3

Micromechanical Analysis of a Lamina

Chapter Objectives

- Develop concepts of volume and weight fraction (mass fraction) of fiber and matrix, density, and void fraction in composites.
- Find the nine mechanical and four hygrothermal constants: four elastic moduli, five strength parameters, two coefficients of thermal expansion, and two coefficients of moisture expansion of a unidirectional lamina from the individual properties of the fiber and the matrix, fiber volume fraction, and fiber packing.
- Discuss the experimental characterization of the nine mechanical and four hygrothermal constants.

3.1 Introduction

In Chapter 2, the stress–strain relationships, engineering constants, and failure theories for an angle lamina were developed using four elastic moduli, five strength parameters, two coefficients of thermal expansion (CTE), and two coefficients of moisture expansion (CME) for a unidirectional lamina. These 13 parameters can be found experimentally by conducting several tension, compression, shear, and hygrothermal tests on unidirectional lamina (laminates). However, unlike in isotropic materials, experimental evaluation of these parameters is quite costly and time consuming because they are functions of several variables: the individual constituents of the composite material, fiber volume fraction, packing geometry, processing, etc. Thus, the need and motivation for developing analytical models to find these parameters are very important. In this chapter, we will develop simple relationships for the these parameters in terms of the stiffnesses, strengths, coefficients of thermal and moisture expansion of the individual constituents of a composite, fiber volume fraction, packing geometry, etc. An understanding of this

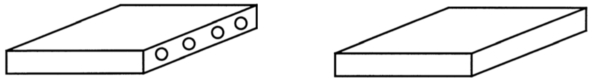


FIGURE 3.1
A nonhomogeneous lamina with fibers and matrix approximated as a homogeneous lamina.

relationship, called micromechanics of lamina, helps the designer to select the constituents of a composite material for use in a laminated structure.

Because this text is for a first course in composite materials, details will be explained only for the simple models based on the mechanics of materials approach and the semi-empirical approach. Results from other methods based on advanced topics such as elasticity are also explained for completeness.

As mentioned in Chapter 2, a unidirectional lamina is not homogeneous. However, one can assume the lamina to be homogeneous by focusing on the average response of the lamina to mechanical and hygrothermal loads (Figure 3.1). The lamina is simply looked at as a material whose properties are different in various directions, but not different from one location to another.

Also, the chapter focuses on a unidirectional continuous fiber-reinforced lamina. This is because it forms the basic building block of a composite structure, which is generally made of several unidirectional laminae placed at various angles. The modeling in the evaluation of the parameters is discussed first. This is followed by examples and experimental methods for finding these parameters.

3.2 Volume and Mass Fractions, Density, and Void Content

Before modeling the 13 parameters of a unidirectional composite, we introduce the concept of relative fraction of fibers by volume. This concept is critical because theoretical formulas for finding the stiffness, strength, and hygrothermal properties of a unidirectional lamina are a function of fiber volume fraction. Measurements of the constituents are generally based on their mass, so fiber mass fractions must also be defined. Moreover, defining the density of a composite also becomes necessary because its value is used in the experimental determination of fiber volume and void fractions of a composite. Also, the value of density is used in the definition of specific modulus and specific strength in Chapter 1.

3.2.1 Volume Fractions

Consider a composite consisting of fiber and matrix. Take the following symbol notations:

$v_{c,f,m}$ = volume of composite, fiber, and matrix, respectively
$\rho_{c,f,m}$ = density of composite, fiber, and matrix, respectively.

Now define the fiber volume fraction V_f and the matrix volume fraction V_m as

$$V_f = \frac{v_f}{v_c},$$

and

$$V_m = \frac{v_m}{v_c}. \tag{3.1a, b}$$

Note that the sum of volume fractions is

$$V_f + V_m = 1,$$

from Equation (3.1) as

$$v_f + v_m = v_c.$$

3.2.2 Mass Fractions

Consider a composite consisting of fiber and matrix and take the following symbol notation: $w_{c,f,m}$ = mass of composite, fiber, and matrix, respectively. The mass fraction (weight fraction) of the fibers (W_f) and the matrix (W_m) are defined as

$$W_f = \frac{w_f}{w_c}, \text{ and}$$

$$W_m = \frac{w_m}{w_c}. \tag{3.2a, b}$$

Note that the sum of mass fractions is

$$W_f + W_m = 1,$$

from Equation (3.2) as

$$w_f + w_m = w_c \ .$$

From the definition of the density of a single material,

$$\begin{aligned} w_c &= r_c v_c, \\ w_f &= r_f v_f, \text{ and} \\ w_m &= r_m v_m. \end{aligned} \tag{3.3a–c}$$

Substituting Equation (3.3) in Equation (3.2), the mass fractions and volume fractions are related as

$$W_f = \frac{\rho_f}{\rho_c} V_f, \text{ and}$$

$$W_m = \frac{\rho_m}{\rho_c} V_m, \tag{3.4a, b}$$

in terms of the fiber and matrix volume fractions. In terms of individual constituent properties, the mass fractions and volume fractions are related by

$$W_f = \frac{\dfrac{\rho_f}{\rho_m}}{\dfrac{\rho_f}{\rho_m} V_f + V_m} V_f,$$

$$W_m = \frac{1}{\dfrac{\rho_f}{\rho_m}(1 - V_m) + V_m} V_m \ . \tag{3.5a, b}$$

One should always state the basis of calculating the fiber content of a composite. It is given in terms of mass or volume. Based on Equation (3.4), it is evident that volume and mass fractions are not equal and that the mismatch between the mass and volume fractions increases as the ratio between the density of fiber and matrix differs from one.

3.2.3 Density

The derivation of the density of the composite in terms of volume fractions is found as follows. The mass of composite w_c is the sum of the mass of the fibers w_f and the mass of the matrix w_m as

$$w_c = w_f + w_m. \tag{3.6}$$

Substituting Equation (3.3) in Equation (3.6) yields

$$\rho_c v_c = \rho_f v_f + \rho_m v_m,$$

and

$$\rho_c = \rho_f \frac{v_f}{v_c} + \rho_m \frac{v_m}{v_c}. \tag{3.7}$$

Using the definitions of fiber and matrix volume fractions from Equation (3.1),

$$\rho_c = \rho_f V_f + \rho_m V_m. \tag{3.8}$$

Now, consider that the volume of a composite v_c is the sum of the volumes of the fiber v_f and matrix (v_m):

$$v_c = v_f + v_m. \tag{3.9}$$

The density of the composite in terms of mass fractions can be found as

$$\frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m}. \tag{3.10}$$

Example 3.1

A glass/epoxy lamina consists of a 70% fiber volume fraction. Use properties of glass and epoxy from Table 3.1* and Table 3.2, respectively, to determine the

* Table 3.1 and Table 3.2 give the typical properties of common fibers and matrices in the SI system of units, respectively. Note that fibers such as graphite and aramids are transversely isotropic, but matrices are generally isotropic. The typical properties of common fibers and matrices are again given in Table 3.3 and Table 3.4, respectively, in the USCS system of units.

TABLE 3.1

Typical Properties of Fibers (SI System of Units)

Property	Units	Graphite	Glass	Aramid
Axial modulus	GPa	230	85	124
Transverse modulus	GPa	22	85	8
Axial Poisson's ratio	—	0.30	0.20	0.36
Transverse Poisson's ratio	—	0.35	0.20	0.37
Axial shear modulus	GPa	22	35.42	3
Axial coefficient of thermal expansion	μm/m/°C	–1.3	5	–5.0
Transverse coefficient of thermal expansion	μm/m/°C	7.0	5	4.1
Axial tensile strength	MPa	2067	1550	1379
Axial compressive strength	MPa	1999	1550	276
Transverse tensile strength	MPa	77	1550	7
Transverse compressive strength	MPa	42	1550	7
Shear strength	MPa	36	35	21
Specific gravity	—	1.8	2.5	1.4

TABLE 3.2

Typical Properties of Matrices (SI System of Units)

Property	Units	Epoxy	Aluminum	Polyamide
Axial modulus	GPa	3.4	71	3.5
Transverse modulus	GPa	3.4	71	3.5
Axial Poisson's ratio	—	0.30	0.30	0.35
Transverse Poisson's ratio	—	0.30	0.30	0.35
Axial shear modulus	GPa	1.308	27	1.3
Coefficient of thermal expansion	μm/m/°C	63	23	90
Coefficient of moisture expansion	m/m/kg/kg	0.33	0.00	0.33
Axial tensile strength	MPa	72	276	54
Axial compressive strength	MPa	102	276	108
Transverse tensile strength	MPa	72	276	54
Transverse compressive strength	MPa	102	276	108
Shear strength	MPa	34	138	54
Specific gravity	—	1.2	2.7	1.2

1. Density of lamina
2. Mass fractions of the glass and epoxy
3. Volume of composite lamina if the mass of the lamina is 4 kg
4. Volume and mass of glass and epoxy in part (3)

Solution

1. From Table 3.1, the density of the fiber is

$$\rho_f = 2500 \ kg / m^3.$$

TABLE 3.3

Typical Properties of Fibers (USCS System of Units)

Property	Units	Graphite	Glass	Aramid
Axial modulus	Msi	33.35	12.33	17.98
Transverse modulus	Msi	3.19	12.33	1.16
Axial Poisson's ratio	—	0.30	0.20	0.36
Transverse Poisson's ratio	—	0.35	0.20	0.37
Axial shear modulus	Msi	3.19	5.136	0.435
Axial coefficient of thermal expansion	µin./in./°F	–0.7222	2.778	–2.778
Transverse coefficient of thermal expansion	µin./in./°F	3.889	2.778	2.278
Axial tensile strength	ksi	299.7	224.8	200.0
Axial compressive strength	ksi	289.8	224.8	40.02
Transverse tensile strength	ksi	11.16	224.8	1.015
Transverse compressive strength	ksi	6.09	224.8	1.015
Shear strength	ksi	5.22	5.08	3.045
Specific gravity	—	1.8	2.5	1.4

TABLE 3.4

Typical Properties of Matrices (USCS System of Units)

Property	Units	Epoxy	Aluminum	Polyamide
Axial modulus	Msi	0.493	10.30	0.5075
Transverse modulus	Msi	0.493	10.30	0.5075
Axial Poisson's ratio	—	0.30	0.30	0.35
Transverse Poisson's ratio	—	0.30	0.30	0.35
Axial shear modulus	Msi	0.1897	3.915	0.1885
Coefficient of thermal expansion	µin./in./°F	35	12.78	50
Coefficient of moisture expansion	in./in./lb/lb	0.33	0.00	0.33
Axial tensile strength	ksi	10.44	40.02	7.83
Axial compressive strength	ksi	14.79	40.02	15.66
Transverse tensile strength	ksi	10.44	40.02	7.83
Transverse compressive strength	ksi	14.79	40.02	15.66
Shear strength	ksi	4.93	20.01	7.83
Specific gravity	—	1.2	2.7	1.2

From Table 3.2, the density of the matrix is

$$\rho_m = 1200 \ kg/m^3.$$

Using Equation (3.8), the density of the composite is

$$\rho_c = (2500)(0.7) + (1200)(0.3)$$
$$= 2110 \ kg/m^3.$$

2. Using Equation (3.4), the fiber and matrix mass fractions are

$$W_f = \frac{2500}{2110} \times 0.3$$
$$= 0.8294$$

$$W_m = \frac{1200}{2110} \times 0.3 \quad .$$
$$= 0.1706$$

Note that the sum of the mass fractions,

$$W_f + W_m = 0.8294 + 0.1706$$
$$= 1.000.$$

3. The volume of composite is

$$v_c = \frac{w_c}{\rho_c}$$

$$= \frac{4}{2110}$$

$$= 1.896 \times 10^{-3} m^3 \ .$$

4. The volume of the fiber is

$$v_f = V_f v_c$$

$$= (0.7)(1.896 \times 10^{-3})$$

$$= 1.327 \times 10^{-3} m^3 \ .$$

The volume of the matrix is

$$v_m = V_m v_c$$

$$=(0.3)(0.1896 \times 10^{-3})$$

$$= 0.5688 \times 10^{-3} \ m^3 \ .$$

The mass of the fiber is

$$w_f = \rho_f v_f$$

$$= (2500)(1.327 \times 10^{-3})$$

$$= 3.318 \ kg \ .$$

The mass of the matrix is

$$w_m = \rho_m v_m$$

$$= (1200)(0.5688 \times 10^{-3})$$

$$= 0.6826 \ kg \ .$$

3.2.4 Void Content

During the manufacture of a composite, voids are introduced in the composite as shown in Figure 3.2. This causes the theoretical density of the composite to be higher than the actual density. Also, the void content of a

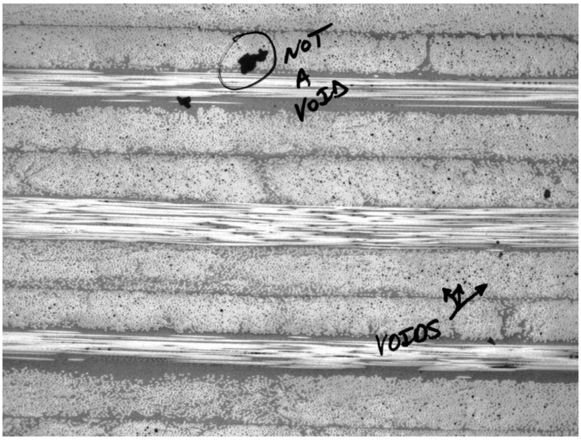


FIGURE 3.2
Photomicrographs of cross-section of a lamina with voids.

composite is detrimental to its mechanical properties. These detriments include lower

- Shear stiffness and strength
- Compressive strengths
- Transverse tensile strengths
- Fatigue resistance
- Moisture resistance

A decrease of 2 to 10% in the preceding matrix-dominated properties generally takes place with every 1% increase in the void content.[1]

For composites with a certain volume of voids V_v the volume fraction of voids V_v is defined as

$$V_v = \frac{v_v}{v_c}. \tag{3.11}$$

Then, the total volume of a composite (v_c) with voids is given by

$$v_c = v_f + v_m + v_v. \tag{3.12}$$

By definition of the experimental density ρ_{ce} of a composite, the actual volume of the composite is

$$v_c = \frac{w_c}{\rho_{ce}}, \tag{3.13}$$

and, by the definition of the theoretical density ρ_{ct} of the composite, the theoretical volume of the composite is

$$v_f + v_m = \frac{w_c}{\rho_{ct}}. \tag{3.14}$$

Then, substituting the preceding expressions (3.13) and (3.14) in Equation (3.12),

$$\frac{w_c}{\rho_{ce}} = \frac{w_c}{\rho_{ct}} + v_v .$$

The volume of void is given by

$$v_v = \frac{w_c}{\rho_{ce}}\left(\frac{\rho_{ct}-\rho_{ce}}{\rho_{ct}}\right). \tag{3.15}$$

Substituting Equation (3.13) and Equation (3.15) in Equation (3.11), the volume fraction of the voids is

$$\begin{aligned} V_v &= \frac{v_v}{v_c} \\ &= \frac{\rho_{ct}-\rho_{ce}}{\rho_{ct}}. \end{aligned} \tag{3.16}$$

Example 3.2

A graphite/epoxy cuboid specimen with voids has dimensions of $a \times b \times c$ and its mass is M_c. After it is put it into a mixture of sulfuric acid and hydrogen peroxide, the remaining graphite fibers have a mass M_f. From independent tests, the densities of graphite and epoxy are ρ_f and ρ_m, respectively. Find the volume fraction of the voids in terms of a, b, c, M_f, M_c, ρ_f, and ρ_m.

Solution

The total volume of the composite v_c is the sum total of the volume of fiber v_f, matrix v_m, and voids v_v:

$$v_c = v_f + v_m + v_v. \tag{3.17}$$

From the definition of density,

$$v_f = \frac{M_f}{\rho_f}, \tag{3.18a}$$

$$v_m = \frac{M_c - M_f}{\rho_m}. \tag{3.18b}$$

The specimen is a cuboid, so the volume of the composite is

$$v_c = abc. \tag{3.19}$$

Substituting Equation (3.18) and Equation (3.19) in Equation (3.17) gives

$$abc = \frac{M_f}{\rho_f} + \frac{M_c - M_f}{\rho_m} + v_v \ ,$$

and the volume fraction of voids then is

$$V_v = \frac{v_v}{abc} = 1 - \frac{1}{abc}\left[\frac{M_f}{\rho_f} + \frac{M_c - M_f}{\rho_m}\right] \tag{3.20}$$

Alternative Solution

The preceding problem can also be solved by using Equation (3.16). The theoretical density of the composite is

$$\rho_{ct} = \rho_f V_f' + \rho_m(1 - V_f') \ , \tag{3.21}$$

where V_f' is the theoretical fiber volume fraction given as

$$V_f' = \frac{volume\ of\ fibers}{volume\ of\ fibers + volume\ of\ matrix}$$

$$V_f' = \frac{\dfrac{M_f}{\rho_f}}{\dfrac{M_f}{\rho_f} + \dfrac{M_c - M_f}{\rho_m}}. \tag{3.22}$$

The experimental density of the composite is

$$\rho_{ce} = \frac{M_c}{abc}. \tag{3.23}$$

Substituting Equation (3.21) through Equation (3.23) in the definition of void volume fractions given by Equation (3.16),

$$V_v = 1 - \frac{1}{abc}\left[\frac{M_f}{\rho_f} + \frac{M_c - M_f}{\rho_m}\right]. \tag{3.24}$$

Experimental determination: the fiber volume fractions of the constituents of a composite are found generally by the burn or the acid digestion tests. These tests involve taking a sample of composite and weighing it. Then the density of the specimen is found by the liquid displacement method in which the sample is weighed in air and then in water. The density of the composite is given by

$$\rho_c = \frac{w_c}{w_c - w_i}\rho_w \,, \tag{3.25}$$

where

w_c = weight of composite
w_i = weight of composite when immersed in water
ρ_w = density of water (1000 kg/m^3 or 62.4 lb/ft^3)

For specimens that float in water, a sinker is attached. The density of the composite is then found by

$$\rho_c = \frac{w_c}{w_c + w_s - w_w}\rho_w \,, \tag{3.26}$$

where

w_c = weight of composite
w_s = weight of sinker when immersed in water
w_w = weight of sinker and specimen when immersed in water

The sample is then dissolved in an acid solution or burned.[2] Glass-based composites are burned, and carbon and aramid-based composites are digested in solutions. Carbon and aramid-based composites cannot be burned because carbon oxidizes in air above 300°C (572°F) and the aramid fiber can decompose at high temperatures. Epoxy-based composites can be digested by nitric acid or a hot mixture of ethylene glycol and potassium hydroxide; polyamide- and phenolic resin-based composites use mixtures of sulfuric acid and hydrogen peroxide. When digestion or burning is complete, the remaining fibers are washed and dried several times and then weighed. The fiber and matrix weight fractions can be found using Equation (3.2). The densities of the fiber and the matrix are known; thus, one can use Equation (3.4) to determine the volume fraction of the constituents of the composite and Equation (3.8) to calculate the theoretical density of the composite.

3.3 Evaluation of the Four Elastic Moduli

As shown in Section 2.4.3, there are four elastic moduli of a unidirectional lamina:

- Longitudinal Young's modulus, E_1
- Transverse Young's modulus, E_2
- Major Poisson's ratio, ν_{12}
- In-plane shear modulus, G_{12}

Three approaches for determining the four elastic moduli are discussed next.

3.3.1 Strength of Materials Approach

From a unidirectional lamina, take a representative volume element* that consists of the fiber surrounded by the matrix (Figure 3.3). This representative volume element (RVE) can be further represented as rectangular blocks. The fiber, matrix, and the composite are assumed to be of the same width, h, but of thicknesses t_f, t_m, and t_c, respectively. The area of the fiber is given by

$$A_f = t_f h \ . \tag{3.27a}$$

The area of the matrix is given by

$$A_m = t_m h, \tag{3.27b}$$

and the area of the composite is given by

$$A_c = t_c h. \tag{3.27c}$$

The two areas are chosen in the proportion of their volume fractions so that the fiber volume fraction is defined as

$$\begin{aligned} V_f &= \frac{A_f}{A_c} \\ &= \frac{t_f}{t_c}, \end{aligned} \tag{3.28a}$$

and the matrix fiber volume fraction V_m is

* A representative volume element (RVE) of a material is the smallest part of the material that represents the material as a whole. It could be otherwise intractable to account for the distribution of the constituents of the material.

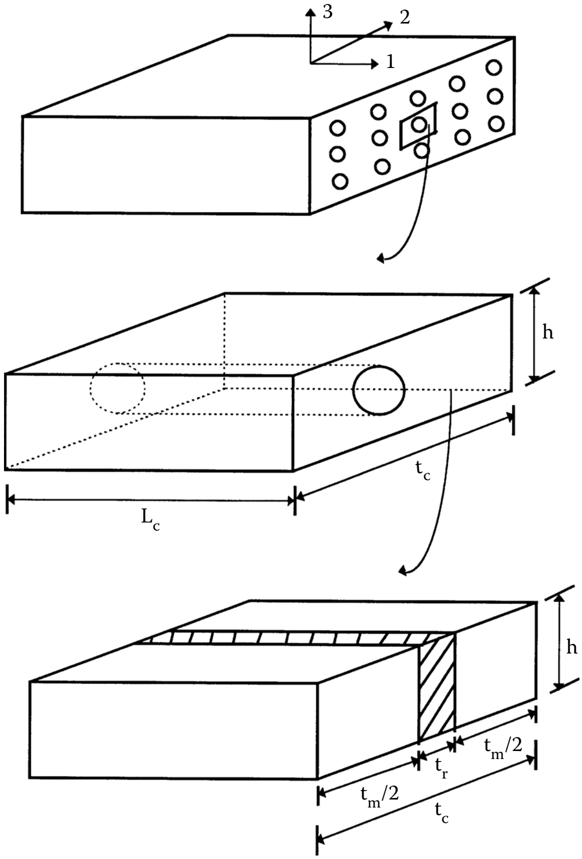


FIGURE 3.3
Representative volume element of a unidirectional lamina.

$$\begin{aligned} V_m &= \frac{A_m}{A_c} \\ &= \frac{t_m}{t_c} \\ &= 1 - V_f. \end{aligned} \tag{3.28b}$$

The following assumptions are made in the strength of materials approach model:

- The bond between fibers and matrix is perfect.
- The elastic moduli, diameters, and space between fibers are uniform.
- The fibers are continuous and parallel.

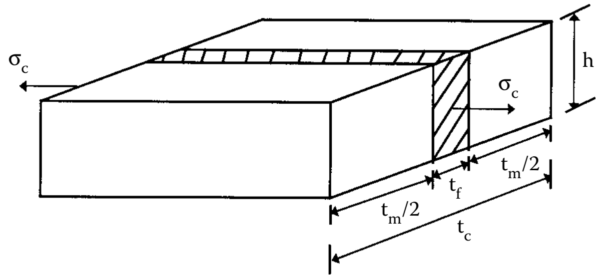


FIGURE 3.4
A longitudinal stress applied to the representative volume element to calculate the longitudinal Young's modulus for a unidirectional lamina.

- The fibers and matrix follow Hooke's law (linearly elastic).
- The fibers possess uniform strength.
- The composite is free of voids.

3.3.1.1 Longitudinal Young's Modulus

From Figure 3.4, under a uniaxial load F_c on the composite RVE, the load is shared by the fiber F_f and the matrix F_m so that

$$F_c = F_f + F_m. \tag{3.29}$$

The loads taken by the fiber, the matrix, and the composite can be written in terms of the stresses in these components and cross-sectional areas of these components as

$$F_c = \sigma_c A_c, \tag{3.30a}$$

$$F_f = \sigma_f A_f, \tag{3.30b}$$

$$F_m = \sigma_m A_m, \tag{3.30c}$$

where

$\sigma_{c,f,m}$ = stress in composite, fiber, and matrix, respectively
$A_{c,f,m}$ = area of composite, fiber, and matrix, respectively

Assuming that the fibers, matrix, and composite follow Hooke's law and that the fibers and the matrix are isotropic, the stress–strain relationship for each component and the composite is

$$\sigma_c = E_1 \varepsilon_c, \tag{3.31a}$$

$$\sigma_f = E_f \varepsilon_f, \tag{3.31b}$$

and

$$\sigma_m = E_m \varepsilon_m, \tag{3.31c}$$

where

$\varepsilon_{c,f,m}$ = strains in composite, fiber, and matrix, respectively
$E_{1,f,m}$ = elastic moduli of composite, fiber, and matrix, respectively

Substituting Equation (3.30) and Equation (3.31) in Equation (3.29) yields

$$E_1 \varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m. \tag{3.32}$$

The strains in the composite, fiber, and matrix are equal ($\varepsilon_c = \varepsilon_f = \varepsilon_m$); then, from Equation (3.32),

$$E_1 = E_f \frac{A_f}{A_c} + E_m \frac{A_m}{A_c}. \tag{3.33}$$

Using Equation (3.28), for definitions of volume fractions,

$$E_1 = E_f V_f + E_m V_m. \tag{3.34}$$

Equation 3.34 gives the longitudinal Young's modulus as a weighted mean of the fiber and matrix modulus. It is also called the rule of mixtures.

