Minimum cost
3. Both minimum mass and minimum cost

You may be unable to minimize mass and cost simultaneously — that is, the design of the pressure vessel for the minimum mass may not be same as for the minimum cost. In that case, give equal weight to cost and mass, and use this as your optimization function:

$$F = \frac{A}{B} + \frac{C}{D}, \tag{5.24}$$

where

A = mass of composite laminate
B = mass of composite laminate if design was based only on minimum mass
C = cost of composite laminate
D = cost of composite laminate if design was based only on minimum cost

Solution

LOADING. For thin-walled cylindrical pressure vessels, the circumferential stress or hoop stress σ_y and the longitudinal or axial stress σ_x is given by

$$\sigma_x = \frac{pr}{2t} \tag{5.25a}$$

$$\sigma_y = \frac{pr}{t}, \tag{5.25b}$$

where

p = internal gage pressure, psi
r = radius of cylinder, in.
t = thickness of cylinder, in.

For our case, we have

$$p = 150 \ psi$$

$$r = \frac{35}{2} = 17.5 \ in$$

giving

$$\sigma_x = \frac{(150)(17.5)}{2t}$$

$$= \frac{1.3125 \times 10^3}{t}$$

$$\sigma_y = \frac{(150)(17.5)}{t}$$

$$= \frac{2.625 \times 10^3}{t}.$$

For the forces per unit length,

$$N_x = \sigma_x t$$

$$= \frac{1.3125 \times 10^3}{t} t \tag{5.26a}$$

$$= 1.3125 \times 10^3 \frac{lb}{in}$$

$$N_y = \sigma_y t$$

$$= \frac{2.625 \times 10^3}{t} t \tag{5.26b}$$

$$= 2.625 \times 10^3 \frac{lb}{in}.$$

MASS OF EACH PLY. The mass of a graphite epoxy ply is

$$m_{Gr/Ep} = V_{Gr/Ep} \rho_{Gr/Ep} \ ,$$

where

$V_{Gr/Ep}$ = volume of a graphite epoxy ply, in.^{3}
$\rho_{Gr/Ep}$ = density of a graphite/epoxy ply, lbm/in.^{3}
$V_{\mathrm{Gr/Ep}} = \pi L d t_p$

where

L = length of the cylinder, in.
d = diameter of the cylinder, in.
t_p = thickness of graphite/epoxy ply, in.

Because L = 6 ft, d = 35 in., and t_p = 0.005 in,

$$V_{Gr/Ep} = \pi(6 \times 12)(35)(0.005)$$

$$= 39.584 \text{ in.}^3$$

The density of a graphite/epoxy lamina is

$$\rho_{Gr/Ep} = \rho_{Gr} V_f + \rho_{Ep} V_m \ .$$

From Table 2.2, the fiber volume fraction, V_f, of the graphite epoxy is 0.7. Thus,

$$V_f = 0.7$$

The matrix volume fraction, V_m, then is

$$V_m = 1 - V_f$$

$$= 1 - 0.7$$

$$= 0.3$$

The specific gravity of graphite and epoxy is given in Table 3.3 and Table 3.4, respectively, as s_{Gr} = 1.8 and s_{Ep} = 1.2; given that the density of water is 3.6127×10^{-2} lbm/in^3,

$$\rho_{Gr/Ep} = (1.8)(3.6127\times10^{-2})(0.7)+(1.2)(3.6127\times10^{-2})(0.3)$$

$$= 5.8526\times10^{-2}\,\frac{lbm}{in^3}$$

Therefore, the mass of a graphite/epoxy lamina is

$$m_{Gr/Ep} = V_{Gr/Ep}\rho_{Gr/Ep}$$

$$= (39.584)(5.8526\times10^{-2})$$

$$= 2.3167\ lbm.$$

The mass of a glass/epoxy ply is

$$m_{Gl/Ep} = V_{Gl/Ep}\rho_{Gl/Ep}$$

where

$V_{Gl/Ep}$ = volume of glass/epoxy, in.^{3}
$\rho_{Gl/Ep}$ = density of glass/epoxy, lbm/in.^{3}
$V_{Gl/Ep} = \pi L d t_p$
= 39.584 in.^{3}

The density of a glass/epoxy lamina is

$$\rho_{Gl/Ep} = \rho_{Gl}V_f + \rho_{Ep}V_m\ .$$

From Table 2.2, the fiber volume fraction V_f of the glass/epoxy is 0.45; thus,

$$V_f = 0.45\ .$$

The matrix volume fraction V_m then is

$$V_m = 1 - V_f$$

$$= 1 - 0.45$$

$$= 0.55.$$

The specific gravity of glass and epoxy is given in Table 3.3 and Table 3.4, respectively, as

$$s_{Gl} = 2.5, s_{Ep} = 1.2$$

and, given that the density of water is 3.6127×10^{-2} lbm/in^3,

$$\rho_{G\ /Ep} = (2.5)(3.6127 \times 10^{-2})(0.45) + (1.2)(3.6127 \times 10^{-2})(0.55)$$

$$= 6.4487 \times 10^{-2} \frac{lbm}{in^3}.$$

Therefore, the mass of glass/epoxy lamina is

$$m_{Gl/Ep} = V_{Gl/Ep}\rho_{Gl/Ep}$$

$$= (39.584)(6.4487 \times 10^{-2})$$

$$= 2.5526\ lbm.$$

COST OF EACH PLY. The cost of a graphite/epoxy ply is

$$C_{Gr/Ep} = m_{Gr/Ep}c_{Gr/Ep}\ ,$$

where

$m_{Gr/Ep}$ = mass of graphite/epoxy ply
$c_{Gr/Ep}$ = unit cost of graphite/epoxy ply

Because

$$m_{Gr/Ep} = 2.3167\ lbm$$

$$c_{Gr/Ep} = 250 \frac{units}{lbm}\ ,$$

the cost of a graphite/epoxy ply is

$$C_{Gr/Ep} = (2.3167)(250)$$

$$= 579.17\ \text{units}.$$

Similarly, the cost of a glass/epoxy ply is

$$C_{Gl/Ep} = m_{Gl/Ep} c_{Gl/Ep} \ .$$

Because

$$m_{Gl/Ep} = 2.5526 \ lbm$$

and

$$c_{Gl/Ep} = 50 \frac{units}{lbm} \ ,$$

the cost of a glass/epoxy ply is

$$C_{Gl/Ep} = (2.5526)(50)$$

$$= 127.63 \text{ units.}$$

1. To find the design for minimum mass, consider a composite laminate made of graphite/epoxy with $[0/90_2]_s$. We simply choose this laminate as $N_y = 2N_x$ and thus choose two 90° plies for every 0° ply. For this laminate, from PROMAL we get the minimum strength ratio as

$$SR = 0.6649.$$

Because the required factor of safety is 1.95, we need

$$\frac{1.95}{0.6649} \times 6$$

$$\cong 18 \text{ plies.}$$

$[0/90_2]_{3s}$ is a possible choice because it gives a strength ratio of 1.995. However, is this laminate with the minimum mass? Choosing some other choices such as laminates with ±60° laminae, a graphite/epoxy $[\pm 60]_{4s}$ laminate gives an SR = 1.192 and that is lower than the required SR of 1.95.

