2

Macromechanical Analysis of a Lamina

Chapter Objectives

- Review definitions of stress, strain, elastic moduli, and strain energy.
- Develop stress–strain relationships for different types of materials.
- Develop stress–strain relationships for a unidirectional/bidirectional lamina.
- Find the engineering constants of a unidirectional/bidirectional lamina in terms of the stiffness and compliance parameters of the lamina.
- Develop stress–strain relationships, elastic moduli, strengths, and thermal and moisture expansion coefficients of an angle ply based on those of a unidirectional/bidirectional lamina and the angle of the ply.

2.1 Introduction

A lamina is a thin layer of a composite material that is generally of a thickness on the order of 0.005 in. (0.125 mm). A laminate is constructed by stacking a number of such laminae in the direction of the lamina thickness (Figure 2.1). Mechanical structures made of these laminates, such as a leaf spring suspension system in an automobile, are subjected to various loads, such as bending and twisting. The design and analysis of such laminated structures demands knowledge of the stresses and strains in the laminate. Also, design tools, such as failure theories, stiffness models, and optimization algorithms, need the values of these laminate stresses and strains.

However, the building blocks of a laminate are single lamina, so understanding the mechanical analysis of a lamina precedes understanding that of a laminate. A lamina is unlike an isotropic homogeneous material. For example, if the lamina is made of isotropic homogeneous fibers and an

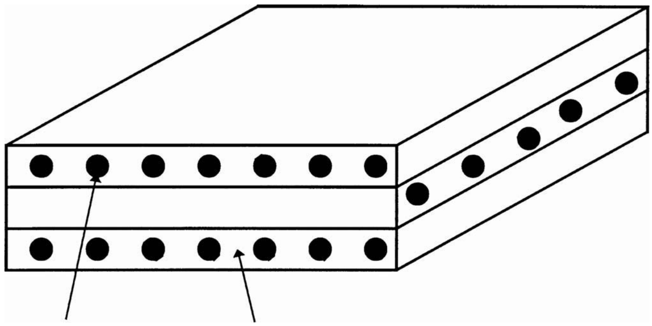


FIGURE 2.1
Typical laminate made of three laminae.

isotropic homogeneous matrix, the stiffness of the lamina varies from point to point depending on whether the point is in the fiber, the matrix, or the fiber–matrix interface. Accounting for these variations will make any kind of mechanical modeling of the lamina very complicated. For this reason, the macromechanical analysis of a lamina is based on average properties and considering the lamina to be homogeneous. Methods to find these average properties based on the individual mechanical properties of the fiber and the matrix, as well as the content, packing geometry, and shape of fibers are discussed in Chapter 3.

Even with the homogenization of a lamina, the mechanical behavior is still different from that of a homogeneous isotropic material. For example, take a square plate of length and width w and thickness t out of a large isotropic plate of thickness t (Figure 2.2) and conduct the following experiments.

Case A: Subject the square plate to a pure normal load P in direction 1. Measure the normal deformations in directions 1 and 2, δ_{1A} and δ_{2A}, respectively.

Case B: Apply the same pure normal load P as in case A, but now in direction 2. Measure the normal deformations in directions 1 and 2, δ_{1B} and δ_{2B}, respectively.

Note that

$$\delta_{1A} = \delta_{2B}\,,$$
$$\delta_{2A} = \delta_{1B}\,. \qquad (2.1a,b)$$

However, taking a unidirectional square plate (Figure 2.3) of the same dimensions $w \times w \times t$ out of a large composite lamina of thickness t and conducting the same case A and B experiments, note that the deformations

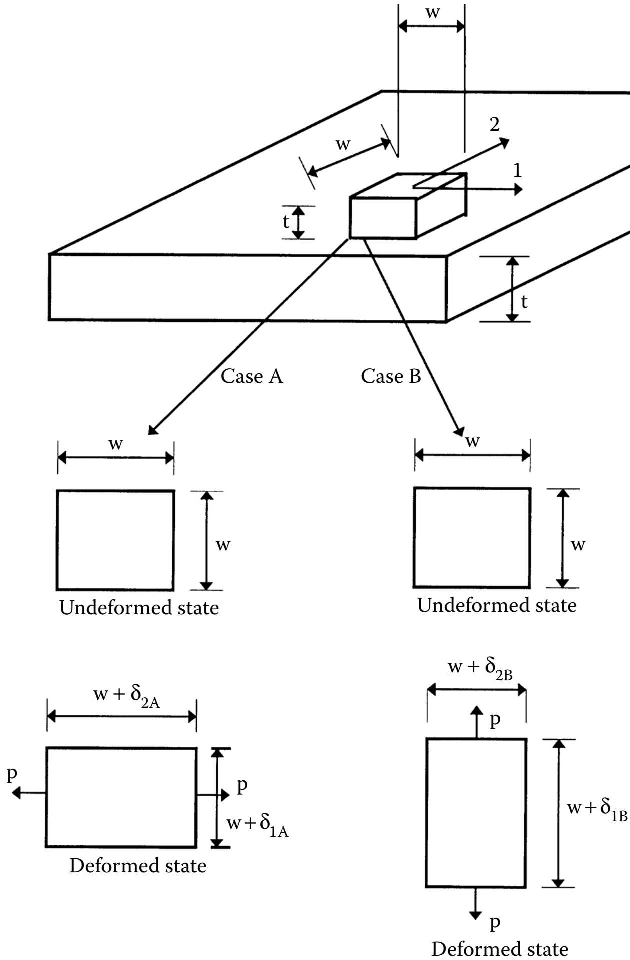


FIGURE 2.2
Deformation of square plate taken from an isotropic plate under normal loads.

$$\delta_{1A} \neq \delta_{2B}\,,$$
$$\delta_{2A} \neq \delta_{1B}\,. \tag{2.2a,b}$$

because the stiffness of the unidirectional lamina in the direction of fibers is much larger than the stiffness in the direction perpendicular to the fibers. Thus, the mechanical characterization of a unidirectional lamina will require more parameters than it will for an isotropic lamina.

Also, note that if the square plate (Figure 2.4) taken out of the lamina has fibers at an angle to the sides of the square plate, the deformations will be different for different angles. In fact, the square plate would not only have

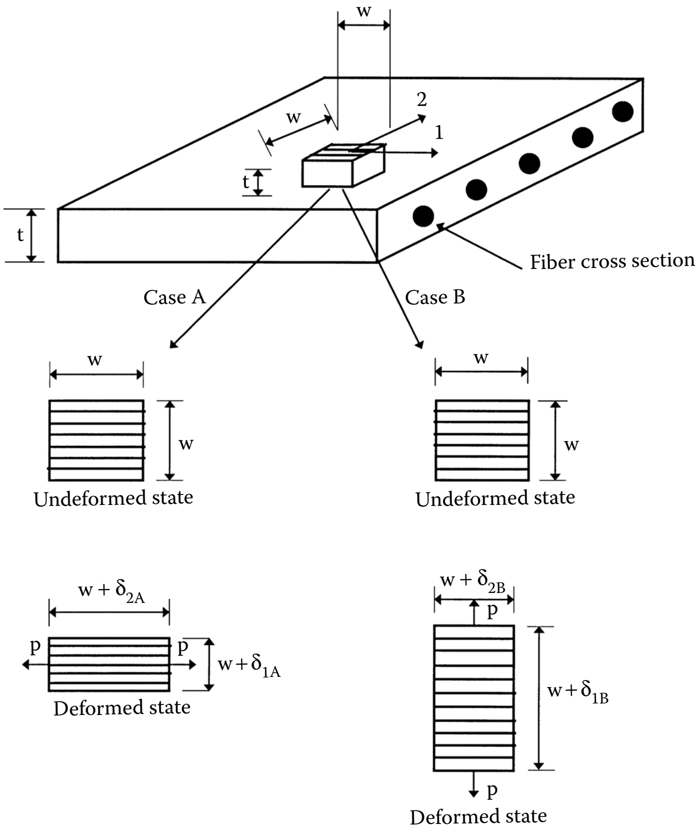


FIGURE 2.3
Deformation of a square plate taken from a unidirectional lamina with fibers at zero angle under normal loads.

deformations in the normal directions but would also distort. This suggests that the mechanical characterization of an angle lamina is further complicated.

Mechanical characterization of materials generally requires costly and time-consuming experimentation and/or theoretical modeling. Therefore, the goal is to find the minimum number of parameters required for the mechanical characterization of a lamina.

Also, a composite laminate may be subjected to a temperature change and may absorb moisture during processing and operation. These changes in temperature and moisture result in residual stresses and strains in the laminate. The calculation of these stresses and strains in a laminate depends on the response of each lamina to these two environmental parameters. In this chapter, the stress–strain relationships based on temperature change and moisture content will also be developed for a single lamina. The effects of temperature and moisture on a laminate are discussed later in Chapter 4.

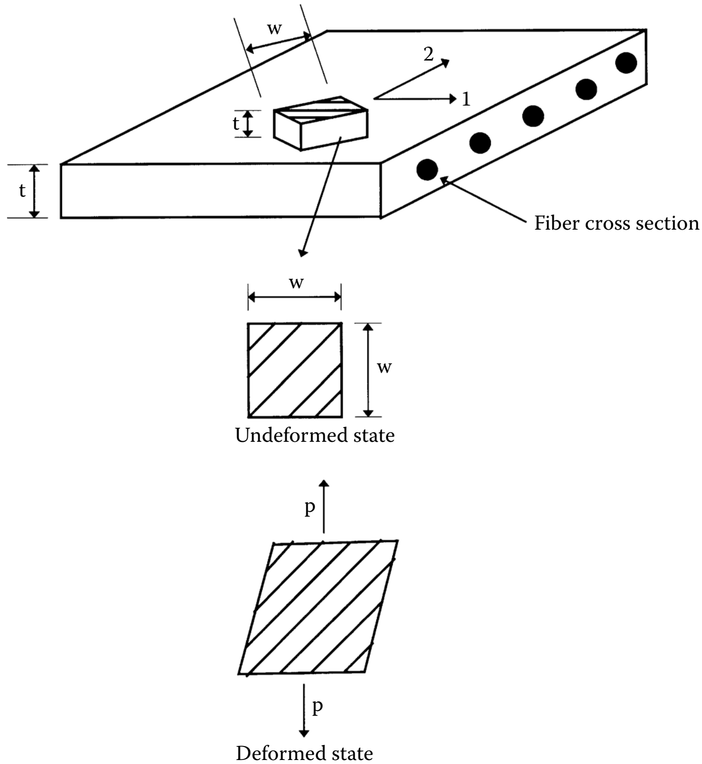


FIGURE 2.4
Deformation of a square plate taken from a unidirectional lamina with fibers at an angle under normal loads.

2.2 Review of Definitions

2.2.1 Stress

A mechanical structure takes external forces, which act upon a body as surface forces (for example, bending a stick) and body forces (for example, the weight of a standing vertical telephone pole on itself). These forces result in internal forces inside the body. Knowledge of the internal forces at all points in the body is essential because these forces need to be less than the strength of the material used in the structure. Stress, which is defined as the intensity of the load per unit area, determines this knowledge because the strengths of a material are intrinsically known in terms of stress.

Imagine a body (Figure 2.5) in equilibrium under various loads. If the body is cut at a cross-section, forces will need to be applied on the cross-sectional area so that it maintains equilibrium as in the original body. At any cross-

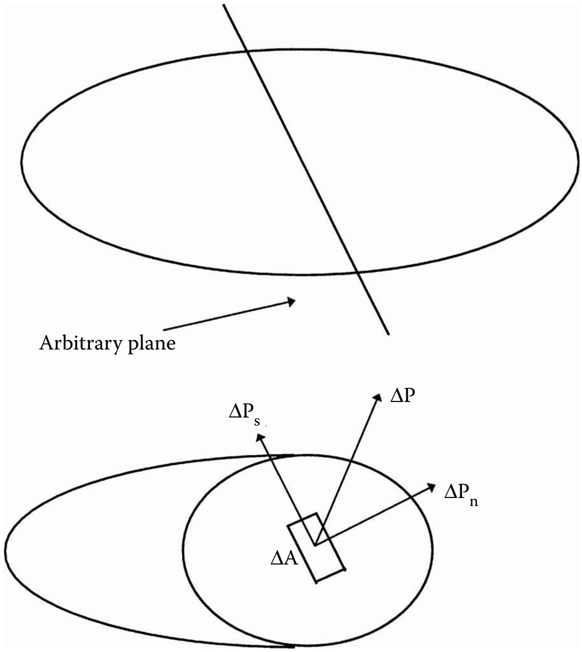


FIGURE 2.5
Stresses on an infinitesimal area on an arbitrary plane.

section, a force ΔP is acting on an area of ΔA. This force vector has a component normal to the surface, ΔP_n, and one parallel to the surface, ΔP_s. The definition of stress then gives

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A},$$

$$\tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A}. \tag{2.3a,b}$$

The component of the stress normal to the surface, σ_n, is called the normal stress and the stress parallel to the surface, τ_s, is called the shear stress. If one takes a different cross-section through the same point, the stress remains unchanged but the two components of stress, normal stress, σ_n, and shear stress, τ_s, will change. However, it has been proved that a complete definition of stress at a point only needs use of any three mutually orthogonal coordinate systems, such as a Cartesian coordinate system.

Take the right-hand coordinate system x–y–z. Take a cross-section parallel to the yz-plane in the body as shown in Figure 2.6. The force vector ΔP acts

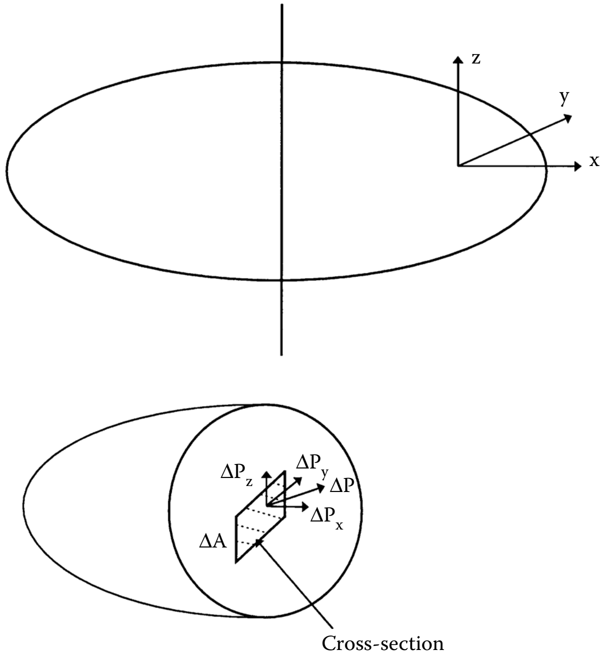


FIGURE 2.6
Forces on an infinitesimal area on the *y–z* plane.

on an area ΔA. The component ΔP_x is normal to the surface. The force vector ΔP_s is parallel to the surface and can be further resolved into components along the y and z axes: ΔP_y and ΔP_z. The definition of the various stresses then is

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A},$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}. \qquad (2.4a–c)$$

Similarly, stresses can be defined for cross-sections parallel to the xy and xz planes. For defining all these stresses, the stress at a point is defined generally by taking an infinitesimal cuboid in a right-hand coordinate system

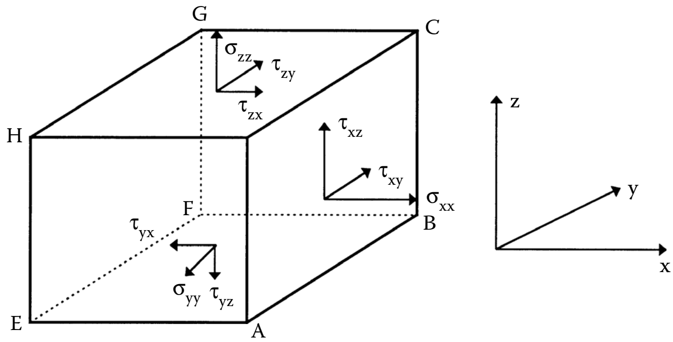


FIGURE 2.7
Stresses on an infinitesimal cuboid.

and finding the stresses on each of its faces. Nine different stresses act at a point in the body as shown in Figure 2.7. The six shear stresses are related as

$$\tau_{xy} = \tau_{yx} \,,$$

$$\tau_{yz} = \tau_{zy} \,,$$

$$\tau_{zx} = \tau_{xz} \,. \qquad (2.5a\text{–}c)$$

The preceding three relations are found by equilibrium of moments of the infinitesimal cube. There are thus six independent stresses. The stresses σ_x, σ_y, and σ_z are normal to the surfaces of the cuboid and the stresses τ_{yz}, τ_{zx}, and τ_{xy} are along the surfaces of the cuboid.

A tensile normal stress is positive, and a compressive normal stress is negative. A shear stress is positive, if its direction and the direction of the normal to the face on which it is acting are both in positive or negative direction; otherwise, the shear stress is negative.

2.2.2 Strain

Similar to the need for knowledge of forces inside a body, knowing the deformations because of the external forces is also important. For example, a piston in an internal combustion engine may not develop larger stresses than the failure strengths, but its excessive deformation may seize the engine. Also, finding stresses in a body generally requires finding deformations. This is because a stress state at a point has six components, but there are only three force-equilibrium equations (one in each direction).

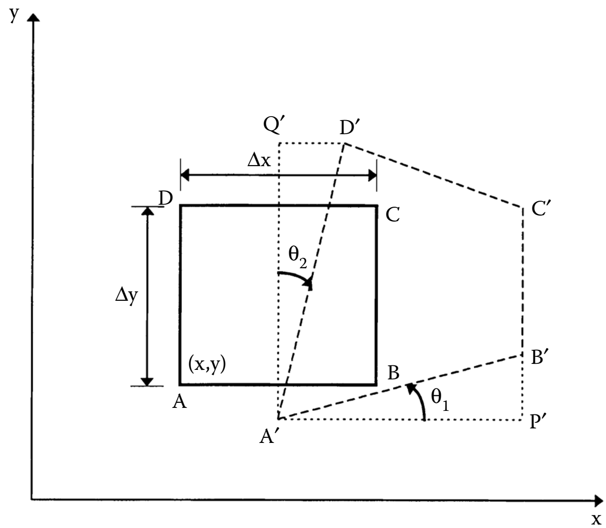


FIGURE 2.8
Normal and shearing strains on an infinitesimal area in the x–y plane.

The knowledge of deformations is specified in terms of strains — that is, the relative change in the size and shape of the body. The strain at a point is also defined generally on an infinitesimal cuboid in a right-hand coordinate system. Under loads, the lengths of the sides of the infinitesimal cuboid change. The faces of the cube also get distorted. The change in length corresponds to a normal strain and the distortion corresponds to the shearing strain. Figure 2.8 shows the strains on one of the faces, *ABCD*, of the cuboid.

The strains and displacements are related to each other. Take the two perpendicular lines *AB* and *AD*. When the body is loaded, the two lines become *A′B′* and *A′D′*. Define the displacements of a point (x,y,z) as

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z)
$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z)
$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)

The normal strain in the x-direction, ε_x, is defined as the change of length of line *AB* per unit length of *AB* as

$$\varepsilon_x = \lim_{AB \to 0} \frac{A'B' - AB}{AB}, \tag{2.6}$$

where

$$A'B' = \sqrt{(A'P')^2 + (B'P')^2},$$

$$= \sqrt{[\Delta x + u(x+\Delta x, y) - u(x,y)]^2 + [v(x+\Delta x, y) - v(x,y)]^2},$$

$$AB = \Delta x. \tag{2.7a,b}$$

Substituting the preceding expressions of Equation (2.7) in Equation (2.6),

$$\varepsilon_x = \lim_{\Delta x \to 0} \left\{ \left[1 + \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right]^2 + \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]^2 \right\}^{1/2} - 1 .$$

Using definitions of partial derivatives

$$\varepsilon_x = \left[\left(1 + \frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]^{1/2} - 1$$

$$\varepsilon_x = \frac{\partial u}{\partial x}, \tag{2.8}$$

because

$$\frac{\partial u}{\partial x} << 1 ,$$

$$\frac{\partial v}{\partial x} << 1 ,$$

for small displacements.

The normal strain in the y-direction, ε_y is defined as the change in the length of line AD per unit length of AD as

$$\varepsilon_y = \lim_{AD \to 0} \frac{A'D' - AD}{AD} , \tag{2.9}$$

where

$$A'D' = \sqrt{(A'Q')^2 + (Q'D')^2},$$

$$A'D' = \sqrt{[\Delta y + v(x, y + \Delta y) - v(x, y)]^2 + [u(x, y + \Delta y) - u(x, y)]^2},$$

$$AD = \Delta y. \tag{2.10a,b}$$

Substituting the preceding expressions of Equation (2.10) in Equation (2.9),

$$\varepsilon_y = \lim_{\Delta y \to 0} \left\{ \left[1 + \frac{v(x, y + \Delta y) - v(x, y)\}}{\Delta y} \right]^2 + \left[\frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \right]^2 \right\}^{1/2} - 1 .$$

Using definitions of partial derivatives,

$$\varepsilon_y = \left[\left(1 + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{1/2} - 1$$

$$\varepsilon_y = \frac{\partial v}{\partial y}, \tag{2.11}$$

because

$$\frac{\partial u}{\partial y} << 1,$$

$$\frac{\partial v}{\partial y} << 1,$$

for small displacements.

A normal strain is positive if the corresponding length increases; a normal strain is negative if the corresponding length decreases.

The shearing strain in the *x–y* plane, γ_{xy} is defined as the change in the angle between sides *AB* and *AD* from 90°. This angular change takes place by the inclining of sides *AB* and *AD*. The shearing strain is thus defined as

$$\gamma_{xy} = \theta_1 + \theta_2, \tag{2.12}$$

where

$$q_1 = \lim_{AB \to 0} \frac{P'B'}{A'P'},$$

$$P'B' = v(x+\Delta x, y) - v(x, y),$$

$$A'P' = u(x+\Delta x, y) + \Delta x - u(x, y), \tag{2.13a–c}$$

$$\theta_2 = \lim_{AD \to 0} \frac{Q'D'}{A'Q'},$$

$$Q'D' = u(x, y+\Delta y) - u(x, y),$$

$$A'Q' = v(x, y+\Delta y) + \Delta y - v(x, y). \tag{2.14a–c}$$

Substituting Equation (2.13) and Equation (2.14) in Equation (2.12),

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\dfrac{v(x+\Delta x, y) - v(x, y)}{\Delta x}}{\dfrac{u(x+\Delta x, y) + \Delta x - u(x, y)}{\Delta x}} + \frac{\dfrac{u(x, y+\Delta y) - u(x, y)}{\Delta y}}{\dfrac{v(x, y+\Delta y) + \Delta y - v(x, y)}{\Delta y}}$$

$$= \frac{\dfrac{\partial v}{\partial x}}{1 + \dfrac{\partial u}{\partial x}} + \frac{\dfrac{\partial u}{\partial y}}{1 + \dfrac{\partial u}{\partial y}}$$

$$= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \tag{2.15}$$

because

$$\frac{\partial u}{\partial x} << 1,$$

$$\frac{\partial v}{\partial y} << 1,$$

for small displacements.

The shearing strain is positive when the angle between the sides *AD* and *AB* decreases; otherwise, the shearing strain is negative.

The definitions of the remaining normal and shearing strains can be found by noting the change in size and shape of the other sides of the infinitesimal cuboid in Figure 2.7 as

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

$$\varepsilon_z = \frac{\partial w}{\partial z}. \tag{2.16a–c}$$

Example 2.1

A displacement field in a body is given by

$u = 10^{-5}(x^2 + 6y + 7xy)$
$v = 10^{-5}(yz)$
$w = 10^{-5}(xy + yz^2)$

Find the state of strain at $(x,y,z) = (1,2,3)$.

Solution

From Equation (2.8),

$$\in_x = \frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(10^{-5}\left(x^2 + 6y + 7xz\right)\right)$$

$$= 10^{-5}\left(2x + 7z\right)$$

$$= 10^{-5}\left(2 \times 1 + 7 \times 3\right)$$

$$= 2.300 \times 10^{-4}.$$

From Equation (2.11),

$$\in_y = \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial y}\left(10^{-5}\left(yz\right)\right)$$

$$= 10^{-5}\left(z\right)$$

$$= 10^{-5}\left(3\right)$$

$$= 3.000 \times 10^{-5} \ .$$

From Equation (2.16c),

$$\in_z = \frac{\partial w}{\partial z}$$

$$= \frac{\partial}{\partial z}\left(10^{-5}\left(xy + yz^2\right)\right)$$

$$= 10^{-5}\left(2yz\right)$$

$$= 10^{-5}(2 \times 2 \times 3)$$

$$= 1.2 \times 10^{-4} \ .$$

From Equation (2.15),

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$= \frac{\partial}{\partial y}\left(10^{-5}\left(x^2 + 6y + 7xz\right)\right) + \frac{\partial}{\partial x}\left(10^{-5}\left(yz\right)\right)$$

$$= 10^{-5}(6) + 10^{-5}(0)$$

$$= 6.000 \times 10^{-5}.$$

From Equation (2.16a),

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$= \frac{\partial}{\partial z}\left(10^{-5}(yz)\right) + \frac{\partial}{\partial y}\left(10^{-5}(xy + yz^2)\right)$$

$$= 10^{-5}(y) + 10^{-5}(x + z^2)$$

$$= 10^{-5}(2) + 10^{-5}(1 + 3^2)$$

$$= 1.2 \times 10^{-4}.$$

From Equation (2.16b),

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(10^{-5}(xy + yz^2)\right) + \frac{\partial}{\partial z}\left(10^{-5}(x^2 + 6y + 7xz)\right)$$

$$= 10^{-5}(y) + 10^{-5}(7x)$$

$$= 10^{-5}(2) + 10^{-5}(7 \times 1)$$

$$= 9.000 \times 10^{-5}.$$

2.2.3 Elastic Moduli

As mentioned in Section 2.2.2, three equilibrium equations are insufficient for defining all six stress components at a point. For a body that is linearly

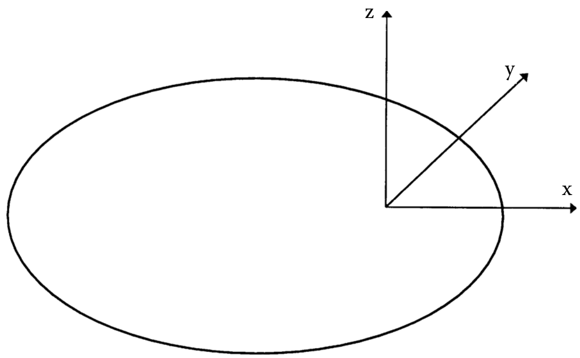


FIGURE 2.9
Cartesian coordinates in a three-dimensional body.

elastic and has small deformations, stresses and strains at a point are related through six simultaneous linear equations called Hooke's law. Note that 15 unknown parameters are at a point: six stresses, six strains, and three displacements. Combined with six simultaneous linear equations of Hooke's law, six strain-displacement relations — given by Equation (2.8), Equation (2.11), Equation (2.15), and Equation (2.16) — and three equilibrium equations give 15 equations for the solution of 15 unknowns.[1] Because strain-displacement and equilibrium equations are differential equations, they are subject to knowing boundary conditions for complete solutions.

For a linear isotropic material in a three-dimensional stress state, the Hooke's law stress–strain relationships at a point in an *x–y–z* orthogonal system (Figure 2.9) in matrix form are

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}, \tag{2.17}$$

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}, \tag{2.18}$$

where ν is the Poisson's ratio. The shear modulus G is a function of two elastic constants, E and ν, as

$$G=\frac{E}{2(1+\nu)}. \tag{2.19}$$

The 6 × 6 matrix in Equation (2.17) is called the compliance matrix [S] of an isotropic material. The 6 × 6 matrix in Equation (2.18), obtained by inverting the compliance matrix in Equation (2.17), is called the stiffness matrix [C] of an isotropic material.