The ratio of the load taken by the fibers F_f to the load taken by the composite F_c is a measure of the load shared by the fibers. From Equation (3.30) and Equation (3.31),

$$\frac{F_f}{F_c} = \frac{E_f}{E_1} V_f. \tag{3.35}$$

In Figure 3.5, the ratio of the load carried by the fibers to the load taken by the composite is plotted as a function of fiber-to-matrix Young's moduli ratio E_f/E_m for the constant fiber volume fraction V_f. It shows that as the fiber to matrix moduli ratio increases, the load taken by the fiber increases tremendously.

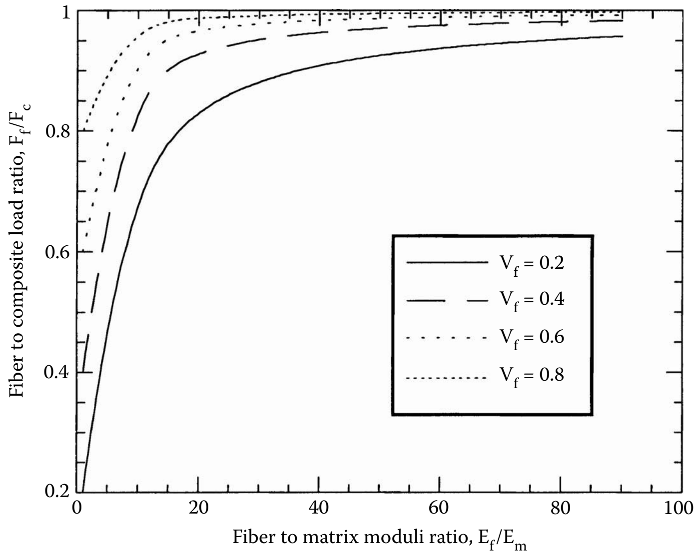


FIGURE 3.5
Fraction of load of composite carried by fibers as a function of fiber volume fraction for constant fiber to matrix moduli ratio.

Example 3.3

Find the longitudinal elastic modulus of a unidirectional glass/epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Also, find the ratio of the load taken by the fibers to that of the composite.

Solution

From Table 3.1, the Young's modulus of the fiber is

$$E_f = 85 \text{ GPa}.$$

From Table 3.2, the Young's modulus of the matrix is

$$E_m = 3.4 \text{ GPa}.$$

Using Equation (3.34), the longitudinal elastic modulus of the unidirectional lamina is

$$\begin{aligned} E_1 &= (85)(0.7) + (3.4)(0.3) \\ &= 60.52 \ GPa. \end{aligned}$$

Using Equation (3.35), the ratio of the load taken by the fibers to that of the composite is

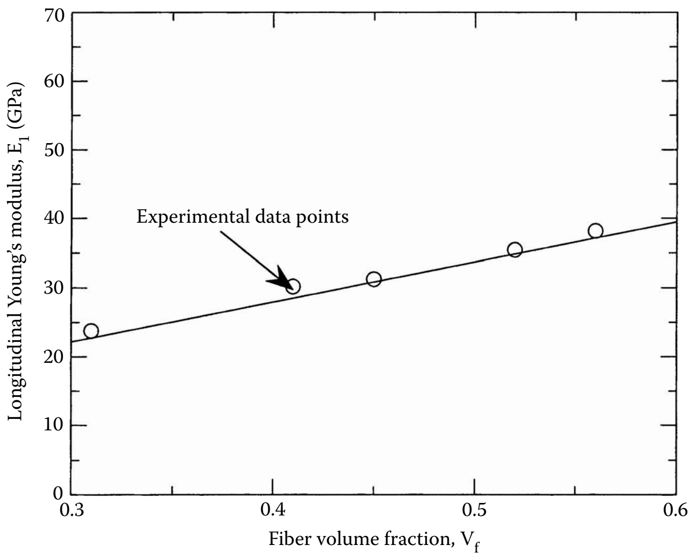


FIGURE 3.6
Longitudinal Young's modulus as function of fiber volume fraction and comparison with experimental data points for a typical glass/polyester lamina. (Experimental data points reproduced with permission of ASM International.)

$$\frac{F_f}{F_c} = \frac{85}{60.52}(0.7)$$

$$= 0.9831.$$

Figure 3.6 shows the linear relationship between the longitudinal Young's modulus of a unidirectional lamina and fiber volume fraction for a typical graphite/epoxy composite per Equation (3.34). It also shows that Equation (3.34) predicts results that are close to the experimental data points.[3]

3.3.1.2 Transverse Young's Modulus

Assume now that, as shown in Figure 3.7, the composite is stressed in the transverse direction. The fibers and matrix are again represented by rectangular blocks as shown. The fiber, the matrix, and composite stresses are equal. Thus,

$$\sigma_c = \sigma_f = \sigma_m, \tag{3.36}$$

where $\sigma_{c,f,m}$ = stress in composite, fiber, and matrix, respectively.

Now, the transverse extension in the composite Δ_c is the sum of the transverse extension in the fiber Δ_f, and that is the matrix, Δ_m.

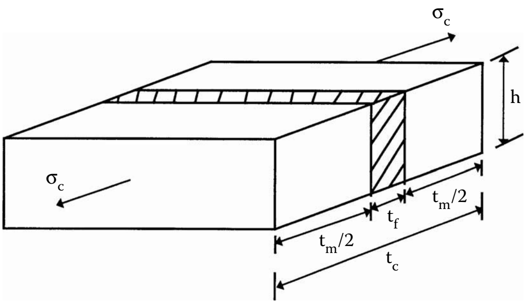


FIGURE 3.7
A transverse stress applied to a representative volume element used to calculate transverse Young's modulus of a unidirectional lamina.

$$\Delta_c = \Delta_f + \Delta_m. \tag{3.37}$$

Now, by the definition of normal strain,

$$\Delta_c = t_c \varepsilon_c, \tag{3.38a}$$

$$\Delta_f = t_f \varepsilon_f, \tag{3.38b}$$

and

$$\Delta_m = t_m \varepsilon_m, \tag{3.38c}$$

where

$t_{c,f,m}$ = thickness of the composite, fiber and matrix, respectively
$\varepsilon_{c,f,m}$ = normal transverse strain in the composite, fiber, and matrix, respectively

Also, by using Hooke's law for the fiber, matrix, and composite, the normal strains in the composite, fiber, and matrix are

$$\varepsilon_c = \frac{\sigma_c}{E_2}, \tag{3.39a}$$

$$\varepsilon_f = \frac{\sigma_f}{E_f}, \tag{3.39b}$$

and

$$\varepsilon_m = \frac{\sigma_m}{E_m}. \tag{3.39c}$$

Substituting Equation (3.38) and Equation (3.39) in Equation (3.37) and using Equation (3.36) gives

$$\frac{1}{E_2} = \frac{1}{E_f}\frac{t_f}{t_c} + \frac{1}{E_m}\frac{t_m}{t_c}. \tag{3.40}$$

Because the thickness fractions are the same as the volume fractions as the other two dimensions are equal for the fiber and the matrix (see Equation 3.28):

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m}. \tag{3.41}$$

Equation (3.41) is based on the weighted mean of the compliance of the fiber and the matrix.

Example 3.4

Find the transverse Young's modulus of a glass/epoxy lamina with a fiber volume fraction of 70%. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

Solution

From Table 3.1, the Young's modulus of the fiber is

$$E_f = 85 \text{ GPa}.$$

From Table 3.2, the Young's modulus of the matrix is

$$E_m = 3.4 \text{ GPa}.$$

Using Equation (3.41), the transverse Young's modulus, E_2, is

$$\frac{1}{E_2} = \frac{0.7}{85} + \frac{0.3}{3.4},$$

$$E_2 = 10.37 \ GPa.$$

Figure 3.8 plots the transverse Young's modulus as a function of fiber volume fraction for constant fiber-to-matrix elastic moduli ratio, E_f/E_m. For metal and ceramic matrix composites, the fiber and matrix elastic moduli are of the same order. (For example, for a SiC/aluminum metal matrix composite, $E_f/E_m = 4$ and for a SiC/CAS ceramic matrix composite, $E_f/E_m = 2$). The transverse Young's modulus of the composite in such cases changes more smoothly as a function of the fiber volume fraction.

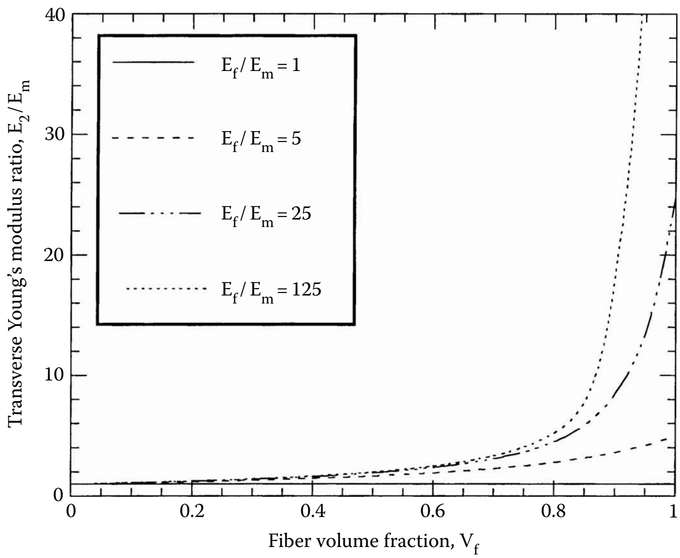


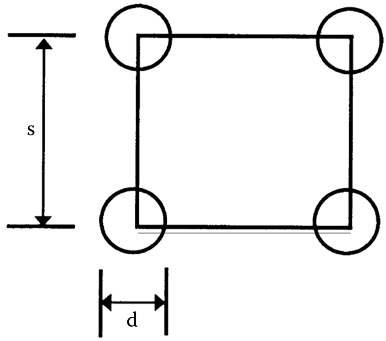
FIGURE 3.8
Transverse Young's modulus as a function of fiber volume fraction for constant fiber to matrix moduli ratio.

For polymeric composites, the fiber-to-matrix moduli ratio is very high. (For example, for a glass/epoxy polymer matrix composite, $E_f/E_m = 25$). The transverse Young's modulus of the composite in such cases changes appreciably only for large fiber volume fractions. Figure 3.8 shows that, for high E_f/E_m ratios, the contribution of the fiber modulus only increases substantially for a fiber volume fraction greater than 80%. These fiber volume fractions are not practical and in many cases are physically impossible due to the geometry of fiber packing. Figure 3.9 shows various possibilities of fiber packing. Note that the ratio of the diameter, d, to fiber spacing, s, d/s varies with geometrical packing. For circular fibers with square array packing (Figure 3.9a),

$$\frac{d}{s} = \left(\frac{4V_f}{\pi}\right)^{1/2}. \tag{3.42a}$$

This gives a maximum fiber volume fraction of 78.54% as $s \geq d$. For circular fibers with hexagonal array packing (Figure 3.9b),

$$\frac{d}{s} = \left(\frac{2\sqrt{3}V_f}{\pi}\right)^{1/2}. \tag{3.42b}$$



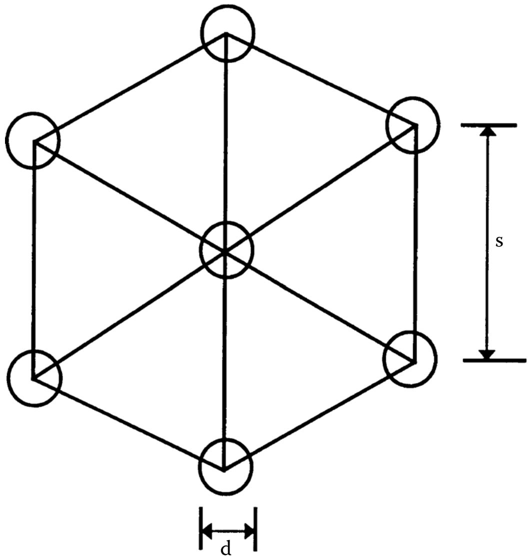


FIGURE 3.9
Fiber to fiber spacing in (a) square packing geometry and (b) hexagonal packing geometry.

This gives a maximum fiber volume fraction of 90.69% because $s \geq d$. These maximum fiber volume fractions are not practical to use because the fibers touch each other and thus have surfaces where the matrix cannot wet out the fibers.

In Figure 3.10, the transverse Young's modulus is plotted as a function of fiber volume fraction using Equation (3.41) for a typical boron/epoxy lamina. Also given are the experimental data points.[4] In Figure 3.10, the experimental and analytical results are not as close to each other as they are for the longitudinal Young's modulus in Figure 3.6.

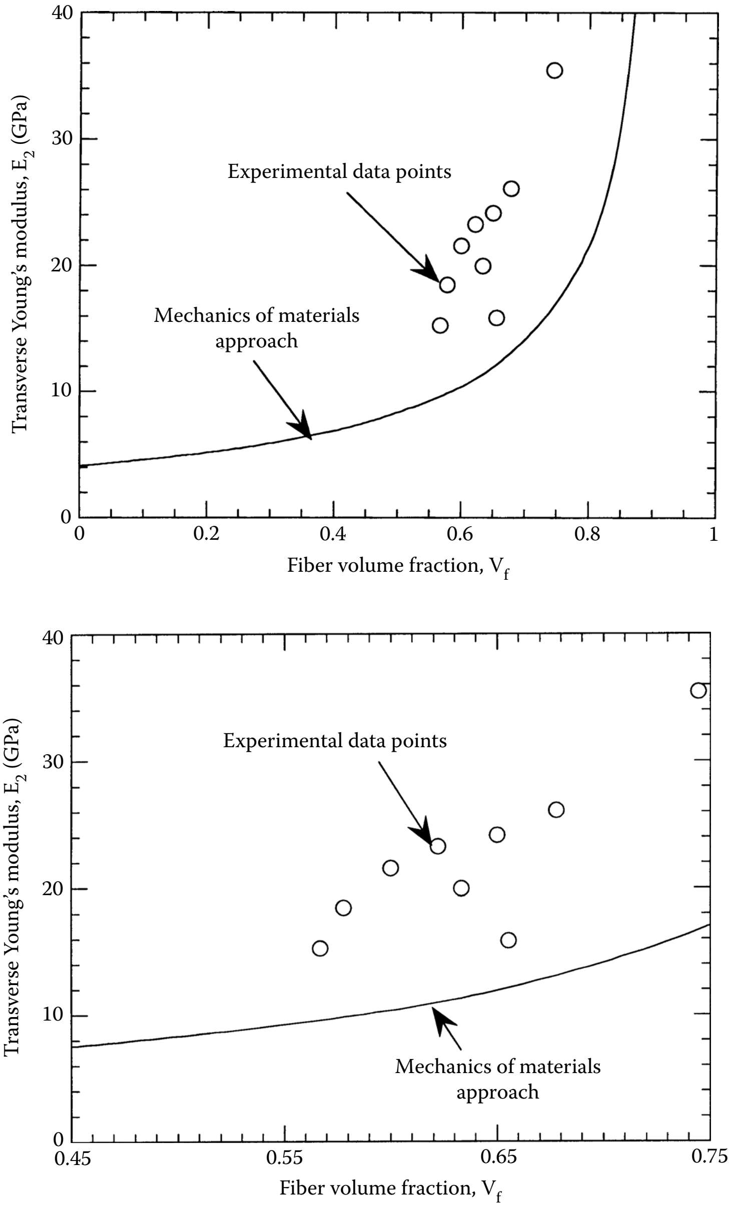


FIGURE 3.10
Theoretical values of transverse Young's modulus as a function of fiber volume fraction for a Boron/Epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35) and comparison with experimental values. Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

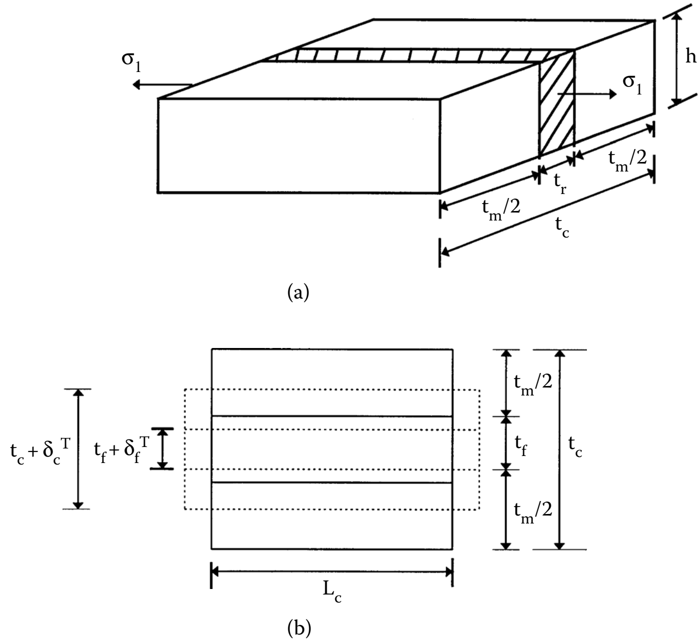


FIGURE 3.11
A longitudinal stress applied to a representative volume element to calculate Poisson's ratio of unidirectional lamina.

3.3.1.3 Major Poisson's Ratio

The major Poisson's ratio is defined as the negative of the ratio of the normal strain in the transverse direction to the normal strain in the longitudinal direction, when a normal load is applied in the longitudinal direction. Assume a composite is loaded in the direction parallel to the fibers, as shown in Figure 3.11. The fibers and matrix are again represented by rectangular blocks. The deformations in the transverse direction of the composite (δ_c^T) is the sum of the transverse deformations of the fiber (δ_f^T) and the matrix (δ_m^T) as

$$\delta_c^T = \delta_f^T + \delta_m^T. \tag{3.43}$$

Using the definition of normal strains,

$$\varepsilon_f^T = \frac{\delta_f^T}{t_f}, \tag{3.44a}$$

$$\varepsilon_m^T = \frac{\delta_m^T}{t_m}, \tag{3.44b}$$

and

$$\varepsilon_c^T = \frac{\delta_c^T}{t_c}, \tag{3.44c}$$

where $\varepsilon_{c,f,m}$ = transverse strains in composite, fiber, and matrix, respectively.
Substituting Equation (3.44) in Equation (3.43),

$$t_c\varepsilon_c^T = t_f\varepsilon_f^T + t_m\varepsilon_m^T. \tag{3.45}$$

The Poisson's ratios for the fiber, matrix, and composite, respectively, are

$$\nu_f = -\frac{\varepsilon_f^T}{\varepsilon_f^L}, \tag{3.46a}$$

$$\nu_m = -\frac{\varepsilon_m^T}{\varepsilon_m^L}, \tag{3.46b}$$

and

$$\nu_{12} = -\frac{\varepsilon_c^T}{\varepsilon_c^L}. \tag{3.46c}$$

Substituting in Equation (3.45),

$$-t_c\nu_{12}\varepsilon_c^L = -t_f\nu_f\varepsilon_f^L - t_m\nu_m\varepsilon_m^L, \tag{3.47}$$

where
$\nu_{12,f,m}v_{12,f,m}$ = Poisson's ratio of composite, fiber, and matrix, respectively
$\varepsilon_{c,f,m}^L$ = longitudinal strains of composite, fiber and matrix, respectively

However, the strains in the composite, fiber, and matrix are assumed to be the equal in the longitudinal direction ($\varepsilon_c^L = \varepsilon_f^L = \varepsilon_m^L$), which, from Equation (3.47), gives

$$t_c\nu_{12} = t_f\nu_f + t_m\nu_m,$$

$$\nu_{12} = \nu_f\frac{t_f}{t_c} + \nu_m\frac{t_m}{t_c}. \tag{3.48}$$

Because the thickness fractions are the same as the volume fractions, per Equation (3.28),

$$\nu_{12} = \nu_f V_f + \nu_m V_m. \tag{3.49}$$

Example 3.5

Find the major and minor Poisson's ratio of a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

Solution

From Table 3.1, the Poisson's ratio of the fiber is

$$\nu_f = 0.2.$$

From Table 3.2, the Poisson's ratio of the matrix is

$$\nu_m = 0.3.$$

Using Equation (3.49), the major Poisson's ratio is

$$\nu_{12} = (0.2)(0.7) + (0.3)(0.3)$$

$$= 0.230.$$

From Example 3.3, the longitudinal Young's modulus is

$$E_1 = 60.52 \text{ GPa}$$

and, from Example 3.4, the transverse Young's modulus is

$$E_2 = 10.37 \text{ GPa}.$$

Then, the minor Poisson's ratio from Equation (2.83) is

$$\nu_{21} = \nu_{12}\frac{E_2}{E_1}$$

$$= 0.230\left(\frac{10.37}{60.52}\right)$$

$$= 0.03941.$$

3.3.1.4 In-Plane Shear Modulus

Apply a pure shear stress τ_c to a lamina as shown in Figure 3.12. The fibers and matrix are represented by rectangular blocks as shown. The resulting

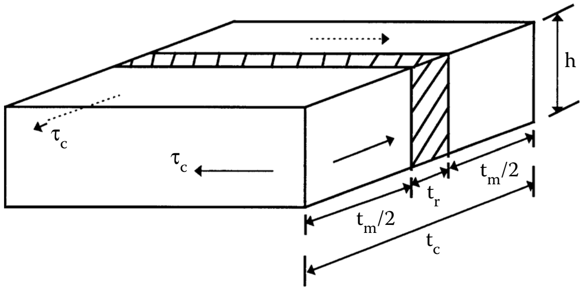


FIGURE 3.12
An in-plane shear stress applied to a representative volume element for finding in-plane shear modulus of a unidirectional lamina.

shear deformations of the composite δ_c the fiber δ_f, and the matrix δ_m are related by

$$\delta_c = \delta_f + \delta_m \; . \tag{3.50}$$

From the definition of shear strains,

$$\delta_c = \gamma_c t_c \; , \tag{3.51a}$$

$$\delta_f = \gamma_f t_f \; , \tag{3.51b}$$

and

$$\delta_m = \gamma_m t_m \; , \tag{3.51c}$$

where

$\gamma_{c,f,m}$ = shearing strains in the composite, fiber, and matrix, respectively
$t_{c,f,m}$ = thickness of the composite, fiber, and matrix, respectively.

From Hooke's law for the fiber, the matrix, and the composite,

$$\gamma_c = \frac{\tau_c}{G_{12}}, \tag{3.52a}$$

$$\gamma_f = \frac{\tau_f}{G_f}, \tag{3.52b}$$

and

$$\gamma_m = \frac{\tau_m}{G_m}, \tag{3.52c}$$

where $G_{12,f,m}$ = shear moduli of composite, fiber, and matrix, respectively.
From Equation (3.50) through Equation (3.52),

$$\frac{\tau_c}{G_{12}} t_c = \frac{\tau_f}{G_f} t_f + \frac{\tau_m}{G_m} t_m. \tag{3.53}$$

The shear stresses in the fiber, matrix, and composite are assumed to be equal ($\tau_c = \tau_f = \tau_m$), giving

$$\frac{1}{G_{12}} = \frac{1}{G_f}\frac{t_f}{t_c} + \frac{1}{G_m}\frac{t_m}{t_c}. \tag{3.54}$$

Because the thickness fractions are equal to the volume fractions, per Equation (3.28),

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}. \tag{3.55}$$

Example 3.6

Find the in-plane shear modulus of a glass/epoxy lamina with a 70% fiber volume fraction. Use properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

Solution

The glass fibers and the epoxy matrix have isotropic properties. From Table 3.1, the Young's modulus of the fiber is

$$E_f = 85 \text{ GPa}$$

and the Poisson's ratio of the fiber is

$$\nu_f = 0.2.$$

The shear modulus of the fiber

$$G_f = \frac{E_f}{2(1+\nu_f)}$$

$$= \frac{85}{2(1+0.2)}$$

$$= 35.42 \; GPa.$$

From Table 3.2, the Young's modulus of the matrix is

$$E_m = 3.4 \text{ GPa}$$

and the Poisson's ratio of the fiber is

$$\nu_m = 0.3.$$

The shear modulus of the matrix is

$$G_m = \frac{E_m}{2(1+\nu_m)}$$

$$= \frac{3.40}{2(1+0.3)}$$

$$= 1.308 \; GPa.$$

From Equation (3.55), the in-plane shear modulus of the unidirectional lamina is

$$\frac{1}{G_{12}} = \frac{0.70}{35.42} + \frac{0.30}{1.308}$$

$$G_{12} = 4.014 \; GPa.$$

Figure 3.13a and Figure 3.13b show the analytical values from Equation (3.55) of the in-plane shear modulus as a function of fiber volume fraction for a typical glass/epoxy lamina. Experimental values[4] are also plotted in the same figure.

3.3.2 Semi-Empirical Models

The values obtained for transverse Young's modulus and in-plane shear modulus through Equation (3.41) and Equation (3.55), respectively, do not agree well with the experimental results shown in Figure 3.10 and Figure 3.13. This establishes a need for better modeling techniques. These techniques include numerical methods, such as finite element and finite difference, and boundary element methods, elasticity solution, and variational principal models.[5] Unfortunately, these models are available only as complicated equations or in graphical form. Due to these difficulties, semi-empirical models have been developed for design purposes. The most useful of these models include those of Halphin and Tsai[6] because they can be used over a wide range of elastic properties and fiber volume fractions.