A $[0/90_2]_{3s}$ laminate made of glass/epoxy gives a strength ratio of SR = 0.5192 and that is also lower than the needed strength ratio of 1.95. Other combinations tried used more than 18 plies. A summary of possible combinations is shown in Table 5.9.

TABLE 5.9

Mass and Cost of Possible Stacking Sequences for Minimum Mass

Stacking sequence	No. plies	Minimum strength ratio	Mass (lbm)	Cost (units)
$[0/90_2]_{3s}$ (Graphite/epoxy)	18	1.995	41.700	10,425
$[\pm 60]_{4s}$ (Graphite/epoxy)	16	1.192	—	—
$[0/90_2]_{3s}$ (Glass/epoxy)	18	0.5192	—	—
$[\pm 60]_{5s}$ (Graphite/epoxy)	20	1.490	—	—
$[\pm 45_2/\pm 60_3]_s$ (Graphite/epoxy)	20	2.332	46.334	11,583

TABLE 5.10

Mass and Cost of Possible Stacking Sequences for Minimum Cost

Stacking sequence	No. plies	Minimum strength ratio	Mass (lbm)	Cost (units)
$[0/90_2]_{3s}$ (Graphite/epoxy)	18	1.995	41.700	10,425
$[\pm 45_2/\pm 60_3]_s$ (Graphite/epoxy)	20	2.291	46.334	11,583
$[0/90_2]_{12s}$ (Glass/epoxy)	72	2.077	183.79	9,189
$[90/\pm 45]_{10s}$ (Glass/epoxy)	60	1.992	153.16	7,658
$[\pm 60]_{15s}$ (Glass/epoxy)	60	2.033	153.16	7,658

Thus, one can say that the laminate for minimum mass is the first stacking sequence in Table 5.9:

Number of plies: 18

Material of plies: graphite/epoxy

Stacking sequence: $[0/90_2]_{3s}$

Mass of laminate = $(18 \times 2.3167) = 41.700$ lbm

Cost of laminate = $(41.700 \times 250) = 10425$ units

2. To find the design for minimum cost, we found in part (1) that the $[0/90_2]_{3s}$ graphite/epoxy laminate is safe, but the same stacking sequence for glass/epoxy gives a $SR = 0.5192$. Therefore, we may need four times more plies of glass/epoxy to keep it safe to obtain a factor of safety of 1.95. If so, would it be cheaper than the $[0/90_2]_{3s}$ graphite/epoxy laminate? Yes, it would because a glass/epoxy costs 127.63 units per ply as opposed to 579.17 units per ply for graphite/epoxy. Choosing $[0/90_2]_{12s}$ glass/epoxy laminate gives $SR = 2.077$. Are there other combinations that give an $SR > 1.95$ but use less than the 72 plies used in $[0/90_2]_{12s}$? Stacking sequences of 60 plies such as $[90/\pm 45]_{10s}$ and $[\pm 60]_{15s}$ were tried and were acceptable designs. The results from some of the stacking sequences are summarized in Table 5.10.

Therefore, we can say that the laminate for minimum cost is as follows

Number of plies = 60

Material of plies: glass/epoxy

Stacking sequence: $[\pm 60]_{15s}$

Mass of laminate = 60 × 2.5526 = 153.16 lbm

Cost of laminate = 153.16 × 50 = 7658 units

3. Now, how do we find the laminate that minimizes cost and mass? We know that the solutions to part (1) and (2) are different. Thus, we need to look at other combinations. However, before doing so, let us find the minimizing function for parts (1) and (2). The minimizing function is given as

$$F = \frac{A}{B} + \frac{C}{D},$$

where

A = mass of composite laminate
B = mass of composite laminate if design was based only on minimum mass
C = cost of composite laminate
D = cost of composite laminate if design was based only on minimum cost

From part (1), B = 41.700 lbm and, from part (2), D = 7658 units; then, the minimizing function is

$$F = \frac{41.700}{41.700} + \frac{10425}{7658} = 2.361$$

for the $[0/90_2]_{3s}$ graphite/epoxy laminate obtained in part (1).

$$F = \frac{153.16}{41.700} + \frac{7658}{7658} = 4.673$$

for the $[\pm 60]_{15s}$ glass/epoxy laminate obtained in part (2).

Therefore, the question is whether a laminate that has an optimizing function value of less than 2.361 can be found. If not, the answer is the same as the laminate in part (1). Table 5.11 gives the summary of some of the laminates that were tried to find minimum value of F. The third stacking sequence in Table 5.11 is the one in which, in the $[0/90_2]_{3s}$ graphite/epoxy laminate of part (1), six of the graphite/

TABLE 5.11

Optimizing Function Values for Different Stacking Sequences

Stacking sequence	Mass (lbm)	Cost	Minimum strength ratio	F
$[0/90_2]_{3s}$ graphite/epoxy (part a)	41.700	10,425	1.995	2.361
$[\pm 60]_{18s}$ glass/epoxy (part b)	153.16	7,658	2.0768	4.672
$[0_{Gr/Ep}/90_{2\ Gr/Ep}/0_{Gr/Ep}/90_{2\ Gr/Ep}/0_{Gl/Ep}/90_{2\ Gl/Ep}/0_{Gl/Ep}/90_{2\ Gl/Ep}/0_{Gl/Ep}/90_{2Gl/Ep}/0_{Gl/Ep}/90_{2\ Gl/Ep}]_s$	89.063	10,013	2.012	3.443

epoxy plies of $0/90_2$ sublaminate group are substituted with 24 glass/epoxy plies of the $0/90_2$ sublaminate group.