2.2.4 Strain Energy

Energy is defined as the capacity to do work. In solid, deformable, elastic bodies under loads, the work done by external loads is stored as recoverable strain energy. The strain energy stored in the body per unit volume is then defined as

$$W=\frac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}). \tag{2.20}$$

Example 2.2

Consider a bar of cross-section A and length L (Figure 2.10). A uniform tensile load P is applied to the two ends of the rod; find the state of stress and strain, and strain energy per unit volume of the body. Assume that the rod is made of a homogeneous isotropic material of Young's modulus, E.

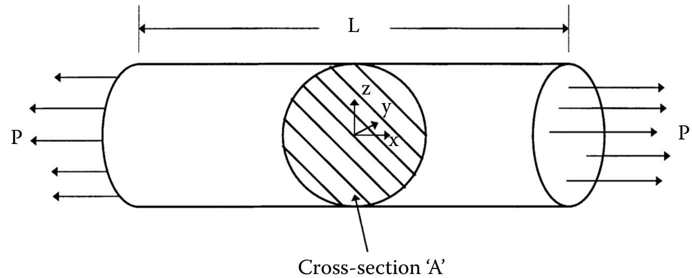


FIGURE 2.10
Cylindrical rod under uniform uniaxial load, P.

Solution

The stress state at any point is given by

$$\sigma_x = \frac{P}{A}, \sigma_y = 0, \sigma_z = 0, \tau_{yz} = 0, \tau_{zx} = 0, \tau_{xy} = 0. \tag{2.21}$$

If the circular rod is made of an isotropic, homogeneous, and linearly elastic material, then the stress–strain at any point is related as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \frac{P}{A} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.22}$$

$$\begin{gathered} \varepsilon_x = \frac{P}{AE}, \varepsilon_y = -\frac{\nu P}{AE}, \varepsilon_z = -\frac{\nu P}{AE}, \\ \gamma_{yz} = 0, \gamma_{zx} = 0, \gamma_{xy} = 0. \end{gathered} \tag{2.23}$$

The strain energy stored per unit volume in the rod, per Equation (2.20), is

$$W = \frac{1}{2}\left[\left(\frac{P}{A}\right)\left(\frac{P}{AE}\right) + (0)\left(-\frac{\nu P}{AE}\right) + (0)\left(-\frac{\nu P}{AE}\right) + (0)\,(0) + (0)(0) + (0)(0)\right]$$

$$= \frac{1}{2}\frac{P^2}{A^2 E}$$

$$= \frac{1}{2}\frac{\sigma_x^2}{E}\,. \tag{2.24}$$

2.3 Hooke's Law for Different Types of Materials

The stress–strain relationship for a general material that is not linearly elastic and isotropic is more complicated than Equation (2.17) and Equation (2.18). Assuming linear and elastic behavior for a composite is acceptable; however, assuming it to be isotropic is generally unacceptable. Thus, the stress–strain relationships follow Hooke's law, but the constants relating stress and strain are more in number than seen in Equation (2.17) and Equation (2.18). The most general stress–strain relationship is given as follows for a three-dimensional body in a 1–2–3 orthogonal Cartesian coordinate system:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}, \tag{2.25}$$

where the 6×6 $[C]$ matrix is called the stiffness matrix. The stiffness matrix has 36 constants.

What happens if one changes the system of coordinates from an orthogonal system 1–2–3 to some other orthogonal system, 1′–2′–3′? Then, new stiffness and compliance constants will be required to relate stresses and strains in the new coordinate system 1′–2′–3′. However, the new stiffness and compliance matrices in the 1′–2′–3′ system will be a function of the stiffness and compliance matrices in the 1–2–3 system and the angle between the axes of the 1′–2′–3′system and the 1–2–3 system.

Inverting Equation (2.25), the general strain–stress relationship for a three-dimensional body in a 1–2–3 orthogonal Cartesian coordinate system is

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}. \tag{2.26}$$

In the case of an isotropic material, relating the preceding strain–stress equation to Equation (2.17), one finds that the compliance matrix is related directly to engineering constants as

$$S_{11} = \frac{1}{E} = S_{22} = S_{33}$$

$$S_{12} = -\frac{\nu}{E} = S_{13} = S_{21} = S_{23} = S_{31} = S_{32}\,, \tag{2.27}$$

$$S_{44} = \frac{1}{G} = S_{55} = S_{66}\,,$$

and S_{ij}, other than in the preceding, are zero.

It can be shown that the 36 constants in Equation (2.25) actually reduce to 21 constants due to the symmetry of the stiffness matrix [C] as follows. The stress–strain relationship (2.25) can also be written as

$$\sigma_i = \sum_{j=1}^{6} C_{ij}\varepsilon_j, \quad i = 1 \ldots 6\,, \tag{2.28}$$

where, in a contracted notation,

$$\sigma_4 = \tau_{23},\ \sigma_5 = \tau_{31},\ \sigma_6 = \tau_{12},$$

$$\varepsilon_4 = \gamma_{23},\ \varepsilon_5 = \gamma_{31},\ \varepsilon_6 = \gamma_{12}. \tag{2.29a–f}$$

The strain energy in the body per unit volume, per Equation (2.20), is expressed as

$$W = \frac{1}{2}\sum_{i=1}^{6} \sigma_i \varepsilon_i. \tag{2.30}$$

Substituting Hooke's law, Equation (2.28), in Equation (2.30),

$$W = \frac{1}{2}\sum_{i=1}^{6}\sum_{j=1}^{6} C_{ij}\varepsilon_j \varepsilon_i. \tag{2.31}$$

Now, by partial differentiation of Equation (2.31),

$$\frac{\partial W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij}, \tag{2.32}$$

and

$$\frac{\partial W}{\partial \varepsilon_j \partial \varepsilon_i} = C_{ji}. \tag{2.33}$$

Because the differentiation does not necessarily need to be in either order,

$$C_{ij} = C_{ji}. \tag{2.34}$$

Equation (2.34) can also be proved by realizing that

$$\sigma_i = \frac{\partial W}{\partial \varepsilon_i}.$$

Thus, only 21 independent elastic constants are in the general stiffness matrix [C] of Equation (2.25). This also implies that only 21 independent constants are in the general compliance matrix [S] of Equation (2.26).

2.3.1 Anisotropic Material

The material that has 21 independent elastic constants at a point is called an anisotropic material. Once these constants are found for a particular point, the stress and strain relationship can be developed at that point. Note that

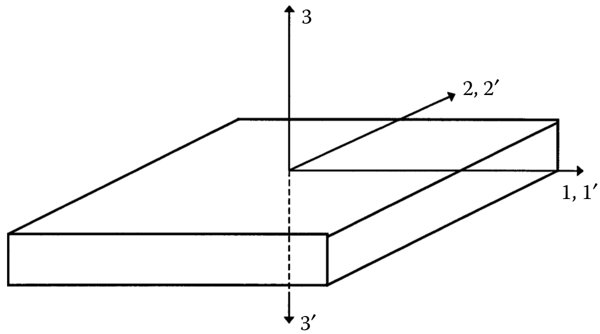


FIGURE 2.11
Transformation of coordinate axes for 1–2 plane of symmetry for a monoclinic material.

these constants can vary from point to point if the material is nonhomogeneous. Even if the material is homogeneous (or assumed to be), one needs to find these 21 elastic constants analytically or experimentally. However, many natural and synthetic materials do possess material symmetry — that is, elastic properties are identical in directions of symmetry because symmetry is present in the internal structure. Fortunately, this symmetry reduces the number of the independent elastic constants by zeroing out or relating some of the constants within the 6 × 6 stiffness [C] and 6 × 6 compliance [S] matrices. This simplifies the Hooke's law relationships for various types of elastic symmetry.

2.3.2 Monoclinic Material

If, in one plane of material symmetry* (Figure 2.11), for example, direction 3 is normal to the plane of material symmetry, then the stiffness matrix reduces to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}. \qquad (2.35)$$

as

* Material symmetry implies that the material and its mirror image about the plane of symmetry are identical.

$$C_{14} = 0, C_{15} = 0, C_{24} = 0, C_{25} = 0, C_{34} = 0, C_{35} = 0, C_{46} = 0, C_{56} = 0.$$

The direction perpendicular to the plane of symmetry is called the *principal direction*. Note that there are 13 independent elastic constants. Feldspar is an example of a monoclinic material.

The compliance matrix correspondingly reduces to

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}. \qquad (2.36)$$

Modifying an excellent example[2] of demonstrating the meaning of elastic symmetry for a monoclinic material given, consider a cubic element of Figure 2.12 taken out of a monoclinic material, in which 3 is the direction perpendicular to the 1–2 plane of symmetry. Apply a normal stress, σ_3, to the element. Then using the Hooke's law Equation (2.26) and the compliance matrix (Equation 2.36) for the monoclinic material, one gets

$$\varepsilon_1 = S_{13}\sigma_3$$

$$\varepsilon_2 = S_{23}\sigma_3$$

$$\varepsilon_3 = S_{33}\sigma_3$$

$$\gamma_{23} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = S_{36}\sigma_3 \,. \qquad (2.37a\text{–}f)$$

The cube will deform in all directions as determined by the normal strain equations. The shear strains in the 2–3 and 3–1 plane are zero, showing that the element will not change shape in those planes. However, it will change

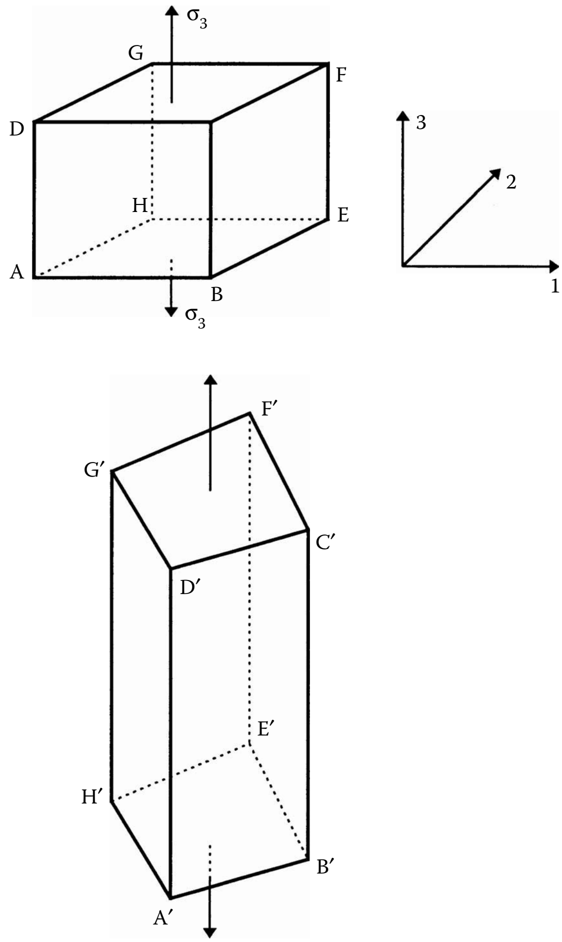


FIGURE 2.12
Deformation of a cubic element made of monoclinic material.

shape in the 1–2 plane. Thus, the faces *ABEH* and *CDFG* perpendicular to the 3 direction will change from rectangles to parallelograms, while the other four faces *ABCD*, *BEFC*, *GFEH*, and *AHGD* will stay as rectangles. This is unlike anisotropic behavior, in which all faces will be deformed in shape, and also unlike isotropic behavior, in which all faces will remain undeformed in shape.

2.3.3 Orthotropic Material (Orthogonally Anisotropic)/Specially Orthotropic

If a material has three mutually perpendicular planes of material symmetry, then the stiffness matrix is given by

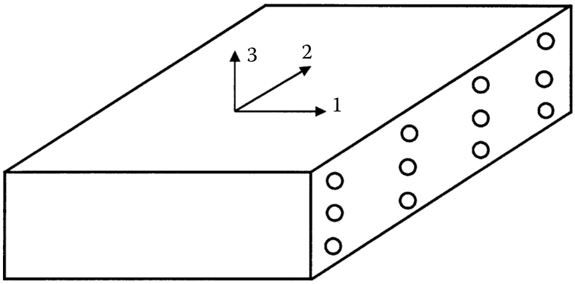


FIGURE 2.13
A unidirectional lamina as a monoclinic material with fibers, arranged in a rectangular array.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.38}$$

The preceding stiffness matrix can be derived by starting from the stiffness matrix [C] for the monoclinic material (Equation 2.35). With two more planes of symmetry, it gives

$$C_{16} = 0, C_{26} = 0, C_{36} = 0, C_{45} = 0 .$$

Three mutually perpendicular planes of material symmetry also imply three mutually perpendicular planes of elastic symmetry. Note that nine independent elastic constants are present. This is a commonly found material symmetry unlike anisotropic and monoclinic materials. Examples of an orthotropic material include a single lamina of continuous fiber composite, arranged in a rectangular array (Figure 2.13), a wooden bar, and rolled steel.

The compliance matrix reduces to

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}. \tag{2.39}$$

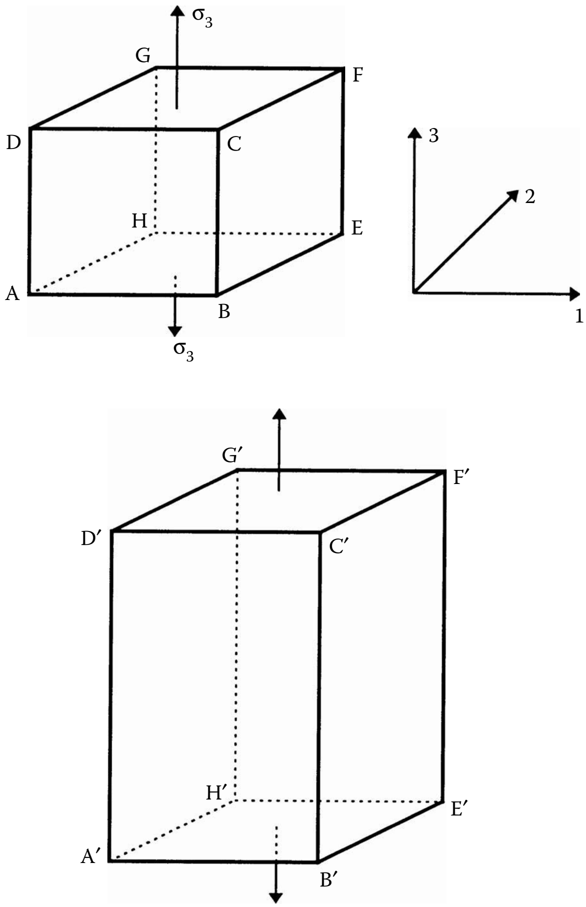


FIGURE 2.14
Deformation of a cubic element made of orthotropic material.

Demonstrating the meaning of elastic symmetry for an orthotropic material is similar to the approach taken for a monoclinic material (Section 2.3.2). Consider a cubic element (Figure 2.14) taken out of the orthotropic material, where 1, 2, and 3 are the principal directions or 1–2, 2–3, and 3–1 are the three mutually orthogonal planes of symmetry. Apply a normal stress, σ_3, to the element. Then, using the Hooke's law Equation (2.26) and the compliance matrix (Equation 2.39) for the orthotropic material, one gets

$$\varepsilon_1 = S_{13}\sigma_3$$

$$\varepsilon_2 = S_{23}\sigma_3$$

$$\varepsilon_3 = S_{33}\sigma_3$$

$$\gamma_{23} = 0 \qquad (2.40a–f)$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0.$$

The cube will deform in all directions as determined by the normal strain equations. However, the shear strains in all three planes (1–2, 2–3, and 3–1) are zero, showing that the element will not change shape in those planes. Thus, the cube will not deform in shape under any normal load applied in the principal directions. This is unlike the monoclinic material, in which two out of the six faces of the cube changed shape.

A cube made of isotropic material would not change its shape either; however, the normal strains, ε_1 and ε_2, will be different in an orthotropic material and identical in an isotropic material.

2.3.4 Transversely Isotropic Material

Consider a plane of material isotropy in one of the planes of an orthotropic body. If direction 1 is normal to that plane (2–3) of isotropy, then the stiffness matrix is given by

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22} - C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}. \qquad (2.41)$$

Transverse isotropy results in the following relations:

$$C_{22} = C_{33}, C_{12} = C_{13}, C_{55} = C_{66}, C_{44} = \frac{C_{22} - C_{23}}{2}.$$

Note the five independent elastic constants. An example of this is a thin unidirectional lamina in which the fibers are arranged in a square array or

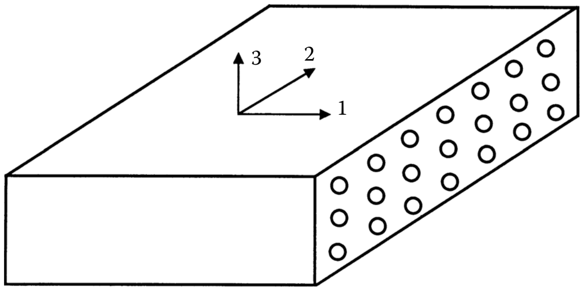


FIGURE 2.15
A unidirectional lamina as a transversely isotropic material with fibers arranged in a square array.

a hexagonal array. One may consider the elastic properties in the two directions perpendicular to the fibers to be the same. In Figure 2.15, the fibers are in direction 1, so plane 2–3 will be considered as the plane of isotropy.

The compliance matrix reduces to

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix}. \quad (2.42)$$

2.3.5 Isotropic Material

If all planes in an orthotropic body are identical, it is an isotropic material; then, the stiffness matrix is given by

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}. \quad (2.43)$$

Isotropy results in the following additional relationships:

$$C_{11} = C_{22}, C_{12} = C_{23}, C_{66} = \frac{C_{22} - C_{23}}{2} = \frac{C_{11} - C_{12}}{2}.$$

This also implies infinite principal planes of symmetry. Note the two independent constants. This is the most common material symmetry available. Examples of isotropic bodies include steel, iron, and aluminum. Relating Equation (2.43) to Equation (2.18) shows that

$$C_{11} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)},$$

$$C_{12} = \frac{\nu E}{(1-2\nu)(1+\nu)}. \tag{2.44a–b}$$

Note that

$$\frac{C_{11} - C_{12}}{2}$$

$$= \frac{1}{2}\left[\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} - \frac{\nu E}{(1-2\nu)(1+\nu)}\right]$$

$$= \frac{E}{2(1+\nu)}$$

$$= G.$$

The compliance matrix reduces to

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) \end{bmatrix}. \tag{2.45}$$

We summarize the number of independent elastic constants for various types of materials:

- Anisotropic: 21
- Monoclinic: 13
- Orthotropic: 9
- Transversely isotropic: 5
- Isotropic: 2

Example 2.3

Show the reduction of anisotropic material stress–strain Equation (2.25) to those of a monoclinic material stress–strain Equation (2.35).

Solution

Assume direction 3 is perpendicular to the plane of symmetry. Now in the coordinate system 1–2–3, Equation (2.25) with $C_{ij} = C_{ji}$ from Equation (2.34) is

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}, \tag{2.46}$$

Also, in the coordinate system 1′–2′–3′ (Figure 2.11),

$$\begin{bmatrix} \sigma_{1'} \\ \sigma_{2'} \\ \sigma_{3'} \\ \tau_{2'3'} \\ \tau_{3'1'} \\ \tau_{1'2'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{1'} \\ \varepsilon_{2'} \\ \varepsilon_{3'} \\ \gamma_{2'3'} \\ \gamma_{3'1'} \\ \gamma_{1'2'} \end{bmatrix}, \tag{2.47}$$

Because there is a plane of symmetry normal to direction 3, the stresses and strains in the 1–2–3 and 1′–2′–3′ coordinate systems are related by

$$\sigma_1 = \sigma_{1'},\ \sigma_2 = \sigma_{2'},\ \sigma_3 = \sigma_{3'}$$

$$\tau_{23} - \tau_{2'3'},\ \tau_{31} = -\tau_{3'1'},\ \tau_{12} = \tau_{1'2'}, \tag{2.48a–f}$$

$$\varepsilon_1 = \varepsilon_{1'}, \varepsilon_2 = \varepsilon_{2'}, \varepsilon_3 = \varepsilon_{3'},$$

$$\gamma_{23} = -\gamma_{2'3'}, \gamma_{31} = -\gamma_{3'1'}, \gamma_{12} = \gamma_{1'2'}. \quad (2.49a–f)$$

The terms in the first equation of Equation (2.46) and Equation (2.47) can be written as

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{14}\gamma_{23} + C_{15}\gamma_{31} + C_{16}\gamma_{12},$$

$$\sigma_{1'} = C_{11}\varepsilon_{1'} + C_{12}\varepsilon_{2'} + C_{13}\varepsilon_{3'} + C_{1'4}\gamma_{2'3'} + C_{15}\gamma_{3'1'}, + C_{16}\gamma_{1'2'} \quad (2.50a–b)$$

Substituting Equation (2.48) and Equation (2.49) in Equation (2.50b),

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 - C_{14}\gamma_{23} - C_{15}\gamma_{31} + C_{16}\gamma_{12}. \quad (2.51)$$

Subtracting Equation (2.51) from Equation (2.50a) gives

$$0 = 2C_{14}\gamma_{23} + 2C_{15}\gamma_{31}. \quad (2.52)$$

Because γ_{23} and γ_{31} are arbitrary,

$$C_{14} = C_{15} = 0. \quad (2.53a)$$

Similarly, one can show that

$$C_{24} = C_{25} = 0,$$

$$C_{34} = C_{35} = 0,$$

$$C_{46} = C_{56} = 0. \quad (2.54b-d)$$

Thus, only 13 independent elastic constants are present in a monoclinic material.

Example 2.4

The stress–strain relation is given in terms of compliance matrix for an orthotropic material in Equation (2.26) and Equation (2.39). Rewrite the compliance matrix equations in terms of the nine engineering constants for

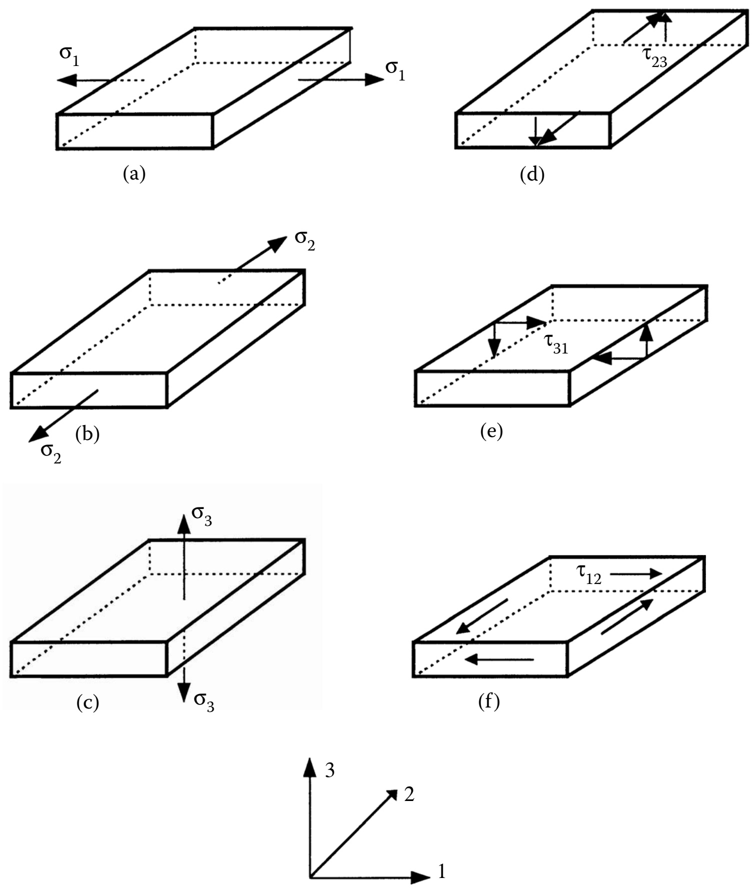


FIGURE 2.16
Application of stresses to find engineering constants of a three-dimensional orthotropic body.

an orthotropic material. What is the stiffness matrix in terms of the engineering constants?

Solution

Let us see how the compliance matrix and engineering constants of an orthotropic material are related. As shown in Figure 2.16a, apply $\sigma_1 \neq 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, $\tau_{23} = 0$, $\tau_{31} = 0$, $\tau_{12} = 0$. Then, from Equation (2.26) and Equation (2.39):

$$\varepsilon_1 = S_{11}\sigma_1$$

$$\varepsilon_2 = S_{12}\sigma_1$$

$$\varepsilon_3 = S_{13}\sigma_1$$

$$\gamma_{23} = 0$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0.$$

The Young's modulus in direction 1, E_1, is defined as

$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}} . \tag{2.55}$$

The Poisson's ratio, ν_{12}, is defined as

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}} . \tag{2.56}$$

In general terms, ν_{ij} is defined as the ratio of the negative of the normal strain in direction j to the normal strain in direction i, when the load is applied in the normal direction i.

The Poisson's ratio ν_{13} is defined as

$$\nu_{13} \equiv -\frac{\varepsilon_3}{\varepsilon_1} = -\frac{S_{13}}{S_{11}} . \tag{2.57}$$

Similarly, as shown in Figure 2.16b, apply $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 \neq 0$, $\tau_{23} = 0$, $\tau_{31} = 0$, $\tau_{12} = 0$. Then, from Equation (2.26) and Equation (2.39),

$$E_2 = \frac{1}{S_{22}} \tag{2.58}$$

$$\nu_{21} = -\frac{S_{12}}{S_{22}} \tag{2.59}$$

$$\nu_{23} = -\frac{S_{23}}{S_{22}} . \tag{2.60}$$

Similarly, as shown in Figure 2.16c, apply $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 \neq 0$, $\tau_{23} = 0$, $\tau_{31} = 0$, $\tau_{12} = 0$. From Equation (2.26) and Equation (2.39),

$$E_3 = \frac{1}{S_{33}} \tag{2.61}$$

$$\nu_{31} = -\frac{S_{13}}{S_{33}} \tag{2.62}$$

$$\nu_{32} = -\frac{S_{23}}{S_{33}} . \tag{2.63}$$

Apply, as shown in Figure 2.16d, $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, $\tau_{23} \neq 0$, $\tau_{31} = 0$, $\tau_{12} = 0$. Then, from Equation (2.26) and Equation (2.39),

$$\varepsilon_1 = 0$$

$$\varepsilon_2 = 0$$

$$\varepsilon_3 = 0$$

$$\gamma_{23} = S_{44}\tau_{23}$$

$$\gamma_{31} = 0$$

$$\gamma_{12} = 0$$

The shear modulus in plane 2–3 is defined as

$$G_{23} \equiv \frac{\tau_{23}}{\gamma_{23}} = \frac{1}{S_{44}} . \tag{2.64}$$

Similarly, as shown in Figure 2.16e, apply $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, $\tau_{23} = 0$, $\tau_{31} \neq 0$, $\tau_{12} = 0$. Then, from Equation (2.26) and Equation (2.39),

$$G_{31} = \frac{1}{S_{55}} . \tag{2.65}$$

Similarly, as shown in Figure 2.16f, apply $\sigma_1 = 0$, $\sigma_2 = 0$, $\sigma_3 = 0$, $\tau_{23} = 0$, $\tau_{31} = 0$, $\tau_{12} \neq 0$. Then, from Equation (2.26) and Equation (2.39),

$$G_{12} = \frac{1}{S_{66}} . \tag{2.66}$$

In Equation (2.55) through Equation (2.66), 12 engineering constants have been defined as follows:

Three Young's moduli, E_1, E_2, and E_3, one in each material axis

Six Poisson's ratios, ν_{12}, ν_{13}, ν_{21}, ν_{23}, ν_{31}, and ν_{32}, two for each plane

Three shear moduli, G_{23}, G_{31}, and G_{12}, one for each plane

However, the six Poisson's ratios are not independent of each other. For example, from Equation (2.55), Equation (2.56), Equation (2.58), and Equation (2.59),

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \; . \tag{2.67}$$

Similarly, from Equation (2.55), Equation (2.57), Equation (2.61), and Equation (2.62),

$$\frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3} \; , \tag{2.68}$$

and from Equation (2.58), Equation (2.60), Equation (2.61), and Equation (2.63),

$$\frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \; . \tag{2.69}$$

Equation (2.67), Equation (2.68), and Equation (2.69) are called reciprocal Poisson's ratio equations. These relations reduce the total independent engineering constants to nine. This is the same number as the number of independent constants in the stiffness or the compliance matrix.