Halphin and Tsai[6] developed their models as simple equations by curve fitting to results that are based on elasticity. The equations are semi-empirical in nature because involved parameters in the curve fitting carry physical meaning.

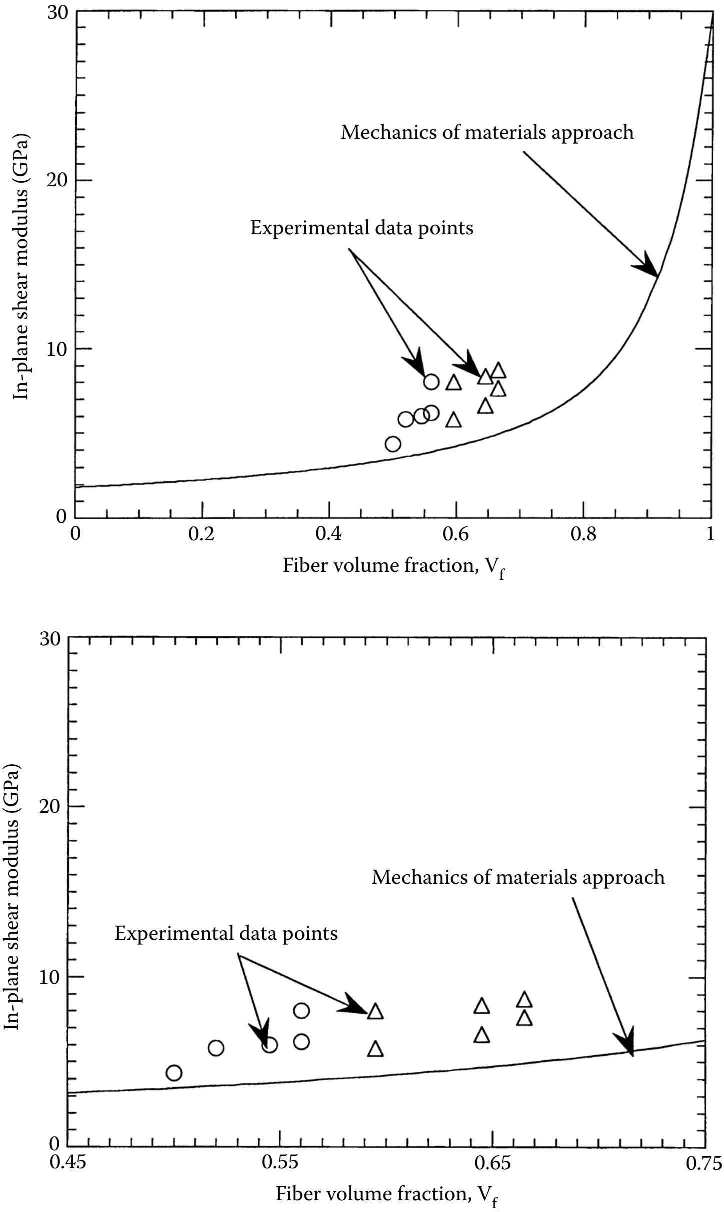


FIGURE 3.13
Theoretical values of in-plane shear modulus as a function of fiber volume fraction and comparison with experimental values for a unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

3.3.2.1 Longitudinal Young's Modulus

The Halphin–Tsai equation for the longitudinal Young's modulus, E_1, is the same as that obtained through the strength of materials approach — that is,

$$E_1 = E_f V_f + E_m V_m. \tag{3.56}$$

3.3.2.2 Transverse Young's Modulus

The transverse Young's modulus, E_2, is given by[6]

$$\frac{E_2}{E_m} = \frac{1 + \xi\eta V_f}{1 - \eta V_f}, \tag{3.57}$$

where

$$\eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi}. \tag{3.58}$$

The term ξ is called the reinforcing factor and depends on the following:

- Fiber geometry
- Packing geometry
- Loading conditions

Halphin and Tsai[6] obtained the value of the reinforcing factor ξ by comparing Equation (3.57) and Equation (3.58) to the solutions obtained from the elasticity solutions. For example, for a fiber geometry of circular fibers in a packing geometry of a square array, $\xi = 2$. For a rectangular fiber cross-section of length a and width b in a hexagonal array, $\xi = 2(a/b)$, where b is in the direction of loading.[6] The concept of direction of loading is illustrated in Figure 3.14.

Example 3.7

Find the transverse Young's modulus for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use Halphin–Tsai equations for a circular fiber in a square array packing geometry.

Solution

Because the fibers are circular and packed in a square array, the reinforcing factor $\xi = 2$. From Table 3.1, the Young's modulus of the fiber is $E_f = 85$ GPa.

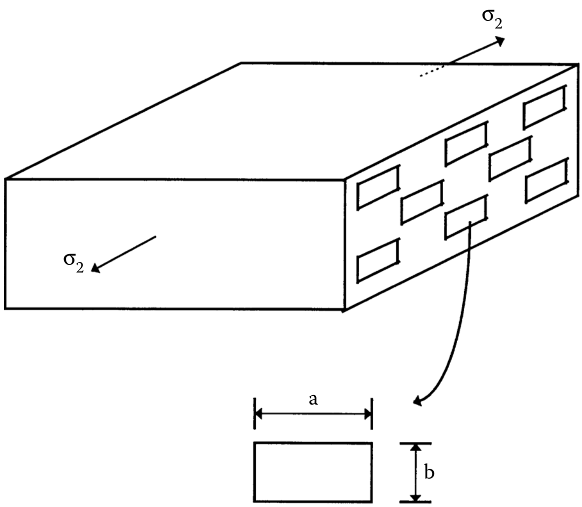


FIGURE 3.14
Concept of direction of loading for calculation of transverse Young's modulus by Halphin–Tsai equations.

From Table 3.2, the Young's modulus of the matrix is E_m = 3.4 GPa. From Equation (3.58),

$$\eta = \frac{(85/3.4) - 1}{(85/3.4) + 2}$$

$$= 0.8889.$$

From Equation (3.57), the transverse Young's modulus of the unidirectional lamina is

$$\frac{E_2}{3.4} = \frac{1 + 2(0.8889)(0.7)}{1 - (0.8889)(0.7)}$$

$$E_2 = 20.20\ GPa.$$

For the same problem, from Example 3.4, this value of E_2 was found to be 10.37 GPa by the mechanics of materials approach.

Figure 3.15a and Figure 3.15b show the transverse Young's modulus as a function of fiber volume fraction for a typical boron/epoxy composite. The Halphin–Tsai equations (3.57) and the mechanics of materials approach Equation (3.41) curves are shown and compared to experimental data points.

As mentioned previously, the parameters ξ and η have a physical meaning. For example,

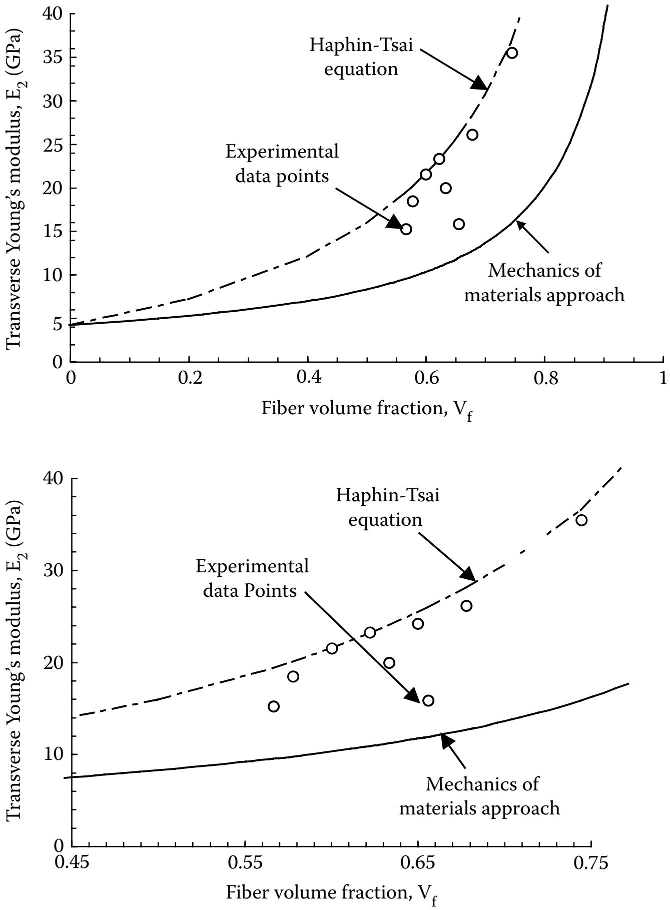


FIGURE 3.15
Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract no. NAS1-8818, November 1970.)

$E_f/E_m = 1$ implies $\eta = 0$, (homogeneous medium)

$E_f/E_m \to \infty$ implies $\eta = 1$ (rigid inclusions)

$$E_f/E_m \to 0 \text{ implies } \eta = -\frac{1}{\xi} \text{ (voids)}$$

3.3.2.3 Major Poisson's Ratio

The Halphin–Tsai equation for the major Poisson's ratio, ν_{12}, is the same as that obtained using the strength of materials approach — that is,

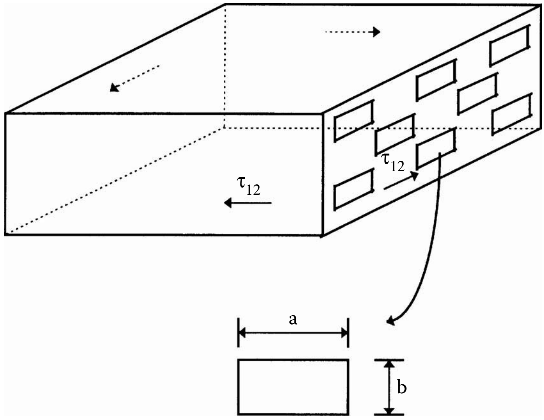


FIGURE 3.16
Concept of direction of loading to calculate in-plane shear modulus by Halphin–Tsai equations.

$$\nu_{12} = \nu_f V_f + \nu_m V_m. \tag{3.59}$$

3.3.2.4 In-Plane Shear Modulus

The Halphin–Tsai[6] equation for the in-plane shear modulus, G_{12}, is

$$\frac{G_{12}}{G_m} = \frac{1+\xi\eta V_f}{1-\eta V_f}, \tag{3.60}$$

where

$$\eta = \frac{(G_f / G_m) - 1}{(G_f / G_m) + \xi}. \tag{3.61}$$

The value of the reinforcing factor, ξ, depends on fiber geometry, packing geometry, and loading conditions. For example, for circular fibers in a square array, $\xi = 1$. For a rectangular fiber cross-sectional area of length a and width b in a hexagonal array, $\xi = \sqrt{3}\log_e(a/b)$, where a is the direction of loading. The concept of the direction of loading[7] is given in Figure 3.16.

The value of $\xi = 1$ for circular fibers in a square array gives reasonable results only for fiber volume fractions of up to 0.5. For example, for a typical glass/epoxy lamina with a fiber volume fraction of 0.75, the value of in-plane shear modulus using the Halphin–Tsai equation with $\xi = 1$ is 30% lower than that given by elasticity solutions. Hewitt and Malherbe[8] suggested choosing a function,

$$\xi = 1 + 40V_f^{10}. \tag{3.62}$$

Example 3.8

Using Halphin–Tsai equations, find the shear modulus of a glass/epoxy composite with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that the fibers are circular and are packed in a square array. Also, get the value of the shear modulus by using Hewitt and Malherbe's[8] formula for the reinforcing factor.

Solution

For Halphin–Tsai's equations with circular fibers in a square array, the reinforcing factor $\xi = 1$. From Example 3.6, the shear modulus of the fiber is

$$G_f = 35.42 \text{ GPa}$$

and the shear modulus of the matrix is

$$G_m = 1.308 \text{ GPa}.$$

From Equation (3.61),

$$\eta = \frac{(35.42/1.308)-1}{(35.42/1.308)+1}$$

$$= 0.9288.$$

From Equation (3.60), the in-plane shear modulus is

$$\frac{G_{12}}{1.308} = \frac{1+(1)(0.9288)(0.7)}{1-(0.9288)(0.7)}$$

$$G_{12} = 6.169 \; GPa.$$

For the same problem, the value of G_{12} = 4.013 GPa was found by the mechanics of materials approach in Example 3.5.

Because the volume fraction is greater than 50%, Hewitt and Mahelbre[8] suggested a reinforcing factor (Equation 3.62):

$$\xi = 1 + 40V_f^{10}$$

$$= 1 + 40(0.7)^{10} \; .$$

$$= 2.130$$

Then, from Equation (3.61),

$$\eta = \frac{(35.42/1.308)-1}{(35.42/1.308)+2.130}.$$

$$= 0.8928$$

From Equation (3.60), the in-plane shear modulus is

$$\frac{G_{12}}{1.308} = \frac{1+(2.130)(0.8928)(0.7)}{1-(0.8928)(0.7)}.$$

$$G = 8.130\ GPa$$

Figure 3.17a and Figure 3.17b show the in-plane shear modulus as a function of fiber volume fraction for a typical glass/epoxy composite. The Halphin–Tsai equation (3.60) and the mechanics of materials approach, Equation (3.55) are shown and compared to the experimental[4] data points.

3.3.3 Elasticity Approach

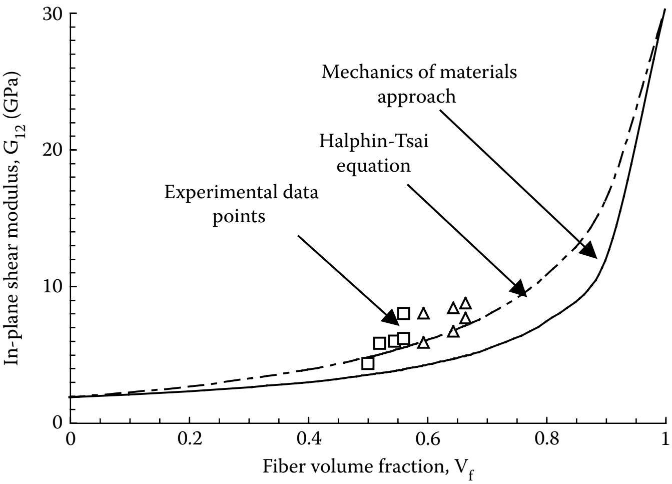
In addition to the strength of materials and semi-empirical equation approaches, expressions for the elastic moduli based on elasticity are also available. Elasticity accounts for equilibrium of forces, compatibility, and Hooke's law relationships in three dimensions; the strength of materials approach may not satisfy compatibility and/or account for Hooke's law in three dimensions, and semi-empirical approaches are just as the name implies — partly empirical.

The elasticity models described here are called composite cylinder assemblage (CCA) models.[4,9–12] In a CCA model, one assumes the fibers are circular in cross-section, spread in a periodic arrangement, and continuous, as shown in Figure 3.18. Then the composite can be considered to be made of repeating elements called the representative volume elements (RVEs). The RVE is considered to represent the composite and respond the same as the whole composite does.

The RVE consists of a composite cylinder made of a single inner solid cylinder (fiber) bonded to an outer hollow cylinder (matrix) as shown in Figure 3.19. The radius of the fiber, a, and the outer radius of the matrix, b, are related to the fiber volume fraction, V_f, as

$$V_f = \frac{a^2}{b^2}. \tag{3.63}$$

Appropriate boundary conditions are applied to this composite cylinder based on the elastic moduli being evaluated.



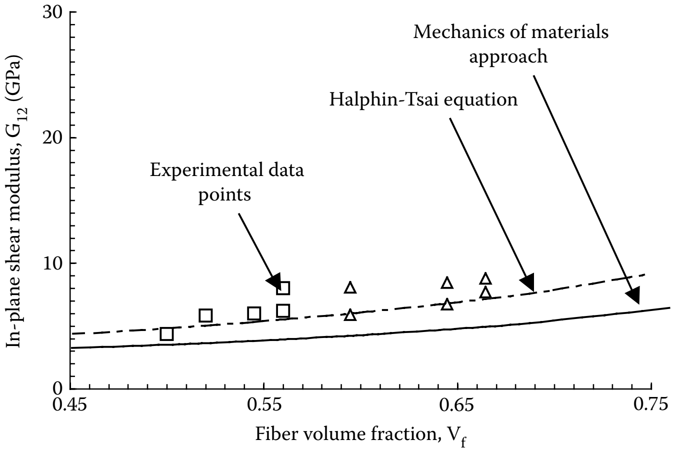


FIGURE 3.17
Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

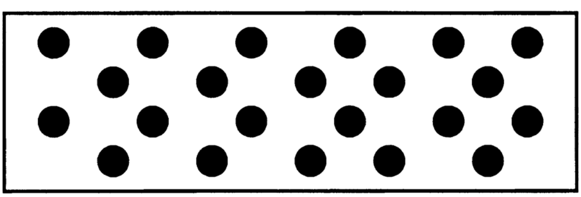


FIGURE 3.18
Periodic arrangement of fibers in a cross-section of a lamina.

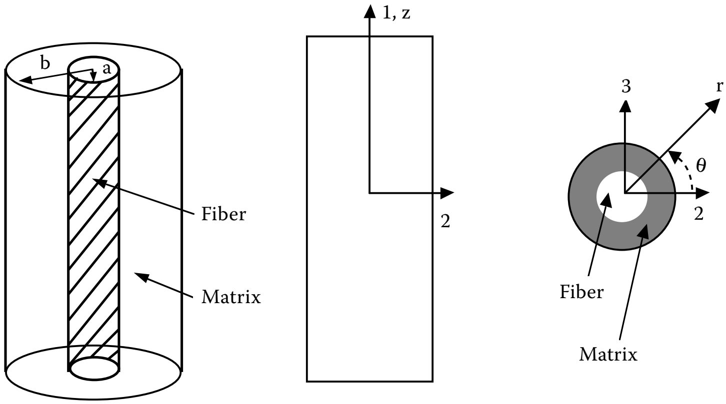


FIGURE 3.19
Composite cylinder assemblage (CCA) model used for predicting elastic moduli of unidirectional composites.

3.3.3.1 Longitudinal Young's Modulus

To find the elastic moduli along the fibers, we will apply an axial load, *P*, in direction 1 (Figure 3.19). The axial stress, σ_1, in direction 1 then is

$$\sigma_1 = \frac{P}{\pi b^2} . \tag{3.64}$$

Now, in terms of Hooke's law,

$$\sigma_1 = E_1 \in_1 \tag{3.65}$$

where

E_1 = longitudinal Young's modulus
$\in_1$ = axial strain in direction 1

Thus, from Equation (3.64) and Equation (3.65), we have

$$E_1 \in_1 = \frac{P}{\pi b^2}$$

$$E_1 = \frac{P}{\pi b^2 \in_1} . \tag{3.66}$$

To find E_1 in terms of elastic moduli of the fiber and the matrix, and the geometrical parameters such as fiber volume fraction, we need to relate the axial load, P, and the axial strain, $\in_1$, in these terms.

Assuming the response of a cylinder is axisymmetric, the equilibrium equation in the radial direction is given by[13]

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \ , \tag{3.67}$$

where

σ_r = radial stress,
σ_θ = hoop stress.

The normal stress–normal strain relationships in polar coordinates, r–θ–z, for an isotropic material with Young's modulus, E, and Poisson's ratio, ν, are given by

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \end{bmatrix} \begin{bmatrix} \in_r \\ \in_\theta \\ \in_z \end{bmatrix} . \tag{3.68}$$

The shear stresses and shear strains are zero in the r–θ–z coordinate system for axisymmetric response.

The strain displacement equations for axisymmetric response are

$$\in_r = \frac{du}{dr} \tag{3.69a}$$

$$\in_\theta = \frac{u}{r} \tag{3.69b}$$

$$\in_z = \frac{dw}{dz} \ , \tag{3.69c}$$

where

u = displacement in radial direction,
w = displacement in axial direction.

Substituting the strain-displacement equations (3.69a-c) in the stress–strain equations (3.68) and noting that $\in_z = \in_1$ everywhere gives

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \end{bmatrix} \begin{bmatrix} \frac{du}{dr} \\ \frac{u}{r} \\ \in_1 \end{bmatrix}, \quad (3.70)$$

which is rewritten for simplicity as

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} \\ C_{12} & C_{11} & C_{12} \\ C_{12} & C_{12} & C_{11} \end{bmatrix} \begin{bmatrix} \frac{du}{dr} \\ \frac{u}{r} \\ \in_1 \end{bmatrix}, \quad (3.71)$$

where the constants of the stiffness matrix are

$$C_{11} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \quad (3.72a)$$

$$C_{12} = \frac{\nu E}{(1-2\nu)(1+\nu)} \,. \quad (3.72b)$$

Substituting Equation (3.71) in the equilibrium equation (3.67) gives

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \,. \quad (3.73)$$

The solution to the linear ordinary differential equation is found by assuming that

$$u = \sum_{n=-\infty}^{\infty} A_n r^n \ . \tag{3.74}$$

Substituting Equation (3.74) in Equation (3.73) gives

$$\sum_{n=-\infty}^{\infty} n(n-1)A_n r^{n-2} + \frac{1}{r}\sum_{n=-\infty}^{\infty} nA_n r^{n-1} - \frac{1}{r^2}\sum_{n=-\infty}^{\infty} A_n r^n = 0$$

$$\sum_{n=-\infty}^{\infty} \left[n(n-1)+n-1\right]A_n r^{n-2} = 0$$

$$\sum_{n=-\infty}^{\infty} (n^2-1)A_n r^{n-2} = 0$$

$$\sum_{n=-\infty}^{\infty} (n-1)(n+1)A_n r^{n-2} = 0 \ . \tag{3.75}$$

The preceding expression (3.75) requires that

$$A_n = 0,\ n = -\infty,\ldots,\infty, \text{ except for } n = 1 \text{ and } n = -1. \tag{3.76}$$

Therefore, the form of the radial displacement is

$$u = A_1 r + \frac{A_{-1}}{r} \ . \tag{3.77}$$

To keep the terminology simple, assume that the form of the radial displacement with different names for the constants,

$$u = Ar + \frac{B}{r} \ . \tag{3.78}$$

The preceding equations are valid for a cylinder with an axisymmetric response. Thus, the radial displacement, u_f and u_m, in the fiber and matrix cylinders, respectively, can be assumed of the form

$$u_f = A_f r + \frac{B_f}{r}, 0 \le r \le a, \tag{3.79}$$

$$u_m = A_m r + \frac{B_m}{r}, a \le r \le b. \tag{3.80}$$

However, because the fiber is a solid cylinder and the radial displacement u_f is finite, $B_f = 0$; otherwise, the radial displacement of the fiber u_f would be infinite. Thus,

$$u_f = A_f r, 0 \le r \le a, \tag{3.81}$$

$$u_m = A_m r + \frac{B_m}{r}, \ a \le r \le b \ . \tag{3.82}$$

Differentiating Equation (3.81) and Equation (3.82) gives

$$\frac{du_f}{dr} = A_f \tag{3.83a}$$

$$\frac{du_m}{dr} = A_m - \frac{B_m}{r^2} \ . \tag{3.83b}$$

Using Equation (3.83a) and Equation (3.83b) in Equation (3.70), the stress–strain relationships for the fiber are

$$\begin{bmatrix} \sigma_r^f \\ \sigma_\theta^f \\ \sigma_z^f \end{bmatrix} = \begin{bmatrix} C_{11}^f & C_{12}^f & C_{12}^f \\ C_{12}^f & C_{11}^f & C_{12}^f \\ C_{12}^f & C_{12}^f & C_{11}^f \end{bmatrix} \begin{bmatrix} A_f \\ A_f \\ \epsilon_1 \end{bmatrix}, \tag{3.84}$$

where the stiffness constants of the fiber are

$$C_{11}^f = \frac{E_f\left(1-\nu_f\right)}{\left(1-2\nu_f\right)\left(1+\nu_f\right)}$$

$$C_{12}^f = \frac{\nu_f E_f}{\left(1-2\nu_f\right)\left(1+\nu_f\right)} \tag{3.85}$$

and the stress–strain relationships for the matrix are

$$\begin{bmatrix} \sigma_r^m \\ \sigma_\theta^m \\ \sigma_z^m \end{bmatrix} = \begin{bmatrix} C_{11}^m & C_{12}^m & C_{12}^m \\ C_{12}^m & C_{11}^m & C_{12}^m \\ C_{12}^m & C_{12}^m & C_{11}^m \end{bmatrix} \begin{bmatrix} A_m - \dfrac{B_m}{r^2} \\ A_m + \dfrac{B_m}{r^2} \\ \epsilon_1 \end{bmatrix}, \tag{3.86}$$

where the stiffness constants of the matrix are

$$C_{11}^m = \frac{E_m\left(1-\nu_m\right)}{\left(1-2\nu_m\right)\left(1+\nu_m\right)} \tag{3.87a}$$

$$C_{12}^m = \frac{\nu_m E_m}{\left(1-2\nu_m\right)\left(1+\nu_m\right)}. \tag{3.87b}$$

How do we now solve for the unknown constants A_f, A_m, B_m, and ε_1? The following four boundary and interface conditions will allow us to do that:

1. The radial displacement is continuous at the interface, $r = a$,

$$u_f\left(r=a\right) = u_m\left(r=a\right). \tag{3.88}$$

Then, from Equation (3.81) and Equation (3.82),

$$A_f a = A_m a + \frac{B_m}{a}. \tag{3.89}$$

2. The radial stress is continuous at $r = a$:

$$\left(\sigma_r^f\right)\left(r=a\right) = \left(\sigma_r^m\right)\left(r=a\right). \tag{3.90}$$

Then, from Equation (3.84) and Equation (3.86),

$$C_{11}^f A_f + C_{12}^f A_f + C_{12}^f \epsilon_1 = C_{11}^m\left(A_m - \frac{B_m}{a^2}\right) + C_{12}^m\left(A_m + \frac{B_m}{a^2}\right) + C_{12}^m \epsilon_1. \tag{3.91}$$

3. Because the surface at $r = b$ is traction free, the radial stress on the outside of matrix, $r = b$, is zero:

$$\left(\sigma_r^m\right)\left(r = b\right) = 0 \ . \tag{3.92}$$

Then, Equation (3.84) gives

$$C_{11}^m\left(A_m - \frac{B_m}{b^2}\right) + C_{12}^m\left(A_m + \frac{B_m}{b^2}\right) + C_{12}^m \in_1 = 0 \ . \tag{3.93}$$

4. The overall axial load over the fiber-matrix cross-sectional area in direction 1 is the applied load, P, then

$$\int_A \sigma_z dA = P \tag{3.94}$$

$$\int_0^b \int_0^{2\pi} \sigma_z r dr d\theta = P \ .$$

Because the axial normal stress, σ_z, is independent of θ,

$$\int_0^b \sigma_z 2\pi r dr = P \ . \tag{3.95}$$

Now,

$$\sigma_z = \sigma_z^f, 0 \le r \le a$$

$$= \sigma_z^m, a \le r \le b \ . \tag{3.96}$$

Then, from Equation (3.84) and Equation (3.86),

$$\begin{aligned} &\int_0^a \left(C_{12}^f A_f + C_{12}^f A_f + C_{11}^f \in_1\right) 2\pi r dr + \\ &\int_a^b \left(C_{12}^m\left(A_m - \frac{B_m}{r^2}\right) + C_{12}^m\left(A_m + \frac{B_m}{r^2}\right) + C_{11}^m \in_1\right) 2\pi r dr = P \end{aligned} \ . \tag{3.97}$$

Solving Equation (3.89), Equation (3.91), Equation (3.93), and Equation (3.97), we get the solution to A_f, A_m, B_m, and ε_1.