Thus, it seems that $[0/90_2]_{3s}$ graphite/epoxy laminate is the answer to part (3). Although more combinations should have been attempted to come to a definite conclusion, it is left to the reader to try other hybrid combinations using the PROMAL program.

Example 5.8

Drive shafts (Figure 5.3) in cars are generally made of steel. An automobile manufacturer is seriously thinking of changing the material to a composite material. The reasons for changing the material to composite materials are that composites

1. Reduce the weight of the drive shaft and thus reduce energy consumption
2. Are fatigue resistant and thus have a long life
3. Are noncorrosive and thus reduce maintenance costs and increase life of the drive shaft
4. Allow single piece manufacturing and thus reduce manufacturing cost

The design constraints are as follows:

1. Based on the engine overload torque of 140 N-m, the drive shaft needs to withstand a torque of 550 N-m.

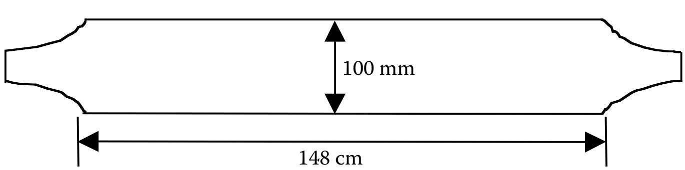

FIGURE 5.3
Fiber composite drive shaft.

2. The shaft needs to withstand torsional buckling.
3. The shaft has a minimum bending natural frequency of at least 80 Hz.
4. Outside radius of drive shaft = 50 mm.
5. Length of drive shaft = 148 cm.
6. Factor of safety = 3.
7. Only 0, +45, –45, +60, –60, and 90° plies can be used.

For steel, use the following properties:

Young's modulus E = 210 GPa,
Poisson's ratio ν = 0.3,
Density of steel ρ = 7800 kg/m^3
Ultimate shear strength τ_{ult} = 80 MPa.

For the composite, use properties of glass/epoxy from Table 2.1 and Table 3.1 and assume that ply thickness is 0.125 mm. Design the drive shaft using

1. Steel
2. Glass/epoxy

Solution

1. STEEL DESIGN.

Torsional strength: The primary load in the drive shaft is torsion. The maximum shear stress, $\tau_{\max}$, in the drive shaft is at the outer radius, r_o, and is given as

$$\tau_{\max} = \frac{Tr_o}{J}, \tag{5.27}$$

where

T = maximum torque applied in drive shaft (N-m)
r_o = outer radius of shaft (m)
J = polar moment of area (m^4)

Because the ultimate shear strength of steel is 80 MPa and the safety factor used is 3, using Equation (5.27) gives

$$\frac{80 \times 10^6}{3} = \frac{(550)(0.050)}{\frac{\pi}{2}(0.050^4 - r_i^4)}$$

$$r_i = 0.04863\ m.$$

Therefore, the thickness of the steel shaft is

$$t = r_o - r_i$$

$$= 0.050 - 0.04863$$

$$= 1.368 \text{ mm.}$$

Torsional buckling: This requirement asks that the applied torsion be less than the critical torsional buckling moment. For a thin, hollow cylinder made of isotropic materials, the critical buckling torsion, T_b, is given by[4]

$$T_b = (2\pi r_m^2 t)(0.272)(E)\left(\frac{t}{r_m}\right)^{2/3}, \tag{5.28}$$

where

r_m = mean radius of the shaft (m)
t = wall thickness of the drive shaft (m)
E = Young's modulus (Pa)

Using the thickness t = 1.368 mm calculated in criterion (1) and the mean radius

$$r_m = \frac{r_o + r_i}{2}$$

$$= \frac{0.050 + 0.04863}{2}$$

$$= 0.049315 \; m,$$

$$T_b = 2(0.049315)^2(0.001368)(0.272)(210 \times 10^9)\left(\frac{0.001368}{0.049315}\right)^{3/2}$$

$$= 109442 \; N\text{-}m.$$

The value of critical torsional buckling moment is larger than the applied torque of 550 N-m.

Natural frequency: The lowest natural frequency for a rotating shaft is given by[5]

$$f_n = \frac{\pi}{2}\sqrt{\frac{EI}{mL^4}}, \tag{5.29}$$

where

g = acceleration due to gravity (m/s^2)
E = Young's modulus of elasticity (Pa)
I = second moment of area (m^4)
m = mass per unit length (kg/m)
L = length of drive shaft (m)

Now the second moment of area, I, is

$$I = \frac{\pi}{4}(r_o^4 - r_i^4)$$

$$= \frac{\pi}{4}(0.050^4 - 0.04863^4)$$

$$-5.162 \times 10^{-7} \; m^4 .$$

The mass per unit length of the shaft is

$$m = \pi \, (r_o^2 - r_i^2) \, \rho$$

$$= \pi \, (0.050^2 - 0.04863^2) \, (7800)$$

$$= 3.307 \text{ kg/m}.$$

Therefore,

$$f_n = \frac{\pi}{2}\sqrt{\frac{(210 \times 10^9)(5.162 \times 10^{-7})}{(3.307)(1.48)^4}}$$

$$= 129.8 \; Hz .$$

This value is greater than the minimum desired natural frequency of 80 Hz.
Thus, the steel design of a hollow shaft of outer radius 50 mm and thickness t = 1.368 mm is an acceptable design.

2. COMPOSITE MATERIALS DESIGN.

Torsional strength: Assuming that the drive shaft is a thin, hollow cylinder, an element in the cylinder can be assumed to be a flat laminate. The only nonzero load on this element is the shear force, N_{xy}. If the average shear stress is $(\tau_{xy})_{\text{average}}$, the applied torque then is

$$T = (\text{shear stress})\,(\text{area})\,(\text{moment arm})$$

$$T = (\tau_{xy})_{average}\pi(r_o^2 - r_i^2)r_m \ . \tag{5.30}$$

The shear force per unit width is given by

$$N_{xy} = (\tau_{xy})_{average} t \ .$$

Because

$$t = r_o - r_i$$

$$r_m = \frac{r_o + r_i}{2} \ ,$$

then

$$N_{xy} = \frac{T}{2\pi r_m^2}$$

$$= \frac{550}{2\pi(0.050)^2} \tag{5.31}$$

$$= 35,014 \ N/m.$$

To find approximately how many layers of glass/epoxy may be needed to resist the shear load, choose a four-ply $[\pm 45]_s$ laminate. Inputting a value of N_{xy} = 35,014 N/m into the PROMAL program, the minimum strength ratio obtained using Tsai–Wu theory is 1.261. A strength ratio of at least 3 is needed, so the number of plies is increased proportionately as $\frac{3}{1.261} \times 4 \cong 10.$ The next laminate chosen is $[\pm 45_2/45]_s$ laminate. A minimum strength ratio of 3.58 is obtained, so it is an acceptable design based on torsional strength criterion.