Rewriting the compliance matrix in terms of the engineering constants gives

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} . \tag{2.70}$$

Inversion of Equation (2.70) would be the compliance matrix [C] and is given by

$$[C] = \begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0 \\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix}, \quad (2.71)$$

where

$$\Delta = \left(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{13}\nu_{31}-2\nu_{21}\nu_{32}\nu_{13}\right)/\left(E_1E_2E_3\right). \quad (2.72)$$

Although nine independent elastic constants are in the compliance matrix [S] and, correspondingly, in the stiffness matrix [C] for orthotropic materials, constraints on the values of these constants exist. Based on the first law of thermodynamics, the stiffness and compliance matrices must be positive definite. Thus, the diagonal terms of [C] and [S] in Equation (2.71) and Equation (2.70), respectively, need to be positive. From the diagonal elements of the compliance matrix [S], this gives

$$E_1 > 0,\ E_2 > 0,\ E_3 > 0,\ G_{12} > 0,\ G_{23} > 0,\ G_{31} > 0 \quad (2.73)$$

and, from the diagonal elements of the stiffness matrix [C], gives

$$1-\nu_{23}\nu_{32} > 0,\ 1-\nu_{31}\nu_{13} > 0,\ 1-\nu_{12}\nu_{21} > 0, \quad (2.74)$$

$$\Delta = 1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{13}\nu_{21}\nu_{32} > 0$$

Using the reciprocal relations given by Equation (2.67) through Equation (2.69),

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \text{ for } i \neq j \text{ and } i,j = 1,2,3,$$

we can rewrite the inequalities as follows.

For example, because

$$1 - \nu_{12}\nu_{21} > 0 \ ,$$

then

$$\nu_{12} < \frac{1}{\nu_{21}} = \frac{E_1}{E_2}\frac{1}{\nu_{12}}$$

$$|\nu_{12}| < \left|\frac{E_1}{E_2}\frac{1}{\nu_{12}}\right|$$

$$|\nu_{12}| < \sqrt{\frac{E_1}{E_2}} \ . \tag{2.75a}$$

Similarly, five other such relationships can be developed to give

$$|\nu_{21}| < \sqrt{\frac{E_2}{E_1}} \tag{2.75b}$$

$$|\nu_{32}| < \sqrt{\frac{E_3}{E_2}} \tag{2.75c}$$

$$|\nu_{23}| < \sqrt{\frac{E_2}{E_3}} \tag{2.75d}$$

$$|\nu_{31}| < \sqrt{\frac{E_3}{E_1}} \tag{2.75e}$$

$$|\nu_{13}| < \sqrt{\frac{E_1}{E_3}} \ . \tag{2.75f}$$

These restrictions on the elastic moduli are important in optimizing properties of a composite because they show that the nine independent properties cannot be varied without influencing the limits of the others.

Example 2.5

Find the compliance and stiffness matrix for a graphite/epoxy lamina. The material properties are given as

$$E_1 = 181 GPa\ ,\ E_2 = 10.3 GPa\ ,\ E_3 = 10.3 GPa$$

$$\nu_{12} = 0.28\ ,\ \nu_{23} = 0.60\ ,\ \nu_{13} = 0.27$$

$$G_{12} = 7.17 GPa\ ,\ G_{23} = 3.0 GPa\ ,\ G_{31} = 7.00 GPa\ .$$

Solution

$$S_{11} = \frac{1}{E_1} = \frac{1}{181 \times 10^9} = 5.525 \times 10^{-12}\ Pa^{-1}$$

$$S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11}\ Pa^{-1}$$

$$S_{33} = \frac{1}{E_3} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11}\ Pa^{-1}$$

$$S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181 \times 10^9} = -1.547 \times 10^{-12}\ Pa^{-1}$$

$$S_{13} = -\frac{\nu_{13}}{E_1} = -\frac{0.27}{181 \times 10^9} = -1.492 \times 10^{-12}\ Pa^{-1}$$

$$S_{23} = -\frac{\nu_{23}}{E_2} = -\frac{0.6}{10.3 \times 10^9} = -5.825 \times 10^{-11}\ Pa^{-1}$$

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{3 \times 10^9} = 3.333 \times 10^{-10}\ Pa^{-1}$$

$$S_{55} = \frac{1}{G_{31}} = \frac{1}{7 \times 10^9} = 1.429 \times 10^{-10}\ Pa^{-1}$$

$$S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17 \times 10^9} = 1.395 \times 10^{-10}\, Pa^{-1}.$$

Thus, the compliance matrix for the orthotropic lamina is given by

$$[S] =$$

$$\begin{bmatrix} 5.525\times10^{-12} & -1.547\times10^{-12} & -1.492\times10^{-12} & 0 & 0 & 0 \\ -1.547\times10^{-12} & 9.709\times10^{-11} & -5.825\times10^{-11} & 0 & 0 & 0 \\ -1.492\times10^{-12} & -5.825\times10^{-11} & 9.709\times10^{-11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.333\times10^{-10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.429\times10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.395\times10^{-10} \end{bmatrix} Pa^{-1}$$

The stiffness matrix can be found by inverting the compliance matrix and is given by

$$[C] = [S]^{-1}$$

$$[C] =$$

$$\begin{bmatrix} 0.1850\times10^{12} & 0.7269\times10^{10} & 0.7204\times10^{10} & 0 & 0 & 0 \\ 0.7269\times10^{10} & 0.1638\times10^{11} & 0.9938\times10^{10} & 0 & 0 & 0 \\ 0.7204\times10^{10} & 0.9938\times10^{10} & 0.1637\times10^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3000\times10^{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6998\times10^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7168\times10^{10} \end{bmatrix} Pa$$

The preceding stiffness matrix $[C]$ can also be found directly by using Equation (2.71).

2.4 Hooke's Law for a Two-Dimensional Unidirectional Lamina

2.4.1 Plane Stress Assumption

A thin plate is a prismatic member having a small thickness, and it is the case for a typical lamina. If a plate is thin and there are no out-of-plane loads, it can be considered to be under plane stress (Figure 2.17). If the upper and lower surfaces of the plate are free from external loads, then $\sigma_3 = 0$, $\tau_{31} = 0$, and $\tau_{23} = 0$. Because the plate is thin, these three stresses within the plate are

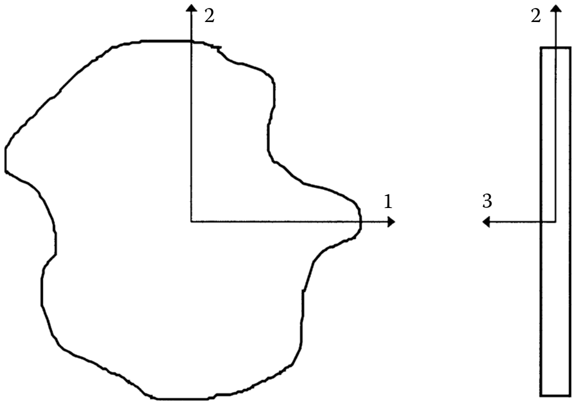


FIGURE 2.17
Plane stress conditions for a thin plate.

assumed to vary little from the magnitude of stresses at the top and the bottom surfaces. Thus, they can be assumed to be zero within the plate also. A lamina is thin and, if no out-of-plane loads are applied, one can assume that it is under plane stress. This assumption then reduces the three-dimensional stress–strain equations to two-dimensional stress–strain equations.

2.4.2 Reduction of Hooke's Law in Three Dimensions to Two Dimensions

A unidirectional lamina falls under the orthotropic material category. If the lamina is thin and does not carry any out-of-plane loads, one can assume plane stress conditions for the lamina. Therefore, taking Equation (2.26) and Equation (2.39) and assuming $\sigma_3 = 0$, $\tau_{23} = 0$, and $\tau_{31} = 0$, then

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2,$$

$$\gamma_{23} = \gamma_{31} = 0. \tag{2.76a,b}$$

The normal strain, ε_3, is not an independent strain because it is a function of the other two normal strains, ε_1 and ε_2. Therefore, the normal strain, ε_3, can be omitted from the stress–strain relationship (2.39). Also, the shearing strains, γ_{23} and γ_{31}, can be omitted because they are zero. Equation (2.39) for an orthotropic plane stress problem can then be written as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}, \tag{2.77}$$

where S_{ij} are the elements of the compliance matrix. Note the four independent compliance elements in the matrix.

Inverting Equation (2.77) gives the stress–strain relationship as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}, \tag{2.78}$$

where Q_{ij} are the reduced stiffness coefficients, which are related to the compliance coefficients as

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}, \tag{2.79a–d}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{66} = \frac{1}{S_{66}}.$$

Note that the elements of the reduced stiffness matrix, Q_{ij}, are not the same as the elements of the stiffness matrix, C_{ij} (see Exercise 2.13).

2.4.3 Relationship of Compliance and Stiffness Matrix to Engineering Elastic Constants of a Lamina

Equation (2.77) and Equation (2.78) show the relationship of stress and strain through the compliance [S] and reduced stiffness [Q] matrices. However, stress and strains are generally related through engineering elastic constants. For a unidirectional lamina, these engineering elastics constants are

E_1 = longitudinal Young's modulus (in direction 1)

E_2 = transverse Young's modulus (in direction 2)

ν_{12} = major Poisson's ratio, where the general Poisson's ratio, ν_{ij} is defined as the ratio of the negative of the normal strain in direction *j* to the normal strain in direction *i*, when the only normal load is applied in direction *i*

G_{12} = in-plane shear modulus (in plane 1–2)

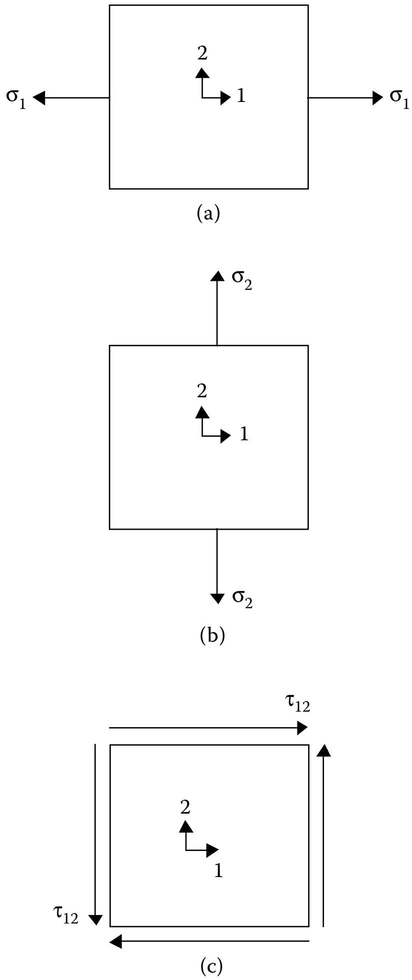


FIGURE 2.18
Application of stresses to find engineering constants of a unidirectional lamina.

Experimentally, the four independent engineering elastic constants are measured as follows and can be related to the four independent elements of the compliance matrix [S] of Equation (2.77).

- Apply a pure tensile load in direction 1 (Figure 2.18a), that is,

$$\sigma_1 \neq 0,\ \sigma_2 = 0,\ \tau_{12} = 0. \tag{2.80}$$

Then, from Equation (2.77),

$$\begin{aligned} \varepsilon_1 &= S_{11}\sigma_1, \\ \varepsilon_2 &= S_{12}\sigma_1, \\ \gamma_{12} &= 0. \end{aligned} \tag{2.81a–c}$$

By definition, if the only nonzero stress is σ_1, as is the case here, then

$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}, \tag{2.82}$$

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}. \tag{2.83}$$

- Apply a pure tensile load in direction 2 (Figure 2.18b), that is

$$\sigma_1 = 0,\ \sigma_2 \neq 0,\ \tau_{12} = 0. \tag{2.84}$$

Then, from Equation (2.77),

$$\begin{aligned} \varepsilon_1 &= S_{12}\sigma_2, \\ \varepsilon_2 &= S_{22}\sigma_2, \\ \gamma_{12} &= 0. \end{aligned} \tag{2.85a–c}$$

By definition, if the only nonzero stress is σ_2, as is the case here, then

$$E_2 \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}}, \tag{2.86}$$

$$\nu_{21} \equiv -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}. \tag{2.87}$$

The ν_{21} term is called the minor Poisson's ratio. From Equation (2.82), Equation (2.83), Equation (2.86), and Equation (2.87), we have the reciprocal relationship

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}. \tag{2.88}$$

- Apply a pure shear stress in the plane 1–2 (Figure 2.18c) — that is,

$$\sigma_1 = 0, \sigma_2 = 0 \text{ and } \tau_{12} \neq 0. \tag{2.89}$$

Then, from Equation (2.77),

$$\varepsilon_1 = 0,$$

$$\varepsilon_2 = 0,$$

$$\gamma_{12} = S_{66}\tau_{12}. \tag{2.90a–c}$$

By definition, if τ_{12} is the only nonzero stress, as is the case here, then

$$G_{12} \equiv \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}}. \tag{2.91}$$

Thus, we have proved that

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \tag{2.92a–d}$$

Also, the stiffness coefficients Q_{ij} are related to the engineering constants through Equation (2.98) and Equation (2.92) as

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}},$$

$$Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}, \text{ and}$$

$$Q_{66} = G_{12}. \tag{2.93a–d}$$

Equation (2.77), Equation (2.78), Equation (2.92), and Equation (2.93) relate stresses and strains through any of the following combinations of four constants.

Q_{11}, Q_{12}, Q_{22}, Q_{66}, or
S_{11}, S_{12}, S_{22}, S_{66}, or
E_1, E_2, ν_{12}, G_{12}

The unidirectional lamina is a *specially orthotropic* lamina because normal stresses applied in the 1–2 direction do not result in any shearing strains in the 1–2 plane because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$. Also, the shearing stresses applied in the 1–2 plane do not result in any normal strains in the 1 and 2 directions because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$.

A woven composite with its weaves perpendicular to each other and short fiber composites with fibers arranged perpendicularly to each other or aligned in one direction also are *specially orthotropic*. Thus, any discussion in this chapter or in Chapter 4 ("Macromechanics of a Laminate") is valid for such a lamina as well. Mechanical properties of some typical unidirectional lamina are given in Table 2.1 and Table 2.2.

Example 2.6

For a graphite/epoxy unidirectional lamina, find the following

1. Compliance matrix
2. Minor Poisson's ratio
3. Reduced stiffness matrix
4. Strains in the 1–2 coordinate system if the applied stresses (Figure 2.19) are

$$\sigma_1 = 2MPa, \ \sigma_2 = -3MPa, \ \tau_{12} = 4MPa.$$

Use the properties of unidirectional graphite/epoxy lamina from Table 2.1.

TABLE 2.1

Typical Mechanical Properties of a Unidirectional Lamina (SI System of Units)

Property	Symbol	Units	Glass/ epoxy	Boron/ epoxy	Graphite/ epoxy
Fiber volume fraction	V_f		0.45	0.50	0.70
Longitudinal elastic modulus	E_1	GPa	38.6	204	181
Transverse elastic modulus	E_2	GPa	8.27	18.50	10.30
Major Poisson's ratio	ν_{12}		0.26	0.23	0.28
Shear modulus	G_{12}	GPa	4.14	5.59	7.17
Ultimate longitudinal tensile strength	$(\sigma_1^T)_{ult}$	MPa	1062	1260	1500
Ultimate longitudinal compressive strength	$(\sigma_1^C)_{ult}$	MPa	610	2500	1500
Ultimate transverse tensile strength	$(\sigma_2^T)_{ult}$	MPa	31	61	40
Ultimate transverse compressive strength	$(\sigma_2^C)_{ult}$	MPa	118	202	246
Ultimate in-plane shear strength	$(\tau_{12})_{ult}$	MPa	72	67	68
Longitudinal coefficient of thermal expansion	α_1	µm/m/°C	8.6	6.1	0.02
Transverse coefficient of thermal expansion	α_2	µm/m/°C	22.1	30.3	22.5
Longitudinal coefficient of moisture expansion	β_1	m/m/kg/kg	0.00	0.00	0.00
Transverse coefficient of moisture expansion	β_2	m/m/kg/kg	0.60	0.60	0.60

Source: Tsai, S.W. and Hahn, H.T., *Introduction to Composite Materials*, CRC Press, Boca Raton, FL, Table 1.7, p. 19; Table 7.1, p. 292; Table 8.3, p. 344. Reprinted with permission.

Solution

From Table 2.1, the engineering elastic constants of the unidirectional graphite/epoxy lamina are

$$E_1 = 181\ GPa,\ E_2 = 10.3\ GPa,\ \nu_{12} = 0.28,\ G_{12} = 7.17\ GPa.$$

1. Using Equation (2.92), the compliance matrix elements are

$$S_{11} = \frac{1}{181 \times 10^9} = 0.5525 \times 10^{-11}\,Pa^{-1},$$

$$S_{12} = -\frac{0.28}{181 \times 10^9} = -0.1547 \times 10^{-11}\,Pa^{-1},$$

TABLE 2.2

Typical Mechanical Properties of a Unidirectional Lamina (USCS System of Units)

Property	Symbol	Units	Glass/ epoxy	Boron/ epoxy	Graphite/ epoxy
Fiber volume fraction	V_f	—	0.45	0.50	0.70
Longitudinal elastic modulus	E_1	Msi	5.60	29.59	26.25
Transverse elastic modulus	E_2	Msi	1.20	2.683	1.49
Major Poisson's ratio	v_{12}		0.26	0.23	0.28
Shear modulus	G_{12}	Msi	0.60	0.811	1.040
Ultimate longitudinal tensile strength	$(\sigma_1^T)_{ult}$	ksi	154.03	182.75	217.56
Ultimate longitudinal compressive strength	$(\sigma_1^C)_{ult}$	ksi	88.47	362.6	217.56
Ultimate transverse tensile strength	$(\sigma_2^T)_{ult}$	ksi	4.496	8.847	5.802
Ultimate transverse compressive strength	$(\sigma_2^C)_{ult}$	ksi	17.12	29.30	35.68
Ultimate in-plane shear strength	$(\tau_{12})_{ult}$	ksi	10.44	9.718	9.863
Longitudinal coefficient of thermal expansion	α_1	µin./in./°F	4.778	3.389	0.0111
Transverse coefficient of thermal expansion	α_2	µin./in./°F	12.278	16.83	12.5
Longitudinal coefficient of moisture expansion	β_1	in./in./lb/lb	0.00	0.00	0.00
Transverse coefficient of moisture expansion	β_2	in./in./lb/lb	0.60	0.60	0.60

Source: Tsai, S.W. and Hahn, H.T., *Introduction to Composite Materials*, CRC Press, Boca Raton, FL, Table 1.7, p. 19; Table 7.1, p. 292; Table 8.3, p. 344. USCS system used for tables reprinted with permission.

$$S_{22} = \frac{1}{10.3 \times 10^9} = 0.9709 \times 10^{-10}\, Pa^{-1},$$

$$S_{66} = \frac{1}{7.17 \times 10^9} = 0.1395 \times 10^{-9}\, Pa^{-1}.$$

2. Using the reciprocal relationship (2.88), the minor Poisson's ratio is

$$\nu_{21} = \frac{0.28}{181 \times 10^9} \times (10.3 \times 10^9) = 0.01593.$$

3. Using Equation (2.93), the reduced stiffness matrix [*Q*] elements are

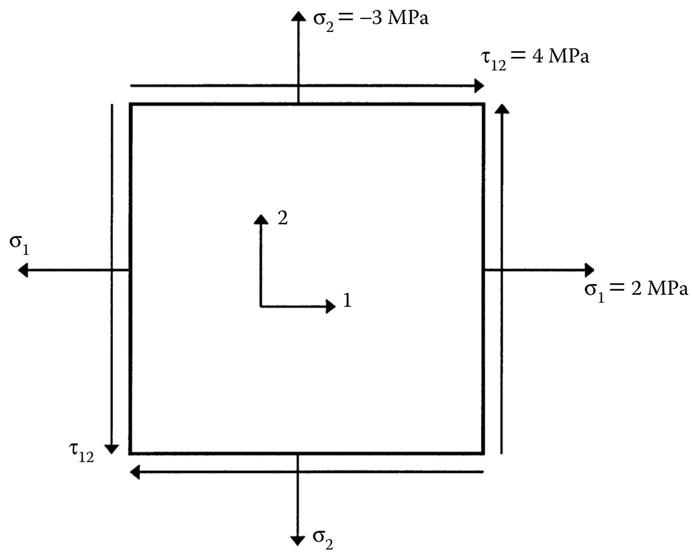


FIGURE 2.19
Applied stresses in a unidirectional lamina in Example 2.6.

$$Q_{11} = \frac{181 \times 10^9}{1 - (0.28)(0.01593)} = 181.8 \times 10^9 \, Pa,$$

$$Q_{12} = \frac{(0.28)(10.3 \times 10^9)}{1 - (0.28)(0.01593)} = 2.897 \times 10^9 \, Pa,$$

$$Q_{22} = \frac{10.3 \times 10^9}{1 - (0.28)(0.01593)} = 10.35 \times 10^9 \, Pa,$$

$$Q_{66} = 7.17 \times 10^9 \, Pa \; .$$

The reduced stiffness matrix [Q] could also be obtained by inverting the compliance matrix [S] of part 1:

$$[Q] = [S]^{-1} = \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix}^{-1} .$$

$$= \begin{bmatrix} 181.8\times10^9 & 2.897\times10^9 & 0 \\ 2.897\times10^9 & 10.35\times10^9 & 0 \\ 0 & 0 & 7.17\times10^9 \end{bmatrix} Pa\ .$$

4. Using Equation (2.77), the strains in the 1–2 coordinate system are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.5525\times10^{-11} & -0.1547\times10^{-11} & 0 \\ -0.1547\times10^{-11} & 0.9709\times10^{-10} & 0 \\ 0 & 0 & 0.1395\times10^{-9} \end{bmatrix} \begin{bmatrix} 2\times10^6 \\ -3\times10^6 \\ 4\times10^6 \end{bmatrix}$$

$$= \begin{bmatrix} 15.69 \\ -294.4 \\ 557.9 \end{bmatrix} (10^{-6}).$$

Thus, the strains in the local axes are

$$\varepsilon_1 = 15.69 \frac{\mu m}{m},$$

$$\varepsilon_2 = 294.4 \frac{\mu m}{m},$$

$$\gamma_{12} = 557.9 \frac{\mu m}{m}.$$

2.5 Hooke's Law for a Two-Dimensional Angle Lamina

Generally, a laminate does not consist only of unidirectional laminae because of their low stiffness and strength properties in the transverse direction. Therefore, in most laminates, some laminae are placed at an angle. It is thus necessary to develop the stress–strain relationship for an angle lamina.

The coordinate system used for showing an angle lamina is as given in Figure 2.20. The axes in the 1–2 coordinate system are called the local axes or the material axes. The direction 1 is parallel to the fibers and the direction 2 is perpendicular to the fibers. In some literature, direction 1 is also called

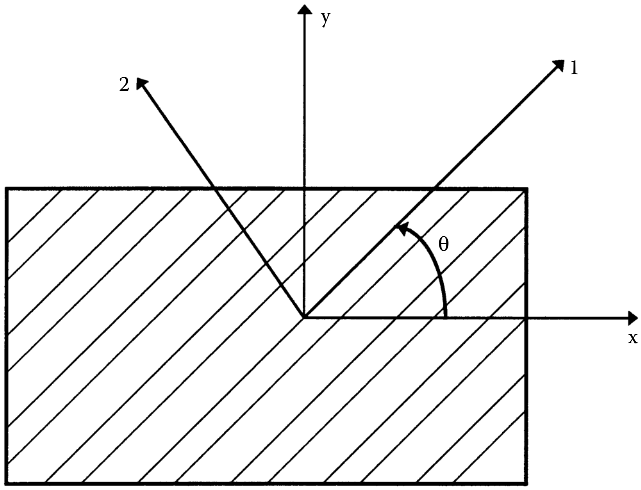


FIGURE 2.20
Local and global axes of an angle lamina.

the longitudinal direction L and the direction 2 is called the transverse direction T. The axes in the x–y coordinate system are called the global axes or the off-axes. The angle between the two axes is denoted by an angle θ. The stress–strain relationship in the 1–2 coordinate system has already been established in Section 2.4 and we are now going to develop the stress–strain equations for the x–y coordinate system.

The global and local stresses in an angle lamina are related to each other through the angle of the lamina, θ (Appendix B):

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}, \tag{2.94}$$

where $[T]$ is called the transformation matrix and is defined as

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}, \tag{2.95}$$

and

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}, \tag{2.96}$$

$$c = \text{Cos}\,(\theta),$$

$$s = \text{Sin}\,(\theta). \tag{2.97a,b}$$

Using the stress–strain Equation (2.78) in the local axes, Equation (2.94) can be written as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}. \tag{2.98}$$

The global and local strains are also related through the transformation matrix (Appendix B):

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} - [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}, \tag{2.99}$$

which can be rewritten as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \tag{2.100}$$

where $[R]$ is the Reuter matrix[3] and is defined as

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.101}$$

Then, substituting Equation (2.100) in Equation (2.98) gives

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \tag{2.102}$$

On carrying the multiplication of the first five matrices on the right-hand side of Equation (2.102),

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \tag{2.103}$$

where $\bar{Q}_{ij}$ are called the elements of the transformed reduced stiffness matrix $[\bar{Q}]$ and are given by

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^2),$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4). \tag{2.104a–f}$$

Note that six elements are in the $[\bar{Q}]$ matrix. However, by looking at Equation (2.104), it can be seen that they are just functions of the four stiffness elements, Q_{11}, Q_{12}, Q_{22}, and Q_{66}, and the angle of the lamina, θ.

Inverting Equation (2.103) gives

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \tag{2.105}$$

where S_{ij} are the elements of the transformed reduced compliance matrix and are given by

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a\text{–}f)$$

From Equation (2.77) and Equation (2.78), for a unidirectional lamina loaded in the material axes directions, no coupling occurs between the normal and shearing terms of strains and stresses. However, for an angle lamina, from Equation (2.103) and Equation (2.105), coupling takes place between the normal and shearing terms of strains and stresses. If only normal stresses are applied to an angle lamina, the shear strains are nonzero; if only shearing stresses are applied to an angle lamina, the normal strains are nonzero. Therefore, Equation (2.103) and Equation (2.105) are stress–strain equations for what is called a *generally orthotropic* lamina.

Example 2.7

Find the following for a 60° angle lamina (Figure 2.21) of graphite/epoxy. Use the properties of unidirectional graphite/epoxy lamina from Table 2.1.