Using the resulting solution for $\in_1$, and using Equation (3.66),

$$E_1 = \frac{P}{\pi b^2 \in_1}$$

$$= E_f V_f + E_m \left(1 - V_f\right)$$

$$- \frac{2 E_m E_f V_f \left(\nu_f - \nu_m\right)^2 \left(1 - V_f\right)}{E_f \left(2\nu_m^2 V_f - \nu_m + V_f \nu_m - V_f - 1\right) + E_m \left(-1 - 2V_f \nu_f^2 + \nu_f - V_f \nu_f + 2\nu_f^2 + V_f\right)} \tag{3.98}$$

Although the preceding expression can be written in a compact form by using definitions of shear and bulk modulus* of the material, we avoid doing so because results given in Equation (3.98) can now be found symbolically by computational systems such as Maple.[14] Note that the first two terms of Equation (3.98) represent the mechanics of materials approach result given by Equation (3.34).

Example 3.9

Find the longitudinal Young's modulus for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use equations obtained using the elasticity model.

Solution

From Table 3.1, the Young's modulus of fiber is

$$E_f = 85 \text{ GPa};$$

the Poisson's ratio of the fiber is

$$\nu_f = 0.2.$$

From Table 3.2, the Young's modulus of matrix is

$$E_m = 3.4 \text{ GPa}$$

* Bulk modulus of an elastic body is defined as the slope of the applied hydrostatic pressure vs. volume dilation curve. Hydrostatic stress is defined as $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, $\tau_{xy} = 0$, $\tau_{yz} = 0$, $\tau_{zx} = 0$ and volume dilation, D_v, is defined as the sum of resulting normal strains. $D_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$. The bulk modulus, K, is used for finding volume changes in a given body subjected to hydrostatic pressure.

and the Poisson's ratio of the matrix is

$$\nu_m = 0.3.$$

Using Equation (3.98), the longitudinal Young's modulus

$$E_1 = (85\times10^9)(0.7)+(3.4\times10^9)(1-0.7)$$

$$-\frac{2(3.4\times10^9)(85\times10^9)(0.7)\left(0.2-0.3\right)^2\left(1-0.7\right)}{\begin{pmatrix}(85\times10^9)\left(2(0.3)^2(0.7)-0.3+(0.7)(0.3)-0.7-1\right)+\\(3.4\times10^9)\left(-1-2(0.7)(0.2)^2+0.2-(0.7)(0.2)+2(0.2)^2+0.7\right)\end{pmatrix}}$$

$$= 60.53\times10^9 \text{ Pa}$$

$$= 60.53 \text{ GPa}.$$

For the same problem, the longitudinal Young's modulus was found to be 60.52 GPa from the mechanics of materials approach as well as the Halphin–Tsai equations.

3.3.3.2 Major Poisson's Ratio

In Section 3.3.3.1, we solved the problems of an axially loaded cylinder. This same problem can be used to determine the axial Poisson's ratio, ν_{12}, because of the definition of major Poisson's ratio as

$$\nu_{12} = -\frac{\epsilon_r}{\epsilon_1}, \tag{3.99}$$

when a body is only under an axial load in direction 1.

From the definition of radial strain from Equation (3.69a) that, at $r = b$,

$$\epsilon_r\,(r=b) = \frac{u_m\left(b\right)}{b}, \tag{3.100}$$

the major Poisson's ratio is

$$\nu_{12} = -\frac{\frac{u_m\left(r=b\right)}{b}}{\epsilon_1}. \tag{3.101}$$

Using Equation (3.101),

$$\nu_{12} = -\frac{\left(A_m + \dfrac{B_m}{b^2}\right)}{\in_1}. \tag{3.102}$$

Using the solution obtained in Section 3.3.3.1 for A_m, B_m, and $\in_1$ by solving Equation (3.89), Equation (3.91), Equation (3.93), and Equation (3.97), we get

$$\nu_{12} = \nu_f V_f + \nu_m V_m$$
$$+\frac{V_f V_m\left(\nu_f - \nu_m\right)\left(2E_f\nu_m^2 + \nu_m E_f - E_f + E_m - E_m\nu_f - 2E_m\nu_f^2\right)}{\left(2\nu_m^2 V_f - \nu_m + \nu_m V_f - 1 - V_f\right)E_f + \left(2\nu_f^2 - V_f\nu_f - 2V_f\nu_f^2 + V_f + \nu_f - 1\right)E_m} \tag{3.103}$$

Although the preceding expression can be written in a compact form by using definitions of shear and bulk modulus of the material, we avoid doing so because results given in Equation (3.103) can be found symbolically by computational systems such as Maple.[14] Note that the first two terms of Equation (3.103) are the same as the mechanics of materials approach result given by Equation (3.34).

Example 3.10

Find the major Poisson's ratio for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use equations obtained using the elasticity model.

Solution

Using Equation (3.103), the major Poisson's ratio is

$$\nu_{12} = (0.2)(0.7) + (0.3)(0.3)$$
$$+\frac{(0.7)(0.3)\left(0.2-0.3\right)\begin{pmatrix}(2)(85\times10^9)(0.3)^2 + (0.3)(85\times10^9) - 85\times10^9 + 3.4\times10^9\\ -\ (3.4\times10^9)(0.2) - (2)(3.4\times10^9)(0.2)^2\end{pmatrix}}{\begin{pmatrix}\left((2)(0.3)^2(0.7) - 0.3 + (0.3)(0.7) - 1 - (0.7)\right)(85\times10^9) +\\ \left(2(0.3)^2 - (0.7)(0.2) - (2)(0.7)(0.2)^2 + 0.7 + 0.2 - 1\right)(3.4\times10^9)\end{pmatrix}}$$
$$= 0.2238.$$

For the same problem, the major Poisson's ratio was found to be 0.2300 from the mechanics of materials approach as well as the Halphin–Tsai equations.

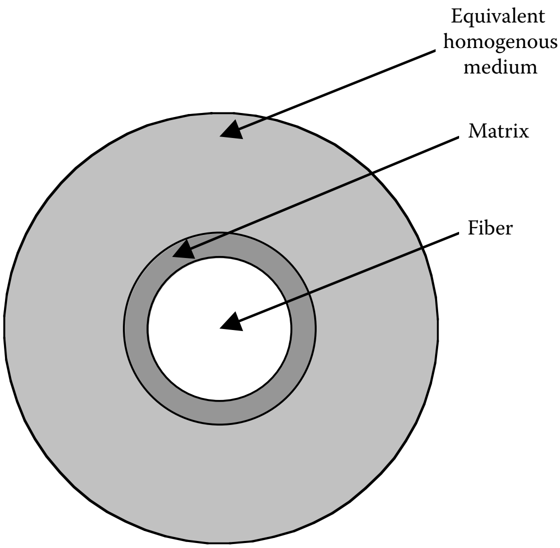


FIGURE 3.20
Three-phase model of a composite.

3.3.3.3 Transverse Young's Modulus

The CCA model only gives lower and upper bounds of the transverse Young's modulus of the composite. However, for the sake of completeness, we will summarize the result from a three-phase model. This model (Figure 3.20), however, yields an exact solution[12] for the transverse shear modulus, G_{23}. However, the transverse Young's modulus can be found as follows.

Assuming that the resulting composite properties are transversely isotropic (a valid assumption for hexagonally arranged fibers; 2–3 plane is isotropic),

$$E_2 = 2\left(1+\nu_{23}\right)G_{23}\,, \tag{3.104}$$

where ν_{23} = transverse Poisson's ratio.

The transverse Poisson's ratio, ν_{23}, is given by[15]

$$\nu_{23} = \frac{K^* - mG_{23}}{K^* + mG_{23}}\,, \tag{3.105}$$

where

$$m = 1 + 4K^* \frac{\nu_{12}^2}{E_1}\,. \tag{3.106}$$

The bulk modulus, K^*, of the composite under longitudinal plane strain is

$$K^* = \frac{K_m\left(K_f + G_m\right)V_m + K_f\left(K_m + G_m\right)V_f}{\left(K_f + G_m\right)V_m + \left(K_m + G_m\right)V_f} \,. \tag{3.107}$$

The bulk modulus K_f of the fiber under longitudinal plane strain is

$$K_f = \frac{E_f}{2\left(1+\nu_f\right)\left(1-2\nu_f\right)} \,. \tag{3.108}$$

The bulk modulus K_m of the matrix under longitudinal plane strain is

$$K_m = \frac{E_m}{2\left(1+\nu_m\right)\left(1-2\nu_m\right)} \,. \tag{3.109}$$

To derive the solution for G_{23} for use in Equation (3.104) is out of scope of this book; however, for the sake of completeness, the final solution is given next. Based on the three-phase model (Figure 3.20) where the fiber is surrounded by matrix, which is then surrounded by a homogeneous material equivalent to the composite, the transverse shear modulus, G_{23}, is given by the acceptable solution of the quadratic equation:[12]

$$A\left(\frac{G_{23}}{G_m}\right)^2 + 2B\left(\frac{G_{23}}{G_m}\right) + C = 0 \,, \tag{3.110}$$

where

$$A = 3V_f\left(1-V_f\right)^2\left(\frac{G_f}{G_m}-1\right)\left(\frac{G_f}{G_m}+\eta_f\right)$$

$$+\left[\frac{G_f}{G_m}\eta_m + \eta_f\eta_m - \left(\frac{G_f}{G_m}\eta_m - \eta_f\right)V_f^3\right]\left[V_f\eta_m\left(\frac{G_f}{G_m}-1\right) - \left(\frac{G_f}{G_m}\eta_m + 1\right)\right]$$

$$B = -3V_f\left(1-V_f\right)^2\left(\frac{G_f}{G_m}-1\right)\left(\frac{G_f}{G_m}+\eta_f\right)$$

$$+\frac{1}{2}\left[\eta_m\frac{G_f}{G_m} + \left(\frac{G_f}{G_m}-1\right)V_f + 1\right]\left[\left(\eta_m - 1\right)\left(\frac{G_f}{G_m}+\eta_f\right)\right.$$

$$\left. -2\left(\frac{G_f}{G_m}\eta_m - \eta_f\right)V_f^3\right] + \frac{V_f}{2}\left(\eta_m + 1\right)\left(\frac{G_f}{G_m}-1\right)\left[\frac{G_f}{G_m} + \eta_f + \left(\frac{G_f}{G_m}\eta_m - \eta_f\right)V_f^3\right]$$

$$C = 3V_f\left(1 - V_f\right)^2\left(\frac{G_f}{G_m} - 1\right)\left(\frac{G_f}{G_m} + \eta_f\right)$$

$$+\left[\eta_m\frac{G_f}{G_m} + \left(\frac{G_f}{G_m} - 1\right)V_f + 1\right]\left[\frac{G_f}{G_m} + \eta_f + \left(\frac{G_f}{G_m}\eta_m - \eta_f\right)V_f^3\right] \tag{3.111}$$

$$\eta_m = 3 - 4\nu_m\ ,$$

$$\eta_f = 3 - 4\nu_f\ . \tag{3.112}$$

Then, using Equation (3.104) through Equation (3.109), we get the transverse Young's modulus, E_2.

Example 3.11

Find the transverse Young's modulus for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use equations obtained using the elasticity model.

Solution

From Equation (3.112),

$$\eta_f = 3 - 4(0.2)$$

$$= 2.2$$

$$\eta_m = 3 - 4(0.3)$$

$$= 1.8\ .$$

From Equation (3.108) and Equation (3.109),

$$K_f = \frac{85\times10^9}{2(1+0.2)(1-2\times0.2)}$$

$$= 59.03\times10^9\,Pa$$

$$K_m = \frac{3.4 \times 10^9}{2(1+0.3)(1-2\times 0.3)}$$

$$= 3.269 \times 10^9 \, Pa \; .$$

From Equation (3.107),

$$K^* = \frac{\begin{pmatrix} 3.269 \times 10^9 \left(59.03 \times 10^9 + 1.308 \times 10^9\right)(0.3) + \\ 59.03 \times 10^9 \left(3.269 \times 10^9 + 1.308 \times 10^9\right)(0.7) \end{pmatrix}}{\left(59.03 \times 10^9 + 1.308 \times 10^9\right)(0.3) + \left(3.269 \times 10^9 + 1.308 \times 10^9\right)(0.7)}$$

$$= 11.66 \times 10^9 \, Pa \; .$$

The three constants of the quadratic Equation (3.110) are given by Equation (3.111) as

$$A = 3(0.7)(1-0.7)^2 \left(\frac{35.42 \times 10^9}{1.308 \times 10^9} - 1\right)\left(\frac{35.42 \times 10^9}{1.308 \times 10^9} + 2.2\right)$$

$$+\left[\frac{35.42 \times 10^9}{1.308 \times 10^9}(1.8) + 2.2 \times 1.8 - \left(\frac{35.42 \times 10^9}{1.308 \times 10^9}(1.8) - 2.2\right)0.7^3\right]$$

$$\left[(0.7)(1.8)\left(\frac{35.42 \times 10^9}{1.308 \times 10^9} - 1\right) - \left(\frac{35.42 \times 10^9}{1.308 \times 10^9}(1.8) + 1\right)\right]$$

$$= -476.0 \; .$$

$$B = -3(0.7)(1-0.7)^2 \left(\frac{35.42 \times 10^9}{1.308 \times 10^9} - 1\right)\left(\frac{35.42 \times 10^9}{1.308 \times 10^9} + 2.2\right)$$

$$+\frac{1}{2}\left[1.8\left(\frac{35.42 \times 10^9}{1.308 \times 10^9}\right) + \left(\frac{35.42 \times 10^9}{1.308 \times 10^9} - 1\right)(0.7) + 1\right]$$

$$\left[(1.8-1)\left(\frac{35.42\times10^9}{1.308\times10^9}+2.2\right)-2\left(\frac{35.42\times10^9}{1.308\times10^9}(1.8)-2.2\right)0.7^3\right]+\frac{0.7}{2}$$

$$\left[(1.8+1)\left(\frac{35.42\times10^9}{1.308\times10^9}-1\right)+\left(\frac{35.42\times10^9}{1.308\times10^9}+1.8+\frac{35.42\times10^9}{1.308\times10^9}(1.8)-2.2\right)0.7^3\right]$$

$$=723.0.$$

$$C=3(0.7)(1-0.7)^2\left(\frac{35.42\times10^9}{1.308\times10^9}-1\right)\left(\frac{35.42\times10^9}{1.308\times10^9}+2.2\right)$$

$$+\left[1.8\frac{35.42\times10^9}{1.308\times10^9}+\left(\frac{35.42\times10^9}{1.308\times10^9}-1\right)(0.7)+1\right]$$

$$\left[\frac{35.42\times10^9}{1.308\times10^9}+2.2+\left(\frac{35.42\times10^9}{1.308\times10^9}(1.8)-2.2\right)0.7^3\right]$$

$$=3222.$$

Substituting values of A, B, and C in Equation (3.110),

$$-476.0\left(\frac{G_{23}}{1.308\times10^9}\right)^2+2(723.0)\left(\frac{G_{23}}{1.308\times10^9}\right)+3222=0$$

$$-278.4\times10^{-18}G_{23}^2+1106\times10^{-9}G_{23}+3222=0$$

gives G_{23} = 5.926 × 10^9 Pa, –1.953 × 10^9 Pa. Thus, the acceptable solution is

$$G_{23}=5.926\times10^9\,Pa\ .$$

From Equation (3.106),

$$m=1+4(11.66\times10^9)\frac{0.2238^2}{60.53\times10^9}$$

$$=1.039\ .$$

From Equation (3.105),

$$\nu_{23} = \frac{11.66 \times 10^9 - 1.039\left(5.926 \times 10^9\right)}{11.66 \times 10^9 + 1.039\left(5.926 \times 10^9\right)}$$

$$= 0.3089 \ .$$

From Equation (3.104),

$$E_2 = 2\left(1 + 0.3089\right)\left(5.926 \times 10^9\right)$$

$$= 15.51 \times 10^9 \, Pa$$

$$= 15.51 \ GPa.$$

For the same problem, the transverse Young's modulus was found to be 10.37 GPa from the mechanics of materials approach and 20.20 GPa from the Halphin–Tsai equations.

Figure 3.21a and Figure 3.21b show the transverse Young's modulus as a function of fiber volume fraction for a typical boron/epoxy unidirectional lamina. The elasticity equation (3.104), Halphin–Tsai equation (3.60), and the mechanics of materials approach (Equation 3.55) are shown and compared to the experimental data points.

3.3.3.4 Axial Shear Modulus

To find the axial shear modulus, G_{12}, of a unidirectional composite, we consider the same concentric cylinder model (Figure 3.19). Consider a long fiber of radius, a, and shear modulus, G_f, surrounded by a long concentric cylinder of matrix of outer radius, b, and shear modulus, G_m. The composite cylinder (Figure 3.19) is subjected to a shear strain, γ_{12}^0 , in the 1–2 plane.

Following the derivation,[4,12,16] the normal displacements in the 1, 2, 3 direction for the fiber or matrix are assumed of the following form:

$$u_1 = -\frac{\gamma_{12}^0}{2} x_2 + F\left(x_2, x_3\right)$$

$$u_2 = \frac{\gamma_{12}^0}{2} x_1$$

$$u_3 = 0 \ , \qquad (3.113a, b, c)$$

where γ_{12}^0 is the applied shear strain to the boundary.

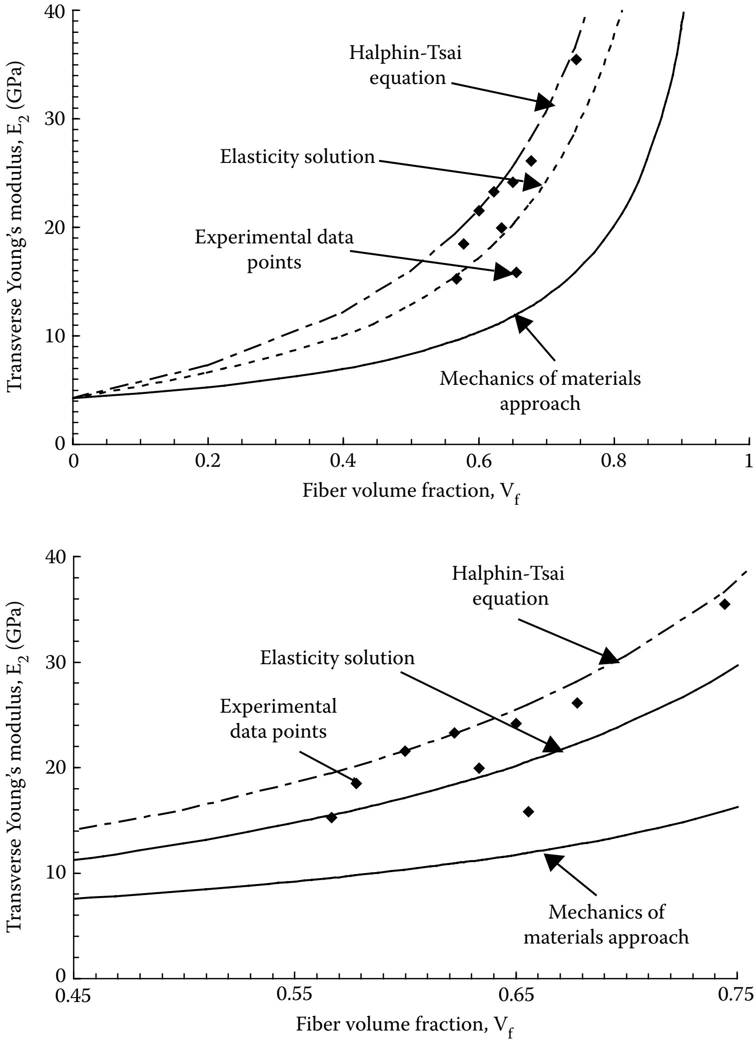


FIGURE 3.21
Theoretical values of transverse Young's modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina (E_f = 414 GPa, ν_f = 0.2, E_m = 4.14 GPa, ν_m = 0.35). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

The preceding assumption of the form of the displacements is based on a semi-inverse method[17] that is beyond the scope of this book. Individual expressions for displacement of the fiber and matrix will be shown later in the derivation.

From the strain-displacement[13] equations and the expressions for the displacement field in Equation (3.113a, b, c),

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$= 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$= 0$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$= 0$$

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}$$

$$= 0$$

$$\gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$$

$$= \frac{\partial F}{\partial x_2}$$

$$\gamma_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}$$

$$= \frac{\partial F}{\partial x_3} \ . \tag{3.114 a–f}$$

Because all normal strains in the 1, 2, and 3 directions are zero, all the normal stresses in 1, 2, 3 directions are also zero. Also, $\tau_{23} = 0$ because $\gamma_{23} = 0$.

Using Equation (3.114e) and Equation (3.114f), the only possible nonzero stresses are

$$\tau_{12} = G\gamma_{12}$$

$$= G\frac{\partial F}{\partial x_2} \tag{3.115a}$$

$$\tau_{13} = G\gamma_{13}$$

$$= G\frac{\partial F}{\partial x_3}\ , \tag{3.115b}$$

where G is the shear modulus of the material.