Torsional buckling: An orthotropic thin hollow cylinder will buckle torsionally if the applied torque is greater than the critical torsional buckling load given by[4]

$$T_c = (2\pi r_m^2 t)(0.272)(E_x E_y^3)^{1/4}\left(\frac{t}{r_m}\right)^{3/2} . \tag{5.32}$$

From PROMAL, the longitudinal Young's moduli E_x and the transverse Young's moduli E_y of the $[\pm 45_2/45]_s$ glass/epoxy laminate based on properties from Table 2.1 are

$$E_x = 12.51 \text{ GPa}$$

$$E_y = 12.51 \text{ GPa}$$

Because lamina thickness is 0.125 mm, the thickness of the ten-ply $[\pm 45_2/45]_s$ laminate, t, is

$$t = 10 \times 0.125 = 1.25 \text{ mm}.$$

The mean radius, r_m, is

$$r_m = r_o - \frac{t}{2}$$

$$= 50 - \frac{1.25}{2}$$

$$= 49.375\ mm.$$

Therefore,

$$T_c = 2\pi(0.049375)^2(0.00125)(0.272)\times$$

$$[(12.51\times 10^9)(12.51\times 10^9)^3]^{1/4}\left(\frac{0.00125}{0.049375}\right)^{2/3}$$

$$= 262\ N\text{-}m.$$

This is less than the applied torque of 550 N-m. Thus, the $[\pm 45_2/45]_s$ laminate would torsionally buckle. Per the formula, the torsional buckling is proportional to $E_y^{3/4}$ and $E_x^{1/4}$. Because the modulus in the y-direction is more effective in increasing the critical torsional buckling load, it will be necessary to substitute by or add 90° plies.

Natural frequency: Although the $[\pm 45_2/45]_s$ laminate is inadequate, per the torsional buckling criterion, let us still find the minimum natural frequency of the drive shaft, which is given by[5]

$$f_n = \frac{\pi}{2}\sqrt{\frac{E_x I}{mL^4}} \, . \tag{5.33}$$

Now,

$$E_x = 12.51 \; GPa$$

$$I = \frac{\pi}{4}(r_0^4 - r_i^4)$$

$$= \frac{\pi}{4}(0.050^4 - 0.04875^4)$$

$$= 4.728 \times 10^{-7} \; m^4 .$$

The mass per unit length of the beam is

$$m = \frac{\pi(r_o^2 - r_i^2)L\rho}{L}$$

$$= \pi(r_o^2 - r_i^2)\rho$$

$$= (0.05^2 - 0.04875^2)(1785)$$

$$= 0.6922 \frac{kg}{m} .$$

Thus,

$$f_n = \frac{\pi}{2}\sqrt{\frac{(12.51 \times 10^9)(4.728 \times 10^{-7})}{0.6922 \times 1.48^4}}$$

$$= 66.3 \; Hz.$$

Because the minimum bending natural frequency is required to be 80 Hz, this requirement is also not met by the $[\pm 45_2/45]_s$ laminate. The minimum natural frequency can be increased by increasing the value of E_x because the natural frequency f_n is proportional to $\sqrt{E_x}$. To achieve this, 0° plies can be added or substituted.

TABLE 5.12

Acceptable and Nonacceptable Designs of Drive Shaft Based on Three Criteria of Torsional Strength, Critical Torsional Buckling Load, and Minimum Natural Frequency

Laminate stacking sequence	No. plies	Minimum strength ratio	Critical torsional buckling load (N-m)	E_x (GPa)	E_y (GPa)	Minimum natural frequency (Hz)	Acceptable design
$[0/\pm45_2/45/90]_s$	14	3.982	797.8	16.44	16.44	***75.6***	No
$[0_2/\pm45_2/90]_s$	14	3.248	828.8	20.16	16.16	83.7	Yes
$[0/\pm45_2/90]_s$	12	3.006	564.1	17.07	17.07	***77.2***	No
$[0/\pm45_2]_s$	10	***2.764***	***291.2***	17.86	12.76	***79.2***	No
$[45/90_3/0_2]_s$	12	4.127	763.5	19.44	24.47	82.4	Yes
Design constraints		***>3***	***>550***			***>80***	

Note: Numbers given in bold italics to show the reason for unacceptable designs.

From the three criteria, we see that ±45° plies increase the torsional strength, 90° plies increase the critical torsional buckling load, and the 0° plies increase the natural frequency of the drive shaft. Therefore, having ±45°, 90°, and 0° plies may be the key to an optimum design.

In Table 5.12, several other combinations have been evaluated to find an acceptable design.

The last stacking sequence $[45/90_3/0_2]_s$ is a 12-ply laminate and meets the three requirements of torsional strength, critical torsional buckling load, and minimum natural frequency.

MASS SAVINGS. The savings in the mass of the drive shaft are calculated as follows:

$$\text{Mass of steel drive shaft} = \pi\,(r_o^2 - r_i^2)\,L\,\rho$$

$$= \pi\,(0.050^2 - 0.04863^2)\,(1.48)\,(7800)$$

$$= 4.894 \text{ kg.}$$

The thickness, t, of the $[45/90_3/0_2]_s$ glass/epoxy shaft is

$$t = 12 \times 0.125$$

$$= 1.5\ mm.$$

The inner radius of the $[45/90_3/0_2]_s$ glass/epoxy shaft then is

$$r_i = r_o - t$$

$$= 0.05 - 0.0015$$

$$= 0.0485 \text{ m.}$$

Mass of $[45/90_3/0_2]_s$ glass/epoxy shaft is

$$= \pi \, (r_o^2 - r_i^2) \, L \, \rho$$

$$= \pi \, (0.05^2 - 0.0485^2) \, (1.48) \, (1758)$$

$$= 1.226 \text{ kg.}$$

Percentage mass saving over steel is

$$= \frac{4.894 - 1.226}{4.894} \times 100$$

$$= 75\%.$$

Would an 11-ply, $[45/90_4/\overline{90}]_s$ glass/epoxy laminate meet all the requirements?