1. Transformed compliance matrix
2. Transformed reduced stiffness matrix

If the applied stress is $\sigma_x = 2$ MPa, $\sigma_y = -3$ MPa, and $\tau_{xy} = 4$ MPa, also find

3. Global strains
4. Local strains
5. Local stresses
6. Principal stresses
7. Maximum shear stress

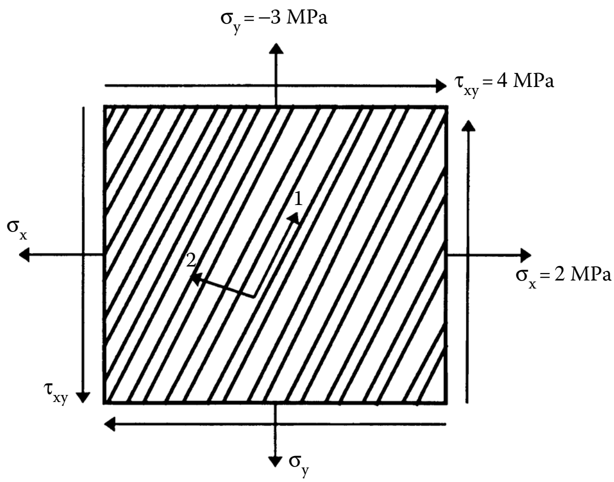


FIGURE 2.21
Applied stresses to an angle lamina in Example 2.7.

8. Principal strains
9. Maximum shear strain

Solution

c = Cos(60°) = 0.500
s = Sin(60°) = 0.866

1. From Example 2.6,

$$S_{11} = 0.5525 \times 10^{-11} \frac{1}{Pa},$$

$$S_{22} = 0.9709 \times 10^{-10} \frac{1}{Pa},$$

$$S_{12} = -0.1547 \times 10^{-11} \frac{1}{Pa},$$

$$S_{66} = 0.1395 \times 10^{-9} \frac{1}{Pa}.$$

Now, using Equation (2.106a),

$$\bar{S}_{11} = 0.5525 \times 10^{-11}(0.500)^4 + [2(-0.1547 \times 10^{-11})$$

$$+ 0.1395 \times 10^{-9}](0.866)^2(0.5)^2 + 0.9709 \times 10^{-10}(0.866)^4$$

$$= 0.8053 \times 10^{-10}\,\frac{1}{Pa}$$

Similarly, using Equation (2.106b–f), one can evaluate

$$\bar{S}_{12} = -0.7878 \times 10^{-11}\,\frac{1}{Pa},$$

$$\bar{S}_{16} = -0.3234 \times 10^{-10}\,\frac{1}{Pa},$$

$$\bar{S}_{22} = 0.3475 \times 10^{-10}\,\frac{1}{Pa},$$

$$\bar{S}_{26} = -0.4696 \times 10^{-10}\,\frac{1}{Pa},$$

$$\bar{S}_{66} = 0.1141 \times 10^{-9}\,\frac{1}{Pa}.$$

2. Invert the transformed compliance matrix $[\bar{S}]$ to obtain the transformed reduced stiffness matrix $[\bar{Q}]$:

$$[\bar{Q}] = \begin{bmatrix} 0.8053 \times 10^{-10} & -0.7878 \times 10^{-11} & -0.3234 \times 10^{-10} \\ -0.7878 \times 10^{-11} & 0.3475 \times 10^{-10} & -0.4696 \times 10^{-10} \\ -0.3234 \times 10^{-10} & -0.4696 \times 10^{-10} & 0.1141 \times 10^{-9} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0.2365 \times 10^{11} & 0.3246 \times 10^{11} & 0.2005 \times 10^{11} \\ 0.3246 \times 10^{11} & 0.1094 \times 10^{12} & 0.5419 \times 10^{11} \\ 0.2005 \times 10^{11} & 0.5419 \times 10^{11} & 0.3674 \times 10^{11} \end{bmatrix} Pa.$$

3. The global strains in the x–y plane are given by Equation (2.105) as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.8053\times10^{-10} & -0.7878\times10^{-11} & -0.3234\times10^{-10} \\ -0.7878\times10^{-11} & 0.3475\times10^{-10} & -0.4696\times10^{-10} \\ -0.3234\times10^{-10} & -0.4696\times10^{-10} & 0.1141\times10^{-9} \end{bmatrix} \begin{bmatrix} 2\times10^{6} \\ -3\times10^{6} \\ 4\times10^{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5534\times10^{-4} \\ -0.3078\times10^{-3} \\ 0.5328\times10^{-3} \end{bmatrix}.$$

4. Using transformation Equation (2.99), the local strains in the lamina are

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.500 \end{bmatrix} \begin{bmatrix} 0.5534\times10^{-4} \\ -0.3078\times10^{-3} \\ 0.5328\times10^{-3}/2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 0.1367\times10^{-4} \\ -0.2662\times10^{-3} \\ -0.5809\times10^{-3} \end{bmatrix}.$$

5. Using transformation Equation (2.94), the local stresses in the lamina are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.500 \end{bmatrix} \begin{bmatrix} 2\times10^{6} \\ -3\times10^{6} \\ 4\times10^{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714\times10^{7} \\ -0.2714\times10^{7} \\ -0.4165\times10^{7} \end{bmatrix} Pa.$$

6. The principal normal stresses are given by[4]

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{2.107}$$

$$= \frac{2\times10^6 - 3\times10^6}{2} \pm \sqrt{\left(\frac{2\times10^6 + 3\times10^6}{2}\right)^2 + (4\times10^6)^2}$$

$$= 4.217, -5.217\ MPa.$$

The value of the angle at which the maximum normal stresses occur is[4]

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \tag{2.108}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2(4\times10^6)}{2\times10^6 + 3\times10^6}\right)$$

$$= 29.00^0\ .$$

Note that the principal normal stresses do not occur along the material axes. This should be also evident from the nonzero shear stresses in the local axes.

7. The maximum shear stress is given by[4]

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{2\times10^6 - 3\times10^6}{2}\right)^2 + (4\times10^6)^2} \tag{2.109}$$

$$= 4.717\ MPa.$$

The angle at which the maximum shear stress occurs is[4]

$$\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) \tag{2.110}$$

$$= \frac{1}{2}\tan^{-1}\left(-\frac{2\times 10^6 + 3\times 10^6}{2(4\times 10^6)}\right)$$

$$= 16.00^0$$

8. The principal strains are given by[4]

$$\varepsilon_{\max,\min} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$= \frac{0.5534\times 10^{-4} + 0.3078\times 10^{-3}}{2} \pm \sqrt{\left(\frac{0.5534\times 10^{-4} + 0.3078\times 10^{-3}}{2}\right)^2 + \left(\frac{0.5328\times 10^{-3}}{2}\right)^2} \tag{2.111}$$

$$= 1.962\times 10^{-4}, -4.486\times 10^{-4}\ .$$

The value of the angle at which the maximum normal strains occur is[4]

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{0.5328\times 10^{-3}}{0.5534\times 10^{-4} + 0.3078\times 10^{-3}}\right) \tag{2.112}$$

$$= 27.86^0.$$

Note that the principal normal strains do not occur along the material axes. This should also be clear from the nonzero shear strain in the local axes. In addition, the axes of principal normal stresses and principal normal strains do not match, unlike in isotropic materials.

9. The maximum shearing strain is given by[4]

$$\gamma_{max} = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

$$= \sqrt{(0.5534 \times 10^{-4} + 0.3078 \times 10^{-3})^2 + (0.532 \times 10^{-3})^2} \tag{2.113}$$

$$= 6.448 \times 10^{-4}.$$

The value of the angle at which the maximum shearing strain occurs is[4]

$$\theta_s = \frac{1}{2}\tan^{-1}\left(-\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(-\frac{0.5534 \times 10^{-4} + 0.3078 \times 10^{-3}}{0.5328 \times 10^{-3}}\right) \tag{2.114}$$

$$= -17.14^0.$$

Example 2.8

As shown in Figure 2.22, a 60° angle graphite/epoxy lamina is subjected only to a shear stress τ_{xy} = 2 MPa in the global axes. What would be the value of the strains measured by the strain gage rosette — that is, what

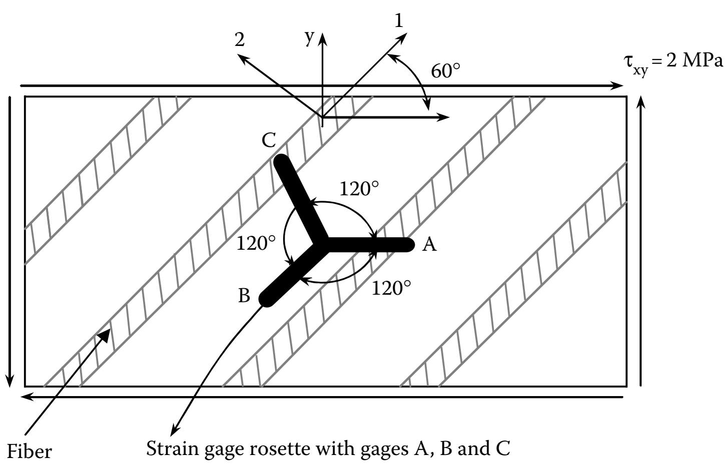


FIGURE 2.22
Strain gage rosette on an angle lamina.

would be the normal strains measured by strain gages A, B, and C? Use the properties of unidirectional graphite/epoxy lamina from Table 2.1.

Solution

Per Example 2.7, the reduced compliance matrix [$\bar{S}$] is

$$\begin{bmatrix} 0.8053\times10^{-10} & -0.7878\times10^{-11} & -0.3234\times10^{-10} \\ -0.7878\times10^{-11} & 0.3475\times10^{-10} & -0.4696\times10^{-10} \\ -0.3234\times10^{-10} & -0.4696\times10^{-10} & 0.1141\times10^{-9} \end{bmatrix} \frac{1}{Pa} .$$

The global strains in the x–y plane are given by Equation (2.105) as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 0.8053\times10^{-10} & -0.7878\times10^{-11} & -0.3234\times10^{-10} \\ -0.7878\times10^{-11} & 0.3475\times10^{-10} & -0.4696\times10^{-10} \\ -0.3234\times10^{-10} & -0.4696\times10^{-10} & 0.1141\times10^{-9} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2\times10^{6} \end{bmatrix}$$

$$= \begin{bmatrix} -6.468\times10^{-5} \\ -9.392\times10^{-5} \\ 2.283\times10^{-4} \end{bmatrix}.$$

For a strain gage placed at an angle, ϕ, to the x-axis, the normal strain recorded by the strain gage is given by Equation (B.15) in Appendix B.

$$\varepsilon_\phi = \varepsilon_x \text{Cos}^2\phi + \varepsilon_y \ \text{Sin}^2\phi + \gamma_{xy}\,\text{Sin}\,\phi\,\text{Cos}\,\phi \ .$$

For strain gage A, ϕ = 0°:

$$\varepsilon_A = -6.468\times10^{-5}\,Cos^2\,0° + (-9.392\times10^{-5})\,\text{Sin}^2\,0° + 2.283\times10^{-4}\,\text{Sin}\,0°\,\text{Cos}\,0°$$

$$= -6.468\times10^{-5} \ .$$

For strain gage B, ϕ = 240°:

$$\varepsilon_B = -6.468\times10^{-5}\,\text{Cos}^2\,240° + (-9.392\times10^{-5})\,\text{Sin}^2\,240°$$
$$+ 2.283\times10^{-4}\,\text{Sin}\,240°\,\text{Cos}\,240°$$

$$= 1.724 \times 10^{-4}\ .$$

For strain gage C, $\phi = 120°$:

$$\varepsilon_C = -6.468 \times 10^{-5}\,\mathrm{Cos}^2\,120° + (-9.392 \times 10^{-5})\,\mathrm{Sin}^2\,120°$$
$$+ 2.283 \times 10^{-4}\,\mathrm{Sin}\,120°\,\mathrm{Cos}\,120°$$

$$= 1.083 \times 10^{-5}\ .$$

2.6 Engineering Constants of an Angle Lamina

The engineering constants for a unidirectional lamina were related to the compliance and stiffness matrices in Section 2.4.3. In this section, similar techniques are applied to relate the engineering constants of an angle ply to its transformed stiffness and compliance matrices.

1. For finding the engineering elastic moduli in direction x (Figure 2.23a), apply

$$\sigma_x \neq 0,\ \sigma_y = 0,\ \tau_{xy} = 0. \tag{2.115}$$

Then, from Equation (2.105),

$$\varepsilon_x = \bar{S}_{11}\sigma_x,$$

$$\varepsilon_y = \bar{S}_{12}\sigma_x,$$

$$\gamma_{xy} = \bar{S}_{16}\sigma_x\ . \tag{2.116a–c}$$

The elastic moduli in direction x is defined as

$$E_x \equiv \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\bar{S}_{11}}. \tag{2.117}$$

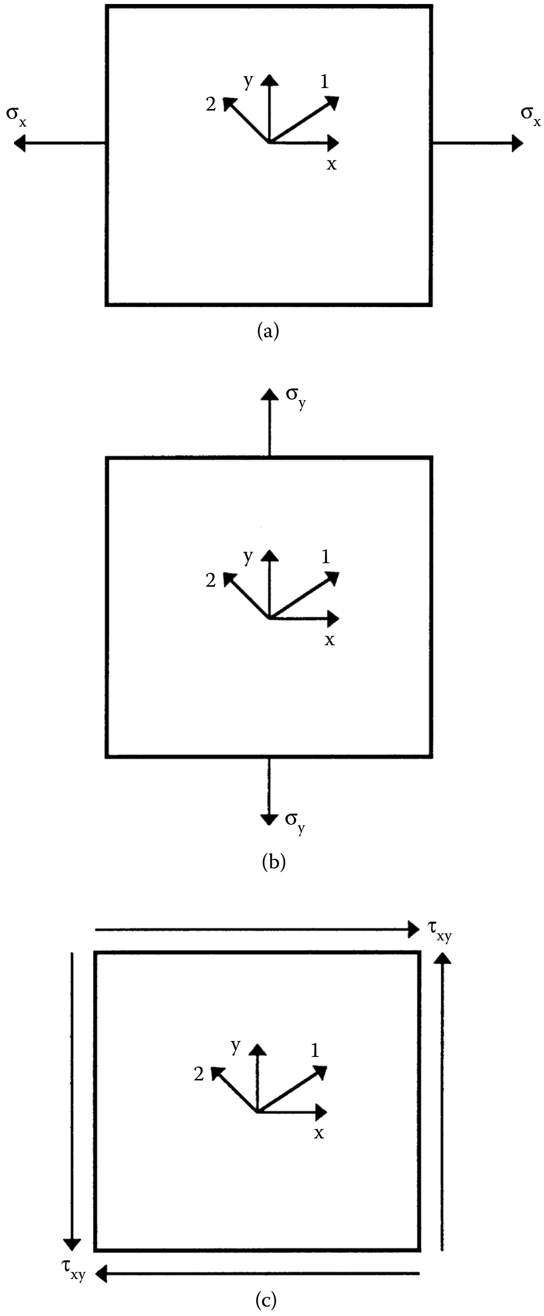


FIGURE 2.23
Application of stresses to find engineering constants of an angle lamina.

Also, the Poisson's ratio, ν_{xy}, is defined as

$$\nu_{xy} \equiv -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}}. \tag{2.118}$$

In an angle lamina, unlike in a unidirectional lamina, interaction also occurs between the shear strain and the normal stresses. This is called shear coupling. The shear coupling term that relates the normal stress in the x-direction to the shear strain is denoted by m_x and is defined as

$$\frac{1}{m_x} \equiv -\frac{\sigma_x}{\gamma_{xy}E_1} = -\frac{1}{\bar{S}_{16}E_1}. \tag{2.119}$$

Note that m_x is a nondimensional parameter like the Poisson's ratio. Later, note that the same parameter, m_x, relates the shearing stress in the x–y plane to the normal strain in direction-x.

The shear coupling term is particularly important in tensile testing of angle plies. For example, if an angle lamina is clamped at the two ends, it will not allow shearing strain to occur. This will result in bending moments and shear forces at the clamped ends.[5]

2. Similarly, by applying stresses

$$\sigma_x = 0,\ \sigma_y \neq 0,\ \tau_{xy} = 0\ , \tag{2.120}$$

as shown in Figure 2.23b, it can be found

$$E_y = \frac{1}{\bar{S}_{22}}, \tag{2.121}$$

$$\nu_{yx} = -\frac{\bar{S}_{12}}{\bar{S}_{22}},\ \text{and} \tag{2.122}$$

$$\frac{1}{m_y} = -\frac{1}{\bar{S}_{26}E_1}. \tag{2.123}$$

The shear coupling term m_y relates the normal stress σ_y to the shear strain γ_{xy}. In the following section (3), note that the same parameter m_y relates the shear stress τ_{xy} in the x–y plane to the normal strain ε_y.

From Equation (2.117), Equation (2.118), Equation (2.121), and Equation (2.122), the reciprocal relationship is given by

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \,. \tag{2.124}$$

3. Also, by applying the stresses

$$\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} \neq 0\,, \tag{2.125}$$

as shown in Figure 2.23c, it is found that

$$\frac{1}{m_x} = -\frac{1}{\bar{S}_{16} E_1}, \tag{2.126}$$

$$\frac{1}{m_y} = -\frac{1}{\bar{S}_{26} E_1}, \text{ and} \tag{2.127}$$

$$G_{xy} = \frac{1}{\bar{S}_{66}}. \tag{2.128}$$

Thus, the strain–stress Equation (2.105) of an angle lamina can also be written in terms of the engineering constants of an angle lamina in matrix form as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \tag{2.129}$$

The preceding six engineering constants of an angle ply can also be written in terms of the engineering constants of a unidirectional ply using Equation (2.92) and Equation (2.106) in Equation (2.117) through Equation (2.119), Equation (2.121), Equation (2.123), and Equation (2.128):

$$\frac{1}{E_x} = \bar{S}_{11}$$

$$= S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4.$$

$$= \frac{1}{E_1}c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)s^2c^2 + \frac{1}{E_2}s^4, \tag{2.130}$$

$$\nu_{xy} = -E_x\bar{S}_{12}$$

$$= -E_x[S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2]$$

$$= E_x\left[\frac{\nu_{12}}{E_1}\left(s^4 + c^4\right) - \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)s^2c^2\right], \tag{2.131}$$

$$\frac{1}{E_y} = \bar{S}_{22}$$

$$= S_{11}s^4 + (2S_{12} + S_{66})c^2s^2 + S_{22}c^4$$

$$= \frac{1}{E_1}s^4 + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)c^2s^2 + \frac{1}{E_2}c^4 \ , \tag{2.132}$$

$$\frac{1}{G_{xy}} = \bar{S}_{66}$$

$$= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4)$$

$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^2c^2 + \frac{1}{G_{12}}(s^4 + c^4), \tag{2.133}$$

$$m_x = -\bar{S}_{16}E_1$$

$$= -E_1[(S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)sc^3 + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c\right], \tag{2.134}$$

$$m_y = -\bar{S}_{26} E_1$$

$$= -E_1[(2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)s^3c + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)sc^3\right]. \qquad (2.135)$$

Example 2.9

Find the engineering constants of a 60° graphite/epoxy lamina. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1.

Solution

From Example 2.7, we have

$$\bar{S}_{11} = 0.8053 \times 10^{-10} \frac{1}{Pa},$$

$$\bar{S}_{12} = -0.7878 \times 10^{-11} \frac{1}{Pa},$$

$$\bar{S}_{16} = -0.3234 \times 10^{-10} \frac{1}{Pa},$$

$$\bar{S}_{22} = 0.3475 \times 10^{-10} \frac{1}{Pa},$$

$$\bar{S}_{26} = -0.4696 \times 10^{-10} \frac{1}{Pa}, \text{ and}$$

$$\bar{S}_{66} = 0.1141 \times 10^{-9} \frac{1}{Pa}.$$

From Equation (2.117),

$$E_x = \frac{1}{0.8053 \times 10^{-10}}$$

$$= 12.42\,GPa.$$

From Equation (2.118),

$$\nu_{xy} = -\frac{-0.7878 \times 10^{-11}}{0.8053 \times 10^{-10}}$$

$$= 0.09783.$$

From Equation (2.119),

$$\frac{1}{m_x} = -\frac{1}{(-0.3234 \times 10^{-10})(181 \times 10^{9})}$$

$$m_x = 5.854\ .$$

From Equation (2.121),

$$E_y = \frac{1}{0.3475 \times 10^{-10}}$$

$$= 28.78\ GPa.$$

From Equation (2.123),

$$\frac{1}{m_y} = -\frac{1}{(-0.4696 \times 10^{-10})(181 \times 10^{9})}$$

$$m_y = 8.499.$$

From Equation (2.128),

$$G_{xy} = \frac{1}{0.1141 \times 10^{-9}}$$

$$= 8.761\ GPa.$$

The variations of the six engineering elastic constants are shown as a function of the angle for the preceding graphite/epoxy lamina in Figure 2.24 through Figure 2.29.

The variations of the Young's modulus, E_x and E_y are inverses of each other. As the fiber orientation (angle of ply) varies from 0° to 90°, the value of E_x

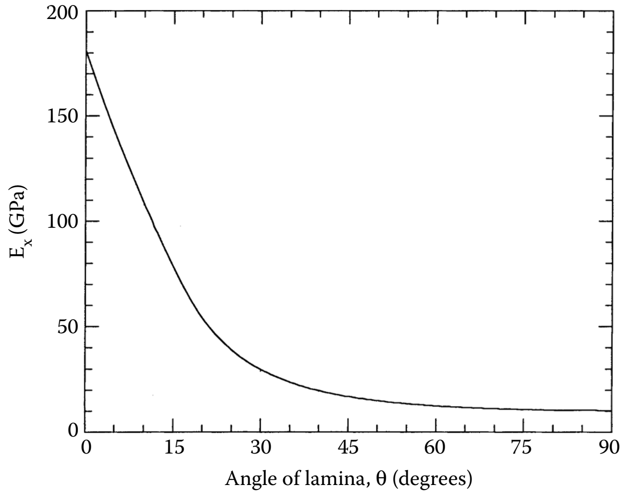


FIGURE 2.24
Elastic modulus in direction-x as a function of angle of lamina for a graphite/epoxy lamina.

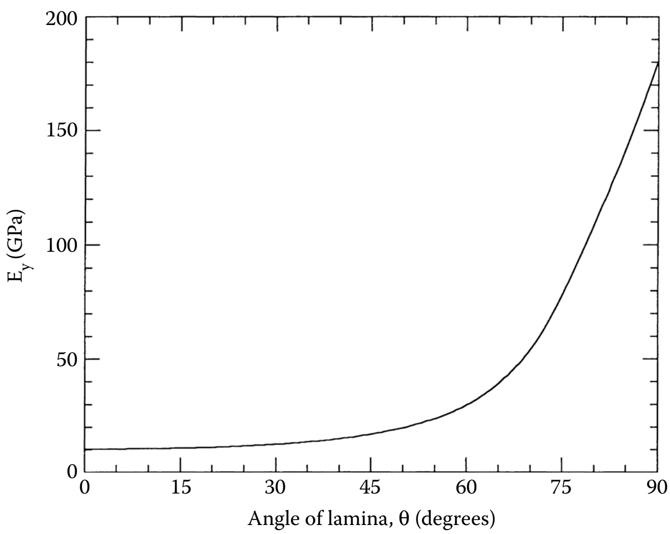


FIGURE 2.25
Elastic modulus in direction-y as a function of angle of lamina for a graphite/epoxy lamina.

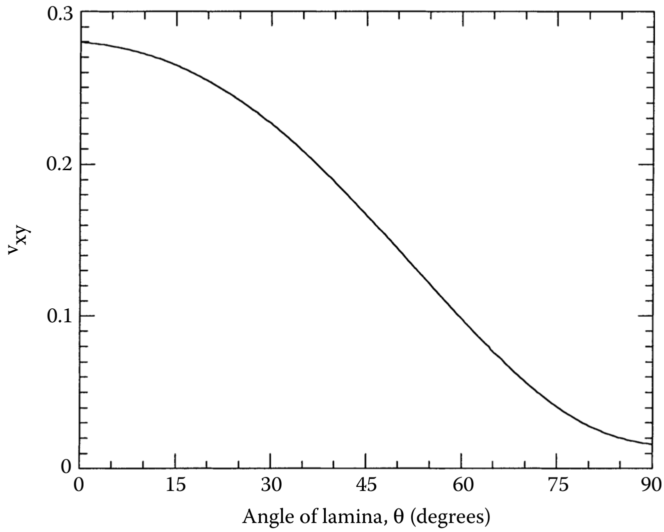


FIGURE 2.26
Poisson's ratio ν_{xy} as a function of angle of lamina for a graphite/epoxy lamina.

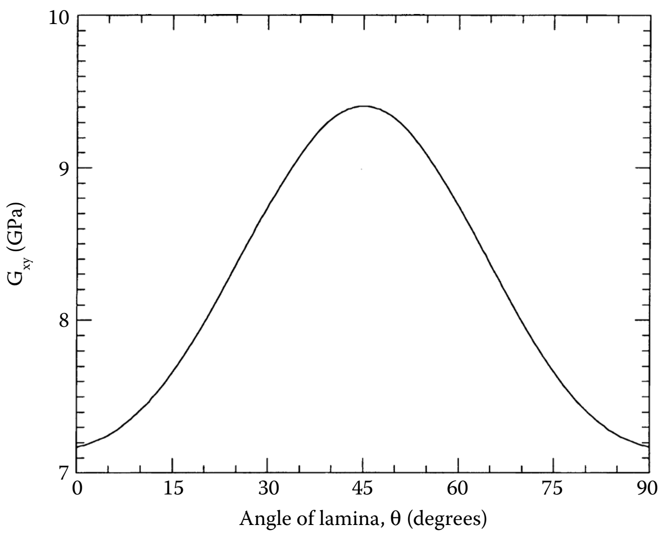


FIGURE 2.27
In-plane shear modulus in *xy*-plane as a function of angle of lamina for a graphite/epoxy lamina.

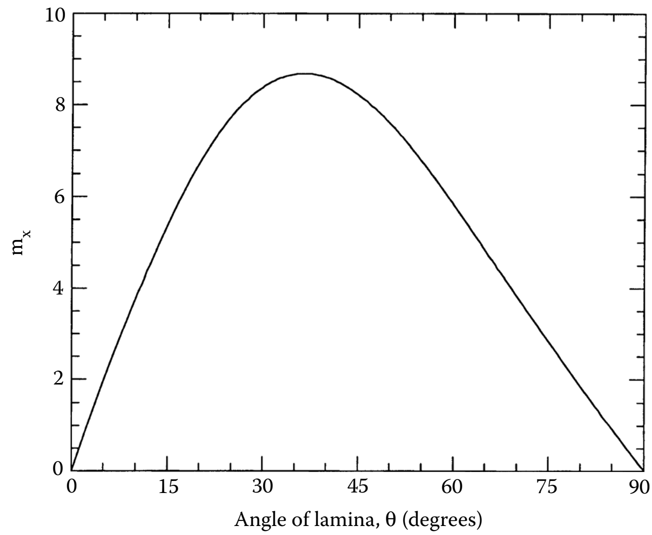


FIGURE 2.28
Shear coupling coefficient m_x as a function of angle of lamina for a graphite/epoxy lamina.

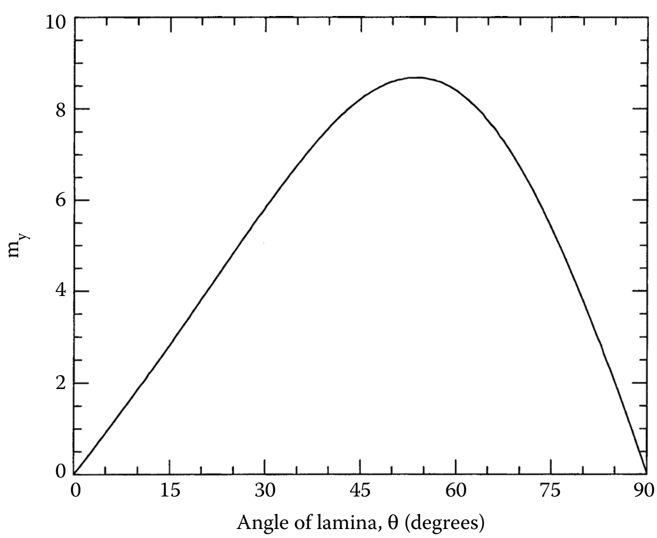


FIGURE 2.29
Shear coupling coefficient m_y as a function of angle of lamina for a graphite/epoxy lamina.