The equilibrium condition derived from the fact that the sum of the forces in direction 1 is zero gives[13]

$$\frac{\partial\sigma_1}{\partial x_1} + \frac{\partial\tau_{12}}{\partial x_2} + \frac{\partial\tau_{13}}{\partial x_3} = 0\ . \tag{3.116}$$

With Equation (3.115a) and Equation (3.115b) and $\sigma_1 = 0$, the equilibrium equation (3.116) reduces it to

$$\frac{\partial^2 F}{\partial x_2^2} + \frac{\partial^2 F}{\partial x_3^2} = 0\ . \tag{3.117}$$

Converting Equation (3.117) to polar coordinates needs the following:

$$x_2 = rCos\theta\ , \tag{3.118}$$

$$x_3 = rSin\theta \tag{3.119}$$

give

$$r^2 = x_2^2 + x_3^2 \tag{3.120a}$$

$$\theta = \tan^{-1}\frac{x_3}{x_2}\ . \tag{3.120b}$$

From Equation (3.118), Equation (3.119), and Equation (3.120a, b),

$$2r\frac{\partial r}{\partial x_2} = 2x_2$$

$$\frac{\partial r}{\partial x_2} = \frac{x_2}{r}$$

$$= Cos\theta \tag{3.121a}$$

$$2r\frac{\partial r}{\partial x_3} = 2x_3$$

$$\frac{\partial r}{\partial x_3} = \frac{x_3}{r}$$

$$= Sin\theta \tag{3.121b}$$

$$\frac{\partial \theta}{\partial x_2} = \frac{1}{1+\left(\frac{x_3}{x_2}\right)^2}\left(-\frac{x_3}{x_2^2}\right)$$

$$= -\frac{x_3}{x_2^2 + x_3^2}$$

$$= -\frac{rSin\theta}{r^2}$$

$$= -\frac{Sin\theta}{r} \tag{3.121c}$$

$$\frac{\partial \theta}{\partial x_3} = \frac{1}{1+\left(\frac{x_3}{x_2}\right)^2}\left(\frac{1}{x_2}\right)$$

$$= \frac{x_2}{x_2^2 + x_3^2}$$

$$= \frac{r Cos\theta}{r}$$

$$= \frac{Cos\theta}{r} . \tag{3.121d}$$

Now, using the chain rule for derivatives,

$$\frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x_2} , \tag{3.122}$$

and using Equation (3.121a) and Equation (3.121c),

$$\frac{\partial F}{\partial x_2} = Cos\theta \frac{\partial F}{\partial r} - \frac{Sin\theta}{r}\frac{\partial F}{\partial \theta} . \tag{3.123}$$

Repeating a similar chain of rule of derivatives on Equation (3.122),

$$\frac{\partial^2 F}{\partial x_2^2} = Cos^2\theta \frac{\partial^2 F}{\partial r^2} + Sin^2\theta\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) - 2 Sin\theta Cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) . \tag{3.124a}$$

Similarly,

$$\frac{\partial^2 F}{\partial x_3^2} = Sin^2\theta \frac{\partial^2 F}{\partial r^2} + Cos^2\theta\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) + 2 Sin\theta Cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right) . \tag{3.124b}$$

Substituting Equation (3.124a) and Equation (3.124b) in Equation (3.117) yields

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = 0 . \tag{3.125}$$

The solution of Equation (3.125) is given by

$$F(r,\theta) = \left(Ar + \frac{B}{r}\right) Cos\theta \tag{3.126}$$

only after noting that the complete solution to Equation (3.125) is of the form

$$F(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right)\left[C_n Sin(n\theta) + D_n Cos(n\theta)\right], \quad (3.127)$$

but that the surface, $r = b$, of the composite cylinder is only subjected to displacements:

$$u_{1m}(r=b) = \frac{\gamma_{12}^0}{2} x_2 \Big|_{r=b}$$

$$= \frac{\gamma_{12}^0}{2} bCos\theta \quad (3.128)$$

$$u_{2m}(r=b) = \frac{\gamma_{12}^0}{2} x_1 \Big|_{r=b}$$

$$= \frac{\gamma_{12}^0}{2} bSin\theta \quad (3.129)$$

$$u_{3m}(r=b) = 0 \ . \quad (3.130)$$

Thus, the function $F(r,\theta)$ of Equation (3.126) for the fiber F_f and matrix F_m is given by

$$F_f(r,\theta) = \left(A_1 r + \frac{B_1}{r}\right) Cos\theta \quad (3.131)$$

$$F_m(r,\theta) = \left(A_2 r + \frac{B_2}{r}\right) Cos\theta \ . \quad (3.132)$$

How do we find A_1, B_1, A_2, and B_2? The following boundary and interface conditions are applied to find these four unknowns:

1. The axial displacements of the fiber u_{1f} and the matrix u_{1m} at the interface, $r = a$, are continuous:

$$u_{1f}(r=a) = u_{1m}(r=a) \ . \quad (3.133)$$

Now, from Equation (3.113a),

$$u_{1f} = -\frac{\gamma_{12}^0}{2} x_2 + F_f\left(x_2, x_3\right)$$

$$= -\frac{\gamma_{12}^0}{2} rCos\theta + F_f\left(rCos\theta, rSin\theta\right) . \tag{3.134}$$

At $r = a$,

$$u_{1f}\left(r = a\right) = -\frac{\gamma_{12}^0}{2} aCos\theta + \left(A_1 a + \frac{B_1}{a}\right) Cos\theta . \tag{3.135}$$

Similarly, from Equations (3.313a), (3.318), and (3.131),

$$u_{1m}\left(r = a\right) = -\frac{\gamma_{12}^0}{2} aCos\theta + \left(A_2 a + \frac{B_2}{a}\right) Cos\theta . \tag{3.136}$$

Equating Equation (3.135) and Equation (3.136) per Equation (3.133) gives

$$A_1 a + \frac{B_1}{a} = A_2 a + \frac{B_2}{a} . \tag{3.137}$$

2. The displacement of the fiber u_{1f} is given by Equations (3.313a), (3.318), and (3.131) as

$$u_{1f} = -\frac{\gamma_{12}^0}{2} rCos\theta + \left(A_1 r + \frac{B_1}{r}\right) Cos\theta . \tag{3.138}$$

Because $r = 0$ is a point on the fiber and displacement in the fiber is finite,

$$B_1 = 0 . \tag{3.139}$$

3. The shear stress in the fiber τ_{1rf} and that in the matrix τ_{1rm} are continuous at the interface $r = a$:

$$\tau_{1r,f}\left(r = a\right) = \tau_{1r,m}\left(r = a\right) . \tag{3.140}$$

First, we need to derive an expression for τ_{1r} from transforming stresses between 1–r and 1–3 coordinates:

$$\tau_{1r} = Cos\theta\ \tau_{12} + Sin\theta\ \tau_{13}\ . \tag{3.141}$$

Using Equation (3.115a, b) in Equation (3.141),

$$\tau_{1r} = Cos\theta\ G\ \frac{\partial F}{\partial x_2} + Sin\theta\ G\ \frac{\partial F}{\partial x_3} \tag{3.142}$$

$$= G\left(Cos\theta\frac{\partial F}{\partial x_2} + Sin\theta\frac{\partial F}{\partial x_3}\right). \tag{3.143}$$

Substituting Equation (3.121a) and Equation (3.121b) in Equation (3.143),

$$\tau_{1r} = G\left(\frac{\partial x_2}{\partial r}\frac{\partial F}{\partial x_2} + \frac{\partial x_3}{\partial r}\frac{\partial F}{\partial x_3}\right),$$

gives

$$\tau_{1r} = G\frac{\partial F}{\partial r}\ . \tag{3.144}$$

Thus, in the fiber from Equation (3.131)

$$\tau_{1r,f} = G_f\ \frac{\partial F_f}{\partial r}$$

$$= G_f\left(A_1 - \frac{B_1}{r^2}\right)Cos\theta \tag{3.145}$$

and in the matrix from Equation (3.132),

$$\tau_{1r,m} = G_m\ \frac{\partial F_m}{\partial r}$$

$$= G_m \left(A_2 - \frac{B_2}{r^2} \right) Cos\theta \,. \tag{3.146}$$

Equating Equation (3.145) and Equation (3.146) at $r = a$, per Equation (3.140), gives

$$G_f \left(A_1 - \frac{B_1}{a^2} \right) = G_m \left(A_2 - \frac{B_2}{a^2} \right). \tag{3.147}$$

4. The displacement due to the applied shear strain of γ_{12}^0 at the boundary $r = b$ of the composite cylinder is given by

$$u_{1m}\left(r = b\right) = \frac{\gamma_{12}^0}{2} x_2 \Big|_{r=b} \tag{3.148a}$$

$$= \frac{\gamma_{12}^0}{2} bCos\theta \,. \tag{3.148b}$$

Based on Equation (3.113a) and Equation (3.132),

$$u_{1m}\left(r = b\right) = -\frac{\gamma_{12}^0}{2} x_2 + F_m\left(x_2, x_3\right)\Big|_{r=b}$$

$$= -\frac{\gamma_{12}^0}{2} bCos\theta + \left(A_2 b + \frac{B_2}{b} \right) Cos\theta \,. \tag{3.149}$$

From Equation (3.148b) and Equation (3.149), we get

$$A_2 b + \frac{B_2}{b} = \gamma_{12}^0 b \,. \tag{3.150}$$

Solving the three simultaneous equations (Equation 3.137, Equation 3.147, and Equation 3.150) to find A_1, A_2, and B_2, ($B_1 = 0$ from Equation 3.139), we get

$$A_1 = \frac{2G_m}{G_m\left(1+V_f\right)+G_f\left(1-V_f\right)}\gamma_{12}^0 \tag{3.151}$$

$$A_2 = \frac{\left(G_f+G_m\right)}{G_m\left(1+V_f\right)+G_f\left(1-V_f\right)}\gamma_{12}^0 \tag{3.152}$$

$$B_2 = -\frac{a^2\left(-G_m+G_f\right)}{G_m\left(1+V_f\right)+G_f\left(1-V_f\right)}\gamma_{12}^0\ , \tag{3.153}$$

where, from Equation (3.63), the fiber volume fraction V_f is substituted for $\frac{a^2}{b^2}$.

The shear modulus G_{12} can be now be found as

$$G_{12} \equiv \frac{\tau_{12,m}\,|_{r=b}}{\gamma_{12}^0}\ , \tag{3.154}$$

where

$$\tau_{12,m}\,|_{r=b} = \text{ shear stress at } r=b$$

because, based on Equation (3.115a),

$$\tau_{12,m} = G_m\frac{\partial F_m}{\partial x_2}$$

$$= G_m\left(\frac{\partial F_m}{\partial r}\frac{\partial r}{\partial x_2}+\frac{\partial F_m}{\partial \theta}\frac{\partial \theta}{\partial x_2}\right). \tag{3.155}$$

Using Equation (3.121a) and Equation (3.121b),

$$\tau_{12,m} = G_m\left[\frac{\partial F_m}{\partial r}Cos\theta+\frac{\partial F_m}{\partial \theta}\left(-\frac{Sin\theta}{r}\right)\right]. \tag{3.156}$$

Using Equation (3.131) and Equation (3.132) in Equation (3.156) gives

$$\tau_{12,m} = G_m \left[\left(A_2 - \frac{B_2}{r^2} \right) Cos\theta Cos\theta + \left(A_2 r + \frac{B_2}{r} \right) (-Sin\theta) \cdot \left(-\frac{Sin\theta}{r} \right) \right]$$

$$= G_m \left[\left(A_2 - \frac{B_2}{r^2} \right) Cos^2\theta + \left(A_2 + \frac{B_2}{r^2} \right) Sin^2\theta \right]. \tag{3.157}$$

At $r = b$, $\theta = 0$

$$\tau_{12,m} \Big|_{r=b,\theta=0} = G_m \left(A_2 - \frac{B_2}{b^2} \right). \tag{3.158}$$

Substituting values of A_2 and B_2 from Equation (3.152) and Equation (3.153), respectively, in Equation (3.158) yields

$$\tau_{12,m} \Big|_{r=b,\theta=0} = G_m \left[\frac{G_f (1+V_f) + G_m (1-V_f)}{G_f (1-V_f) + G_m (1+V_f)} \right] \gamma_{12}^0 \tag{3.159}$$

and the shear modulus, G_{12}, can be found as

$$G_{12} \equiv \frac{\tau_{12,m} \big|_{r=b,\theta=0}}{\gamma_{12}^0}.$$

This gives

$$G_{12} = G_m \left[\frac{G_f (1+V_f) + G_m (1-V_f)}{G_f (1-V_f) + G_m (1+V_f)} \right]. \tag{3.160}$$

Example 3.12

Find the shear modulus, G_{12}, for a glass/epoxy composite with 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use the equations obtained using the elasticity model.

Solution

From Example 3.6, G_f = 35.42 GPa and G_m = 1.308 GPa. Using Equation (3.160), the in-plane shear modulus is

$$G_{12} = 1.308 \times 10^9 \left[\frac{(35.42 \times 10^9)\left(1+0.7\right) + (1.308 \times 10^9)\left(1-0.7\right)}{(35.42 \times 10^9)\left(1-0.7\right) + (1.308 \times 10^9)\left(1+0.7\right)} \right]$$

$$= 6.169 \times 10^9 \ Pa$$

$$= 6.169 \ GPa.$$

For the same problem, the shear modulus, G_{12}, is found to be 4.014 GPa from the mechanics of materials approach and 6.169 GPa from the Halphin-Tsai equations.

Figure 3.22a and Figure 3.22b show the in-plane shear modulus as a function of fiber volume fraction for a typical glass/epoxy unidirectional lamina. The elasticity equation (3.160), Halphin-Tsai equation (3.60), and the mechanics of materials approach (Equation 3.55) are shown and compared to the experimental data points.

A comparison of the elastic moduli from the mechanics of materials approach, the Halphin-Tsai equations, and elasticity models (Example 3.3 through Example 3.11) is given in Table 3.5.

3.3.4 Elastic Moduli of Lamina with Transversely Isotropic Fibers

Glass, aramids, and graphite are the three most common types of fibers used in composites; among these, aramids and graphite are transversely isotropic. From the definition of transversely isotropic materials in Chapter 2, such fibers have five elastic moduli.

If L represents the longitudinal direction along the length of the fiber and T represents the plane of isotropy (Figure 3.23) perpendicular to the longitudinal direction, the five elastic moduli of the transversely isotropic fiber are

E_{fL} = longitudinal Young's modulus

E_{fT} = Young's modulus in plane of isotropy

ν_{fL} = Poisson's ratio characterizing the contraction in the plane of isotropy when longitudinal tension is applied

ν_{fT} = Poisson's ratio characterizing the contraction in the longitudinal direction when tension is applied in the plane of isotropy

G_{fT} = in-plane shear modulus in the plane perpendicular to the plane of isotropy

The elastic moduli using strength of materials approach for lamina with transversely isotropic fibers[18] are

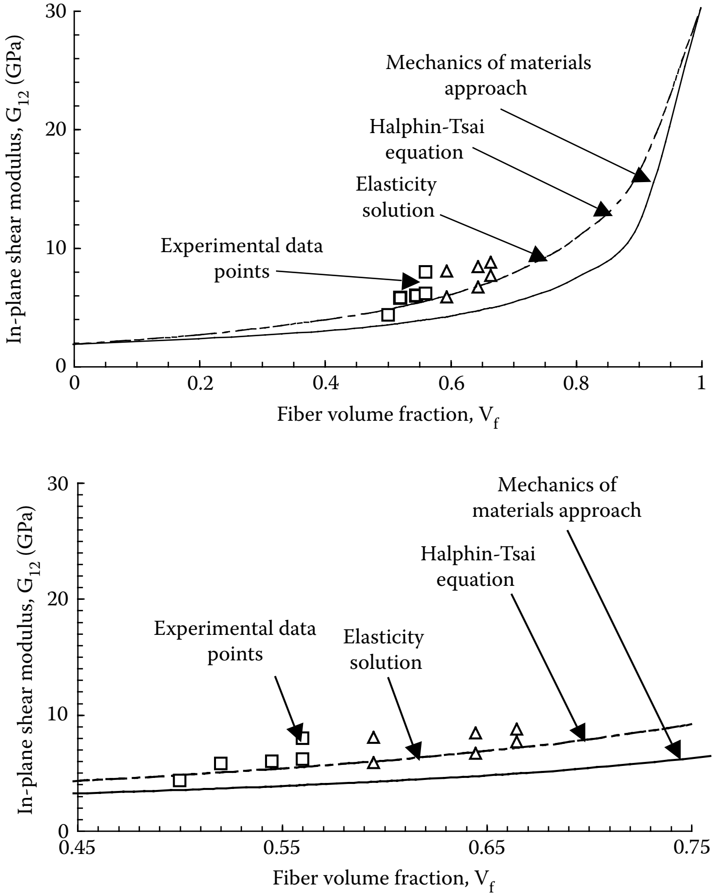


FIGURE 3.22
Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina (G_f = 30.19 GPa, G_m = 1.83 GPa). Figure (b) zooms figure (a) for fiber volume fraction between 0.45 and 0.75. (Experimental data from Hashin, Z., NASA tech. rep. contract No. NAS1-8818, November 1970.)

TABLE 3.5
Comparison of Predicted Elastic Moduli

Method	E_1 (GPa)	E_2 (GPa)	ν_{12}	G_{12} (GPa)
Mechanics of materials	60.52	10.37	0.2300	4.014
Halphin–Tsai	60.52	20.20	0.2300	6.169
Elasticity	60.53	15.51	0.2238	6.169[a]

[a] The Halphin–Tsai equations and the elasticity model equations give the same value for the shear modulus. Can you show that this is not a coincidence?

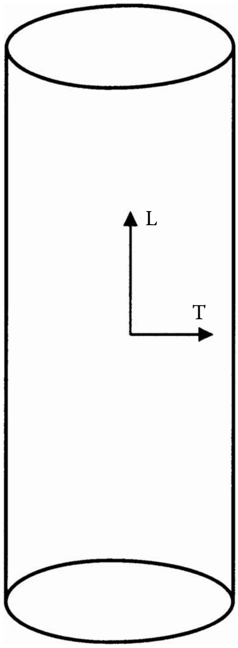


FIGURE 3.23
Longitudinal and transverse direction in a transversely isotropic fiber.

$$E_1 = E_{fL} V_f + E_m V_m,$$

$$\frac{1}{E_2} = \frac{V_f}{E_{fT}} + \frac{V_m}{E_m},$$

$$\nu_{12} = \nu_{fT} V_f + \nu_m V_m,$$

and

$$\frac{1}{G_{12}} = \frac{V_f}{G_{fT}} + \frac{V_m}{G_m}. \tag{3.161a–d}$$

The preceding expressions are similar to those of a lamina with isotropic fibers. The only difference is that appropriate transverse or longitudinal properties of the fiber are used. In composites such as carbon–carbon, the matrix is also transversely isotropic. In that case, the preceding equations cannot be used and are given elsewhere.[15,19]

3.4 Ultimate Strengths of a Unidirectional Lamina

As shown in Chapter 2, one needs to know five ultimate strength parameters for a unidirectional lamina:

- Longitudinal tensile strength $(\sigma_1^T)_{ult}$
- Longitudinal compressive strength $(\sigma_1^C)_{ult}$
- Transverse tensile strength $(\sigma_2^T)_{ult}$
- Transverse compressive strength $(\sigma_2^C)_{ult}$
- In-plane shear strength $(\tau_{12})_{ult}$

In this section, we will see whether and how these parameters can be found from the individual properties of the fiber and matrix by using the mechanics of materials approach. The strength parameters for a unidirectional lamina are much harder to predict than the stiffnesses because the strengths are more sensitive to the material and geometric nonhomogeneities, fiber–matrix interface, fabrication process, and environment. For example, a weak interface between the fiber and matrix may result in premature failure of the composite under a transverse tensile load, but may increase its longitudinal tensile strength. For these reasons of sensitivity, some theoretical and empirical models are available for some of the strength parameters. Eventually, the experimental evaluation of these strengths becomes important because it is direct and reliable. These experimental techniques are also discussed in this section.

3.4.1 Longitudinal Tensile Strength

A simple mechanics of materials approach model is presented (Figure 3.24). Assume that

- Fiber and matrix are isotropic, homogeneous, and linearly elastic until failure.
- The failure strain for the matrix is higher than for the fiber, which is the case for polymeric matrix composites. For example, glass fibers fail at strains of 3 to 5%, but an epoxy fails at strains of 9 to 10%.

Now, if

$(\sigma_f)_{ult}$ = ultimate tensile strength of fiber

E_f = Young's modulus of fiber

$(\sigma_m)_{ult}$ = ultimate tensile strength of matrix

E_m = Young's modulus of matrix

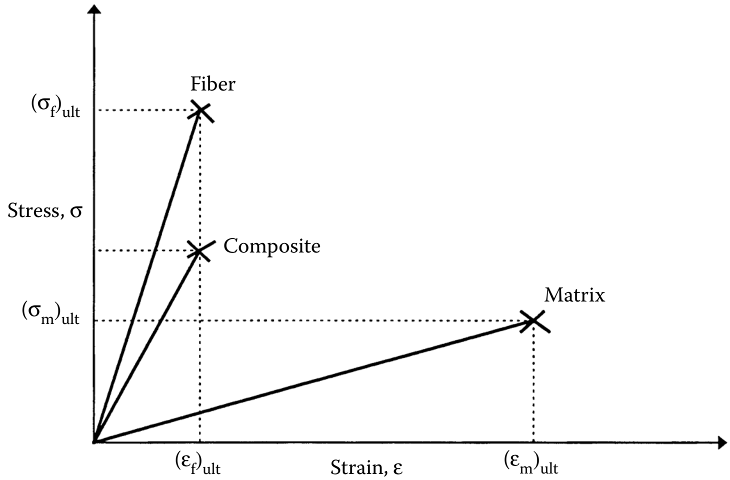


FIGURE 3.24
Stress–strain curve for a unidirectional composite under uniaxial tensile load along fibers.

then the ultimate failure strain of the fiber is

$$(\varepsilon_f)_{ult} = \frac{(\sigma_f)_{ult}}{E_f}, \tag{3.162}$$

and the ultimate failure strain of the matrix is

$$(\varepsilon_m)_{ult} = \frac{(\sigma_m)_{ult}}{E_m}. \tag{3.163}$$

Because the fibers carry most of the load in polymeric matrix composites, it is assumed that, when the fibers fail at the strain of $(\varepsilon_f)_{ult}$, the whole composite fails. Thus, the composite tensile strength is given by

$$(\sigma_1^T)_{ult} = (\sigma_f)_{ult} V_f + (\varepsilon_f)_{ult} E_m (1 - V_f). \tag{3.164}$$

Once the fibers have broken, can the composite take more load? The stress that the matrix can take alone is given by (σ_{mult}) $(1 - V_f)$. Only if this stress is greater than $(\sigma_1^T)_{ult}$ (Equation 3.164), is it possible for the composite to take more load. The volume fraction of fibers for which this is possible is called the minimum fiber volume fraction, $(V_f)_{\text{minimum}}$, and is

$$(\sigma_m)_{ult}\left[1-(V_f)_{minimum}\right] > (\sigma_f)_{ult}(V_f)_{minimum} + (\varepsilon_f)_{ult}E_m\left[1-(V_f)_{minimum}\right]$$

$$(V_f)_{minimum} < \frac{(\sigma_m)_{ult} - E_m(\varepsilon_f)_{ult}}{(\sigma_f)_{ult} - E_m(\varepsilon_f)_{ult} + (\sigma_m)_{ult}}. \tag{3.165}$$

It is also possible that, by adding fibers to the matrix, the composite will have lower ultimate tensile strength than the matrix. In that case, the fiber volume fraction for which this is possible is called the critical fiber volume fraction, $(V_f)_{critical}$, and is

$$(\sigma_m)_{ult} > (\sigma_f)_{ult}(V_f)_{critical} + (\varepsilon_f)_{ult}E_m\left[1-(V_f)_{critical}\right]$$

$$(V_f)_{critical} < \frac{(\sigma_m)_{ult} - E_m(\varepsilon_f)_{ult}}{(\sigma_f)_{ult} - E_m(\varepsilon_f)_{ult}}. \tag{3.166}$$

Example 3.13

Find the ultimate tensile strength for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Also, find the minimum and critical fiber volume fractions.

Solution

From Table 3.1,

$$E_f = 85\ GPa$$

$$(\sigma_f)_{ult} = 1550\ MPa.$$

Thus,

$$(\varepsilon_f)_{ult} = \frac{1550 \times 10^6}{85 \times 10^9}$$

$$= 0.1823 \times 10^{-1}.$$

From Table 3.2,

$$E_m = 3.4\ GPa$$

$$(\sigma_m)_{ult} = 72\ MPa.$$

Thus,

$$(\varepsilon_m)_{ult} = \frac{72 \times 10^6}{3.4 \times 10^9}$$

$$= 0.2117 \times 10^{-1}.$$

Applying Equation (3.164), the ultimate longitudinal tensile strength is

$$(\sigma_1^T)_{ult} = (1550 \times 10^6)(0.7) + (0.1823 \times 10^{-1})(3.4 \times 10^9)(1 - 0.7)$$

$$= 1104 \ MPa.$$

Applying Equation (3.165), the minimum fiber volume fraction is

$$(V_f)_{miminum} = \frac{72 \times 10^6 - (3.4 \times 10^9)(0.1823 \times 10^{-1})}{1550 \times 10^6 - (3.4 \times 10^9)(0.1823 \times 10^{-1}) + 72 \times 10^6}$$

$$= 0.6422 \times 10^{-2}$$

$$= 0.6422\%.$$

This implies that, if the fiber volume fraction is less than 0.6422%, the matrix can take more loading after all the fibers break. Applying Equation (3.166), the critical fiber volume fraction is

$$(V_f)_{critical} = \frac{72 \times 10^6 - (3.4 \times 10^9)(0.1823 \times 10^{-1})}{1550 \times 10^6 - (3.4 \times 10^9)(0.1823 \times 10^{-1})}$$

$$= 0.6732 \times 10^{-2}$$

$$= 0.6732\%.$$

This implies that, if the fiber volume fraction were less than 0.6732%, the composite longitudinal tensile strength would be less than that of the matrix.

Experimental evaluation: The general test method recommended for tensile strength is the ASTM test method for tensile properties of fiber–resin composites (D3039) (Figure 3.25). A tensile test geometry (Figure 3.26) to find

FIGURE 3.25
Tensile coupon mounted in the test frame for finding the tensile strengths of a unidirectional lamina. (Photo courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH.)