5.5 Other Mechanical Design Issues

5.5.1 Sandwich Composites

One group of laminated composites used extensively is sandwich composites. Sandwich panels consist of thin facings (also called skin) sandwiching a core. The facings are made of high-strength material, such as steel, and composites such as graphite/epoxy; the core is made of thick and lightweight materials such as foam, cardboard, plywood, etc. (Figure 5.4).

The motivation in doing this is twofold. First, if a plate or beam is bent, the maximum stresses occur at the top and bottom surfaces. Therefore, it makes sense to use high-strength materials only at the top and bottom and low- and lightweight strength materials in the middle. The strong and stiff facings also support axial forces. Second, the resistance to bending of a rectangular cross-sectional beam/plate is proportional to the cube of the thickness. Thus, increasing the thickness by adding a core in the middle increases this resistance. Note that the shear forces are maximum in the

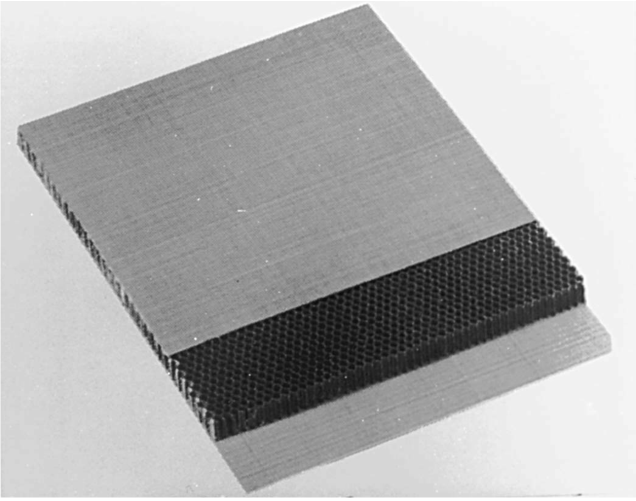

FIGURE 5.4
Fiberglass facings with a Nomex7 honeycomb core. (Picture Courtesy of M.C. Gill Corporation, http://www.mcgillcorp.com).

middle of the sandwich panel, thus requiring the core to support shear. This advantage in weight and bending stiffness makes sandwich panels more attractive than other materials. Sandwich panels are evaluated based on strength, safety, weight, durability, corrosion resistance, dent and puncture resistance, weatherability, and cost.[6]

The most commonly used facing materials are aluminum alloys and fiber-reinforced plastics. Aluminum has high specific modulus, but it corrodes without treatment and is prone to denting. Fiber-reinforced plastics such as graphite/epoxy and glass/epoxy are becoming popular as facing materials because of their high specific modulus and strength and corrosion resistance. Fiber-reinforced plastics may be unidirectional or woven laminae.

The most commonly used core materials are balsa wood, foam, and honeycombs. Balsa wood has high compressive strength (1500 psi), good fatigue life, and high shear strength (200 psi). Foams are low-density polymers such as polyuretherane, phenolic, and polystyrene. Honeycombs are made of plastic, paper, cardboard, etc. The strength and stiffness of honeycomb depend on the material and its cell size and thickness.

Adhesives join the facing and core materials and thus are critical in the overall integrity of the sandwich panel. Adhesives come in forms of film, paste, and liquid. Common examples include vinyl phenolic, modified epoxy, and urethane.

5.5.2 Long-Term Environmental Effects

Section 4.5 has already discussed the effects caused by temperature and moisture, such as residual stresses and strains. What effect do these and other environmental factors such as corrosive atmospheres and temperatures and humidity variations have over the long term on composites? These elements may lessen the adhesion of the fiber-matrix interface, such as between glass and epoxy. Epoxy matrices soften at high temperatures, affecting properties dominated by the matrix, such as transverse and in-plane shear stiffness and strength, and flexural strength. For example, Quinn[7] found that a glass/epoxy composite rod absorbed as much as 0.4% of water over 150 days of immersion. The effect of this moisture absorption on flexural modulus is shown in Figure 5.5.

5.5.3 Interlaminar Stresses

Due to the mismatch of elastic moduli and angle between the layers of a laminated composite, interlaminar stresses are developed between the layers. These stresses, which are normal and shear, can be high enough to cause edge delamination between the layers.[8–10] Delamination eventually limits the life of the laminated structure. Delamination can be further caused due to nonoptimum curing and introduction of foreign bodies in the structure.[11]

In Figure 5.6, theoretical interlaminar shear and normal stresses are plotted as a function of normalized distance — zero at the center line and one at the free edge — from the center line of a $[\pm 45]_s$ graphite/epoxy laminate. The interlaminar stresses given are for the bottom surface of the top ply of the laminate and are found by using equations of elasticity.[9] Away from the edges, these stresses are the same as predicted by the classical lamination theory discussed in Chapter 4. However, near the edges, the normal shear stress τ_{xy} decreases to zero, and the out-of-plane shear stress τ_{xz} becomes infinite (not shown). The classical lamination theory and elasticity results give different results because the former violates equilibrium and boundary conditions at the interface. For example, for a simple state of stress on the $[\pm 45]_s$ laminate, the classical lamination theory predicts nonzero values for the stresses σ_{xx}, σ_{yy}, and τ_{xy} for each ply. This is not true at the edges, where σ_y and τ_{xy} are actually zero because they are free boundaries.