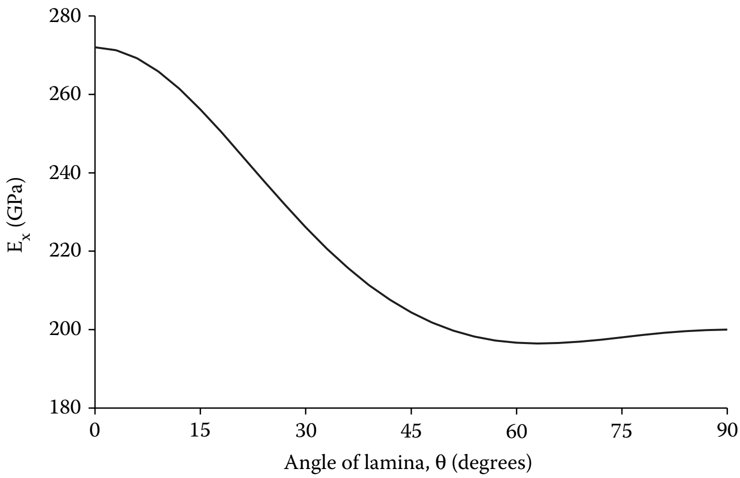


FIGURE 2.30
Variation of elastic modulus in direction-x as a function of angle of lamina for a typical $SCS-6/Ti6-Al-4V$ lamina.

varies from the value of the longitudinal (E_1) to the transverse Young's modulus E_2. However, the maximum and minimum values of E_x do not necessarily exist for $\theta = 0°$ and $\theta = 90°$, respectively, for every lamina.

Consider the case of a metal matrix composite such as a typical $SCS-6/Ti6-Al-4V$ composite. The elastic moduli of such a lamina with a 55% fiber volume fraction is

$$E_1 = 272 \text{ GPa}$$

$$E_2 = 200 \text{ GPa}$$

$$\nu_{12} = 0.2770$$

$$G_{12} = 77.33 \text{ GPa}$$

In Figure 2.30, the lowest modulus value of E_x is found for $\theta = 63°$. In fact, the angle of 63° at which E_x is minimum is independent of the fiber volume fraction, if one uses the "mechanics of materials approach" (Section 3.3.1) to evaluate the preceding four elastic moduli of a unidirectional lamina. See Exercise 3.13.

In Figure 2.27, the shear modulus G_{xy} is maximum for $\theta = 45°$ and is minimum for 0 and 90° plies. The shear modulus G_{xy} becomes maximum for 45° because the principal stresses for pure shear load on a 45° ply are along the material axis.

From Equation (2.133), the expression for G_{xy} for a 45° ply is

$$G_{xy/45^\circ} = \frac{E_1}{\left(1 + 2\nu_{12} + \frac{E_1}{E_2}\right)}. \quad (2.136)$$

In Figure 2.28 and Figure 2.29, the shear coupling coefficients m_x and m_y are maximum at θ = 36.2° and θ = 53.78°, respectively. The values of these coefficients are quite extreme, showing that the normal-shear coupling terms have a stronger effect than the Poisson's effect. This phenomenon of shear coupling terms is missing in isotropic materials and unidirectional plies, but cannot be ignored in angle plies.

2.7 Invariant Form of Stiffness and Compliance Matrices for an Angle Lamina

Equation (2.104) and Equation (2.106) for the [$\bar{Q}$] and [$\bar{S}$] matrices are not analytically convenient because they do not allow a direct study of the effect of the angle of the lamina on the [$\bar{Q}$] and [$\bar{S}$] matrices. The stiffness elements can be written in invariant form as[6]

$$\bar{Q}_{11} = U_1 + U_2 \operatorname{Cos} 2\theta + U_3 \operatorname{Cos} 4\theta,$$

$$\bar{Q}_{12} = U_4 - U_3 \operatorname{Cos} 4\theta,$$

$$\bar{Q}_{22} = U_1 - U_2 \operatorname{Cos} 2\theta + U_3 \operatorname{Cos} 4\theta$$

$$\bar{Q}_{16} = \frac{U_2}{2} \operatorname{Sin} 2\theta + U_3 \operatorname{Sin} 4\theta,$$

$$\bar{Q}_{26} = \frac{U_2}{2} \operatorname{Sin} 2\theta - U_3 \operatorname{Sin} 4\theta,$$

$$\bar{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \operatorname{Cos} 4\theta, \quad (2.137a\text{–}f)$$

where

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22}),$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}),$$

$$U_4 = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}). \qquad (2.138a–d)$$

The terms U_1, U_2, U_3, and U_4 are the four invariants and are combinations of the Q_{ij}, which are invariants as well.

The transformed reduced compliance [$\bar{S}$] matrix can similarly be written as

$$\bar{S}_{11} = V_1 + V_2 \text{Cos}\, 2\theta + V_3 \text{Cos}\, 4\theta,$$

$$\bar{S}_{12} = V_4 - V_3 \text{Cos}\, 4\theta,$$

$$\bar{S}_{22} = V_1 - V_2 \text{Cos}\, 2\theta + V_3 \text{Cos}\, 4\theta,$$

$$\bar{S}_{16} = V_2 \text{Sin}\, 2\theta + 2V_3 \text{Sin}\, 4\theta,$$

$$\bar{S}_{26} = V_2 \text{Sin}\, 2\theta - 2V_3 \text{Sin}\, 4\theta, \text{ and}$$

$$\bar{S}_{66} = 2(V_1 - V_4) - 4V_3 \text{Cos}\, 4\theta, \qquad (2.139a–f)$$

where

$$V_1 = \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}),$$

$$V_2 = \frac{1}{2}(S_{11} - S_{22}),$$

$$V_3 = \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}),$$

$$V_4 = \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66}). \tag{2.140a–d}$$

The terms V_1, V_2, V_3, and V_4 are invariants and are combinations of S_{ij}, which are also invariants.

The main advantage of writing the equations in this form is that one can easily examine the effect of the lamina angle on the reduced stiffness matrix elements. Also, formulas given by Equation (2.137) and Equation (2.139) are easier to manipulate for integration, differentiation, etc. The concept is mainly important in deriving the laminate stiffness properties in Chapter 4.

The elastic moduli of quasi-isotropic laminates that behave like isotropic material are directly given in terms of these invariants. Because quasi-isotropic laminates have the minimum stiffness of any laminate, these can be used as a comparative measure of the stiffness of other types of laminates.[7]

Example 2.10

Starting with the expression for $\bar{Q}_{11}$ from Equation (2.104a), $\bar{Q}_{11} = Q_{11}\,\mathrm{Cos}^4\,\theta,$ $+ Q_{22}\,\mathrm{Sin}^4\,\theta + 2(Q_{12} + 2Q_{66})\mathrm{Sin}^2\,\theta\,\mathrm{Cos}^2\,\theta$, reduce it to the expression for $\bar{Q}_{11}$ of Equation (2.137a) — that is,

$$\bar{Q}_{11} = U_1 + U_2\,\mathrm{Cos}\,2\theta + U_3\,\mathrm{Cos}\,4\theta$$

Solution

Given

$$\bar{Q}_{11} = Q_{11}\,\mathrm{Cos}^4\,\theta + Q_{22}\,\mathrm{Sin}^4\,\theta + 2(Q_{12} + 2Q_{66})\mathrm{Sin}^2\,\theta\,\mathrm{Cos}^2\,\theta\,,$$

and substituting

$$\mathrm{Cos}^2\,\theta = \frac{1 + \mathrm{Cos}\,2\theta}{2},$$

$$\mathrm{Sin}^2\,\theta = \frac{1 - \mathrm{Cos}\,2\theta}{2},$$

$$\cos^2 2\theta = \frac{1 + \cos 4\theta}{2}, \text{ and}$$

$$2 \sin\theta \cos\theta = \sin 2\theta,$$

$$\sin^2 2\theta = \frac{1 - \cos 4\theta}{2},$$

we get

$$\bar{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta ,$$

where

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}),$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) .$$

Example 2.11

Evaluate the four compliance and four stiffness invariants for a graphite/epoxy angle lamina. Use the properties for a unidirectional graphite/epoxy lamina from Table 2.1.

Solution

From Example 2.6, the compliance matrix [S] elements are

$$S_{11} = 0.5525 \times 10^{-11} \frac{1}{Pa},$$

$$S_{12} = -0.1547 \times 10^{-11} \frac{1}{Pa},$$

$$S_{22} = 0.9709 \times 10^{-10} \frac{1}{Pa},$$

$$S_{66} = 0.1395 \times 10^{-9} \frac{1}{Pa}.$$

The stiffness matrix $[Q]$ elements are

$$[Q] = [S]^{-1},$$

$$Q_{11} = 0.1818 \times 10^{12}\ Pa,$$

$$Q_{12} = 0.2897 \times 10^{10}\ Pa,$$

$$Q_{22} = 0.1035 \times 10^{11}\ Pa,$$

$$Q_{66} = 0.7170 \times 10^{10}\ Pa.$$

Using Equation (2.138),

$$U_1 = \frac{1}{8}[3(0.1818 \times 10^{12}) + 3(0.1035 \times 10^{11}) + 2(0.2897 \times 10^{10}) + 4(0.7171 \times 10^{10})]$$

$$= 0.7637 \times 10^{11}\ Pa,$$

$$U_2 = \frac{1}{2}(0.1818 \times 10^{12} - 0.1035 \times 10^{11})$$

$$= 0.8573 \times 10^{11}\ Pa,$$

$$U_3 = \frac{1}{8}[0.1818 \times 10^{12} + 0.1035 \times 10^{11} - 2(0.2897 \times 10^{10}) - 4(0.7171 \times 10^{10})]$$

$$= 0.1971 \times 10^{11}\ Pa,$$

$$U_4 = \frac{1}{8}[0.1818 \times 10^{12} + 0.1035 \times 10^{11} + 6(0.2897 \times 10^{10}) - 4(0.7171 \times 10^{10})]$$

$$= 0.2261 \times 10^{11}\ Pa.$$

Using Equation (2.140),

$$V_1 = \frac{1}{8}[3(0.5525 \times 10^{-11}) + 3(-0.1547 \times 10^{-11}) + 2(0.9709 \times 10^{-10}) + 0.1395 \times 10^{-9}]$$

$$= 0.5553 \times 10^{-10} \frac{1}{Pa},$$

$$V_2 = \frac{1}{2}[(0.5525 \times 10^{-11} - (-0.1547 \times 10^{-11})]$$

$$= -0.4578 \times 10^{-10} \frac{1}{Pa},$$

$$V_3 = \frac{1}{8}[0.5525 \times 10^{-11} + 0.9709 \times 10^{-10} - 2(0.1547 \times 10^{-11}) - 0.1395 \times 10^{-9}]$$

$$= -0.4220 \times 10^{-11} \frac{1}{Pa},$$

$$V_4 = \frac{1}{8}[0.5525 \times 10^{-11} + 0.9709 \times 10^{-10} + 6(0.1547 \times 10^{-11}) - 0.1395 \times 10^{-9}]$$

$$= -0.5767 \times 10^{-11} \frac{1}{Pa}.$$

2.8 Strength Failure Theories of an Angle Lamina

A successful design of a structure requires efficient and safe use of materials. Theories need to be developed to compare the state of stress in a material to failure criteria. It should be noted that failure theories are only stated and their application is validated by experiments.

For a laminate, the strength is related to the strength of each individual lamina. This allows for a simple and economical method for finding the strength of a laminate. Various theories have been developed for studying the failure of an angle lamina. The theories are generally based on the normal and shear strengths of a unidirectional lamina.

An isotropic material, such as steel, generally has two strength parameters: normal strength and shear strength. In some cases, such as concrete or gray cast iron, the normal strengths are different in the tension and compression. A simple failure theory for an isotropic material is based on finding the principal normal stresses and the maximum shear stresses. These maximum stresses, if greater than any of the corresponding ultimate strengths, indicate failure in the material.

Example 2.12

A cylindrical rod made of gray cast iron is subjected to a uniaxial tensile load, *P*. Given:

Cross-sectional area of rod = 2 in.^{2}
Ultimate tensile strength = 25 ksi
Ultimate compressive strength = 95 ksi
Ultimate shear strength = 35 ksi
Modulus of elasticity = 10 Msi

Find the maximum load, *P*, that can be applied using maximum stress failure theory.

Solution

At any location, the stress state in the rod is $\sigma = P/2$. From a typical Mohr's circle analysis, the maximum principal normal stress is $P/2$. The maximum shear stress is $P/4$ and acts at a cross-section 45° to the plane of maximum normal stress. Comparing these maximum stresses to the corresponding ultimate strengths, we have

$$\frac{P}{2} < 25 \times 10^3 \ \ or \ \ P < 50{,}000 \text{ lb},$$

and

$$\frac{P}{4} < 35 \times 10^3 \ \ or \ \ P < 140{,}000 \text{ lb}.$$

Thus, the maximum load is 50,000 lb.

However, in a lamina, the failure theories are not based on principal normal stresses and maximum shear stresses. Rather, they are based on the stresses in the material or local axes because a lamina is orthotropic and its properties are different at different angles, unlike an isotropic material.

In the case of a unidirectional lamina, there are two material axes: one parallel to the fibers and one perpendicular to the fibers. Thus, there are four normal strength parameters for a unidirectional lamina, one for tension and one for compression, in each of the two material axes directions. The fifth strength parameter is the shear strength of a unidirectional lamina. The shear stress, whether positive or negative, does not have an effect on the reported shear strengths of a unidirectional lamina. However, we will find later that the sign of the shear stress does affect the strength of an angle lamina. The five strength parameters of a unidirectional lamina are therefore

$(\sigma_1^T)_{ult}$ = Ultimate longitudinal tensile strength (in direction 1),
$(\sigma_1^C)_{ult}$ = Ultimate longitudinal compressive strength (in direction 1),
$(\sigma_2^T)_{ult}$ = Ultimate transverse tensile strength (in direction 2),
$(\sigma_2^C)_{ult}$ = Ultimate transverse compressive strength (in direction 2), and
$(\tau_{12})_{ult}$ = Ultimate in-plane shear strength (in plane 12).

Unlike the stiffness parameters, these strength parameters cannot be transformed directly for an angle lamina. Thus, the failure theories are based on first finding the stresses in the local axes and then using these five strength parameters of a unidirectional lamina to find whether a lamina has failed. Four common failure theories are discussed here. Related concepts of strength ratio and the development of failure envelopes are also discussed.

2.8.1 Maximum Stress Failure Theory

Related to the maximum normal stress theory by Rankine and the maximum shearing stress theory by Tresca, this theory is similar to those applied to isotropic materials. The stresses acting on a lamina are resolved into the normal and shear stresses in the local axes. Failure is predicted in a lamina, if any of the normal or shear stresses in the local axes of a lamina is equal to or exceeds the corresponding ultimate strengths of the unidirectional lamina.

Given the stresses or strains in the global axes of a lamina, one can find the stresses in the material axes by using Equation (2.94). The lamina is considered to be failed if

$$-(\sigma_1^C)_{ult} < \sigma_1 < (\sigma_1^T)_{ult}, \ or$$

$$-(\sigma_2^C)_{ult} < \sigma_2 < (\sigma_2^T)_{ult}, \ or$$

$$-(\tau_{12})_{ult} < \tau_{12} < (\tau_{12})_{ult} \tag{2.141a–c}$$

is violated. Note that all five strength parameters are treated as positive numbers, and the normal stresses are positive if tensile and negative if compressive.

Each component of stress is compared with the corresponding strength; thus, each component of stress does not interact with the others.

Example 2.13

Find the maximum value of $S > 0$ if a stress of $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to the 60° lamina of graphite/epoxy. Use maximum stress failure theory and the properties of a unidirectional graphite/epoxy lamina given in Table 2.1.

Solution

Using Equation (2.94), the stresses in the local axes are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714 \times 10^1 \\ -0.2714 \times 10^1 \\ -0.4165 \times 10^1 \end{bmatrix} S.$$

From Table 2.1, the ultimate strengths of a unidirectional graphite/epoxy lamina are

$$(\sigma_1^T)_{ult} = 1500 \text{ MPa}$$

$$(\sigma_1^C)_{ult} = 1500 \text{ MPa}$$

$$(\sigma_2^T)_{ult} = 40 \text{ MPa}$$

$$(\sigma_2^C)_{ult} = 246 \text{ MPa}$$

$$(\tau_{12})_{ult} = 68 \text{ MPa}$$

Then, using the inequalities (2.141) of the maximum stress failure theory,

$$-1500 \times 10^6 < 0.1714 \times 10^1 S < 1500 \times 10^6$$

$$-246 \times 10^6 < -0.2714 \times 10^1 S < 40 \times 10^6$$

$$-68 \times 10^6 < -0.4165 \times 10^1 S < 68 \times 10^6$$

or

$$-875.1 \times 10^6 < S < 875.1 \times 10^6$$

$$-14.73 \times 10^6 < S < 90.64 \times 10^6$$

$$-16.33 \times 10^6 < S < 16.33 \times 10^6.$$

All the inequality conditions (and $S > 0$) are satisfied if $0 < S < 16.33$ MPa. The preceding inequalities also show that the angle lamina will fail in shear. The maximum stress that can be applied before failure is

$$\sigma_x = 32.66\, MPa, \sigma_y = -48.99\, MPa, \tau_{xy} = 65.32\, MPa.$$

Example 2.14

Find the off-axis shear strength of a 60° graphite/epoxy lamina. Use the properties of unidirectional graphite/epoxy from Table 2.1 and apply the maximum stress failure theory.

Solution

The off-axis shear strength of a lamina is defined as the minimum of the magnitude of positive and negative shear stress (Figure 2.31) that can be applied to an angle lamina before failure.

Assume the following stress state

$$\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} = \tau.$$

Then, using the transformation Equation (2.94),

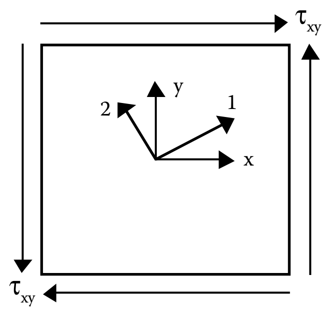
$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \tau \end{bmatrix}$$

$$\sigma_1 = 0.866\tau$$

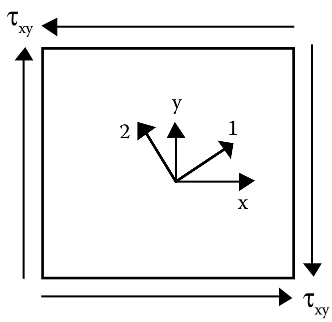
$$\sigma_2 - 0.866\tau$$

$$\tau_{12} = -0.500\tau\ .$$

Using the inequalities (2.141) of the maximum stress failure theory, we have



(a) Positive shear stress



(b) Negative shear stress

FIGURE 2.31
Positive and negative shear stresses applied to an angle lamina.

$$-1500 < 0.866\tau < 1500 \text{ or } -1732 < \tau < 1732$$

$$-246 < -0.866\tau < 40 \text{ or } -46.19 < \tau < 284.1$$

$$-68 < -0.500\tau < 68 \text{ or } -136.0 < \tau < 136.0,$$

which shows that τ_{xy} = 46.19 MPa is the largest magnitude of shear stress that can be applied to the 60° graphite/epoxy lamina. However, the largest positive shear stress that could be applied is τ_{xy} = 136.0 MPa, and the largest negative shear stress is τ_{xy} = –46.19 MPa.

This shows that the maximum magnitude of allowable shear stress in other than the material axes' direction depends on the sign of the shear stress. This is mainly because the local axes' stresses in the direction perpendicular to the fibers are opposite in sign to each other for opposite signs of shear stress ($\sigma_2 = -0.866\tau$ for positive τ_{xy} and $\sigma_2 = 0.866\tau$ for negative τ_{xy}). Because the tensile strength perpendicular to the fiber direction is much lower than the compressive strength perpendicular to the fiber direction, the two limiting values of τ_{xy} are different.

TABLE 2.3

Effect of Sign of Shear Stress as a Function of Angle of Lamina

Angle, Degrees	Positive τ_{xy} MPa	Negative τ_{xy} MPa	Shear strength MPa
0	68.00 (S)	68.00 (S)	68.00
15	78.52 (S)	78.52 (S)	78.52
30	136.0 (S)	46.19 (2T)	46.19
45	246.0 (2C)	40.00 (2T)	40.00
60	136.0 (S)	46.19 (2T)	46.19
75	78.52 (S)	78.52 (S)	78.52
90	68.00 (S)	68.00 (S)	68.00

Note: The notation in the parentheses denotes the mode of failure of the angle lamina as follows:
(1T) — longitudinal tensile failure;
(1C) — longitudinal compressive failure;
(2T) — transverse tensile failure;
(2C) — transverse compressive failure;
(S) — shear failure.

Table 2.3 shows the maximum negative and positive values of shear stress that can be applied to different angle plies of graphite/epoxy of Table 2.1. The minimum magnitude of the two stresses is the shear strength of the angle lamina.

2.8.2 Strength Ratio

In a failure theory such as the maximum stress failure theory of Section 2.8.1, it can be determined whether a lamina has failed if any of the inequalities of Equation (2.141) are violated. However, this does not give the information about how much the load can be increased if the lamina is safe or how much the load should be decreased if the lamina has failed. The definition of strength ratio (SR) is helpful here. The strength ratio is defined as

$$SR = \frac{Maximum\ Load\ Which\ Can\ Be\ Applied}{Load\ Applied}. \tag{2.142}$$

The concept of strength ratio is applicable to any failure theory. If SR > 1, then the lamina is safe and the applied stress can be increased by a factor of SR. If SR < 1, the lamina is unsafe and the applied stress needs to be reduced by a factor of SR. A value of SR = 1 implies the failure load.

Example 2.15

Assume that one is applying a load of

$$\sigma_x = 2\,MPa, \sigma_y = -3\,MPa, \tau_{xy} = 4\,MPa$$

to a 60° angle lamina of graphite/epoxy. Find the strength ratio using the maximum stress failure theory.

Solution

If the strength ratio is R, then the maximum stress that can be applied is

$$\sigma_x = 2R,\ \sigma_y = -3R,\ \tau_{xy} = 4R\ .$$

Following Example 2.13 for finding the local stresses gives

$$\sigma_1 0.1714 \times 10^1\, R$$

$$\sigma_2 = -0.2714 \times 10^1\, R$$

$$\tau_{12} = -0.4165 \times 10^1\, R\ .$$

Using the maximum stress failure theory as given by Equation (2.141) yields

$$R = 16.33.$$

Thus, the load that can be applied just before failure is

$$\sigma_x = 16.33 \times 2\ MPa,\ \sigma_y = 16.33 \times (-3)\ MPa,\ \tau_{xy} = 16.33 \times 4\ Mpa,$$

$$\sigma_x = 32.66\ MPa,\ \sigma_y = -48.99\ MPa,\ \tau_{xy} = 65.32\ MPa.$$

Note that all the components of the stress vector must be multiplied by the strength ratio.

2.8.3 Failure Envelopes

A failure envelope is a three-dimensional plot of the combinations of the normal and shear stresses that can be applied to an angle lamina just before failure. Because drawing three dimensional graphs can be time consuming, one may develop failure envelopes for constant shear stress τ_{xy} and then use the two normal stresses σ_x and σ_y as the two axes. Then, if the applied stress is within the failure envelope, the lamina is safe; otherwise, it has failed.

Example 2.16

Develop a failure envelope for the 60° lamina of graphite/epoxy for a constant shear stress of τ_{xy} = 24 MPa. Use the properties for the unidirectional graphite/epoxy lamina from Table 2.1.

Solution

From Equation (2.94), the stresses in the local axes for a 60° lamina are given by

$$\sigma_1 = 0.2500\,\sigma_x + 0.7500\,\sigma_y + 20.78\,MPa,$$

$$\sigma_2 = 0.7500\,\sigma_x + 0.2500\,\sigma_y - 20.78\,MPa,$$

$$\tau_{12} = -0.4330\,\sigma_x + 0.4330\,\sigma_y - 12.00\,MPa,$$

where σ_x and σ_y are also in units of MPa.

Using the preceding inequalities,

$$-1500 < 0.2500\,\sigma_x + 0.7500\,\sigma_y + 20.78 < 1500$$

$$-246 < 0.7500\,\sigma_x + 0.2500\,\sigma_y - 20.78 < 40$$

$$-68 < -0.4330\,\sigma_x + 0.4330\,\sigma_y - 12.00 < 68\ .$$

Various combinations of (σ_x, σ_y) can be found to satisfy the preceding inequalities. However, the objective is to find the points on the failure envelope. These are combinations of σ_x and σ_y, where one of the three inequalities is just violated and the other two are satisfied. Some of the values of (σ_x, σ_y) obtained on the failure envelope are given in Table 2.4.

Several methods can be used to obtain the points on the failure envelope for a constant shear stress. One way is to fix the value of σ_x and find the maximum value of σ_y that can be applied without violating any of the conditions. For example, for σ_x = 100 MPa, from the inequalities we have

$$-2061 < \sigma_y < 1939,$$

$$-1201 < \sigma_y < -56.88,$$

$$-29.33 < \sigma_y < 284.80.$$

TABLE 2.4

Typical Values of (σ_x, σ_y) on the Failure Envelope for Example 2.16

σ_x (MPa)	σ_y (MPa)
50.0	93.1
50.0	–79.3
–50.0	179
–50.0	–135
25.0	168
25.0	–104
–25.0	160
–25.0	–154

The preceding three inequalities show no allowable value of σ_y for this value of σ_x = 100 MPa.

As another example, for σ_x = 50 MPa, we have from inequalities,

$$-2044 < \sigma_y < 1956,$$

$$-1051 < \sigma_y < 93.12,$$

$$-79.33 < \sigma_y < 234.80.$$

The preceding three inequalities show two maximum allowable values of the normal stress, σ_y. These are σ_y = 93.12 MPa and σ_y = –79.33 MPa. The failure envelope for τ_{xy} = 24 MPa is shown in Figure 2.32.