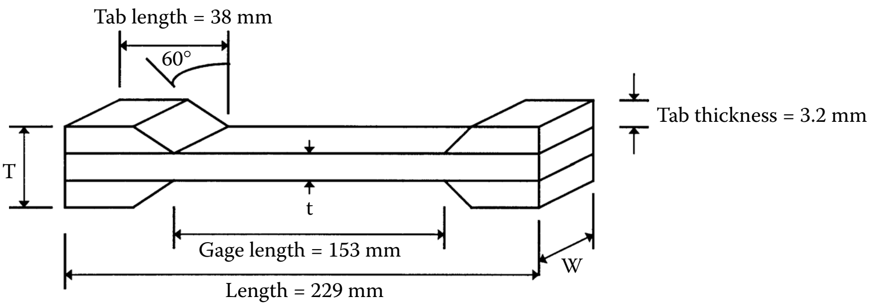


FIGURE 3.26
Geometry of a longitudinal tensile strength specimen.

the longitudinal tensile strength consists of six to eight 0° plies that are 12.5 mm (1/2 in.) wide and 229 mm (10 in.) long. The specimen is mounted with strain gages in the longitudinal and transverse directions. Tensile stresses are applied on the specimen at a rate of about 0.5 to 1 mm/min (0.02 to 0.04 in./min). A total of 40 to 50 data points for stress and strain is taken until a specimen fails. The stress in the longitudinal direction is plotted as a function

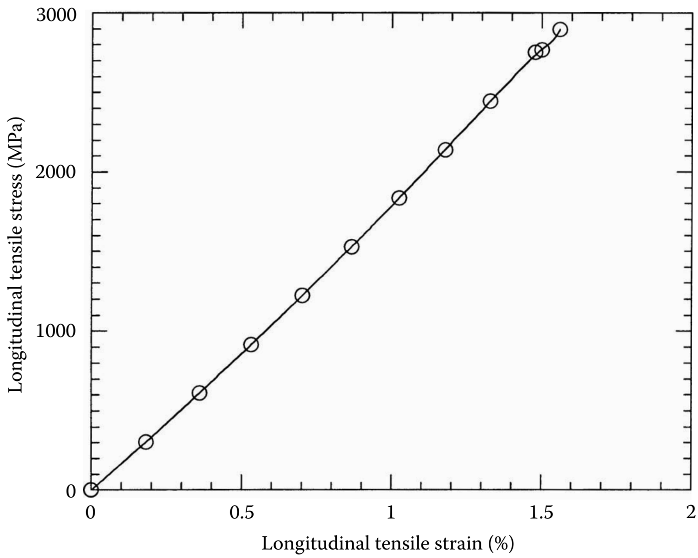


FIGURE 3.27
Stress–strain curve for a $[0]_8$ laminate under a longitudinal tensile load. (Data courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH).

of longitudinal strain, as shown in Figure 3.27. The data are reduced using linear regression. The longitudinal Young's modulus is the initial slope of the σ_1 vs. ε_1 curve.

From Figure 3.27, the following values are obtained:

$$E_1 = 187.5\ GPa,$$

$$(\sigma_1^T)_{ult} = 2896\ MPa,$$

$$(\varepsilon_1^T)_{ult} = 1.560\%.$$

Discussion: Failure of a unidirectional ply under a longitudinal tensile load takes place with

1. Brittle fracture of fibers
2. Brittle fracture of fibers with pullout
3. Fiber pullout with fiber–matrix debonding

The three failure modes are shown in Figure 3.28. The mode of failure depends on the fiber–matrix bond strength and fiber volume fraction.[20] For low fiber volume fractions, $0 < V_f < 0.40$, a typical glass/epoxy composite

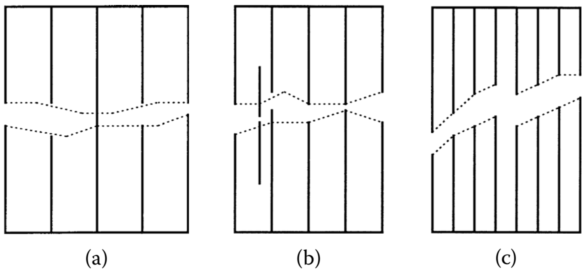


FIGURE 3.28
Modes of failure of unidirectional lamina under a longitudinal tensile load.

exhibits a mode (1) type failure. For intermediate fiber volume fractions, $0.4 < V_f < 0.65$, mode (2) type failure occurs. For high fiber volume fractions, $V_f > 0.65$, it exhibits mode (3) type of failure.

3.4.2 Longitudinal Compressive Strength

The model used for calculating the longitudinal tensile strength for a unidirectional lamina cannot also be used for its longitudinal compressive strength because the failure modes are different. Three typical failure modes are shown in Figure 3.29:

- Fracture of matrix and/or fiber–matrix bond due to tensile strains in the matrix and/or bond
- Microbuckling of fibers in shear or extensional mode
- Shear failure of fibers

Ultimate tensile strains in matrix failure mode: A mechanics of materials approach model based on the failure of the composite in the transverse direction due to transverse tensile strains is given next.[20] Assuming that one is applying a longitudinal compressive stress of magnitude σ_1, then the magnitude of longitudinal compressive strain is given by

$$\left|\varepsilon_1\right| = \frac{\left|\sigma_1\right|}{E_1}. \tag{3.167}$$

Because the major Poisson's ratio is ν_{12}, the transverse strain is tensile and is given by

$$\left|\varepsilon_2\right| = \nu_{12}\frac{\left|\sigma_1\right|}{E_1}. \tag{3.168}$$

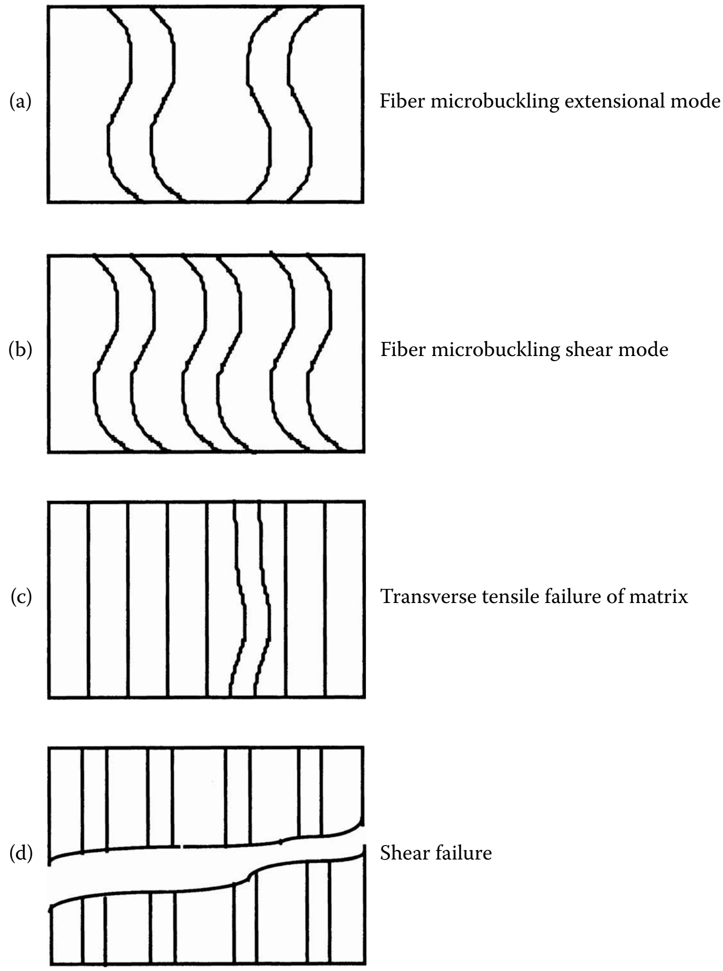


FIGURE 3.29
Modes of failure of a unidirectional lamina under a longitudinal compressive load.

Using maximum strain failure theory, if the transverse strain exceeds the ultimate transverse tensile strain, $(\varepsilon_2^T)_{ult}$, the lamina is considered to have failed in the transverse direction. Thus,

$$(\sigma_1^c)_{ult} = \frac{E_1(\varepsilon_2^T)_{ult}}{\nu_{12}}. \tag{3.169}$$

The value of the longitudinal modulus, E_1, and the major Poisson's ratio, v_{12}, can be found from Equation (3.34) and Equation (3.49), respectively. However, for the value of $(\varepsilon_2^T)_{ult}$, one can use the empirical formula,

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}(1 - V_f^{1/3}), \tag{3.170}$$

or the mechanics of materials formula,

$$(\varepsilon_2^T)_{ult} = (\varepsilon_m^T)_{ult}\left[\frac{d}{s}\left(\frac{E_m}{E_f} - 1\right) + 1\right], \tag{3.171}$$

where
$(\varepsilon_m^T)_{ult}$ = ultimate tensile strain of the matrix
d = diameter of the fibers
s = center-to-center spacing between the fibers

Equation (3.170) and Equation (3.171) will be discussed later in Section 3.4.3.

Shear/extensional fiber microbuckling failure mode: local buckling models for calculating longitudinal compressive strengths have been developed.[21,22] Because these results are based on advanced topics, only the final expressions are given:

$$(\sigma_1^c)_{ult} = \min[S_1^c, S_2^c], \tag{3.172}$$

where

$$S_1^c = 2\left[V_f + (1 - V_f)\frac{E_m}{E_f}\right]\sqrt{\frac{V_f E_m E_f}{3(1 - V_f)}}, \tag{3.173a}$$

and

$$S_2^c = \frac{G_m}{1 - V_f}. \tag{3.173b}$$

Note that the extensional mode buckling stress (S_1^c) is higher than the shear mode buckling stress (S_2^c) for most cases. Extensional mode buckling is prevalent only in low fiber volume fraction composites.

Shear stress failure of fibers mode: A unidirectional composite may fail due to direct shear failure of fibers. In this case, the rule of mixtures gives the shear strength of the unidirectional composite as

$$(\tau_{12})_{ult} = (\tau_f)_{ult} V_f + (\tau_m)_{ult} V_m, \tag{3.174}$$

where

$(\tau_f)_{ult}$ = ultimate shear strength of the fiber
$(\tau_m)_{ult}$ = ultimate shear strength of the matrix

The maximum shear stress in a lamina under a longitudinal compressive load σ_1^c is $(\sigma_1^c)/2$ at 45° to the loading axis. Thus,

$$(\sigma_1^c)_{ult} = 2\left[(\tau_f)_{ult}V_f + (\tau_m)_{ult}V_m\right]. \tag{3.175}$$

Three models based on each of the preceding failure modes were discussed to find the magnitude of the ultimate longitudinal compressive strength. One may caution that these models are not found to match the experimental results as is partially evident in the comparison of experimental and predicted values[23] of longitudinal compressive strength given in Table 3.6. Comparison with other equations (3.169) and (3.175) is not available because the properties of constituents are not given in the reference,[23] although fiber buckling is the most probable mode of failure in advanced polymer matrix composites.

Several factors may contribute to this discrepancy, including

- Irregular spacing of fibers causing premature failure in matrix-rich areas
- Less than perfect bonding between the fibers and the matrix
- Poor alignment of fibers
- Not accounting for Poisson's ratio mismatch between the fiber and the matrix
- Not accounting for the transversely isotropic nature of fibers such as aramids and graphite

In addition, there is controversy concerning the techniques used in measuring compressive strengths.

TABLE 3.6

Comparison of Experimental and Predicted Values of Longitudinal Compressive Strength of Unidirectional Laminae[a]

Material	Experimental strength	Equation (3.78a) (MPa)	Equation (3.78b) (MPa)
Glass/polyester	600–1000	8700	2200
Type I carbon/epoxy	700–900	22,800	2900
Kevlar 49/epoxy	240–290	13,200	2900

[a] $V_f = 0.50$.

Source: Hull, D., *Introduction to Composite Materials*, Cambridge University Press, 1981, Table 7.2. Reprinted with the permission of Cambridge University Press.

Example 3.14

Find the longitudinal compressive strength of a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that fibers are circular and are in a square array.

Solution

From Table 3.1, the Young's modulus for the fiber is

$$E_f = 85 \; GPa$$

and the Poisson's ratio of the fiber is

$$\nu_f = 0.20.$$

The ultimate tensile strength of the fiber is

$$(\sigma_f)_{ult} = 1550 \; MPa$$

and the ultimate shear strength of the fiber is

$$(\tau_f)_{ult} = 35 \; MPa.$$

From Table 3.2, the Young's modulus of the matrix is

$$E_m = 3.4 \; GPa$$

and the Poisson's ratio of the matrix is

$$\nu_m = 0.30.$$

The ultimate normal strength of the matrix is

$$(\sigma_m)_{ult} = 72 \; MPa$$

and the ultimate shear strength of the matrix is

$$(\tau_m)_{ult} = 34 \; MPa.$$

From Example 3.3, the longitudinal Young's modulus of the unidirectional lamina is

$$E_1 = 60.52 \; GPa.$$

From Example 3.5, the major Poisson's ratio of the unidirectional lamina is

$$\nu_{12} = 0.23.$$

Using Equation (3.42a), the fiber diameter to fiber spacing ratio is

$$\frac{d}{s} = \left[\frac{4(0.7)}{\pi}\right]^{1/2}$$

$$= 0.9441.$$

The ultimate tensile strain of the matrix is

$$(\varepsilon_m)_{ult} = \frac{72\times10^6}{3.40\times10^9}$$

$$= 0.2117\times10^{-1}.$$

Using the transverse ultimate tensile strain failure mode formula (3.76),

$$(\varepsilon_2^T)_{ult} = 0.2117\times10^{-1}\left[0.9441\left(\frac{3.4\times10^9}{85\times10^9}-1\right)+1\right]$$

$$= 0.1983\times10^{-2}.$$

From the empirical Equation (3.170),

$$(\varepsilon_2^T)_{ult} = (0.2117\times10^{-1})(1-0.7^{1/3})$$

$$= 0.2373\times10^{-2}.$$

Using the lesser of these two values of ultimate transverse tensile strain, $(\varepsilon_2^T)_{ult}$, and Equation (3.169),

$$(\sigma_1^C)_{ult} = \frac{(60.52\times10^9)(0.1983\times10^{-2})}{0.23}$$

$$= 521.8\ MPa.$$

Using shear/extensional fiber microbuckling failure mode formulas (3.173a),

$$S_1^C = 2\left[0.7+(1-0.7)\frac{3.4\times10^9}{85\times10^9}\right]\sqrt{\frac{(0.7)(3.4\times10^9)(85\times10^9)}{3(1-0.7)}}$$

$$= 21349\ MPa.$$

From Example 3.6, the shear modulus of the matrix is

$$G_m = 1.308\ GPa.$$

Using Equation (3.173b),

$$S_2^C = \frac{1.308 \times 10^9}{1 - 0.7}$$
$$= 4360\ MPa.$$

Thus, from Equation (3.172), the ultimate longitudinal compressive strength is

$$(\sigma_1^C)_{ult} = \min(21349, 4360) = 4360\ MPa.$$

Using shear stress failure of fibers mode, the ultimate longitudinal compressive strength from Equation (3.175) is

$$(\sigma_1^C)_{ult} = 2\left[(35 \times 10^6)(0.7) + (34 \times 10^6)(0.3)\right]$$
$$= 69.4\ MPa.$$

Taking the minimum value of the preceding, the ultimate longitudinal compressive strength is predicted as

$$(\sigma_1^C)_{ult} = 69.4\ MPa$$

Experimental evaluation: The compressive strength of a lamina has been found by several different methods. A highly recommended method is the IITRI (Illinois Institute of Technology Research Institute), compression test.[24] Figure 3.30 shows the (ASTM D3410 Celanese) IITRI fixture mounted in a test frame. A specimen (Figure 3.31) consists generally of 16 to 20 plies of 0° lamina that are 6.4 mm (1/4 in.) wide and 127 mm (5 in.) long. Strain gages are mounted in the longitudinal direction on both faces of the specimen to check for parallelism of the edges and ends. The specimen is compressed at a rate of 0.5 to 1 mm/min (0.02 to 0.04 in./min). A total of 40 to 50 data points for stress and strain are taken until the specimen fails. The stress in the longitudinal direction is plotted as a function of longitudinal strain and is shown for a typical graphite/epoxy lamina in Figure 3.32. The data are reduced using linear regression and the modulus is the initial slope of the stress–strain curve. From Figure 3.32, the following values are obtained:

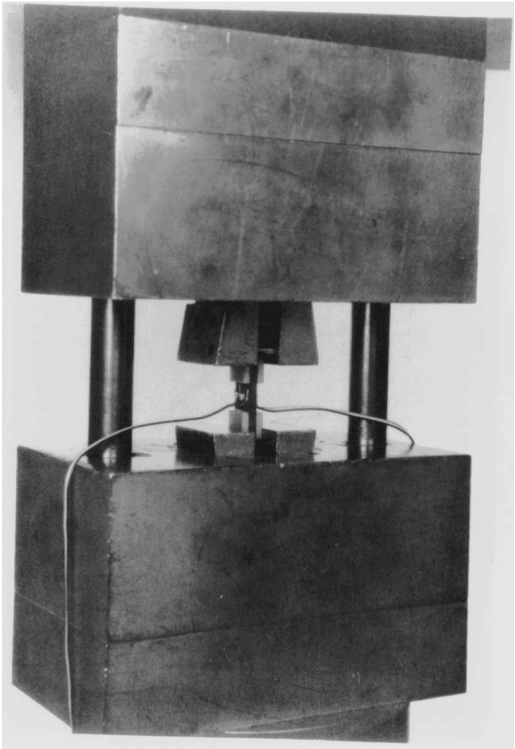


FIGURE 3.30
IITRI fixture mounted in a test frame for finding the compressive strengths of a lamina. (Data reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 76. Copyright CRC Press, Boca Raton, FL.)

$$E_1^c = 199\ GPa,$$

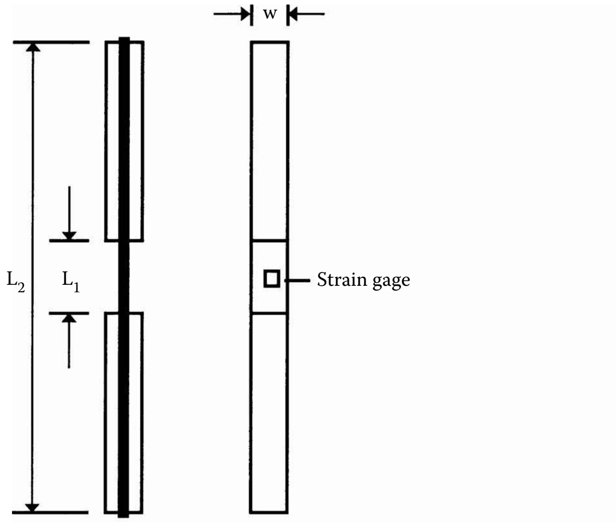
$$(\sigma_1^c)_{ult} = 1908\ MPa, \text{ and}$$

$$(\varepsilon_1^c)_{ult} = 0.9550\%.$$

3.4.3 Transverse Tensile Strength

A mechanics of materials approach model for finding the transverse tensile strength of a unidirectional lamina is given next.[25] Assumptions used in the model include

- A perfect fiber–matrix bond
- Uniform spacing of fibers



Specimen dimensions		
L_1, mm	L_2, mm	w^*, mm
12.7 ± 1	12.7 ± 1.5	12.7 ± 0.1 or 6.4 ± 0.1

FIGURE 3.31
Geometry of a longitudinal compressive strength specimen. (Data reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 76. Copyright CRC Press, Boca Raton, FL.)

- The fiber and matrix follow Hooke's law
- There are no residual stresses

Assume a plane model of a composite as shown by the shaded portion in Figure 3.33. In this case,

s = distance between center of fibers

d = diameter of fibers

The transverse deformations of the fiber, δ_f, the matrix, δ_m, and the composite, δ_c, are related by

$$\delta_c = \delta_f + \delta_m. \tag{3.176}$$

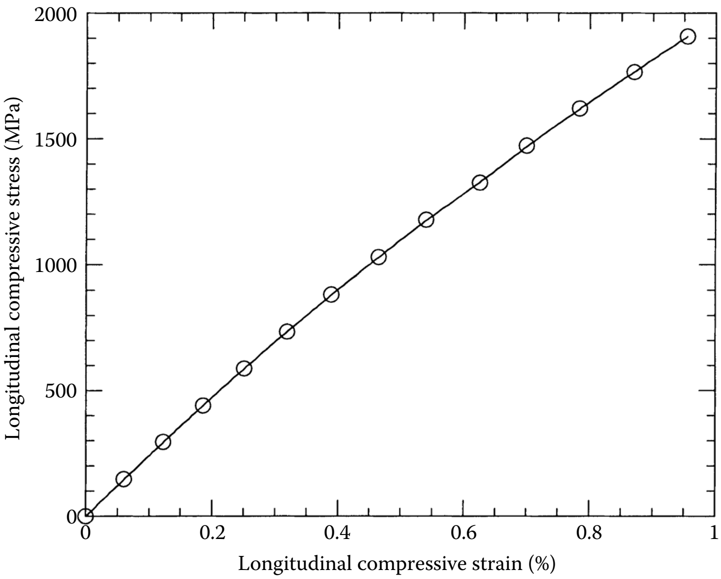


FIGURE 3.32
Stress–strain curve for a $[0]_{24}$ graphite/epoxy laminate under a longitudinal compressive load. (Data courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH.)

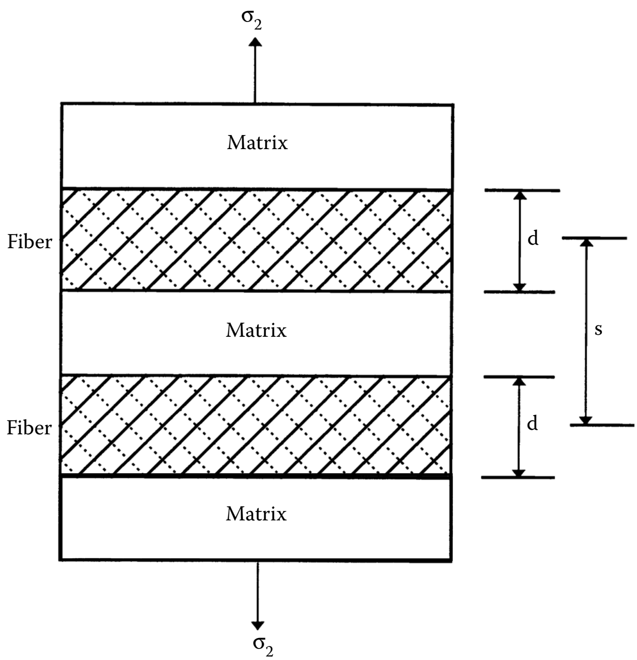


FIGURE 3.33
Representative volume element to calculate transverse tensile strength of a unidirectional lamina.

Now, by the definition of strain, the deformations are related to the transverse strains,

$$\delta_c = s\varepsilon_c \ , \tag{3.177a}$$

$$\delta_f = d\varepsilon_f \ , \tag{3.177b}$$

$$\delta_m = (s-d)\varepsilon_m \ , \tag{3.177c}$$

where $\varepsilon_{c,f,m}$ = the transverse strain in the composite, fiber, and matrix, respectively.

Substituting the expressions in Equation (3.82) in Equation (3.81), we get

$$\varepsilon_c = \frac{d}{s}\varepsilon_f + \left(1 - \frac{d}{s}\right)\varepsilon_m. \tag{3.178}$$

Now, under transverse loading, one assumes that the stresses in the fiber and matrix are equal (see derivation of transverse Young's modulus in Section 3.2.1.2). Then, the strains in the fiber and matrix are related through Hooke's law as

$$E_f\varepsilon_f = E_m\varepsilon_m. \tag{3.179}$$

Substituting the expression for the transverse strain in the fiber, ε_f, in Equation (3.178), the transverse strain in the composite

$$\varepsilon_c = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right]\varepsilon_m. \tag{3.180}$$

If one assumes that the transverse failure of the lamina is due to the failure of the matrix, then the ultimate transverse failure strain is

$$(\varepsilon_2^T)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right](\varepsilon_m^T)_{ult}, \tag{3.181}$$

where $(\varepsilon_m^T)_{ult}$ = ultimate tensile failure strain of the matrix.

The ultimate transverse tensile strength is then given by

$$(\sigma_2^T)_{ult} = E_2(\varepsilon_2^T)_{ult}, \tag{3.182}$$

where $(\varepsilon_2^T)_{ult}$ is given by Equation (3.181). The preceding expression assumes that the fiber is perfectly bonded to the matrix. If the adhesion between the fiber and matrix is poor, the composite transverse strength will be further reduced.

Example 3.15

Find the ultimate transverse tensile strength for a unidirectional glass/epoxy lamina with a 70% fiber volume fraction. Use properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that the fibers are circular and arranged in a square array.

Solution

From Example 3.14, the ultimate transverse tensile strain of the lamina

$$(\varepsilon_2^T)_{ult} = 0.1983 \times 10^{-2}$$

is the lower estimate from using Equation (3.170) and Equation (3.171).