Interlaminar stresses pose a challenge to the designer and there are some ways to counter their effects. Pagano and Pipes[9] found theoretically that keeping the angle, symmetry, and number of plies the same but changing the stacking sequence influences the interlaminar stresses. The key to changing the stacking sequence is to decrease the interlaminar shear stresses without increasing the tensile (if any) interlaminar normal stresses. For example, a laminate stacking sequence of $[\pm 30/90]_s$ produces tensile interlaminar normal stresses under a uniaxial tensile load however, if the sequence is just changed to $[90/\pm 30]_s$, it produces compressive interlaminar normal stresses. This makes the latter sequence less prone to delamination. Other techniques to improve tolerance to delamination include using toughened resin systems[12] and interleaved systems in which a discrete layer of resin with high toughness and strain to failure is added on top of a layer.[13,14]

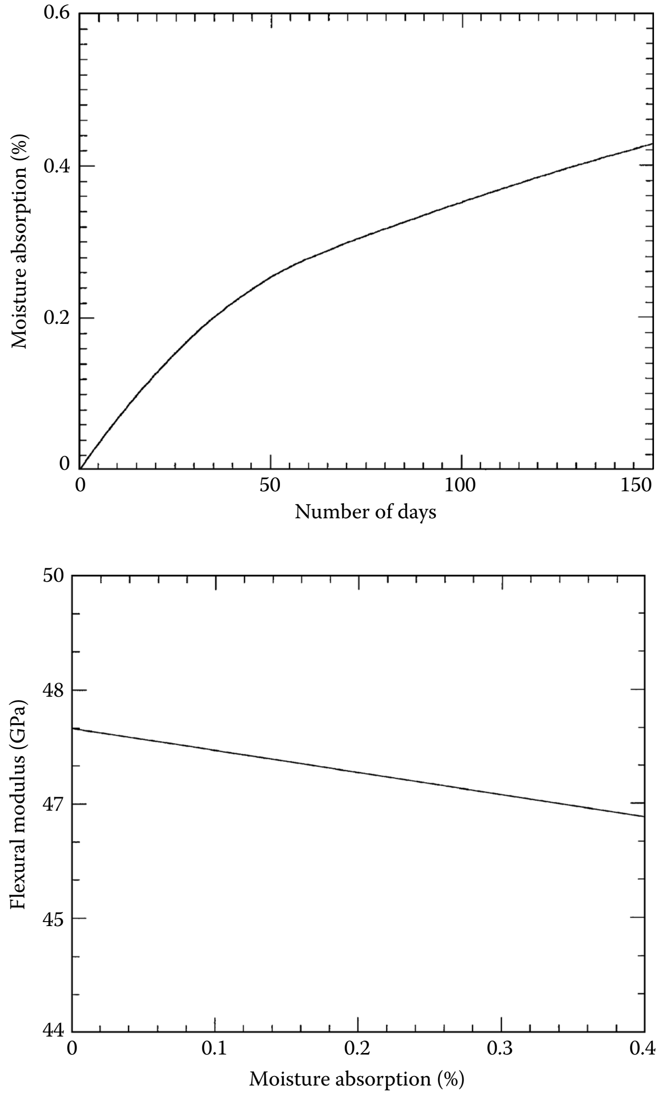

FIGURE 5.5
Moisture absorption as a function of time and its effect on flexural modulus of a glass/polyester composite rod. (Reprinted from Quinn, J.A., in *Design with Advanced Composite Materials*, Phillips, L.N., Ed., 1990, Figure 3.10 (p. 91) and Figure 3.11 (p. 92), Springer–Verlag, Heidelberg.)

5.5.4 Impact Resistance

The resistance to impact of laminated composites is important in applications such as a bullet hitting a military aircraft structure or even the contact of a composite leaf spring in a car to runaway stones on a gravel road. The resistance to impact depends on several factors of the laminate, such as the

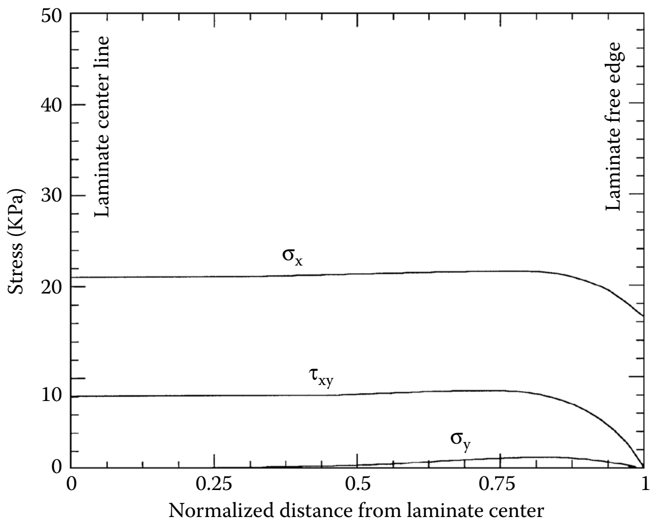

FIGURE 5.6
Normal and shear stresses at the interface of bottom surface of top ply in a four-ply laminate. (Reprinted from Pagano, N.J. and Soni, S.R., in *Interlaminar Response of Composite Materials*, Pagano, N.J., Ed., 1989, p. 9, Elsevier Science, New York, with kind permission from authors.)

material system; interlaminar strengths; stacking sequence; and nature of the impact, such as velocity, mass, and size of the impacting object. Impact reduces strengths of the laminate and also initiates delamination in composites. Delamination becomes more problematic because, many times, visual inspection cannot find it. Solutions for increasing impact resistance and residual impact strength have included toughened epoxies and interleaved laminates. In the former, epoxies are toughened by liquid rubber and, in the latter case, a discrete toughened layer is added to the laminae at selected places.

5.5.5 Fracture Resistance

When a crack develops in an isotropic material, the stresses at the crack tip are infinite. The intensity of these infinite stresses is called the stress intensity factor. If the stress intensity factor is greater than the critical stress intensity factor for that material, the crack is considered to grow catastrophically. Another parameter, called the strain energy release rate, is also used in determining fracture resistance. This is the rate of the energy release as the crack grows. If this rate is greater than the critical strain energy release rate of the material, the crack will grow catastrophically. The strain energy release rate and stress intensity factor are related to each other in isotropic materials.

In composites, the mechanics of fracture is not as simple. First, cracks can grow in the form of fiber breaks, matrix breaks, debonding between fiber and matrix, and debonding between layers. Second, no single critical stress intensity factors and strain energy release rates can determine the fracture mechanics process.

Fiber breaks may occur because of the brittle nature of fibers. Some fibers may break because, statistically, some fibers are weaker than others and thus fail at low strains. The matrix may then break because of high strains caused by the fiber breaks. In ceramic matrix composites, the matrix failure strain is lower than that of the fiber. Therefore, matrix breaks precede fiber breaks. In fact, fiber breaks are seen to occur only close to the ultimate failure of the composite. Also, matrix breaks may keep occurring parallel to the crack length.