2.8.4 Maximum Strain Failure Theory

This theory is based on the maximum normal strain theory by St. Venant and the maximum shear stress theory by Tresca as applied to isotropic materials. The strains applied to a lamina are resolved to strains in the local axes. Failure is predicted in a lamina, if any of the normal or shearing strains in the local axes of a lamina equal or exceed the corresponding ultimate strains of the unidirectional lamina. Given the strains/stresses in an angle lamina, one can find the strains in the local axes. A lamina is considered to be failed if

$$-(\varepsilon_1^C)_{ult} < \varepsilon_1 < (\varepsilon_1^T)_{ult}, \; or$$

$$-(\varepsilon_2^C)_{ult} < \varepsilon_2 < (\varepsilon_2^T)_{ult}, \; or$$

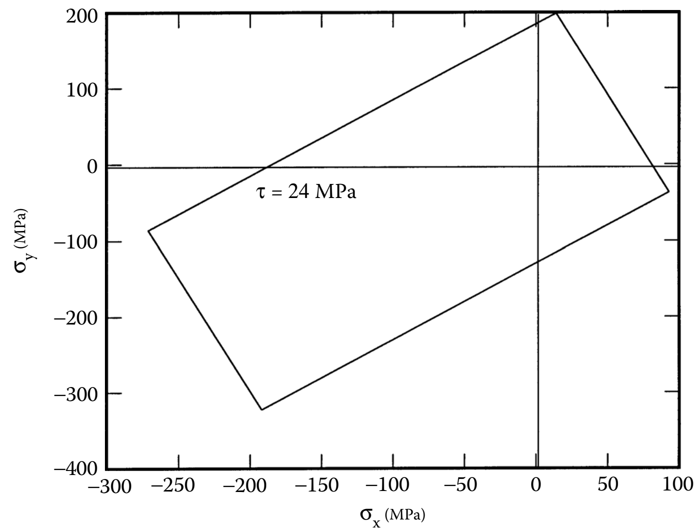


FIGURE 2.32
Failure envelopes for constant shear stress using maximum stress failure theory.

$$-(\gamma_{12})_{ult} < \gamma_{12} < (\gamma_{12})_{ult} \quad (2.143a\text{–}c)$$

is violated, where

$(\varepsilon_1^T)_{ult}$ = ultimate longitudinal tensile strain (in direction 1)
$(\varepsilon_1^C)_{ult}$ = ultimate longitudinal compressive strain (in direction 1)
$(\varepsilon_2^T)_{ult}$ = ultimate transverse tensile strain (in direction 2)
$(\varepsilon_2^C)_{ult}$ = ultimate transverse compressive strain (in direction 2)
$(\gamma_{12})_{ult}$ = ultimate in-plane shear strain (in plane 1–2)

The ultimate strains can be found directly from the ultimate strength parameters and the elastic moduli, assuming the stress–strain response is linear until failure. The maximum strain failure theory is similar to the maximum stress failure theory in that no interaction occurs between various components of strain. However, the two failure theories give different results because the local strains in a lamina include the Poisson's ratio effect. In fact, if the Poisson's ratio is zero in the unidirectional lamina, the two failure theories will give identical results.

Example 2.17

Find the maximum value of $S > 0$ if a stress, $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$, is applied to a 60° graphite/epoxy lamina. Use maximum strain failure theory. Use the properties of the graphite/epoxy unidirectional lamina given in Table 2.1.

Solution

In Example 2.6, the compliance matrix [S] was obtained and, in Example 2.13, the local stresses for this problem were obtained. Then, from Equation (2.77),

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [S] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\ -0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\ 0 & 0 & 0.1395 \times 10^{-9} \end{bmatrix} \begin{bmatrix} 0.1714 \times 10^{1} \\ -0.2714 \times 10^{1} \\ -0.4165 \times 10^{1} \end{bmatrix} S$$

$$= \begin{bmatrix} 0.1367 \times 10^{-10} \\ -0.2662 \times 10^{-9} \\ -0.5809 \times 10^{-9} \end{bmatrix} S.$$

Assume a linear relationship between all the stresses and strains until failure; then the ultimate failure strains are

$$(\varepsilon_1^T)_{ult} = \frac{(\sigma_1^T)_{ult}}{E_1} = \frac{1500 \times 10^6}{181 \times 10^9} = 8.287 \times 10^{-3},$$

$$(\varepsilon_1^C)_{ult} = \frac{(\sigma_1^C)_{ult}}{E_1} = \frac{1500 \times 10^6}{181 \times 10^9} = 8.287 \times 10^{-3},$$

$$(\varepsilon_2^T)_{ult} = \frac{(\sigma_2^T)_{ult}}{E_2} = \frac{40 \times 10^6}{10.3 \times 10^9} = 3.883 \times 10^{-3},$$

$$(\varepsilon_2^C)_{ult} = \frac{(\sigma_2^C)_{ult}}{E_2} = \frac{246 \times 10^6}{10.3 \times 10^9} = 2.388 \times 10^{-2},$$

$$(\gamma_{12})_{ult} = \frac{(\tau_{12})_{ult}}{G_{12}} = \frac{68 \times 10^6}{7.17 \times 10^6} = 9.483 \times 10^{-3}.$$

The preceding values for the ultimate strains also assume that the compressive and tensile stiffnesses are identical. Using the inequalities (2.143) and recognizing that $S > 0$,

$$-8.287 \times 10^{-3} < 0.1367 \times 10^{-10} S < 8.287 \times 10^{-3},$$

$$-2.388 \times 10^{-2} < -0.2662 \times 10^{-9} S < 3.883 \times 10^{-3},$$

$$-9.483 \times 10^{-3} < -0.5809 \times 10^{-9} S < 9.483 \times 10^{-3},$$

or

$$-606.2 \times 10^{6} < S < 606.2 \times 10^{6},$$

$$-14.58 \times 10^{6} < S < 89.71 \times 10^{6}$$

$$-16.33 \times 10^{6} < S < 16.33 \times 10^{6},$$

which give

$$0 < S < 16.33\, MPa.$$

The maximum value of S before failure is 16.33 MPa. The same maximum value of S = 16.33 MPa is also found using maximum stress failure theory. There is no difference between the two values because the mode of failure is shear. However, if the mode of failure were other than shear, a difference in the prediction of failure loads would have been present due to the Poisson's ratio effect, which couples the normal strains and stresses in the local axes.

Neither the maximum stress failure theory nor the maximum strain failure theory has any coupling among the five possible modes of failure. The following theories are based on the interaction failure theory.

2.8.5 Tsai–Hill Failure Theory

This theory is based on the distortion energy failure theory of Von-Mises' distortional energy yield criterion for isotropic materials as applied to anisotropic materials. Distortion energy is actually a part of the total strain energy in a body. The strain energy in a body consists of two parts; one due to a change in volume and is called the dilation energy and the second is due to a change in shape and is called the distortion energy. It is assumed that failure in the material takes place only when the distortion energy is greater than the failure distortion energy of the material. Hill[8] adopted the Von-Mises' distortional energy yield criterion to anisotropic materials. Then, Tsai[7] adapted it to a unidirectional lamina. Based on the distortion energy theory, he proposed that a lamina has failed if

$$(G_2+G_3)\sigma_1^2+(G_1+G_3)\sigma_2^2+(G_1+G_2)\sigma_3^2-2G_3\sigma_1\sigma_2-2G_2\sigma_1\sigma_3 \quad (2.144)$$

$$-\ 2G_1\sigma_2\sigma_3+2G_4\tau_{23}^2+2G_5\tau_{13}^2+2G_6\tau_{12}^2<1$$

is violated. The components G_1, G_2, G_3, G_4, G_5, and G_6 of the strength criterion depend on the failure strengths and are found as follows.

1. Apply $\sigma_1=(\sigma_1^T)_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (2.144) reduces to

$$(G_2+G_3)(\sigma_1^T)_{ult}^2=1. \quad (2.145)$$

2. Apply $\sigma_2=(\sigma_2^T)_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (2.144) reduces to

$$(G_1+G_3)(\sigma_2^T)_{ult}^2=1. \quad (2.146)$$

3. Apply $\sigma_3=(\sigma_2^T)_{ult}$ to a unidirectional lamina and, assuming that the normal tensile failure strength is same in directions (2) and (3), the lamina will fail. Thus, Equation (2.144) reduces to

$$(G_1+G_2)(\sigma_2^T)_{ult}^2=1. \quad (2.147)$$

4. Apply τ_{12} = $(\tau_{12})_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (2.144) reduces to

$$2G_6(\tau_{12})_{ult}^2=1. \quad (2.148)$$

From Equation (2.145) to Equation (2.148),

$$G_1 = \frac{1}{2}\left(\frac{2}{[(\sigma_2^T)_{ult}]^2} - \frac{1}{[(\sigma_1^T)_{ult}]^2}\right),$$

$$G_2 = \frac{1}{2}\left(\frac{1}{[(\sigma_1^T)_{ult}]^2}\right),$$

$$G_3 = \frac{1}{2}\left(\frac{1}{[(\sigma_1^T)_{ult}]^2}\right),$$

$$G_6 = \frac{1}{2}\left(\frac{1}{[(\tau_{12})_{ult}]^2}\right). \qquad (2.149a–d)$$

Because the unidirectional lamina is assumed to be under plane stress — that is, $\sigma_3 = \tau_{31} = \tau_{23} = 0$, then Equation (2.144) reduces through Equation (2.149) to

$$\left[\frac{\sigma_1}{(\sigma_1^T)_{ult}}\right]^2 - \left[\frac{\sigma_1\sigma_2}{(\sigma_1^T)_{ult}^2}\right] + \left[\frac{\sigma_2}{(\sigma_2^T)_{ult}}\right]^2 + \left[\frac{\tau_{12}}{(\tau_{12})_{ult}}\right]^2 < 1. \qquad (2.150)$$

Given the global stresses in a lamina, one can find the local stresses in a lamina and apply the preceding failure theory to determine whether the lamina has failed.

Example 2.18

Find the maximum value of $S > 0$ if a stress of $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to a 60° graphite/epoxy lamina. Use Tsai–Hill failure theory. Use the unidirectional graphite/epoxy lamina properties given in Table 2.1.

Solution

From Example 2.13,

$$\sigma_1 = 1.714\ S,$$

$$\sigma_2 = -2.714\ S,$$

$$\tau_{12} = -4.165\ S.$$

Using the Tsai–Hill failure theory from Equation (2.150),

$$\left(\frac{1.714S}{1500\times10^6}\right)^2 - \left(\frac{1.714S}{1500\times10^6}\right)\left(\frac{-2.714S}{1500\times10^6}\right) + \left(\frac{-2.714S}{40\times10^6}\right)^2 + \left(\frac{-4.165S}{68\times10^6}\right)^2 < 1$$

$$S < 10.94\ MPa$$

1. Unlike the maximum strain and maximum stress failure theories, the Tsai–Hill failure theory considers the interaction among the three unidirectional lamina strength parameters.
2. The Tsai–Hill failure theory does not distinguish between the compressive and tensile strengths in its equations. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories. For the load of σ_x = 2 MPa, σ_y = –3 MPa, and τ_{xy} = 4 MPa, as found in Example 2.15, Example 2.17, and Example 2.18, the strength ratios are given by

   SR = 10.94 (Tsai–Hill failure theory)

   SR = 16.33 (maximum stress failure theory)

   SR = 16.33 (maximum strain failure theory)

Tsai–Hill failure theory underestimates the failure stress because the transverse tensile strength of a unidirectional lamina is generally much less than its transverse compressive strength. The compressive strengths are not used in the Tsai–Hill failure theory, but it can be modified to use corresponding tensile or compressive strengths in the failure theory as follows

$$\left[\frac{\sigma_1}{X_1}\right]^2 - \left[\left(\frac{\sigma_1}{X_2}\right)\left(\frac{\sigma_2}{X_2}\right)\right] + \left[\frac{\sigma_2}{Y}\right]^2 + \left[\frac{\tau_{12}}{S}\right]^2 < 1\,, \tag{2.151}$$

where

$$X_1 = (\sigma_1^T)_{ult} \text{ if } \sigma_1 > 0$$

$$= (\sigma_1^C)_{ult} \text{ if } \sigma_1 < 0;$$

$$X_2 = (\sigma_1^T)_{ult} \text{ if } \sigma_2 > 0$$

$$= (\sigma_1^C)_{ult} \text{ if } \sigma_2 < 0;$$

$$Y = (\sigma_2^T)_{ult} \text{ if } \sigma_2 > 0$$

$$= (\sigma_2^C)_{ult} \text{ if } \sigma_2 < 0$$

$$S = (\tau_{12})_{ult}.$$

For Example 2.18, the modified Tsai–Hill failure theory given by Equation (2.151) now gives

$$\left(\frac{1.714\sigma}{1500\times10^6}\right)^2-\left(\frac{1.714\sigma}{1500\times10^6}\right)\left(\frac{-2.714\sigma}{1500\times10^6}\right)+\left(\frac{-2.714\sigma}{246\times10^6}\right)^2+\left(\frac{-4.165\sigma}{68\times10^6}\right)^2<1$$

$$\sigma < 16.06 \; MPa,$$

which implies that the strength ratio is $SR = 16.06$ (modified Tsai–Hill failure theory). This value is closer to the values obtained using maximum stress and maximum strain failure theories.

3. The Tsai–Hill failure theory is a unified theory and thus does not give the mode of failure like the maximum stress and maximum strain failure theories do. However, one can make a reasonable guess of the failure mode by calculating $\left|\sigma_1/(\sigma_1^T)_{ult}\right|$, $\left|\sigma_2/(\sigma_2^T)_{ult}\right|$ and $|\tau_{12}/(\tau_{12})_{ult}|$. The maximum of these three values gives the associated mode of failure. In the modified Tsai–Hill failure theory, calculate the maximum of $|\sigma_1/X_1|$, $|\sigma_2/Y|$, and $|\tau_{12}/S|$ for the associated mode of failure.

2.8.6 Tsai–Wu Failure Theory

This failure theory is based on the total strain energy failure theory of Beltrami. Tsai-Wu[9] applied the failure theory to a lamina in plane stress. A lamina is considered to be failed if

$$H_1\sigma_1 + H_2\sigma_2 + H_6\tau_{12} + H_{11}\,\sigma_1^2 + H_{22}\,\sigma_2^2 + H_{66}\,\tau_{12}^2 + 2H_{12}\sigma_1\sigma_2 < 1 \quad (2.152)$$

is violated. This failure theory is more general than the Tsai–Hill failure theory because it distinguishes between the compressive and tensile strengths of a lamina.

The components H_1, H_2, H_6, H_{11}, H_{22}, and H_{66} of the failure theory are found using the five strength parameters of a unidirectional lamina as follows:

1. Apply $\sigma_1 = (\sigma_1^T)_{ult}$, $\sigma_2 = 0$, $\tau_{12} = 0$ to a unidirectional lamina; the lamina will fail. Equation (2.152) reduces to

$$H_1(\sigma_1^T)_{ult} + H_{11}(\sigma_1^T)_{ult}^2 = 1. \tag{2.153}$$

2. Apply $\sigma_1 = -(\sigma_1^C)_{ult}$, $\sigma_2 = 0$, $\tau_{12} = 0$ to a unidirectional lamina; the lamina will fail. Equation (2.152) reduces to

$$-H_1(\sigma_1^C)_{ult} + H_{11}(\sigma_1^C)_{ult}^2 = 1. \tag{2.154}$$

From Equation (2.153) and Equation (2.154),

$$H_1 = \frac{1}{(\sigma_1^T)_{ult}} - \frac{1}{(\sigma_1^C)_{ult}}, \tag{2.155}$$

$$H_{11} = \frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}. \tag{2.156}$$

3. Apply $\sigma_1 = 0$, $\sigma_2 = (\sigma_2^T)_{ult}$, $\tau_{12} = 0$ to a unidirectional lamina; the lamina will fail. Equation (2.152) reduces to

$$H_2(\sigma_2^T)_{ult} + H_{22}(\sigma_2^T)_{ult}^2 = 1. \tag{2.157}$$

4. Apply $\sigma_1 = 0$, $\sigma_2 = -(\sigma_2^C)_{ult}$, $\tau_{12} = 0$ to a unidirectional lamina; the lamina will fail. Equation (2.152) reduces to

$$-H_2(\sigma_2^C)_{ult} + H_{22}(\sigma_2^C)_{ult}^2 = 1. \tag{2.158}$$

From Equation (2.157) and Equation (2.158),

$$H_2 = \frac{1}{(\sigma_2^T)_{ult}} - \frac{1}{(\sigma_2^C)_{ult}}, \tag{2.159}$$

$$H_{22} = \frac{1}{(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}. \tag{2.160}$$

5. Apply $\sigma_1 = 0$, $\sigma_2 = 0$, and $\tau_{12} = (\tau_{12})_{ult}$ to a unidirectional lamina; it will fail. Equation (2.152) reduces to

$$H_6(\tau_{12})_{ult} + H_{66}(\tau_{12})^2_{ult} = 1. \tag{2.161}$$

6. Apply $\sigma_1 = 0$, $\sigma_2 = 0$, and $\tau_{12} = -(\tau_{12})_{ult}$ to a unidirectional lamina; the lamina will fail. Equation (2.152) reduces to

$$-H_6(\tau_{12})_{ult} + H_{66}(\tau_{12})^2_{ult} = 1. \tag{2.162}$$

From Equation (2.161) and Equation (2.162),

$$H_6 = 0, \tag{2.163}$$

$$H_{66} = \frac{1}{(\tau_{12})^2_{ult}}. \tag{2.164}$$

The only component of the failure theory that cannot be found directly from the five strength parameters of the unidirectional lamina is H_{12}. This can be found experimentally by knowing a biaxial stress at which the lamina fails and then substituting the values of σ_1, σ_2, and τ_{12} in the Equation (2.152). Note that σ_1 and σ_2 need to be nonzero to find H_{12}. Experimental methods to find H_{12} include the following.

1. Apply equal tensile loads along the two material axes in a unidirectional composite. If $\sigma_x = \sigma_y = \sigma$, $\tau_{xy} = 0$ is the load at which the lamina fails, then

$$(H_1 + H_2)\sigma + (H_{11} + H_{22} + 2H_{12})\sigma^2 = 1. \tag{2.165}$$

The solution of Equation (2.165) gives

$$H_{12} = \frac{1}{2\sigma^2}[1 - (H_1 + H_2)\sigma - (H_{11} + H_{22})\sigma^2]. \tag{2.166}$$

It is not necessary to pick tensile loads in the preceding biaxial test, but one may apply any combination of

$$\sigma_1 = \sigma,\ \sigma_2 = \sigma,$$

$$\sigma_1 = -\sigma, \ \sigma_2 = -\sigma,$$

$$\sigma_1 = \sigma, \ \sigma_2 = -\sigma,$$

$$\sigma_1 = -\sigma, \ \sigma_2 = \sigma. \tag{2.167}$$

This will give four different values of H_{12}, each corresponding to the four tests.

2. Take a 45° lamina under uniaxial tension σ_x. The stress σ_x at failure is noted. If this stress is $\sigma_x = \sigma$, then, using Equation (2.94), the local stresses at failure are

$$\sigma_1 = \frac{\sigma}{2},$$

$$\sigma_2 = \frac{\sigma}{2}, \tag{2.168a–c}$$

$$\tau_{12} = -\frac{\sigma}{2}.$$

Substituting the preceding local stresses in Equation (2.152),

$$(H_1 + H_2)\frac{\sigma}{2} + \frac{\sigma^2}{4}(H_{11} + H_{22} + H_{66} + 2H_{12}) = 1. \tag{2.169}$$

$$H_{12} = \frac{2}{\sigma^2} - \frac{(H_1 + H_2)}{\sigma} - \frac{1}{2}(H_{11} + H_{22} + H_{66}). \tag{2.170}$$

Some empirical suggestions for finding the value of H_{12} include

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2}, \text{ per Tsai–Hill failure theory}^8 \tag{2.171a–c}$$

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}, \text{ per Hoffman criterion}^{10}$$

$$H_{12} = -\frac{1}{2}\sqrt{\frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}}, \text{ per Mises–Hencky criterion.}^{11}$$

Example 2.19

Find the maximum value of $S > 0$ if a stress $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$ are applied to a 60° lamina of graphite/epoxy. Use Tsai–Wu failure theory. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1.

Solution

From Example 2.13,

$$\sigma_1 = 1.714S,$$

$$\sigma_2 = -2.714S,$$

$$\tau_{12} = -4.165S.$$

From Equations (2.155), (2.156), (2.159), (2.160), (2.163), and (2.164),

$$H_1 = \frac{1}{1500 \times 10^6} - \frac{1}{1500 \times 10^6} = 0 \ Pa^{-1},$$

$$H_2 = \frac{1}{40 \times 10^6} - \frac{1}{246 \times 10^6} = 2.093 \times 10^{-8} \ Pa^{-1},$$

$$H_6 = 0 \ Pa^{-1},$$

$$H_{11} = \frac{1}{(1500 \times 10^6)(1500 \times 10^6)} = 4.4444 \times 10^{-19} \ Pa^{-2},$$

$$H_{22} = \frac{1}{(40 \times 10^6)(246 \times 10^6)} = 1.0162 \times 10^{-16} \ Pa^{-2},$$

$$H_{66} = \frac{1}{(68 \times 10^6)^2} = 2.1626 \times 10^{-16} \ Pa^{-2} \ .$$

Using the Mises–Hencky criterion for evaluation of H_{12}, (Equation 2.165c),

$$H_{12} = -\frac{1}{2}\sqrt{\frac{1}{(1500 \times 10^6)(1500 \times 10^6)(40 \times 10^6)(246 \times 10^6)}} = 3.360 \times 10^{-18} \ Pa^{-2}.$$

Substituting these values in Equation (2.152), we obtain

$$(0)(1.714S) + (2.093 \times 10^{-8})(-2.714S)$$

$$+ (0)(-4.165S) + (4.444 \times 10^{-19})(1.714S)^2$$

$$+ (1.0162 \times 10^{-16})(-2.714S)^2 + (2.1626 \times 10^{-16})(-4.165S)^2$$

$$+ 2(-3.360 \times 10^{-18})(1.714S)(-2.714S) < 1,$$

or

$$S < 22.39 \ MPa \ .$$

If one uses the other two empirical criteria for H_{12}, per Equation (2.171), this yields

$$S < 22.49 \ MPa \text{ for } H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}^2},$$

$$S < 22.49 \ MPa \ \text{\textit{for}} \ H_{12} = -\frac{1}{2}\frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}.$$

Summarizing the four failure theories for the same stress state, the value of S obtained is

S = 16.33 (maximum stress failure theory)
S = 16.33 (maximum strain failure theory)
S = 10.94 (Tsai–Hill failure theory)
S = 16.06 (modified Tsai–Hill failure theory)
S = 22.39 (Tsai–Wu failure theory)

2.8.7 Comparison of Experimental Results with Failure Theories

Tsai[7] compared the results from various failure theories to some experimental results. He considered an angle lamina subjected to a uniaxial load in the x-direction, σ_x, as shown in Figure 2.33. The failure stresses were obtained experimentally for tensile and compressive stresses for various angles of the lamina.

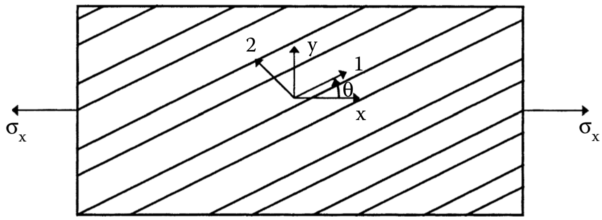


FIGURE 2.33
Off-axis loading in the *x*-direction in Figure 2.34 to Figure 2.37.

The experimental results can be compared with the four failure theories by finding the stresses in the material axes for an arbitrary stress, σ_x, for an angle lamina with an angle, θ, between the fiber and loading direction as

$$\sigma_1 = \sigma_x \mathrm{Cos}^2\theta,$$

$$\sigma_2 = \sigma_x \mathrm{Sin}^2\theta, \tag{2.172}$$

$$\tau_{12} = -\sigma_x \mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta,$$

per Equation (2.94).

The corresponding strains in the material axes are

$$\varepsilon_1 = \frac{1}{E_1}(\mathrm{Cos}^2\theta - \nu_{12}\mathrm{Sin}^2\theta)\sigma_x,$$

$$\varepsilon_2 = \frac{1}{E_2}(\mathrm{Sin}^2\theta - \nu_{21}\mathrm{Cos}^2\theta)\sigma_x, \tag{2.173}$$

$$\gamma_{12} = -\frac{1}{G_{12}}(\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta)\sigma_x,$$

per Equation (2.99).

Using the preceding local strains and stresses in the four failure theories given by Equation (2.141), Equation (2.143), Equation (2.150), and Equation (2.152), one can find the ultimate off-axis load, σ_x, that can be applied as a function of the angle of the lamina.

The following values were used in the failure theories for the unidirectional lamina stiffnesses and strengths:

$$E_1 = 7.8\ Msi,$$

$$E_2 = 2.6\ Msi,$$

$$\nu_{12} = 0.25,$$

$$G_{12} = 1.3\ Msi,$$

$$(\sigma_1^T)_{ult} = 150\ Ksi,$$

$$(\sigma_1^C)_{ult} = 150\ Ksi,$$

$$(\sigma_2^T)_{ult} = 4\ Ksi,$$

$$(\sigma_2^C)_{ult} = 20\ Ksi,$$

$$(\tau_{12})_{ult} = 6\ Ksi.$$

The comparison for the four failure theories is shown in Figure 2.34 through Figure 2.37. Observations from the figures are:

- The difference between the maximum stress and maximum strain failure theories and the experimental results is quite pronounced.
- Tsai–Hill and Tsai–Wu failure theories' results are in good agreement with experimentally obtained results.
- The variation of the strength of the angle lamina as a function of angle is smooth in the Tsai–Hill and Tsai–Wu failure theories, but has cusps in the maximum stress and maximum strain failure theories. The cusps correspond to the change in failure modes in the maximum stress and maximum strain failure theories.