From Example 3.4, the transverse Young's modulus of the lamina is E_2 = 10.37 GPa.

Using Equation (3.182), the ultimate transverse tensile strength of the lamina is

$$(\sigma_2^T)_{ult} = (10.37 \times 10^9)(0.1983 \times 10^{-2})$$
$$= 20.56 \; MPa.$$

Experimental evaluation: The procedure for finding the transverse tensile strength is the same as for finding the longitudinal tensile strength. Only the specimen dimensions differ. The standard width of the specimen is 25.4 mm (1 in.) and 8 to 16 plies are used. This is mainly done to increase the amount of load required to break the specimen. Figure 3.34 shows the typical stress–strain curve for a 90° graphite/peek laminate. From Figure 3.34, the following data are obtained:

$$E_2 = 9.963 \text{ GPa}$$

$$(\sigma_2^T)_{ult} = 53.28 \; MPa$$

$$(\varepsilon_2^T)_{ult} = 0.5355\%$$

Discussion: Predicting transverse tensile strength is quite complicated. Under a transverse tensile load, factors other than the individual properties of the fiber and matrix are important. These include the bond strength between the fiber and the matrix, the presence of voids, and the presence

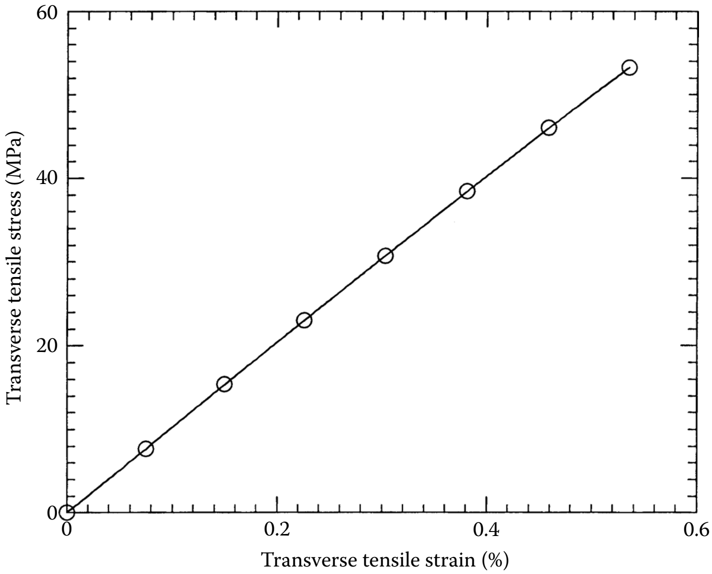


FIGURE 3.34
Stress–strain curve for a $[90]_{16}$ graphite/epoxy laminate under a transverse tensile load. (Data courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH.)

of residual stresses due to thermal expansion mismatch between the fiber and matrix. Possible modes of failure under transverse tensile stress include matrix tensile failure accompanied by fiber matrix debonding and/or fiber splitting.

3.4.4 Transverse Compressive Strength

Equation (3.182), which was developed for evaluating transverse tensile strength, can be used to find the transverse compressive strengths of a lamina. The actual compressive strength is again lower due to imperfect fiber/matrix interfacial bond and longitudinal fiber splitting. Using compressive parameters in Equation (3.182),

$$(\sigma_2^C)_{ult} = E_2(\varepsilon_2^C)_{ult}, \tag{3.183}$$

where

$$(\varepsilon_2^C)_{ult} = \left[\frac{d}{s}\frac{E_m}{E_f} + \left(1 - \frac{d}{s}\right)\right](\varepsilon_m^C)_{ult}, \tag{3.184}$$

$(\varepsilon_m^C)_{ult}$ = ultimate compressive failure strain of matrix.

Example 3.16

Find the ultimate transverse compressive strength of a glass/epoxy lamina with 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that the fibers are circular and packed in a square array.

Solution

From Table 3.1, the Young's modulus of the fiber is E_f = 85 GPa.

From Table 3.2, the Young's modulus of the matrix is E_m = 3.4 GPa and the ultimate compressive strength of the matrix is

$$(\sigma_m^C)_{ult} = 102 \ MPa.$$

From Example 3.4, the transverse Young's modulus is E_2 = 10.37 GPa.

From Example 3.14, the fiber diameter to fiber spacing ratio is

$$\frac{d}{s} = 0.9441.$$

The ultimate compressive strain of the matrix is

$$(\varepsilon_m^C) = \frac{102 \times 10^6}{3.4 \times 10^9}$$
$$= 0.03.$$

From Equation (3.184), the ultimate transverse compressive strain of the lamina is

$$(\varepsilon_2^C) = \left[0.9441 \frac{3.4 \times 10^9}{85 \times 10^9} + (1 - 0.9441)\right](0.03)$$
$$= 0.2810 \times 10^{-2},$$

and from Equation (3.183), the ultimate transverse compressive strength is

$$(\sigma_2^C)_{ult} = (10.37 \times 10^9)(0.2810 \times 10^{-2}) = 29.14 \ MPa.$$

Experimental evaluation: The procedure for finding the transverse compressive strength is the same as that for finding the longitudinal compressive strength. The only difference is in the specimen dimensions. The width of the specimen is 12.7 mm (1/2 in.) and 30 to 40 plies are used in the test.

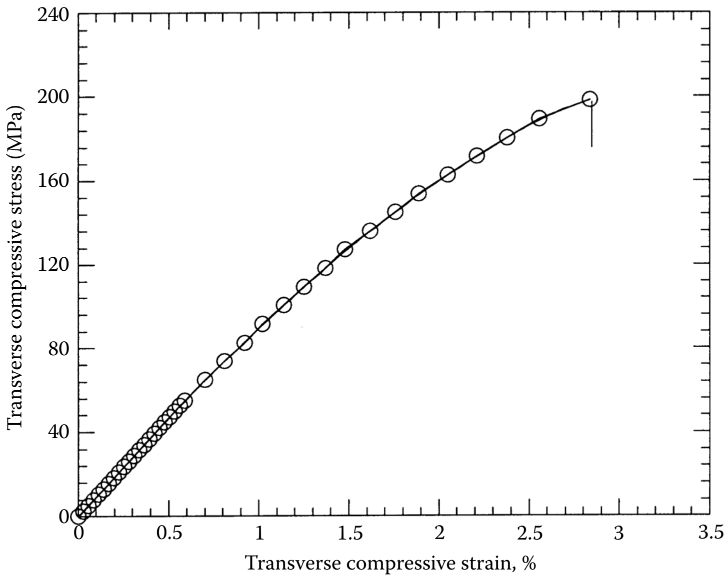


FIGURE 3.35
Stress–strain curve for a $[90]_{40}$ graphite/epoxy laminate under a transverse compressive load perpendicular to the fibers. (Data reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 79.)

Figure 3.35 shows the typical stress–strain curve for a 90° graphite/epoxy laminate. From Figure 3.35, the following data are obtained[26]:

$$E_2^c = 93\ GPa,$$

$$(\sigma_2^c)_{ult} = 198\ MPa,$$

$$(\varepsilon_2^c)_{ult} = 2.7\%.$$

Discussion: Methods for predicting transverse compressive strength are also not yet satisfactory. Several modes of failure possible under a transverse compressive stress include matrix compressive failure, matrix shear failure, and matrix shear failure with fiber–matrix debonding and/or fiber crushing.

3.4.5 In-Plane Shear Strength

The procedure for finding the ultimate shear strength for a unidirectional lamina using a mechanics of materials approach follows that described in Section 3.4.3. Assume that one is applying a shear stress of magnitude τ_{12}

and then that the shearing deformation in the representative element is given by the sum of the deformations in the fiber and matrix,

$$\Delta_c = \Delta_f + \Delta_m. \tag{3.185}$$

By definition of shearing strain,

$$\Delta_c = s(\gamma_{12})_c, \tag{3.186a}$$

$$\Delta_f = d(\gamma_{12})_f, \tag{3.186b}$$

and

$$\Delta_m = (s-d)(\gamma_{12})_m, \tag{3.186c}$$

where $(\gamma_{12})_{c,f,m}$ = the in-plane shearing strains in the composite, fiber, and matrix, respectively.

Substituting the Equation (3.186a-c) in Equation (3.185),

$$(\gamma_{12})_c = \frac{d}{s}(\gamma_{12})_f + \left(1-\frac{d}{s}\right)(\gamma_{12})_m. \tag{3.187}$$

Now, under shearing stress loading, one assumes that the shear stress in the fiber and matrix are equal (see derivation of shear modulus in Section 3.3.1.4). Then, the shearing strains in the fiber and matrix are related as

$$(\gamma_{12})_m G_m = (\gamma_{12})_f G_f. \tag{3.188}$$

Substituting the expression for $(\gamma_{12})_f$ from Equation (3.188) in Equation (3.187),

$$(\gamma_{12})_c = \left[\frac{d}{s}\frac{G_m}{G_f} + \left(1-\frac{d}{s}\right)\right](\gamma_{12})_m. \tag{3.189}$$

If one assumes that the shear failure is due to failure of the matrix, then

$$(\gamma_{12})_{ult} = \left[\frac{d}{s}\frac{G_m}{G_f} + \left(1-\frac{d}{s}\right)\right](\gamma_{12})_{mult}, \tag{3.190}$$

where $(\gamma_{12})_{m\,ult}$ = ultimate shearing strain of the matrix.

The ultimate shear strength is then given by

$$
\begin{aligned}
(\tau_{12})_{ult} &= G_{12}(\gamma_{12})_{ult} \\
&= G_{12}\left[\frac{d}{s}\frac{G_m}{G_f} + \left(1 - \frac{d}{s}\right)\right](\gamma_{12})_{mult}.
\end{aligned}
\tag{3.191}
$$

Example 3.17

Find the ultimate shear strength for a glass/epoxy lamina with 70% fiber volume fraction. Use properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that the fibers are circular and arranged in a square array.

Solution

From Example 3.6, the shear modulus of the fiber is

$$G_f = 35.42\ GPa,$$

the shear modulus of the matrix is

$$G_m = 1.308\ GPa,$$

and the in-plane shear modulus of the lamina is

$$G_{12} = 4.014\ GPa.$$

From Example 3.14, the fiber diameter to fiber spacing ratio is

$$\frac{d}{s} = 0.9441.$$

From Table 3.2, the ultimate shear strength of the matrix is

$$(\tau_{12})_{mult} = 34\ MPa.$$

Then, the ultimate shearing strain of the matrix is

$$
\begin{aligned}
(\gamma_{12})_{mult} &= \frac{34 \times 10^6}{1.308 \times 10^9} \\
&= 0.2599 \times 10^{-1}.
\end{aligned}
$$

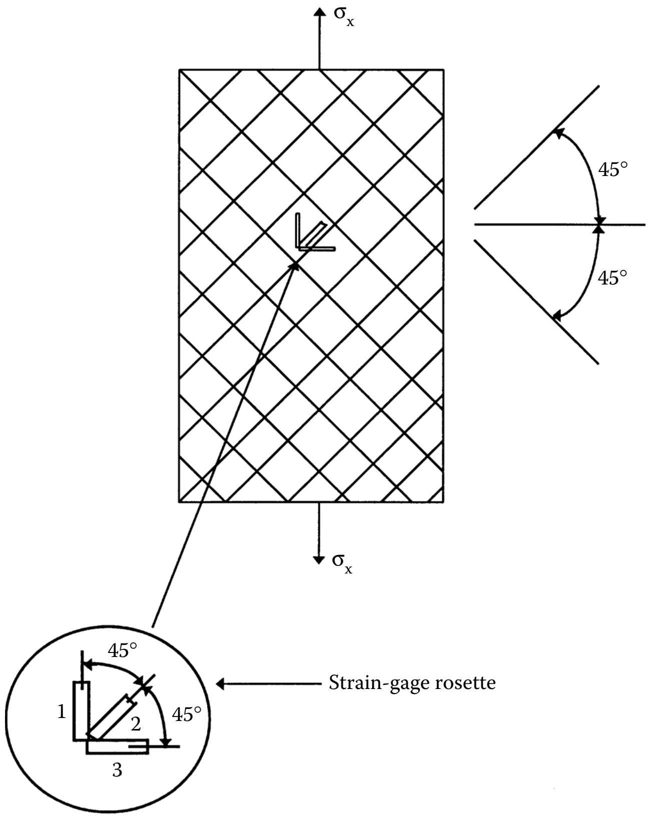


FIGURE 3.36
Schematic of a $[\pm 45]_{2S}$ laminate shear test.

Using Equation (3.191), the ultimate in-plane shear strength of the unidirectional lamina is

$$(\tau_{12})_{ult} = (4.014 \times 10^9)\left[0.9441 \frac{1.308 \times 10^9}{35.42 \times 10^9} + (1 - 0.9441)\right](0.2599 \times 10^{-1})$$

$$= 9.469 \ MPa.$$

Experimental determination: One of the most recommended methods[27] for calculating the in-plane shear strength is the $[\pm 45]_{2S}$ laminated tensile coupon* (Figure 3.36). A $[\pm 45]_{2S}$ laminate is an eight-ply laminate with [+45/–45/+45/–45/–45/+45/–45/+45] distribution of plies on top of each other.

* See Section 4.2 of Chapter 4 for an explanation on laminate codes.

An axial stress σ_x is applied to the eight-ply laminate; the axial strain ε_x and transverse strain ε_y are measured. If the laminate fails at a load of $(\sigma_x)_{ult}$, the ultimate shear strength of a unidirectional lamina is given by

$$(\tau_{12})_{ult} = \frac{(\sigma_x)_{ult}}{2}, \tag{3.192}$$

and the ultimate shear strain of a unidirectional lamina is

$$(\gamma_{12})_{ult} = \left(|\varepsilon_x|\right)_{ult} + \left(|\varepsilon_y|\right)_{ult}. \tag{3.193}$$

An eight-ply $[\pm 45]_{2S}$ laminate is used for several reasons. First, according to maximum stress and strain failure theories of Chapter 2, each lamina fails in the shear mode and at the same load. The stress at which it fails is simply twice the shear strength of a unidirectional lamina and is independent of the other mechanical properties of the lamina, as reflected in Equation (3.192). Second, the shear strain is measured simply by strain gages in two perpendicular directions and does not require the values of elastic constants of the lamina.

Equation (3.192) and Equation (3.193) can be derived using concepts from Chapter 4 and Chapter 5. The in-plane shear strength is simply half of the maximum uniaxial stress that can be applied to the laminate. The initial slope of the τ_{12} vs. γ_{12} curve gives the shear modulus, G_{12}. A total of 40 to 50 points are taken for the stress and strains until the specimen fails. From Figure 3.37, the following values are obtained for a typical graphite/epoxy lamina:

$$G_{12} = 5.566 \ GPa,$$

$$(\tau_{12}) = 87.57 \ MPa,$$

$$(\gamma_{12})_{ult} = 2.619\%.$$

Discussion: The prediction of the ultimate shear strength is complex. Similar parameters, such as weak interfaces, the presence of voids, and Poisson's ratio mismatch, make modeling quite complex.

Theoretical methods for obtaining the strength parameters also include statistical and advanced methods. Statistical methods include accounting for variations in fiber strength, fiber–matrix adhesion, voids, fiber spacing, fiber diameter, alignment of fibers, etc. Advanced methods use elasticity, finite element methods, boundary element methods, finite difference methods, etc.

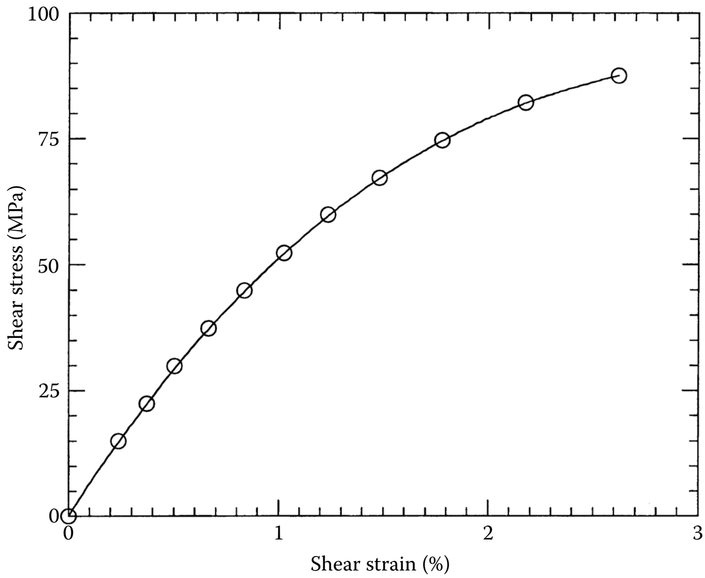


FIGURE 3.37
Shear stress–shear strain curve obtained from a $[\pm 45]_{2S}$ graphite/epoxy laminate under a tensile load. (Data courtesy of Dr. R.Y. Kim, University of Dayton Research Institute, Dayton, OH.)

3.5 Coefficients of Thermal Expansion

When a body undergoes a temperature change, its dimensions relative to its original dimensions change in proportion to the temperature change. The coefficient of thermal expansion is defined as the change in the linear dimension of a body per unit length per unit change of temperature.

For a unidirectional lamina, the dimensions changes differ in the two directions 1 and 2. Thus, the two coefficients of thermal expansion are defined as

α_1 = linear coefficient of thermal expansion in direction 1, m/m/°C (in./in./°F)

α_2 = linear coefficient of thermal expansion in direction 2, m/m/°C (in./in./°F)

The following are the expressions developed for the two thermal expansion coefficients using the thermoelastic extremum principle[28]:

$$\alpha_1 = \frac{1}{E_1}(\alpha_f E_f V_f + \alpha_m E_m V_m), \tag{3.194}$$

$$\alpha_2 = (1+\nu_f)\alpha_f V_f + (1+\nu_m)\alpha_m V_m - \alpha_1\nu_{12}, \tag{3.195}$$

where α_f and α_m are the coefficients of thermal expansion for the fiber and the matrix, respectively.

3.5.1 Longitudinal Thermal Expansion Coefficient

As an example, Equation (3.194) can be derived using the mechanics of materials approach.[29] Consider the expansion of a unidirectional lamina in the longitudinal direction under a temperature change of ΔT. If only the temperature ΔT is applied, the unidirectional lamina has zero overall load, F_1, in the longitudinal direction. Then

$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m \tag{3.196}$$

$$\sigma_f V_f + \sigma_m V_m = 0\ , \tag{3.197}$$

where

$A_{c,f,m}$ = the cross-sectional area of composite, fiber, and matrix, respectively

$\sigma_{1,f,m}$ = the stress in composite, fiber, and matrix, respectively

Although the overall load in the longitudinal direction 1 is zero, stresses are caused in the fiber and the matrix by the thermal expansion mismatch between the fiber and the matrix. These stresses are

$$\sigma_f = E_f(\varepsilon_f - \alpha_f \Delta T), \tag{3.198a}$$

and

$$\sigma_m = E_m(\varepsilon_m - \alpha_m \Delta T)\ . \tag{3.198b}$$

Substituting Equation (3.198a) and Equation (3.198b) in Equation (3.197) and realizing that the strains in the fiber and matrix are equal ($\varepsilon_f = \varepsilon_m = \varepsilon_1$),

$$\varepsilon_f = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m}\Delta T. \tag{3.199}$$

For free expansion in the composite in the longitudinal direction 1, the longitudinal strain is

$$\varepsilon_1 = \alpha_1 \Delta T. \tag{3.200}$$

Because the strains in the fiber and composite are also equal ($\varepsilon_1 = \varepsilon_f$), from Equation (3.199) and Equation (3.200),

$$\alpha_1 = \frac{\alpha_f E_f V_f + \alpha_m E_m V_m}{E_f V_f + E_m V_m}.$$

Using Equation (3.34) for the definition of longitudinal Young's modulus,

$$\alpha_1 = \frac{1}{E_1}(\alpha_f E_f V_f + \alpha_m E_m V_m)\ . \tag{3.201}$$

The longitudinal coefficient of thermal expansion can be rewritten as

$$\alpha_1 = \left(\frac{\alpha_f E_f}{E_1}\right) V_f + \left(\frac{\alpha_m E_m}{E_1}\right) V_m\ , \tag{3.202}$$

which shows that it also follows the rule of mixtures based on the weighted mean of $\alpha E/E_1$ of the constituents.

3.5.2 Transverse Thermal Expansion Coefficient

Due to temperature change, ΔT, assume that the compatibility condition that the strain in the fiber and matrix is equal in direction 1 — that is,

$$\in_m = \in_f = \in_1\ . \tag{3.203}$$

Now, the stress in the fiber in the longitudinal direction 1 is

$$\begin{aligned}\left(\sigma_f\right)_1 &= E_f\left(\in_f\right)_1 \\ &= E_f \in_1 \\ &= E_f\left(\alpha_1 - \alpha_f\right)\Delta T\end{aligned} \tag{3.204}$$

and the stress in the matrix in longitudinal direction 1 is

$$\begin{aligned}\left(\sigma_m\right)_1 &= E_m\left(\in_m\right)_1 \\ &= E_m \in_1 \\ &= -E_m\left(\alpha_m - \alpha_1\right)\Delta T.\end{aligned} \tag{3.205}$$

The strains in the fiber and matrix in the transverse direction 2 are given by using Hooke's law:

$$\left(\in_f\right)_2 = \alpha_f \Delta T - \frac{\nu_f\left(\sigma_f\right)_1}{E_f} \tag{3.206}$$

$$\left(\in_m\right)_2 = \alpha_m \Delta T - \frac{\nu_m\left(\sigma_m\right)_1}{E_m} . \tag{3.207}$$

The transverse strain of the composite is given by the rule of mixtures as

$$\in_2 = \left(\in_f\right)_2 V_f + \left(\in_m\right)_2 V_m . \tag{3.208}$$

Substituting Equation (3.206) and Equation (3.207) in Equation (3.208),

$$\in_2 = \left[\alpha_f \Delta T - \frac{\nu_f E_f\left(\alpha_1 - \alpha_f\right)\Delta T}{E_f}\right] V_f$$

$$+ \left[\alpha_m \Delta T + \frac{\nu_m E_m\left(\alpha_m - \alpha_1\right)\Delta T}{E_m}\right] V_m \tag{3.209}$$

and, because

$$\in_2 = \alpha_2 \Delta T , \tag{3.210}$$

we get

$$\alpha_2 = \left[\alpha_f - \nu_f\left(\alpha_1 - \alpha_f\right)\right] V_f + \left[\alpha_m + \nu_m\left(\alpha_m - \alpha_1\right)\right] V_m . \tag{3.211}$$

Substituting

$$\nu_{12} = \nu_f V_f + \nu_m V_m \tag{3.212}$$

in the preceding equation, it can be rewritten as

$$\alpha_2 = \left(1 + \nu_f\right)\alpha_f V_f + \left(1 + \nu_m\right)\alpha_m V_m - \alpha_1 \nu_{12} . \tag{3.213}$$

Example 3.18

Find the coefficients of thermal expansion for a glass/epoxy lamina with 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

Solution

From Table 3.1, the Young's modulus of the fiber is

$$E_f = 85 \ GPa$$

and the Poisson's ratio of the fiber is

$$\nu_f = 0.2.$$

The coefficient of thermal expansion of the fiber is

$$\alpha_f = 5 \times 10^{-6} \ m/m/{}^\circ C.$$

From Table 3.2, the Young's modulus of the matrix is

$$E_m = 3.4 \ GPa,$$

the Poisson's ratio of the matrix is

$$\nu_m = 0.3,$$

and the coefficient of thermal expansion of the matrix is

$$\alpha_m = 63 \times 10^{-6} \ m/m/{}^\circ C.$$

From Example 3.3, the longitudinal Young's modulus of the unidirectional lamina is

$$E_1 = 60.52 \ GPa.$$

From Example 3.5, the major Poisson's ratio of the unidirectional lamina is

$$\nu_{12} = 0.2300.$$

Now, substituting the preceding values in Equation (3.194) and Equation (3.195), the coefficients of thermal expansion are

$$\alpha_1 = \frac{1}{60.52 \times 10^9}\left[(5\times10^{-6})(85\times10^9)(0.7) + (63\times10^{-6})(3.4\times10^9)(0.3)\right]$$

$$= 5.978 \times 10^{-6}\ m/m/°C,$$

$$\alpha_2 = (1+0.2)(5.0\times10^{-6})(0.7) + (1+0.3)(63\times10^{-6})(0.3) - (5.978\times10^{-6})(0.23)$$

$$= 27.40 \times 10^{-6}\ m/m/°C$$

In Figure 3.38, the two coefficients of thermal expansion of glass/epoxy are plotted as a function of fiber volume fraction.

It should be noted that the longitudinal thermal expansion coefficient is lower than the transverse thermal expansion coefficient in polymeric matrix composites. Also, in some cases, the thermal expansion coefficient of the fibers is negative, and it is thus possible for a lamina to have zero thermal expansions in the fiber directions. This property is widely used in the manufacturing of antennas, doors, etc., when dimensional stability in the presence of wide temperature fluctuations is desired.