When a fiber or matrix breaks, the crack does not grow in a self-similar fashion. It may grow along the interface that blunts the crack and improves the fracture resistance of the composites, or it may grow into the next constituent, resulting in uncontrolled failure. The competition between whether a crack grows along the interface or jumps to the adjoining constituent depends on the material properties of the fiber, matrix, and the interface, as well as the fiber volume fraction.

Fracture mechanics in composites is still an open field because there are several mechanisms of failure and developing uniform criteria for the materials looks quite impossible.

5.5.6 Fatigue Resistance

Structures over time are subjected to repeated cyclic loading, such as the fluctuating loads on an aircraft wing. This cyclic loading weakens the material and gives it a finite life. For example, a composite helicopter blade may have a service life of 10,000 hours.

Fatigue data for composite materials are collected using several different data, such as plotting the peak stress applied during the loading as a function of the number of cycles. The allowable peak stress decreases as the number of cycles to failure is increased. The peak stress is compared to the static strength of the composite structure. If these peak stresses are comparably larger than the allowable ultimate strength of the composite, fatigue does not influence the design of the composite structure. This is the case in graphite/epoxy composites in which the allowable ultimate strength is low due to its low impact resistance.

Other factors that influence the fatigue properties are the laminate stacking sequence, fiber and matrix properties, fiber volume fraction, interfacial bonding, etc. For example, for quasi-isotropic laminates, S–N curves are quite different from those of unidirectional laminates. In this case, the 90° plies develop transverse cracks, which influence the elastic moduli and strength of the laminate. Although the influence is limited because 90° plies do not contribute to the static stiffness and strength in the first place, the stress concentrations caused by these cracks may lead to damage in the 0° plies. Other damage modes include fiber and matrix breaks, interfacial and interlaminar debonding, etc. Laminate stacking sequence influences the onset of edge delamination. For example, Foye and Baker[15] conducted tensile fatigue

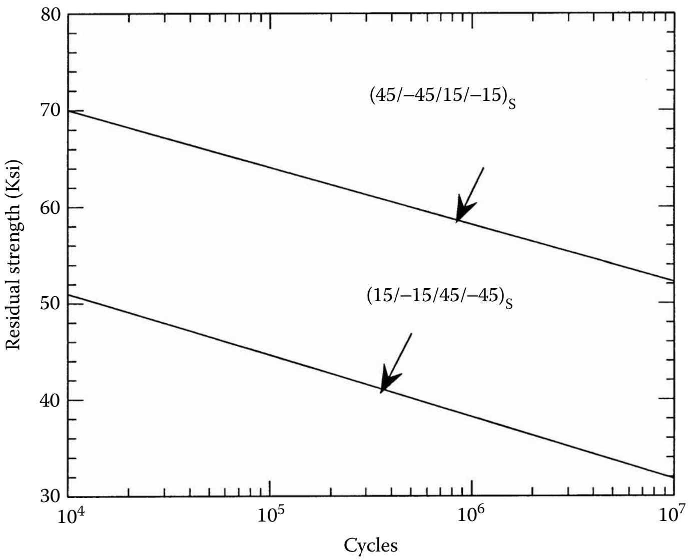

FIGURE 5.7
Comparison of residual strength as a function of number of cycles for two laminates. (Reprinted from Pagano, N.J. and Soni, S.R., in *Interlaminar Response of Composite Materials*, Pagano, N.J., Ed., 1989, p. 12, Elsevier Science, New York, with kind permission from authors.)

testing of boron/epoxy laminates and found the dependence of fatigue life on stacking sequence. A $[\pm 45/\pm 15]_s$ laminate had a higher fatigue life than a $[\pm 15/\pm 45]_s$ laminate (Figure 5.7). Both laminates have the same number and angle of plies, and only the stacking sequence has been changed.

Loading factors such as tension and/or compression, temperature, moisture, and frequency of loading also determine the fatigue behavior of composites. For example, for compressive fatigue loading or tension-compressive fatigue loading, carbon/epoxy composites have very low peak strains because compression can cause layer buckling, etc. In such cases, the dominance of fiber effects is not present, but matrix, fiber-matrix interfaces, and the layers play a more important role.

Nonmechanical issues are also important in design of composite structures. These include fire resistance, smoke emission, lightning strikes, electrical and thermal conductivity, recycling potential, electromagnetic interference, etc.

5.6 Summary

In this chapter, we introduced the special case of laminates and their effect on the stiffness matrices, and response to external loads. We established failure criteria for laminates using the ply-by-ply failure theory. Examples of designing laminated structures such as plates, thin pressure vessels, and drive shafts were given. Other mechanical design issues such as environmental effects, interlaminar stresses, impact resistance, fracture resistance, and fatigue resistance were discussed.

Key Terms

Special laminates
Cross-ply laminates
Angle ply laminates
Antisymmetric laminates
Balanced laminates
Quasi-isotropic laminates
Failures of laminates
Design of laminates
Sandwich composites
Environmental effects
Interlaminar stresses
Impact resistance
Fracture resistance
Fatigue resistance

Exercise Set

5.1 Classify the following laminates:

[–30/45/–45/–30]

[–30/30/–30/30]

[30/–30/30]

[45/30/–30/–45]

[0/90/0/90/0/90/90]

[0/90/90/90/90/0]

[0/18/36/54/72/90/–18/–36/–54/–72]

5.2 Write an example of laminate code for the following:

Symmetric laminate

Antisymmetric laminate

Symmetric cross-ply laminate

Symmetric angle-ply laminate

Balanced angle-ply laminate

5.3 Give an example of a laminate with zero coupling stiffness matrix [*B*].

5.4 Is a nonzero [*B*] matrix attributed to the orthotropy of layers?

5.5 Is a nonzero [*B*] matrix attributed to the unsymmetrical stacking of laminae in a laminate?

5.6 Show numerically that a [0/90] laminate is not a quasi-isotropic laminate. Use the properties of unidirectional glass/epoxy lamina from Table 2.2.

5.7 Does a symmetric quasi-isotropic laminate have [*A*], [*B*], and [*D*] stiffness matrices like that of an isotropic material?