2.9 Hygrothermal Stresses and Strains in a Lamina

Composite materials are generally processed at high temperatures and then cooled down to room temperatures. For polymeric matrix composites, this temperature difference is in the range of 200 to 300°C; for ceramic matrix composites, it may be as high as 1000°C. Due to mismatch of the coefficients of thermal expansion of the fiber and matrix, residual stresses result in a lamina when it is cooled down. Also, the cooling down induces expansional

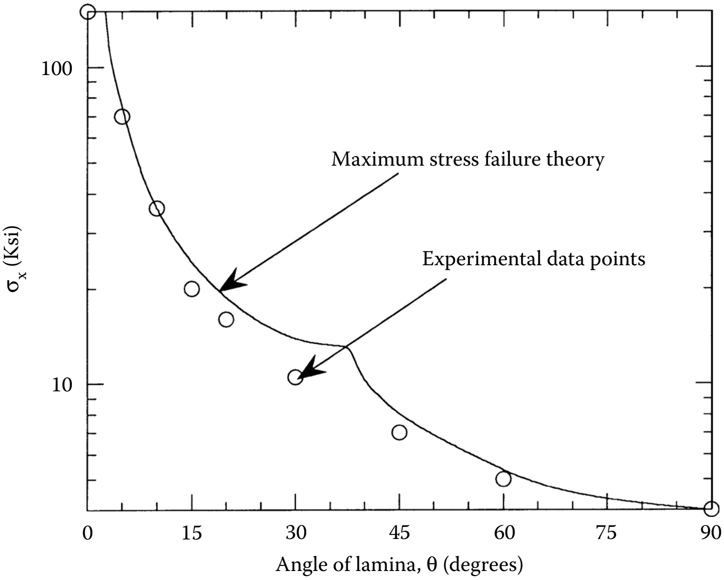


FIGURE 2.34
Maximum normal tensile stress in the x-direction as a function of angle of lamina using maximum stress failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

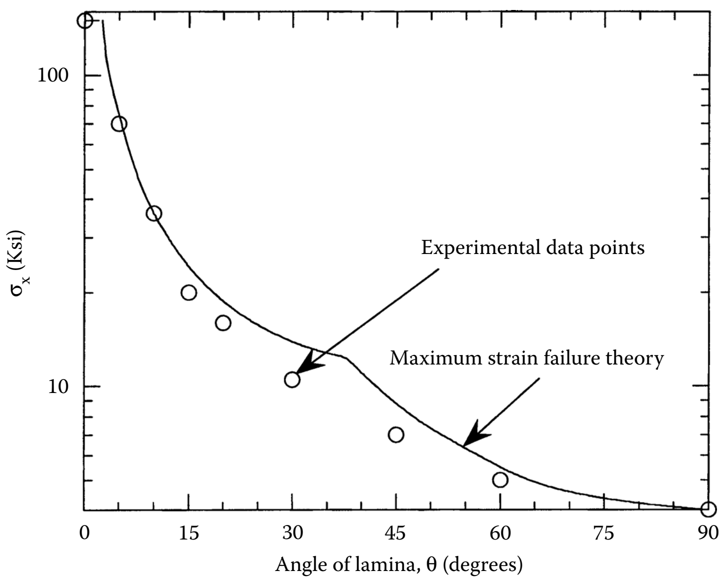


FIGURE 2.35
Maximum normal tensile stress in the x-direction as a function of angle of lamina using maximum strain failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

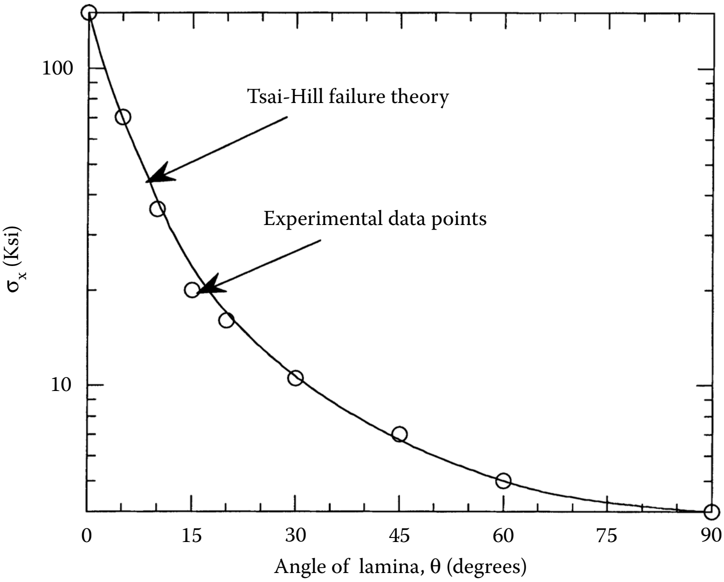


FIGURE 2.36
Maximum normal tensile stress in the x-direction as a function of angle of lamina using Tsai–Hill failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

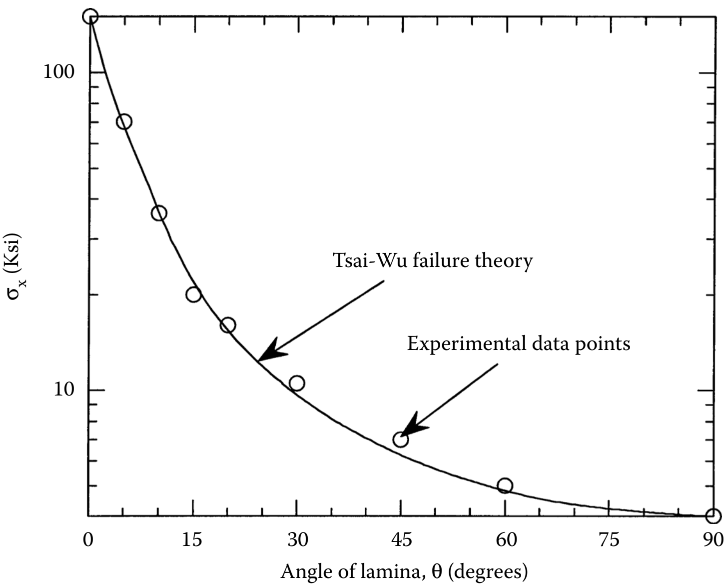


FIGURE 2.37
Maximum normal tensile stress in the x-direction as a function of angle of lamina using Tsai–Wu failure theory. (Experimental data reprinted with permission from *Introduction to Composite Materials*, Tsai, S.W. and Hahn, H.T., 1980, CRC Press, Boca Raton, FL, 301.)

strains in the lamina. In addition, most polymeric matrix composites can absorb or deabsorb moisture. This moisture change leads to swelling strains and stresses similar to those due to thermal expansion. Laminates in which laminae are placed at different angles have residual stresses in each lamina due to differing hygrothermal expansion of each lamina. The hygrothermal strains are not equal in a lamina in the longitudinal and transverse directions because the elastic constants and the thermal and moisture expansion coefficients of the fiber and matrix are different. In the following sections, stress–strain relationships are developed for unidirectional and angle laminae subjected to hygrothermal loads.

2.9.1 Hygrothermal Stress–Strain Relationships for a Unidirectional Lamina

For a unidirectional lamina, the stress–strain relationship with temperature and moisture difference gives

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix}, \tag{2.174}$$

where the subscripts T and C are used to denote temperature and moisture, respectively. Note that the temperature and moisture change do not have any shearing strain terms because no shearing strains are induced in the material axes. The thermally induced strains are given by

$$\begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ 0 \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}, \tag{2.175}$$

where α_1 and α_2 are the longitudinal and transverse coefficients of thermal expansion, respectively, and ΔT is the temperature change. The moisture-induced strains are given by

$$\begin{bmatrix} \varepsilon_1^C \\ \varepsilon_2^C \\ 0 \end{bmatrix} = \Delta C \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix}, \tag{2.176}$$

where β_1 and β_2 are the longitudinal and transverse coefficients of moisture, respectively, and ΔC is the weight of moisture absorption per unit weight of the lamina.

Equation (2.174) can be inverted to give

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 - \varepsilon_1^T - \varepsilon_1^C \\ \varepsilon_2 - \varepsilon_2^T - \varepsilon_2^C \\ \gamma_{12} \end{bmatrix}. \tag{2.177}$$

2.9.2 Hygrothermal Stress–Strain Relationships for an Angle Lamina

The stress–strain relationship for an angle lamina takes the following form:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} + \begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix}, \tag{2.178}$$

where

$$\begin{bmatrix} \varepsilon_x^T \\ \varepsilon_y^T \\ \gamma_{xy}^T \end{bmatrix} = \Delta T \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix}, \tag{2.179}$$

and

$$\begin{bmatrix} \varepsilon_x^C \\ \varepsilon_y^C \\ \gamma_{xy}^C \end{bmatrix} = \Delta C \begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix}. \tag{2.180}$$

The terms α_x, α_y, and α_{xy} are the coefficients of thermal expansion for an angle lamina and are given in terms of the coefficients of thermal expansion for a unidirectional lamina as

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix}. \tag{2.181}$$

Similarly, β_x, β_y, and β_{xy} are the coefficients of moisture expansion for an angle lamina and are given in terms of the coefficients of moisture expansion for a unidirectional lamina as

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{bmatrix}. \tag{2.182}$$

From Equation (2.174), if no constraints are placed on a lamina, no mechanical strains will be induced in it. This also implies then that no mechanical stresses are induced. However, in a laminate, even if the laminate has no constraints, the difference in the thermal/moisture expansion coefficients of the various layers induces different thermal/moisture expansions in each layer. This difference results in residual stresses and will be explained fully in Chapter 4.

Example 2.20

Find the following for a 60° angle lamina of glass/epoxy:

1. Coefficients of thermal expansion
2. Coefficients of moisture expansion
3. Strains under a temperature change of –100°C and a moisture absorption of 0.02 kg/kg.

Use properties of unidirectional glass/epoxy lamina from Table 2.1.

Solution

1. From Table 2.1,

$$\alpha_1 = 8.6 \times 10^{-6} \text{ m/m/°C},$$

$$\alpha_2 = 22.1 \times 10^{-6} \text{ m/m/°C}.$$

Using Equation (2.181) gives

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy}/2 \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & -0.8660 \\ 0.7500 & 0.2500 & 0.8660 \\ 0.4330 & -0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 8.6 \times 10^{-6} \\ 22.1 \times 10^{-6} \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{bmatrix} = \begin{bmatrix} 18.73 \times 10^{-6} \\ 11.98 \times 10^{-6} \\ -11.69 \times 10^{-6} \end{bmatrix} \text{m/m/°C.}$$

2. From Table 2.1,

$$\beta_1 = 0 \text{ m/m/kg/kg},$$

$$\beta_2 = 0.6 \text{ m/m/kg/kg}.$$

Using Equation (2.182) gives

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy}/2 \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & -0.8660 \\ 0.7500 & 0.2500 & 0.8660 \\ 0.4330 & -0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \beta_x \\ \beta_y \\ \beta_{xy} \end{bmatrix} = \begin{bmatrix} 0.4500 \\ 0.1500 \\ -0.5196 \end{bmatrix} \text{m/m/kg/kg} .$$

3. Now, use Equation (2.179) and Equation (2.180) to calculate the strains as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 18.73 \times 10^{-6} \\ 11.98 \times 10^{-6} \\ -11.69 \times 10^{-6} \end{bmatrix} (-100) + \begin{bmatrix} 0.4500 \\ 0.1500 \\ -0.5196 \end{bmatrix} (0.02)$$

$$= \begin{bmatrix} 0.7127 \times 10^{-2} \\ 0.1802 \times 10^{-2} \\ -0.9223 \times 10^{-2} \end{bmatrix} \text{m/m.}$$

2.10 Summary

After reviewing the definitions of stress, strain, elastic moduli, and strain energy, we developed the three-dimensional stress–strain relationships for different materials. These materials range from anisotropic to isotropic. The number of independent constants ranges from 21 for anisotropic to 2 for isotropic materials, respectively. Using plane stress assumptions, we reduced the three-dimensional problem to a two-dimensional problem and developed a stress–strain relationship for a unidirectional/bidirectional lamina. These relationships were then found for an angle lamina, using transformation of strains and stresses. We introduced failure theories of an angle lamina in terms of strengths of unidirectional lamina. Finally, we developed stress–strain equations for an angle lamina under thermal and moisture loads. In the appendix of this chapter, we review matrix algebra and the transformation of stresses and strains.

Key Terms

Mechanical characterization
Stress
Strain
Elastic moduli
Strain energy
Anisotropic material
Monoclinic material
Orthotropic material
Transversely isotropic material
Isotropic material
Plane stress
Compliance matrix
Stiffness matrix
Angle ply
Engineering constants
Invariant stiffness and compliance
Failure theories
Maximum stress failure theory
Maximum strain failure theory
Tsai–Hill theory
Tsai–Wu theory
Failure envelopes
Hygrothermal stresses
Hygrothermal loads

Exercise Set

2.1 Write the number of independent elastic constants for three-dimensional anisotropic, monoclinic, orthotropic, transversely isotropic, and isotropic materials.

2.2 The engineering constants for an orthotropic material are found to be

$$E_1 = 4\,Msi,\ E_3 = 3\,Msi,\ E_3 = 3.1\,Msi,$$

$$\nu_{12} = 0.2,\ \nu_{23} = 0.4,\ \nu_{31} = 0.6,$$

$$G_{12} = 6\,Msi,\ G_{23} = 7\,Msi,\ G_{31} = 2\,Msi$$

Find the stiffness matrix [C] and the compliance matrix [S] for the preceding orthotropic material.

2.3 Consider an orthotropic material with the stiffness matrix given by

$$[C] = \begin{bmatrix} -0.67308 & -1.8269 & -1.0577 & 0 & 0 & 0 \\ -1.8269 & -0.67308 & -1.4423 & 0 & 0 & 0 \\ -1.0577 & -1.4423 & 0.48077 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \end{bmatrix} GPa$$

Find:

1. The stresses in the principal directions of symmetry if the strains in the principal directions of symmetry at a point in the material are $\varepsilon_1 = 1\ \mu m/m$, $\varepsilon_2 = 3\ \mu m/m$, $\varepsilon_3 = 2\ \mu m/m$; $\gamma_{23} = 0$, $\gamma_{31} = 5\ \mu m/m$, $\gamma_{12} = 6\ \mu m/m$
2. The compliance matrix [S]
3. The engineering constants E_1, E_2, E_3, ν_{12}, ν_{23}, ν_{31}, G_{12}, G_{23}, G_{31}
4. The strain energy per unit volume at the point where strains are given in part (1.)

2.4 Reduce the monoclinic stress–strain relationships to those of an orthotropic material.

2.5 Show the difference between monoclinic and orthotropic materials by applying normal stress in principal directions and shear stress in principal planes, one at a time and studying the resulting nonzero and zero strains.

2.6 Write down the compliance matrix of a transversely isotropic material (where 2–3 is the plane of isotropy) in terms of the following engineering constants:

E is the Young's modulus in the plane of isotropy 2–3

E' is the Young's modulus in direction 1 that is perpendicular to plane of isotropy 2–3

ν is the Poisson's ratio in the plane of isotropy 2–3

ν' is the Poisson's ratio in the 1–2 plane

G' is the shear modulus in the 1–2 plane

2.7 Find the relationship between the engineering constants of a three-dimensional orthotropic material and its compliance matrix.

2.8 What are the values of stiffness matrix elements C_{11} and C_{12} in terms of the Young's modulus and Poisson's ratio for an isotropic material?

2.9 Are ν_{12} and ν_{21} independent of each other for a unidirectional orthotropic lamina?

2.10 Find the reduced stiffness [Q] and the compliance [S] matrices for a unidirectional lamina of boron/epoxy. Use the properties of a unidirectional boron/epoxy lamina from Table 2.1.

2.11 Find the strains in the 1–2 coordinate system (local axes) in a unidirectional boron/epoxy lamina, if the stresses in the 1–2 coordinate system applied to are $\sigma_1 = 4$ MPa, $\sigma_2 = 2$ MPa, and $\tau_{12} = -3$ MPa. Use the properties of a unidirectional boron/epoxy lamina from Table 2.1.

2.12. Write the reduced stiffness and the compliance matrix for an isotropic lamina.

2.13 Show that for an orthotropic material $Q_{11} \neq C_{11}$. Explain why. Also, show $Q_{66} = C_{66}$. Explain why.

2.14 Consider a unidirectional continuous fiber composite. Start from [σ] = [Q] [ε] and follow the procedure in Section 2.4.3 to get

$$E_1 = Q_{11} - \frac{Q_{12}^2}{Q_{22}} \qquad \nu_{12} = \frac{Q_{12}}{Q_{22}}$$

$$E_2 = Q_{22} - \frac{Q_{12}^2}{Q_{11}} \qquad \nu_{21} = \frac{Q_{12}}{Q_{11}} \qquad G_{12} = Q_{66}.$$

2.15 The reduced stiffness matrix [Q] is given for a unidirectional lamina is given as follows:

$$[Q] = \begin{bmatrix} 5.681 & 0.3164 & 0 \\ 0.3164 & 1.217 & 0 \\ 0 & 0 & 0.6006 \end{bmatrix} Msi .$$

What are the four engineering constants, E_1, E_2, ν_{12}, and G_{12}, of the lamina?

2.16 The stresses in the global axes of a 30° ply are given as σ_x = 4 MPa, σ_y = 2 MPa, and τ_{xy} = –3 MPa. Find the stresses in the local axes. Are the stresses in the local axes independent of elastic moduli? Why or why not?

2.17 The strains in the global axes of a 30° ply are given as ε_x = 4 μin./in., ε_y = 2 μin./in., and γ_{xy} = –3 μin./in. Find the strains in the local axes. Are the strains independent of material properties? Why or why not?

2.18 Find the transformed reduced stiffness matrix $[\bar{Q}]$ and transformed compliance matrix $[\bar{S}]$ for a 60° angle lamina of a boron/epoxy lamina. Use the properties of a unidirectional boron/epoxy lamina from Table 2.1.

2.19 What is the relationship between the elements of the transformed compliance matrix $[\bar{S}]$ for a 0 and 90° lamina?

2.20 For a 60° angle lamina of boron/epoxy under stresses in global axes as σ_x = 4 MPa, σ_y = 2 MPa, and τ_{xy} = –3 MPa, and using the properties of a unidirectional boron/epoxy lamina from Table 2.1, find the following

1. Global strains
2. Local stresses and strains
3. Principal normal stresses and principal normal strains
4. Maximum shear stress and maximum shear strain

2.21 An angle glass/epoxy lamina is subjected to a shear stress τ_{xy} = 0.4 ksi in the global axes resulting in a shear strain γ_{xy} = 468.3 μin./in. in the global axes. What is the angle of the ply? Use the properties of unidirectional glass/epoxy lamina from Table 2.2.

2.22 Find the six engineering constants for a 60° boron/epoxy lamina. Use the properties of unidirectional boron/epoxy lamina from Table 2.2.

2.23 A bidirectional woven composite ply may yield equal longitudinal and transverse Young's modulus but is still orthotropic. Determine the angles of the ply for which the shear modulus (G_{xy}) are maximum and minimum. Also find these maximum and minimum values. Given: E_1 = 69 GPa, E_2 = 69 GPa, ν_{12} = 0.3, G_{12} = 20 GPa.

2.24 A strain gage measures normal strain in a component. Experiments[12] suggest that errors due to strain gage misalignment are more appreciable for angle plies of composite materials than isotropic materials.

1. Take a graphite/epoxy angle ply of 8° under a uniaxial stress, σ_x = 4 Msi. Estimate the strain, ε_x, as measured by a strain gage aligned in the x-direction. Now, if the strain gage is misaligned by +3° to the x-axis, estimate the measured strain. Find the percentage of error due to misalignment. Use properties of unidirectional graphite/epoxy lamina from Table 2.2.
2. Take an aluminum layer under a uniaxial stress, σ_x = 4 Msi. Estimate the strain, ε_x, as measured by a strain gage in the x-direction. Now, if the strain gage is misaligned by +3° to the x-axis, estimate the measured strain. Find the percentage of error due to misalignment. Assume E = 10 Msi, v = 0.3 for aluminum.

2.25 A uniaxial load is applied to a 10° ply. The linear stress–strain curve along the line of load is related as $\sigma_x = 123\varepsilon_x$, where the stress is measured in GPa and strain in m/m. Given E_1 = 180 GPa, E_2 = 10 GPa and ν_{12} = 0.25, find the value of (1) shear modulus, G_{12};and (2) modulus E_x for a 60° ply.

2.26 The tensile modulus of a 0°, 90°, and 45° graphite/epoxy ply is measured as follows to give E_1 = 26.25 Msi, E_2 = 1.494 Msi, E_x = 2.427 Msi for the 45° ply, respectively.

1. What is the value E_x for a 30° ply?
2. Can you calculate the values of ν_{12} and G_{12} from the previous three measured values of elastic moduli?

2.27 Can the value of the modulus, E_x, of an angle lamina be less than both the longitudinal and transverse Young's modulus of a unidirectional lamina?

2.28 Can the value of the modulus, E_x, of an angle lamina be greater than both the longitudinal and transverse Young's modulus of a unidirectional lamina?

2.29 Is the ν_{xy} for a lamina maximum for a 45° boron/epoxy ply? Use properties of unidirectional boron/epoxy lamina from Table 2.2.

2.30 In finding the value of the Young's modulus, E_x, for an angle ply, length-to-width (L/W) ratio of the specimen affects the measured value of E_x. The Young's modulus E_x^1 for a finite length-to-width ratio specimen is related to the Young's modulus, E_x, for an infinite length-to-width ratio specimen by[5]

$$E_x^1 = \frac{E_x}{1-\zeta},$$

where

$$\zeta = \frac{1}{\bar{S}_{11}}\left[\frac{3\bar{S}_{16}^2}{3\bar{S}_{66} + 2\bar{S}_{11}(L/W)^2}\right].$$

Tabulate the values of ζ for $L/W = 2, 8, 16$, and 64 for a 30° glass/epoxy. Use properties of unidirectional glass/epoxy lamina from Table 2.2.

2.31 Starting from the expression for the reduced stiffness element

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)\ ,$$

derive the expression

$$\bar{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \operatorname{Cos} 4\theta\ .$$

2.32 Initial stress–strain data are given for a uniaxial tensile test of a 45° angle ply. Find the in-plane shear modulus of the unidirectional lamina, G_{12}. Use linear regression analysis for finding slopes of curves.

σ_x (KPa)	ε_x (%)	$-\varepsilon_y$ (%)
210	0.1	0.08
413	0.2	0.16
644	0.3	0.25
847	0.4	0.33
1092	0.5	0.42

If similar data were given for a 35° angle ply, would it be sufficient to find the in-plane shear modulus of the unidirectional lamina, G_{12}?

2.33 Calculate the four stiffness invariants, U_1, U_2, U_3, and U_4, and the four compliance invariants V_1, V_2, V_3, and V_4, for a boron/epoxy lamina. Use the properties of a unidirectional boron/epoxy lamina from Table 2.2.

2.34 Show that $\bar{Q}_{11} + \bar{Q}_{22} + \bar{Q}_{12} + \bar{Q}_{66}$ is not a function of the angle of ply.

2.35 Find the off-axis shear strength and mode of failure of a 60° boron/epoxy lamina. Use the properties of a unidirectional boron/epoxy lamina from Table 2.1. Apply the maximum stress failure, maximum strain, Tsai–Hill, and Tsai–Wu failure theories.

2.36 Give one advantage of the maximum stress failure theory over the Tsai–Wu failure theory.

2.37 Give one advantage of the Tsai–Wu failure theory over the maximum stress failure theory.

2.38 Find the maximum biaxial stress, $\sigma_x = -\sigma$, $\sigma_y = -\sigma$, $\sigma > 0$, that one can apply to a 60° lamina of graphite/epoxy. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1. Use maximum strain and Tsai–Wu failure theories.

2.39 Using Mohr's circle, show why the maximum shear stress that can be applied to angle laminae differs with the shear stress sign. Take a 45° graphite/epoxy lamina as an example. Use the properties of a unidirectional graphite/epoxy lamina from Table 2.1.

2.40 Reduce the Tsai–Wu failure theory for an isotropic material with equal ultimate tensile and compressive strengths and a shear strength that is 40% of the ultimate tensile strength.

2.41 An off-axis test is used to find the value of H_{12} for use in the Tsai–Wu failure theory for a boron/epoxy system. The five lamina strengths of a unidirectional boron/epoxy system are given as follows:

$$(\sigma_1^T)_{ult} = 188 \text{ ksi},\ (\sigma_1^C)_{ult} = 361 \text{ ksi},\ (\sigma_2^T)_{ult} = 9 \text{ ksi},\ (\sigma_2^C)_{ult} = 45 \text{ ksi},$$
$$\text{and } (\tau_{12})_{ult} = 10 \text{ ksi}.$$

A 15° specimen fails at a uniaxial load of 33.546 ksi. Find the value of H_{12}. Does it satisfy the inequality $H_{12}^2 < H_{11}H_{22}$, which is a stability criterion for Tsai–Wu failure theory that says failure surfaces intercept all stress axes and form a closed geometric surface[13]?

2.42 Give the units for the coefficient of thermal expansion in the USCS and SI systems.

2.43 Find the free-expansional strains of a glass/epoxy unidirectional lamina under a temperature change of –100°C and a moisture absorption of 0.002 kg/kg. Also find the temperature change for which the transverse expansional strains vanish for a moisture absorption of 0.002 kg/kg. Use the properties of a unidirectional glass/epoxy lamina from Table 2.1.

2.44 Find the coefficients of thermal expansion of a 60° glass/epoxy lamina. Use the properties of unidirectional glass/epoxy lamina from Table 2.2.

2.45 Give the units for coefficient of moisture expansion in the USCS and SI systems.

2.46 Find the coefficients of moisture expansion of a 60° glass/epoxy lamina. Use the properties of unidirectional glass/epoxy lamina from Table 2.1.

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Appendix A: Matrix Algebra*

What is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions and/or numbers. Matrix [*A*] is denoted by

$$[A]=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}.$$

Look at the following matrix about the sale of tires — given by quarter and make of tires — in a Blowoutr'us store:

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Tirestone	25	20	3	2
Michigan	5	10	15	25
Copper	6	16	7	27

To determine how many Copper tires were sold in quarter 4, we go along the row *Copper* and column *quarter 4* and find that it is 27.

Row i of [A] has n elements and is $\begin{bmatrix} a_{i1} & a_{i2} .... a_{in} \end{bmatrix}$ and

Column j of [A] has m elements and is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \\ a_{mj} \end{bmatrix}$.

Each matrix has rows and columns that define the size of the matrix. If a matrix [A] has m rows and n columns, the size of the matrix is denoted by $m \times n$. The matrix [A] may also be denoted by $[A]_{mxn}$ to show that [A] is a matrix with m rows and n columns.

Each entry in the matrix is called the *entry* or *element* of the matrix and is denoted by a_{ij}, where i is the row number (i = 1, 2,…m) and j is the column number (j = 1, 2, …n) of the element.

The matrix for the tire sales example given earlier could be denoted by the matrix [A] as

* This section on matrix algebra is adapted, with permission, from A.K. Kaw, *Introduction to Matrix Algebra*, E-book, http://numericalmethods.eng.usf.edu/, 2004. At the time of printing, the complete E-book can be downloaded free of charge from the given website.

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}.$$

The size of the matrix is 3×4 because there are three rows and four columns. In the preceding $[A]$ matrix, $a_{34} = 27$.

What are the special types of matrices?

Vector: A vector is a matrix that has only one row or one column. The two types of vectors are row vectors and column vectors.

Row vector: If a matrix has one row, it is called a row vector — $[B] = [b_1, b_2, \ldots b_m]$ and m is the dimension of the row vector.

Column vector: If a matrix has one column, it is called a column vector

$$[C] = \begin{bmatrix} c_1 \\ \\ \\ c_n \end{bmatrix}$$

and n is the dimension of the column vector.

Example A.1

Give an example of a row vector.

Solution

$[B] = [25\ 20\ 3\ 2\ 0]$ is an example of a row vector of dimension 5.

Example A.2

Give an example of a column vector.

Solution

An example of a column vector of dimension 3 is

$$[C] = \begin{bmatrix} 25 \\ 5 \\ 6 \end{bmatrix}.$$

Submatrix: If some row(s) or/and column(s) of a matrix [A] are deleted, the remaining matrix is called a submatrix of [A].

Example A.3

Find some of the submatrices of the matrix

$$[A] = \begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}.$$

Solution

Some submatrices of [A] are

$$\begin{bmatrix} 4 & 6 & 2 \\ 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 6 & 2 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Can you find other submatrices of [A]?

Square matrix: If the number of rows, m, of a matrix is equal to the number of columns, n, of the matrix, ($m = n$), it is called a square matrix. The entries a_{11}, a_{22},…a_{nn} are called the *diagonal elements* of a square matrix. Sometimes the diagonal of the matrix is also called the *principal* or *main* of the matrix.

Example A.4

Give an example of a square matrix.

Solution

Because it has the same number of rows and columns (that is, three),

$$[A] = \begin{bmatrix} 25 & 20 & 3 \\ 5 & 10 & 15 \\ 6 & 15 & 7 \end{bmatrix}$$

is a square matrix.

The diagonal elements of [A] are $a_{11} = 25$, $a_{22} = 10$, and $a_{33} = 7$.

Diagonal matrix: A square matrix with all nondiagonal elements equal to zero is called a diagonal matrix — that is, only the diagonal entries of the square matrix can be nonzero ($a_{ij} = 0$, $i \neq j$).

Example A.5

Give examples of a diagonal matrix.

Solution

An example of a diagonal matrix is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Any or all the diagonal entries of a diagonal matrix can be zero. For example, the following is also a diagonal matrix:

$$[A] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Identity matrix: A diagonal matrix with all diagonal elements equal to one is called an identity matrix ($a_{ij} = 0$, $i \neq j$; and $a_{ii} = 1$ for all i).

Example A.6

Give an example of an identity matrix.

Solution

An identity matrix is

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Zero matrix: A matrix whose entries are all zero is called a zero matrix ($a_{ij} = 0$ for all i and j).

Example A.7

Give examples of a zero matrix.

Solution

Examples of a zero matrix include:

$$[A] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[D] = [0 \quad 0 \quad 0]\,.$$

When are two matrices considered equal?

Two matrices $[A]$ and $[B]$ are equal if

The size of $[A]$ and $[B]$ is the same (number of rows of $[A]$ is same as the number of rows of $[B]$ and the number of columns of $[A]$ is same as number of columns of $[B]$) and

$a_{ij} = b_{ij}$ for all i and j.

Example A.8

What would make

$$[A] = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$$

equal to

$$[B] = \begin{bmatrix} b_{11} & 3 \\ 6 & b_{22} \end{bmatrix}?$$

Solution

The two matrices $[A]$ and $[B]$ would be equal if $b_{11} = 2$, $b_{22} = 7$.