Experimental determinations: The linear coefficients of thermal expansion are determined experimentally by measuring the dimensional changes in a

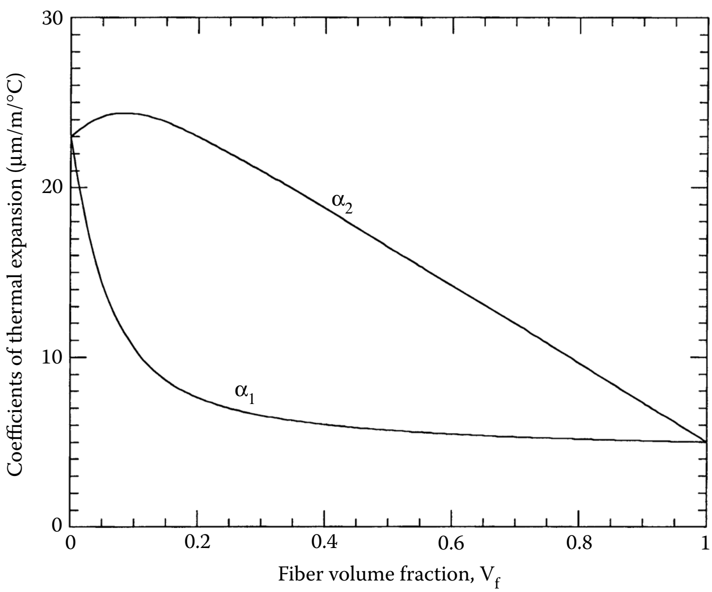


FIGURE 3.38
Longitudinal and transverse coefficients of thermal expansion as a function of fiber volume fraction for a glass/epoxy unidirectional lamina. (Properties of glass and epoxy from Table 3.1 and Table 3.2.)

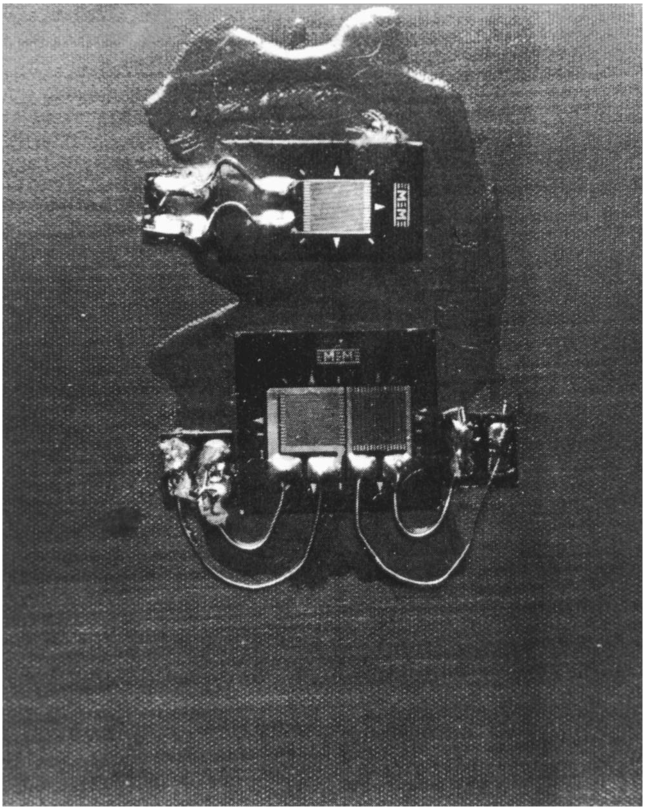


FIGURE 3.39
Unidirectional graphite/epoxy specimen with strain gages and temperature sensors for finding coefficients of thermal expansion. (Reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 98. Copyright CRC Press, Boca Raton, FL.)

lamina that is free of external stresses. A test specimen is made of a 50 × 50 mm (2 in. × 2 in.), eight-ply laminated unidirectional composite (Figure 3.39). Two strain gages are placed perpendicular to each other on the specimen. A temperature sensor is also placed. The specimen is put in an oven and the temperature is slowly increased. Strain and temperature measurements are taken and plotted as a function of each other as given in Figure 3.40. The data are reduced using linear regression. The slope of the two strain-temperature curves directly gives the coefficient of thermal expansion.

From Figure 3.40, the following values are obtained for a typical graphite/epoxy laminate[26]:

$$\alpha_1 = -1.3 \times 10^{-6}\ m/m/°C$$

$$\alpha_2 = 33.9 \times 10^{-6}\ m/m/°C.$$

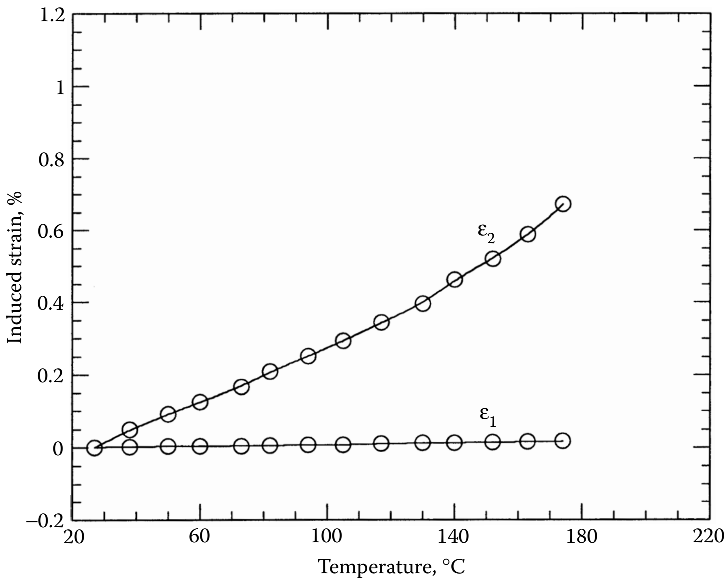


FIGURE 3.40
Induced strain as a function of temperature to find the coefficients of thermal expansion of a unidirectional graphite/epoxy laminate. (Reprinted with permission from *Experimental Characterization of Advanced Composites*, Carlsson, L.A. and Pipes, R.B., Technomic Publishing Co., Inc., 1987, p. 102. Copyright CRC Press, Boca Raton, FL.)

3.6 Coefficients of Moisture Expansion

When a body absorbs water, as is the case for resins in polymeric matrix composites, it expands. The change in dimensions of the body are measured by the coefficient of moisture expansion defined as the change in the linear dimension of a body per unit length per unit change in weight of moisture content per unit weight of the body. Similar to the coefficients of thermal expansion, there are two coefficients of moisture expansion: one in the longitudinal direction 1 and the other in the transverse direction 2:

β_1 = linear coefficient of moisture expansion in direction 1, m/m/kg/kg (in./in./lb/lb)

β_2 = linear coefficient of moisture expansion in direction 2, m/m/kg/kg (in./in./lb/lb)

The following are the expressions for the two coefficients of moisture expansion[30]:

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_1(\Delta C_f \rho_f V_f + \Delta C_m \rho_m V_m)} \rho_c , \tag{3.214}$$

$$\beta_2 = \frac{V_f(1+\nu_f)\Delta C_f \beta_f + V_m(1+\nu_m)\Delta C_m \beta_m}{(V_m \rho_m \Delta C_m + V_f \rho_f \Delta C_f)} \rho_c - \beta_1 \nu_{12} , \tag{3.215}$$

where

ΔC_f = the moisture concentration in the fiber, kg/kg (lb/lb)
ΔC_m = the moisture concentration in the matrix, kg/kg (lb/lb)
β_f = the coefficient of moisture expansion of the fiber, m/m/kg/kg (in./in./lb/lb)
β_m = the coefficient of moisture expansion of the matrix, m/m/kg/kg (in./in./lb/lb)

Note that, unlike the coefficients of thermal expansion, the content of moisture enters into the formula because the moisture absorption capacity in each constituent can be different. However, in most polymeric matrix composites, fibers do not absorb or deabsorb moisture, so the expressions for coefficients of moisture expansion do become independent of moisture contents. Substituting $\Delta C_f = 0$ in Equation (3.214) and Equation (3.215),

$$\beta_1 = \frac{E_m}{E_1} \frac{\rho_c}{\rho_m} \beta_m , \tag{3.216}$$

$$\beta_2 = (1+\nu_m) \frac{\rho_c}{\rho_m} \beta_m - \beta_1 \nu_{12} . \tag{3.217}$$

Further simplification for composites such as graphite/epoxy with high fiber-to-matrix moduli ratio (E_f/E_m) and no moisture absorption by fibers leads to

$$\beta_1 = 0, \text{ and} \tag{3.218}$$

$$\beta_2 = (1+\nu_m) \frac{\rho_c}{\rho_m} \beta_m . \tag{3.219}$$

Similar to the derivation for the longitudinal coefficient of thermal expansion in Section 3.5, Equation (3.214) can be derived using the mechanics of materials approach. Consider the expansion of a unidirectional lamina in the longitudinal direction because of change in moisture content in the composite. The overall load in the composite, F_1, is zero — that is,

$$F_1 = \sigma_1 A_c = 0 = \sigma_f A_f + \sigma_m A_m, \text{ and}$$

$$\sigma_f V_f + \sigma_m V_m = 0 , \tag{3.220}$$

where

$A_{c,f,m}$ = the cross-sectional areas of the fiber, matrix, and composite, respectively

$\sigma_{1,f,m}$ = the stresses in the fiber, matrix, and composite, respectively

The stresses in the fiber and matrix caused by moisture are

$$\sigma_f = E_f(\varepsilon_f - \beta_f \Delta C_f), \tag{3.221}$$

$$\sigma_m = E_m(\varepsilon_m - \beta_m \Delta C_m) . \tag{3.222}$$

Substituting Equation (3.221) and (3.222) in Equation (3.220) and knowing that the strains in the fiber and matrix are equal ($\varepsilon_f = \varepsilon_m$),

$$\varepsilon_f = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_f V_f + E_m V_m} . \tag{3.223}$$

For free expansion of the composite in the longitudinal direction, the longitudinal strain is

$$\varepsilon_1 = \beta_1 \Delta C_c , \tag{3.224}$$

where ΔC_c = the moisture concentration in composite.

Because the strains in the fiber and the matrix are equal,

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m)\Delta C_c}. \tag{3.225}$$

Equation (3.225) can be simplified by relating the moisture concentration in the composite (ΔC_c) to the moisture concentration in the fiber (ΔC_f) and the matrix (ΔC_m).

The moisture content in the composite is the sum of the moisture contents in the fiber and the matrix,

$$\Delta C_c w_c = \Delta C_f w_f + \Delta C_m w_m , \tag{3.226}$$

where $w_{c,f,m}$ = the mass of composite, fiber, and matrix, respectively. Thus,

$$\Delta C_c = \Delta C_f W_f + \Delta C_m W_m \ , \tag{3.227}$$

where $W_{f,m}$ = the mass fractions of the fiber and matrix, respectively.

Substituting Equation (3.227) in Equation (3.225),

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{(E_f V_f + E_m V_m)(\Delta C_f W_f + \Delta C_m W_m)}. \tag{3.228}$$

Using Equation (3.4) and Equation (3.34), one can rewrite Equation (3.228) in terms of fiber volume fractions and the longitudinal Young's modulus as

$$\beta_1 = \frac{\beta_f \Delta C_f V_f E_f + \beta_m \Delta C_m V_m E_m}{E_1(\Delta C_f \rho_f V_f + \Delta C_m \rho_m V_m)} \rho_c. \tag{3.229}$$

Example 3.19

Find the two coefficients of moisture expansion for a glass/epoxy lamina with 70% fiber volume fraction. Use properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that glass does not absorb moisture.

Solution

From Table 3.1, the density of the fiber is

$$\rho_f = 2500 \ kg/m^3.$$

From Table 3.2, the density of the matrix is

$$\rho_m = 1200 \ kg/m^3,$$

the swelling coefficient of the matrix is

$$\beta_m = 0.33 \ m/m/kg/kg,$$

and the Young's modulus of the matrix is

$$E_m = 3.4 \ GPa.$$

The Poisson's ratio of the matrix is

$$\nu_m = 0.3.$$

From Example 3.1, the density of the composite is

$$\rho_c = 2110 \; kg/m^3.$$

From Example 3.3, the longitudinal Young's modulus of the lamina is

$$E_1 = 60.52 \; GPa.$$

From Example 3.5, the major Poisson's ratio is

$$\nu_{12} = 0.230.$$

Thus, the longitudinal coefficient of moisture expansion from Equation (3.216) is

$$\beta_1 = \frac{3.4 \times 10^9}{60.52 \times 10^9} \frac{2110}{1200}(0.33)$$
$$= 0.3260 \times 10^{-1} \; m/m/kg/kg,$$

and the transverse coefficient of moisture expansion from Equation (3.217) is

$$\beta_2 = (1 + 0.3)\frac{2110}{1200}(0.33) - (0.3260 \times 10^{-1})(0.230)$$
$$= 0.7468 \; m/m/kg/kg.$$

Experimental determination: A specimen is placed in water and the moisture expansion strain is measured in the longitudinal and transverse directions. Because moisture attacks strain gage adhesives, micrometers are used to find the swelling strains.

3.7 Summary

After developing the concepts of fiber volume and weight fractions, we developed equations for density and void content. We found the four elastic moduli constants of a unidirectional lamina using three analytical approaches: strength of materials, Halphin–Tsai, and elasticity. Analytical models and experimental techniques for the five strength parameters, the two coefficients of thermal expansion, and the two coefficients of moisture expansion for a unidirectional lamina were discussed.

Key Terms

Volume fraction
Weight (mass) fraction
Density
Void volume fraction
Void content
Elastic moduli
Array packing
Halphin–Tsai equations
Elasticity models
Transversely isotropic fibers
Strength
ASTM standards
Failure modes
IITRI compression test
Shear test
Coefficient of thermal expansion
Coefficient of moisture expansion

Exercise Set

3.1 The weight fraction of glass in a glass/epoxy composite is 0.8. If the specific gravity of glass and epoxy is 2.5 and 1.2, respectively, find the

1. Fiber and matrix volume fractions
2. Density of the composite

3.2 A hybrid lamina uses glass and graphite fibers in a matrix of epoxy for its construction. The fiber volume fractions of glass and graphite are 40 and 20%, respectively. The specific gravity of glass, graphite, and epoxy is 2.6, 1.8, and 1.2, respectively. Find

1. Mass fractions
2. Density of the composite

3.3 The acid digestion test left 2.595 g of fiber from a composite specimen weighing 3.697 g. The composite specimen weighs 1.636 g in water. If the specific gravity of the fiber and matrix is 2.5 and 1.2, respectively, find the

1. Theoretical volume fraction of fiber and matrix
2. Theoretical density of composite

3. Experimental density
4. Weight fraction of fiber and matrix
5. Void fraction

3.4 A resin hybrid lamina is made by reinforcing graphite fibers in two matrices: resin A and resin B. The fiber weight fraction is 40%; for resin A and resin B, the weight fraction is 30% each. If the specific gravity of graphite, resin A, and resin B is 1.2, 2.6, and 1.7, respectively, find

1. Fiber volume fraction
2. Density of composite

3.5 Find the elastic moduli of a glass/epoxy unidirectional lamina with 40% fiber volume fraction. Use the properties of glass and epoxy from Table 3.3 and Table 3.4, respectively.

3.6 Show that

$$G_{12} = \frac{G_m}{1 - V_f}$$

if the fibers are much stiffer than the matrix — that is, $G_f >> G_m$.

3.7 Assume that fibers in a unidirectional lamina are circularly shaped and in a square array. Calculate the ratio of fiber diameter to fiber center-to-center spacing ratio in terms of the fiber volume fraction.

3.8 Circular graphite fibers of 10 μm diameter are packed in a hexagonal array in an epoxy matrix. The fiber weight fraction is 50%. Find the fiber-to-fiber spacing between the centers of the fibers. The density of graphite fibers is 1800 kg/m^3 and epoxy is 1200 kg/m^3.

3.9 Find the elastic moduli for problem 3.5 using Halphin–Tsai equations. Assume that the fibers are circularly shaped and are in a square array. Compare your results with those of problem 3.5.

3.10 A unidirectional glass/epoxy lamina with a fiber volume fraction of 70% is replaced by a graphite/epoxy lamina with the same longitudinal Young's modulus. Find the fiber volume fraction required in the graphite/epoxy lamina. Use properties of glass, graphite, and epoxy from Table 3.1 and Table 3.2.

3.11 Sometimes, the properties of a fiber are determined from the measured properties of a unidirectional lamina. As an example, find the experimentally determined value of the Poisson's ratio of an isotropic fiber from the following measured properties of a unidirectional lamina:

1. Major Poisson's ratio of composite = 0.27
2. Poisson's ratio of the matrix = 0.35
3. Fiber volume fraction = 0.65

3.12 Using elasticity model equations, find the elastic moduli of a glass/epoxy unidirectional lamina with 40% fiber volume fraction. Use the properties of glass and epoxy from Table 3.3 and Table 3.4, respectively. Compare your results with those obtained by using the strength of materials approach and the Halphin–Tsai approach. Assume that the fibers are circularly shaped and are in a square array for the Halphin–Tsai approach.

3.13 A measure of degree of orthotropy of a material is given by the ratio of the longitudinal to transverse Young's modulus. Given the properties of glass, graphite, and epoxy from Table 3.1 and Table 3.2 and using the mechanics of materials approach to find the longitudinal and transverse Young's modulus, find the fiber volume fraction at which the degree of orthotropy is maximum for graphite/epoxy and glass/epoxy unidirectional laminae.

3.14 What are three common modes of failure of a unidirectional composite subjected to longitudinal tensile load?

3.15 Do high fiber volume fractions increase the transverse strength of a unidirectional lamina?

3.16 Find the five strength parameters of a unidirectional glass/epoxy lamina with 40% fiber volume fraction. Use the properties of glass and epoxy from Table 3.3 and Table 3.4.

3.17 A rod is designed to carry a uniaxial tensile load of 1400 N with a factor of safety of two. The designer has two options for the materials: steel or 66% fiber volume fraction graphite/epoxy. Use the properties of graphite and epoxy from Table 3.1 and Table 3.2. Assume the following properties for steel:

- Young's modulus of steel = 210 GPa
- Poisson's ratio of steel = 0.3
- Tensile strength of steel = 450 MPa
- Specific gravity of steel = 7.8

The cost of graphite/epoxy is five times that of steel by weight. List your material of choice if the criterion depends on just

1. Mass
2. Cost

3.18 Find the coefficients of thermal expansion for a 60% unidirectional glass/epoxy lamina with a 60% fiber volume fraction. Use properties of glass and epoxy from Table 3.3 and Table 3.4, respectively.

3.19 If one plots the transverse coefficient of thermal expansion, α_2, as a function of fiber volume fraction, V_f, for a unidirectional glass/epoxy lamina, $\alpha_2 > \alpha_m$ for a certain fiber volume fraction. Find this range of fiber volume fraction. Use properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

3.20 Find the fiber volume fraction for which the unidirectional glass/epoxy lamina transverse thermal expansion coefficient is a maximum. Use properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

3.21 Prove[31] that it is possible to have the transverse coefficient of thermal expansion of a unidirectional lamina greater than the coefficient of thermal expansion of the matrix ($\alpha_2 > \alpha_m$) only if

$$\frac{E_f}{E_m} > \frac{1+\nu_f}{\nu_m} \text{ or } \frac{E_f}{E_m} < \frac{1+\nu_f}{1+\nu_m}$$

3.22 The coefficient of thermal expansion perpendicular to the fibers of a unidirectional glass/epoxy lamina is given as 28 μm/m/°C. Use the properties of glass and epoxy from Table 3.3 and Table 3.4 to find the coefficient of thermal expansion of the unidirectional glass/epoxy lamina in the direction parallel to the fibers.

3.23 There are large excursions of temperature in space and thus composites with zero or near zero thermal expansion coefficients are attractive. Find the volume fraction of the graphite fibers for which the thermal expansion coefficient is zero in the longitudinal direction of a graphite/epoxy unidirectional lamina. Use all the properties of graphite and epoxy from Table 3.1 and Table 3.2, respectively, but assume that the longitudinal coefficient of thermal expansion of graphite fiber is -1.3×10^{-6} m/m/°C.

3.24 Find the coefficients of moisture expansion of a glass/epoxy lamina with 40% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

3.25 Assume a 60% fiber volume fraction glass/epoxy lamina of cuboid dimensions 25 cm (along the fibers) × 10 cm × 0.125 mm. Epoxy absorbs water as much as 8% of its weight. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively, and find

1. Maximum mass of water that the specimen can absorb
2. Change in volume of the lamina if the maximum possible water is absorbed

Assume that the coefficient of moisture expansion through the thickness is the same as the coefficient of moisture expansion in the transverse direction and that the glass fibers absorb no moisture.

References

1. Judd, N.C.W. and Wright, W.W., Voids and their effects on the mechanical properties of composites — an appraisal, *SAMPE J.*, 10, 14, 1978.

2. Geier, M.H., *Quality Handbook for Composite Materials*, Chapman & Hall, London, 1994.
3. Adams, R.D., Damping properties analysis of composites, in *Engineered Materials Handbook*, vol. 1, ASM International, Metals Park, OH, 1987.
4. Hashin, Z., Theory of fiber reinforced materials, NASA tech. rep. contract no: NAS1-8818, November 1970.
5. Chamis, L.C. and Sendeckyj, G.P., Critique on theories predicting thermoelastic properties of fibrous composites, *J. Composite Mater.*, 2, 332, 1968.
6. Halphin, J.C. and Tsai, S.W., Effect of environment factors on composite materials, Air Force tech. rep. AFML-TR-67-423, June 1969.
7. Foye, R.L., An evaluation of various engineering estimates of the transverse properties of unidirectional composite, *SAMPE*, 10, 31, 1966.
8. Hewitt, R.L. and Malherbe, M.D.De, An approximation for the longitudinal shear modulus of continuous fiber composites, *J. Composite Mater.*, 4, 280, 1970.
9. Hashin, Z. and Rosen, B.W., 1964, The elastic moduli of fiber reinforced materials, *ASME J. Appl. Mech.*, 31, 223, 1964.
10. Hashin, Z., Analysis of composite materials — a survey, *ASME J. Appl. Mech.*, 50, 481, 1983.
11. Knott, T.W. and Herakovich, C.T., Effect of fiber orthotropy on effective composite properties, *J. Composite Mater.*, 25, 732, 1991.
12. Christensen, R.M., Solutions for effective shear properties in three phase sphere and cylinder models, *J. Mech. Phys. Solids*, 27, 315, 1979.
13. Timoshenko, S.P. and Goodier, J.N., *Theory of Elasticity*, McGraw–Hill, New York, 1970.
14. Maple 9.0, Advancing mathematics. See http://www.maplesoft.com.
15. Hashin, Z., Analysis of properties of fiber composites with anisotropic constituents, *ASME J. Appl. Mech.*, 46, 543, 1979.
16. Hyer, M.W., *Stress Analysis of Fiber-Reinforced Materials,* WCB McGraw–Hill, New York, 1998.
17. Hashin, Z., *Theory of Fiber Reinforced Materials*, NASA CR-1974, 1972.
18. Whitney, J.M. and Riley, M.B., Elastic properties of fiber reinforced composite materials, *AIAA J.*, 4, 1537, 1966.
19. Hill, R., Theory of mechanical properties of fiber-strengthened materials — I. Elastic behavior, *J. Mech. Phys. Solids*, 12, 199, 1964.
20. Agrawal, B.D. and Broutman, L.J., *Analysis and Performance of Fiber Composites*, John Wiley & Sons, New York, 1990.
21. Dow, N.F. and Rosen, B.W., Evaluations of filament reinforced composites for aerospace structural applications, NASA CR-207, April 1965.
22. Schuerch, H., Prediction of compressive strength in uniaxial boron fiber metal matrix composites, *AIAA J.*, 4, 102, 1966.
23. Hull, D., *An Introduction to Composite Materials*, Cambridge University Press, 1981.
24. Hofer, K.E., Rao, N., and Larsen, D., Development of engineering data on mechanical properties of advanced composite materials, Air Force tech. rep. AFML-TR-72-205, part 1, September 1972.
25. Kies, J.A., Maximum strains in the resin of fiber-glass composites, NRL rep. No. 5752, AD-274560, 1962.
26. Carlsson, L.A. and Pipes, R.B., *Experimental Characterization of Advanced Composite Materials*, Technomic Publishing Company, Inc., Lancaster, PA, 1996.

27. Rosen, B.W., A simple procedure for experimental determination of the longitudinal shear modulus of unidirectional composites, *J. Composite Mater.*, 21, 552, 1972.
28. Schepery, R.A., Thermal expansion coefficients of composite materials based on energy principles, *J. Composite Mater.*, 2, 380, 1968.
29. Greszak, L.B., Thermoelastic properties of filamentary composites, presented at AIAA 6th Structures and Materials Conference, April 1965.
30. Tsai, S.W. and Hahn, H.T., *Introduction to Composite Materials*, Technomic Publishing Company, Inc., Lancaster, PA, 1980.
31. Kaw, A.K., On using a symbolic manipulator in mechanics of composites, *ASEE Computers Educ. J.*, 3, 61, 1993.