5.8 Are [0/60/–60] and [60/–60/60] quasi-isotropic laminates?

5.9 Are midplane strains and/or midplane curvatures always zero for symmetric laminates?

5.10 Find (1) the extensional stiffness matrix and (2) the extensional elastic moduli of the following graphite/epoxy laminate: $[0/18/36/54/72/90/-18/-36/-54/-72]_s$. Use properties of unidirectional graphite/epoxy lamina from Table 2.1.

5.11 Show that $A_{12} = U_4 h$ for a quasi-isotropic laminate.

5.12 A $[0/90]_s$ laminate made of glass/epoxy is subjected to an axial load N_x. Use properties of unidirectional glass/epoxy lamina from Table 2.2 and assume that each layer is 0.005 in. thick.

1. Use the maximum stress failure theory to find the first and last ply failure of the laminate.
2. Draw the stress–strain curve for the laminate till the last ply failure.

5.13 Using Tsai–Wu theory, find the ply-by-ply failure of a $[45/-45]_s$ graphite/epoxy laminate under a pure bending moment, M_x. Use properties of unidirectional graphite/epoxy lamina from Table 2.1 and assume each layer is 0.125 mm thick.

5.14 Repeat the preceding exercise in the presence of a temperature change of $\Delta T = -150°F$ and a moisture content of $\Delta C = 0.4\%$.

5.15 Develop a comparison table to show the elastic moduli E_x, E_y, ν_{xy}, and G_{xy} and the tensile strengths in *x* and *y* directions, shear strength in the *x*–*y* plane of the two laminates $[0/90]_s$ and $[45/-45]_s$ glass/epoxy laminate. Use properties of unidirectional glass/epoxy lamina from Table 2.2 and assume failure based on first ply failure (FPF).

5.16 Find the angle in $[\pm\theta]_{ns}$ graphite/epoxy sublaminate for maximum value of each of the elastic moduli:

1. E_x

2. E_y
3. G_{xy}

Use properties of unidirectional graphite/epoxy lamina from Table 2.1.

5.17 The bending stiffness of a laminate does not decrease substantially by replacing some of the plies at the midplane:

1. Find the percentage decrease in the longitudinal bending modulus of a $[0]_8$ glass/epoxy laminate if four of the plies closest to the midplane are replaced by a core of negligible stiffness.
2. What is the percentage decrease in the longitudinal bending modulus of a $[0/90/-45/45]_s$ glass/epoxy laminate if four of the plies closest to the midplane are replaced by a core of negligible stiffness?

Use properties of unidirectional glass/epoxy lamina from Table 2.1.

5.18 A designer uses a $[0]_8$ glass/epoxy laminate to manufacture a rotating blade. The in-plane longitudinal modulus is adequate, but the in-plane shear modulus is not. A suggestion is to replace the $[0]_8$ glass/epoxy laminate by a $[\pm 45]_{2s}$ graphite/epoxy laminate. Use the properties of unidirectional glass/epoxy lamina and unidirectional graphite/epoxy lamina from Table 2.2 to find:

1. Whether the longitudinal modulus increases or decreases and by how much
2. Percentage increase or decrease in the in-plane shear modulus with the replacement

5.19 Design a symmetric graphite/epoxy cross-ply sublaminate such that the thermal expansion coefficient in the *x*-direction is zero. Use the properties of unidirectional graphite/epoxy laminate from Table 2.1; however, assume that the longitudinal coefficient of thermal expansion is -0.3×10^{-6} m/m/°C.

5.20 1. Find the coefficient of thermal expansion of a symmetric quasi-isotropic graphite/epoxy laminate.

2. If you were able to change the longitudinal Young's modulus of the unidirectional graphite/epoxy lamina without affecting the value of other properties, what value would you choose to get zero thermal expansion coefficient for the quasi-isotropic laminate?

Use the properties of unidirectional graphite/epoxy lamina given in Table 2.1, *except* choose the longitudinal coefficient of thermal expansion as -0.3×10^{-6} m/m/°C.

5.21 Certain laminated structures, such as thin walled hollow drive shafts, are designed for maximum shear stiffness. Find the angle, θ, for a symmetric $[\pm\theta]_{ns}$ graphite/epoxy laminate such that the in-plane shear stiffness is a maximum. Use the properties of unidirectional graphite/epoxy lamina from Table 2.2.

5.22 A thin-walled pressure vessel is manufactured by a filament winding method using glass/epoxy prepregs. Find the optimum angles, θ, if the pressure vessel is made of $[\pm\theta]_{ns}$ sublaminate with

1. Spherical construction for maximum strength
2. Cylindrical construction for maximum strength
3. Cylindrical construction for no change in the internal diameter

Apply Tsai–Wu failure theory. Use properties of unidirectional glass/epoxy lamina from Table 2.2.

5.23 A cylindrical pressure vessel with flat ends of length 6 ft and inner diameter of 35 in. is subjected to an internal gauge pressure of 150 psi. Neglect the end effects and the mass of ends of the pressure vessel in your design. Take the factor of safety as 1.95:

1. Design the radial thickness of the pressure vessel using steel. For steel, assume that the Young's modulus is 30 Msi, Poisson's ratio is 0.3, specific gravity of steel is 7.8, and the ultimate normal tensile and compressive strength is 36 ksi.
2. Find the axial elongation of the steel pressure vessel designed in part (1), assuming plane stress conditions.
3. Find whether graphite/epoxy would be a better material to use for minimizing mass if, in addition to resisting the applied pressure, the axial elongation of the pressure vessel does not exceed that of the steel pressure vessel. The vessel operates at room temperature and curing residual stresses are neglected for simplification. The following are other specifications of the design:

   Only 0°, +45°, –45°, +60°, –60°, and 90° plies can be used.

   The thickness of each lamina is 0.005 in.

   Use specific gravities of the laminae from Example 5.6.

   Use Tsai–Wu failure criterion for calculating strength ratios.

5.24 Revisit the design problem of the drive shaft in Example 5.8. Use graphite/epoxy laminate with ply properties given in Table 2.1 to design the drive shaft.

1. If minimizing mass is still an issue, would a graphite/epoxy laminate be a better choice than glass/epoxy?
2. If cost is the only issue, is glass/epoxy laminate, steel, or graphite/epoxy the best choice? Assume total manufacturing cost of graphite/epoxy is five times that of glass/epoxy on a per-unit-mass basis and that the glass/epoxy and steel cost the same on a per-unit-mass basis.

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