How are two matrices added?

Two matrices $[A]$ and $[B]$ can be added only if they are the same size (number of rows of $[A]$ is same as the number of rows of $[B]$ and the number of columns of $[A]$ is same as number of columns of $[B]$). Then, the addition is shown as $[C] = [A] + [B]$, where $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

Example A.9

Add the two matrices

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}.$$

Solution

$$[C] = [A] + [B]$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 5+6 & 2+7 & 3-2 \\ 1+3 & 2+5 & 7+19 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 1 \\ 4 & 7 & 26 \end{bmatrix}.$$

How are two matrices subtracted?

Two matrices $[A]$ and $[B]$ can be subtracted only if they are the same size (number of rows of $[A]$ is same as the number of rows of $[B]$ and the number of columns of $[A]$ is same as number of columns of $[B]$). The subtraction is given by $[D] = [A] - [B]$, where $d_{ij} = a_{ij} - b_{ij}$ for all i and j.

Example A.10

Subtract matrix [B] from matrix [A] — that is, find [A] – [B].

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}.$$

Solution

$$[C] = [A] - [B]$$

$$= \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 7 & -2 \\ 3 & 5 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 5-6 & 2-7 & 3-(-2) \\ 1-3 & 2-5 & 7-19 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & 5 \\ -2 & -3 & -12 \end{bmatrix}.$$

How are two matrices multiplied?

A matrix [A] can be multiplied by another matrix [B] only if the number of columns of [A] is equal to the number of rows of [B] to give $[C]_{mxn} = [A]_{mxp}[B]_{pxn}$. If [$A$] is an $m \times p$ matrix and [B] is a $p \times n$ matrix, then the size of the resulting matrix [C] is an $m \times n$ matrix.

How does one calculate the elements of [C] matrix?

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots\dots + a_{ip}b_{pj}$$

for each i = 1, 2,…m and j = 1, 2,…n.

To put it in simpler terms, the i^{th} row and j^{th} column of the [C] matrix in [C] = [A][B] is calculated by multiplying the i^{th} row of [A] by the j^{th} column of [B] — that is,

$$c_{ij} = [a_{i1}\ a_{i2}\ \quad a_{ip}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \\ \\ b_{pj} \end{bmatrix}$$

$$= a_{i1}\, b_{1j} + \ a_{i2}\, b_{2j} + \ldots\ldots.. + a_{ip}\, b_{pj}.$$

$$= \sum_{k=1}^{p} a_{ik} b_{kj} \ .$$

Example A.11

Given

$$[A] = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 2 & 7 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 3 & -2 \\ 5 & -8 \\ 9 & -10 \end{bmatrix},$$

find

$$[C] = [A][B] \ .$$

Solution

For example, the element c_{12} of the [C] matrix can be found by multiplying the first row of [A] by the second column of [B]:

$$c_{12} = [5 \quad 2 \quad 3]\begin{bmatrix} -2 \\ -8 \\ -10 \end{bmatrix}$$

$$= (5)(-2) + (2)(-8) + (3)(-10)$$

$$= -56.$$

Similarly, one can find the other elements of [*C*] to give

$$[C] = \begin{bmatrix} 52 & -56 \\ 76 & -88 \end{bmatrix}.$$

What is a scalar product of a constant and a matrix?

If [*A*] is an $n \times n$ matrix and k is a real number, then the scalar product of k and [*A*] is another matrix [*B*], where $b_{ij} = ka_{ij}$.

Example A.12

Let

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}.$$

Find 2 [*A*].

Solution

$$[A] = \begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix};$$

then,

$$2[A] = 2\begin{bmatrix} 2.1 & 3 & 2 \\ 5 & 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(2.1) & (2)(3) & (2)(2) \\ (2)(5) & (2)(1) & (2)(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4.2 & 6 & 4 \\ 10 & 2 & 12 \end{bmatrix}.$$

What is a linear combination of matrices?

If $[A_1]$, $[A_2]$,…,$[A_p]$ are matrices of the same size and k_1, k_2,…,k_p are scalars, then

$$k_1[A_1]+k_2[A_2]+........+k_p[A_p]$$

is called a linear combination of $[A_1]$, $[A_2]$,…,$[A_p]$.

Example A.13

If

$$[A_1]=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix},\ [A_2]=\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix},\ [A_3]=\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix},$$

then find

$$[A_1]+2[A_2]-0.5[A_3]\ .$$

Solution

$$[A_1]+2[A_2]-0.5[A_3]=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+2\begin{bmatrix}2.1 & 3 & 2\\ 5 & 1 & 6\end{bmatrix}-0.5\begin{bmatrix}0 & 2.2 & 2\\ 3 & 3.5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}5 & 6 & 2\\ 3 & 2 & 1\end{bmatrix}+\begin{bmatrix}4.2 & 6 & 4\\ 10 & 2 & 12\end{bmatrix}-\begin{bmatrix}0 & 1.1 & 1\\ 1.5 & 1.75 & 3\end{bmatrix}$$

$$=\begin{bmatrix}9.2 & 10.9 & 5\\ 11.5 & 2.25 & 10\end{bmatrix}.$$

What are some of the rules of binary matrix operations?

Commutative law of addition: If $[A]$ and $[B]$ are $m \times n$ matrices, then

$$[A]+[B]=[B]+[A]\ .$$

Associate law of addition: If $[A]$, $[B]$, and $[C]$ all are $m \times n$ matrices, then

$$[A] + ([B] + [C]) = ([A] + [B]) + [C] \ .$$

Associate law of multiplication: If [A], [B], and [C] are $m \times n$, $n \times p$, and $p \times r$ size matrices, respectively, then

$$[A]([B][C]) = ([A][B])[C]$$

and the resulting matrix size on both sides is $m \times r$.

Distributive law: If [A] and [B] are $m \times n$ size matrices and [C] and [D] are $n \times p$ size matrices, then

$$[A]([C] + [D]) = [A][C] + [A][D]$$

$$([A] + [B])[C] = [A][C] + [B][C]$$

and the resulting matrix size on both sides is $m \times p$.

Example A.14

Illustrate the associative law of multiplication of matrices using

$$[A] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix}, \quad [B] = \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix}, \quad [C] = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}.$$

Solution

$$[B][C] = \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix}$$

$$[A][B][C] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 19 & 27 \\ 36 & 39 \end{bmatrix} = \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}$$

$$[A][B] = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix}$$

$$[A][B][C] = \begin{bmatrix} 20 & 17 \\ 51 & 45 \\ 18 & 12 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 91 & 105 \\ 237 & 276 \\ 72 & 78 \end{bmatrix}.$$

These illustrate the associate law of multiplication of matrices.

Is [*A*][*B*] = [*B*][*A*]?

First, both operations, [*A*][*B*] and [*B*][*A*], are only possible if [*A*] and [*B*] are square matrices of same size. Why? If [*A*][*B*] exists, the number of columns of [*A*] must be the same as the number of rows of [*B*]; if [*B*][*A*] exists, the number of columns of [*B*] must be the same as the number of rows of [*A*].

Even then, in general, [*A*][*B*] ≠ [*B*][*A*].

Example A.15

Illustrate whether [*A*][*B*] = [*B*][*A*] for the following matrices:

$$[A] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}, \quad [B] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}.$$

Solution

$$[A][B] = \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 27 \\ -1 & 29 \end{bmatrix}$$

$$[B][A] = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 \\ 16 & 28 \end{bmatrix}$$

$$[A][B] \neq [B][A].$$

What is the transpose of a matrix?

Let $[A]$ be an $m \times n$ matrix. Then $[B]$ is the transpose of the $[A]$ if $b_{ji} = a_{ij}$ for all i and j. That is, the i^{th} row and the j^{th} column element of $[A]$ is the j^{th} row and i^{th} column element of $[B]$. Note that $[B]$ would be an nxm matrix. The transpose of $[A]$ is denoted by $[A]^T$.

Example A.16

Find the transpose of

$$[A] = \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix}$$

Solution

The transpose of $[A]$ is

$$[A]^T = \begin{bmatrix} 25 & 5 & 6 \\ 20 & 10 & 16 \\ 3 & 15 & 7 \\ 2 & 25 & 27 \end{bmatrix}.$$

Note that the transpose of a row vector is a column vector and the transpose of a column vector is a row vector. Also, note that the transpose of a transpose of a matrix is the matrix — that is, $([A]^T)^T = [A]$. Also, $(A + B)^T = A^T + B^T$; $(cA)^T = cA^T$.

What is a symmetric matrix?

A square matrix $[A]$ with real elements, where $a_{ij} = a_{ji}$ for $i = 1,\ldots,n$ and $j = 1,\ldots,n$, is called a symmetric matrix. This is same as that if $[A] = [A]^T$, then $[A]$ is a symmetric matrix.

Example A.17

Give an example of a symmetric matrix.

Solution

A symmetric matrix is

$$[A] = \begin{bmatrix} 21.2 & 3.2 & 6 \\ 3.2 & 21.5 & 8 \\ 6 & 8 & 9.3 \end{bmatrix}$$

because $a_{12} = a_{21} = 3.2$; $a_{13} = a_{31} = 6$; and $a_{23} = a_{32} = 8$.

What is a skew-symmetric matrix?

A square matrix [A] with real elements, where $a_{ij} = -a_{ji}$ for $i = 1,\ldots,n$ and $j = 1,\ldots,n$, is called a skew symmetric matrix. This is same as that if $[A] = -[A]^T$, then [A] is a skew symmetric matrix.

Example A.18

Give an example of a skew-symmetric matrix.

Solution

A skew-symmetric matrix is

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -5 \\ -2 & 5 & 0 \end{bmatrix}$$

because $a_{12} = -a_{21} = 1$; $a_{13} = -a_{31} = 2$; $a_{23} = -a_{32} = -5$. Because $a_{ii} = -a_{ii}$ only if $a_{ii} = 0$, all the diagonal elements of a skew-symmetric matrix must be zero.

Matrix algebra is used for solving systems of equations. Can you illustrate this concept?

Matrix algebra is used to solve a system of simultaneous linear equations. Let us illustrate with an example of three simultaneous linear equations:

$$25a + 5b + c = 106.8$$

$$64a + 8b + c = 177.2$$

$$144a + 12b + c = 279.2\ .$$

This set of equations can be rewritten in the matrix form as

$$\begin{bmatrix} 25a+ & 5b+ & c \\ 64a+ & 8b+ & c \\ 144a+ & 12b+ & c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}.$$

The preceding equation can be written as a linear combination as follows

$$a\begin{bmatrix} 25 \\ 64 \\ 144 \end{bmatrix} + b\begin{bmatrix} 5 \\ 8 \\ 12 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

and, further using matrix multiplications, gives

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}.$$

For a general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{22}x_2 + \qquad + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + \qquad + a_{2n}x_n = c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = c_m$$

can be rewritten in the matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ & & & & \\ & & & & \\ a_{m1} & a_{m2} & . & . & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_m \end{bmatrix}.$$

Denoting the matrices by [A], [X], and [C], the system of equation is [A] [X] = [C], where [A] is called the *coefficient matrix*, [C] is called the *right-hand side vector*, and [X] is called the *solution vector*.

Sometimes [A] [X] = [C] systems of equations are written in the *augmented form* — that is,

$$[A\ \ C] = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & c_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & c_2 \\ & & & & \\ & & & & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & c_n \end{bmatrix}.$$

Can you divide two matrices because that will help me find the solution vector for a general set of equations given by [*A*] [*X*] = [*C*]?

If [A][B]=[C] is defined, it might seem intuitive that $[A] = \frac{[C]}{[B]}$, but matrix division is not defined. However, an *inverse of a matrix* can be defined for certain types of square matrices. The inverse of a square matrix [A], if existing, is denoted by $[A]^{-1}$ such that $[A][A]^{-1} = [I] = [A]^{-1}[A]$.

In other words, let [A] be a square matrix. If [B] is another square matrix of the same size so that [B][A] = [I], then [B] is the inverse of [A]. [A] is then called *invertible* or *nonsingular*. If $[A]^{-1}$ does not exist, [A] is called *noninvertible* or *singular*.

Example A.19

Show whether

$$[B] = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

is the inverse of

$$[A] = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}.$$

Solution

$$[B][A]=\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=[I].$$

$[B][A] = [I]$, so $[B]$ is the inverse of $[A]$ and $[A]$ is the inverse of $[B]$. However, we can also show that

$$[A][B]=\begin{bmatrix}-3 & 2\\5 & -3\end{bmatrix}\begin{bmatrix}3 & 2\\5 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=[I]$$

to show that $[A]$ is the inverse of $[B]$.

Can I use the concept of the inverse of a matrix to find the solution of a set of equations [*A*][*X*] = [*C*]?

Yes, if the number of equations is the same as the number of unknowns, the coefficient matrix $[A]$ is a square matrix.

Given $[A][X] = [C]$. Then, if $[A]^{-1}$ exists, multiplying both sides by $[A]^{-1}$:

$$[A]^{-1}\,[A][X] = [A]^{-1}\,[C]$$

$$[I][X] = [A]^{-1}[C]$$

$$[X] = [A]^{-1}\,[C].$$

This implies that if we are able to find $[A]^{-1}$, the solution vector of $[A][X] = [C]$ is simply a multiplication of $[A]^{-1}$ and the right-hand side vector, $[C]$.

How do I find the inverse of a matrix?

If $[A]$ is an $n \times n$ matrix, then $[A]^{-1}$ is an $n \times n$ matrix and, according to the definition of inverse of a matrix, $[A][A]^{-1} = [I]$.

Denoting,

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix}$$

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & \cdot & \cdot & a'_{1n} \\ a'_{21} & a'_{22} & \cdot & \cdot & a'_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a'_{n1} & a'_{n2} & \cdot & \cdot & a'_{nm} \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & & & & 0 \\ 0 & & \cdot & & & \cdot \\ \cdot & & & 1 & & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}.$$

Using the definition of matrix multiplication, the first column of the $[A]^{-1}$ matrix can then be found by solving:

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & a_{nn} \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ \cdot \\ \cdot \\ a'_{n1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

Similarly, one can find the other columns of the $[A]^{-1}$ matrix by changing the right-hand side accordingly.

Example A.20

Solve the set of equations:

$$25a + 5b + c = 106.8$$

$$64a + 8b + c = 177.2$$

$$144a + 12b + c = 279.2\ .$$

Solution

In matrix form, the preceding three simultaneous linear equations are written as

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}.$$

First, we will find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

and then use the definition of inverse to find the coefficients a, b, c.

If

$$[A]^{-1} = \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix}$$

is the inverse of $[A]$, then

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives three sets of equations:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solving the preceding three sets of equations separately gives

$$\begin{bmatrix} a'_{11} \\ a'_{21} \\ a'_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

$$\begin{bmatrix} a'_{12} \\ a'_{22} \\ a'_{32} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$$

$$\begin{bmatrix} a'_{13} \\ a'_{23} \\ a'_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}.$$

Therefore,

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix}.$$

Now, $[A][X] = [C]$, where

$$[X] = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[C] = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}.$$

Using the definition of $[A]^{-1}$,

$$[A]^{-1}[A][X] = [A]^{-1}[C]$$

$$[X] = [A]^{-1}[C]$$

$$= \begin{bmatrix} -0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.429 \end{bmatrix} \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}.$$

Computationally and algorithmically more efficient, a set of simultaneous linear equations, such as those given previously, can also be solved by using various numerical techniques. These techniques are explained completely in the source (http://numericalmethods.eng.usf.edu) of this appendix. Some of the common techniques of solving a set of simultaneous linear equations are

Matrix inverse method

Gaussian elimination method

Gauss–Siedel method

LU decomposition method

Key Terms

Matrix
Vector
Row vector
Column vector
Submatrix

Square matrix
Diagonal matrix
Identity matrix
Zero matrix
Equal matrices
Addition of matrices
Subtraction of matrices
Multiplication of matrices
Scalar product of matrices
Linear combination of matrices
Rules of binary matrix operation
Transpose of a matrix
Symmetric matrix
Skew symmetric matrix
Inverse of a matrix

Appendix B: Transformation of Stresses and Strains

Equation (2.100) and Equation (2.94) give the relationship between stresses/strains in the global (x,y) coordinate system and the local (1,2) coordinate system, respectively. Note that the transformation is independent of material properties and depends only on the angle between the x-axis and 1-axis, or the angle through which the coordinate system (1,2) is rotated anticlockwise.

B.1 Transformation of Stress

Consider that σ_x, σ_y, and τ_{xy} are the stresses on the rectangular element at a point O in a two-dimensional body (Figure 2.38). One now wants to find the values of the stresses σ_1, σ_2, and τ_{12} on another rectangular element but at the same point O on the body. To do so, make a cut at an angle θ normal to direction 1. Now the stresses in the local 1–2 coordinate system can be related to those in the global x–y coordinate system.

Summing the forces in the direction 1 gives,

$$\sigma_1 \overline{BC} - \tau_{xy} \overline{AB} \operatorname{Cos}\theta - \sigma_y \overline{AB} \operatorname{Sin}\theta - \tau_{xy} \overline{AC} \operatorname{Sin}\theta - \sigma_x \overline{AC} \operatorname{Cos}\theta = 0$$

$$\sigma_1 = \tau_{xy} \frac{\overline{AB}}{\overline{BC}} \operatorname{Cos}\theta + \sigma_y \frac{\overline{AB}}{\overline{BC}} \operatorname{Sin}\theta + \tau_{xy} \frac{\overline{AC}}{\overline{BC}} \operatorname{Sin}\theta + \sigma_x \frac{\overline{AC}}{\overline{BC}} \operatorname{Cos}\theta \ .$$

Now,

$$\operatorname{Sin}\theta = \frac{\overline{AB}}{\overline{BC}},$$

and

$$\operatorname{Cos}\theta = \frac{\overline{AC}}{\overline{BC}} \ ;$$

we have

$$\sigma_1 \tau_{xy} \operatorname{Sin}\theta \operatorname{Cos}\theta + \sigma_y \operatorname{Sin}^2\theta + \tau_{xy} \operatorname{Cos}\theta \operatorname{Sin}\theta + \sigma_x \operatorname{Cos}^2\theta$$

$$\sigma_1 = \sigma_x \operatorname{Cos}^2\theta + \sigma_y \operatorname{Sin}^2\theta + 2\tau_{xy} \operatorname{Sin}\theta \operatorname{Cos}\theta \ . \tag{B.1}$$

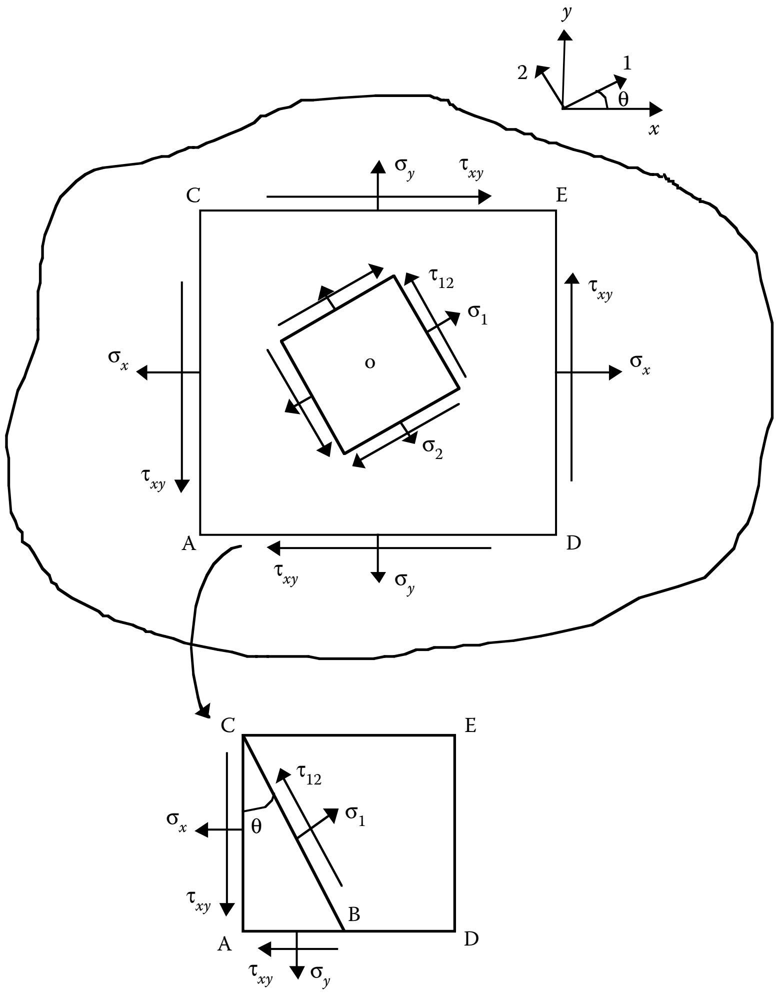

FIGURE 2.38
Free body diagrams for transformation of stresses between local and global axes.

Similarly, summing the forces in direction 2 gives

$$\tau_{12} = -\sigma_x \operatorname{Sin}\theta \operatorname{Cos}\theta + \sigma_y \operatorname{Sin}\theta \operatorname{Cos}\theta + \tau_{xy}(\operatorname{Cos}^2\theta - \operatorname{Sin}^2\theta) \,. \tag{B.2}$$

By making a cut at an angle, θ, normal to direction 2,

$$\sigma_2 = \sigma_x \operatorname{Sin}^2\theta + \sigma_y \operatorname{Cos}^2\theta - 2\tau_{xy} \operatorname{Sin}\theta \operatorname{Cos}\theta \,. \tag{B.3}$$

In matrix form, Equation (B.1), Equation (B.2), and Equation (B.3) relate the local stresses to global stresses as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{B.4}$$

where $c = Cos\ \theta$ and $s = Sin\ \theta$.

The 3×3 matrix in Equation (B.4) is called the transformation matrix $[T]$:

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}. \tag{B.5}$$

By inverting (B.5),

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}. \tag{B.6}$$

This relates the global stresses to local stresses as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}. \tag{B.7}$$

B.2 Transformation of Strains

Consider an arbitrary line, AB, in direction 1 at an angle, θ, to the x-direction. Under loads, the line AB deforms to $A'B'$. By definition of normal strain along AB,

$$\varepsilon_1 = \frac{A'B' - AB}{AB} \tag{B.8}$$

$$= \frac{A'B'}{AB} - 1 .$$

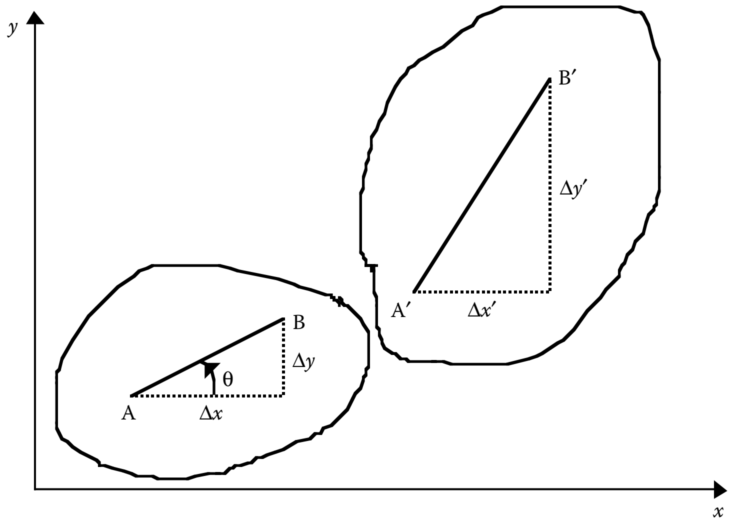

FIGURE 2.39
Line element for transformation of strains between local and global axes.

From Figure 2.39,

$$1 + \varepsilon_1 = \frac{A'B'}{AB}. \tag{B.9}$$

$$(AB)^2 = (\Delta x)^2 + (\Delta y)^2 \tag{B.10}$$

$$(A'B')^2 = (\Delta x')^2 + (\Delta y')^2 \; . \tag{B.11}$$

However, from definition of strain,

$$\Delta x' = \left(1 + \frac{\partial u}{\partial x}\right)\Delta x + \frac{\partial u}{\partial y}\Delta y \tag{B.12}$$

$$\Delta y' = \frac{\partial v}{\partial x}\;\Delta x + \left(1 + \frac{\partial v}{\partial y}\right)\Delta y \; . \tag{B.13}$$

Then, from Equation (B.11) through Equation (B.13),

$$(A'B')^2 = \left[\left(1+\frac{\partial u}{\partial x}\right)\Delta x + \frac{\partial u}{\partial y}(\Delta y)\right]^2 + \left[\frac{\partial v}{\partial x}\Delta x + \left(1+\frac{\partial v}{\partial y}\right)\Delta y\right]^2.$$

Neglecting products and squares of derivatives of strain,

$$(A'B')^2 = \left(1+2\frac{\partial u}{\partial x}\right)(\Delta x)^2 + \left(1+2\frac{\partial v}{\partial y}\right)(\Delta y)^2 + 2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\Delta x\Delta y. \quad \text{(B.14)}$$

From Equation (B.9),

$$(1+\varepsilon_1)^2 = \frac{(A'B')^2}{(AB)^2}$$

$$= \frac{\left(1+2\frac{\partial u}{\partial x}\right)(\Delta x)^2 + \left(1+2\frac{\partial v}{\partial y}\right)(\Delta y)^2 + 2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

$$= \left(1+2\frac{\partial u}{\partial x}\right)\frac{(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2} + \left(1+2\frac{\partial v}{\partial y}\right)\frac{(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}$$

$$+\ 2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\frac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

$$= \left(1+2\frac{\partial u}{\partial x}\right)\mathrm{Cos}^2\theta + \left(1+2\frac{\partial v}{\partial y}\right)\mathrm{Sin}^2\theta + 2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta$$

$$(1+\varepsilon_1)^2 = (1+2\varepsilon_x)\mathrm{Cos}^2\theta + (1+2\varepsilon_y)\mathrm{Sin}^2\theta + 2\gamma_{xy}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta$$

$$1+\varepsilon_1^2+2\varepsilon_1 = 1+2\varepsilon_x\,\mathrm{Cos}^2\theta + 2\varepsilon_y\,\mathrm{Sin}^2\theta + 2\gamma_{xy}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta\,.$$

Neglecting again the squares of the strains,

$$\varepsilon_1 = \varepsilon_x\,\mathrm{Cos}^2\theta + \varepsilon_y\,\mathrm{Sin}^2\theta + \gamma_{xy}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta\,. \quad \text{(B.15)}$$

Similarly, one can take an arbitrary line in direction 2 and prove

$$\varepsilon_2 = \varepsilon_x \,\mathrm{Sin}^2\,\theta + \varepsilon_y \,\mathrm{Cos}^2\,\theta - \gamma_{xy}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta, \tag{B.16}$$

and, by taking two straight lines in direction 1 and 2 (perpendicular to each other), one can prove

$$\gamma_{12} = -2\varepsilon_x \,Sin\,\theta\,Cos\,\theta + 2\varepsilon_y\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta + \gamma_{xy}(\mathrm{Cos}^2\,\theta - \mathrm{Sin}^2\,\theta)\ . \tag{B.17}$$

In matrix form, Equation (B.15), Equation (B.16), and Equation (B.17) relate the local strains to global strains

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}, \tag{B.18}$$

where the 3×3 matrix in Equation (B.18) is the transformation matrix $[T]$ given in Equation (B.5).

Inverting Equation (B.18) gives

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix}, \tag{B.19}$$

where the 3×3 matrix in Equation (B.19) is the inverse of the transformation matrix given in Equation (B.6).

Key Terms

Transformation of stress
Transformation of strain
Free body diagram
Transformation matrix