6

Bending of Beams

Chapter Objectives

- Develop formulas to find the deflection and stresses in a beam made of composite materials.
- Develop formulas for symmetric beams that are narrow or wide.
- Develop formulas for nonsymmetric beams that are narrow or wide.

6.1 Introduction

To study mechanics of beams made of laminated composite materials, we need to review the beam analysis of isotropic materials. Several concepts applied to beams made of isotropic materials will help in understanding beams made of composite materials. We are limiting our study to beams with transverse loading or applied moments.

The bending stress in an isotropic beam (Figure 6.1 and Figure 6.2) under an applied bending moment, M, is given by[1,2]

$$\sigma = \frac{Mz}{I}, \tag{6.1}$$

where

z = distance from the centroid
I = second moment of area

The bending deflections, w, are given by solving the differential equation

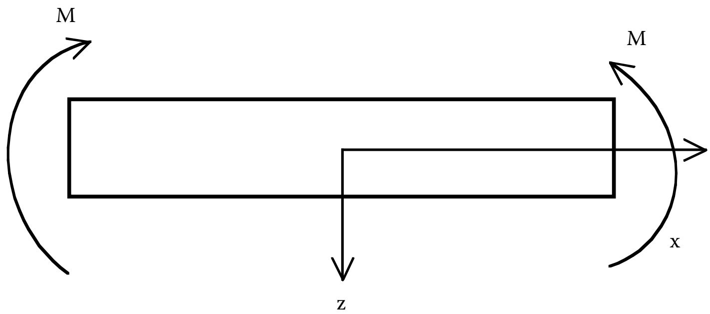

FIGURE 6.1
Bending of a beam.

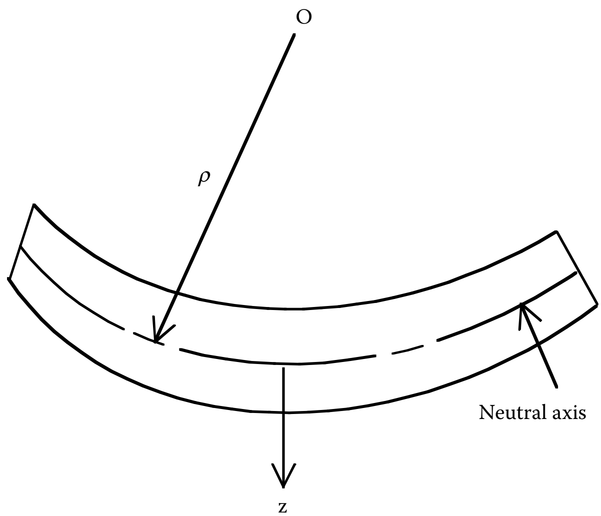

FIGURE 6.2
Curvature of a bended beam.

$$EI \frac{d^2 w}{dx^2} = -M \ , \tag{6.2}$$

where E = Young's modulus of the beam material.

The term of $\frac{d^2 w}{dx^2}$ is defined as the curvature

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2} \ , \tag{6.3}$$

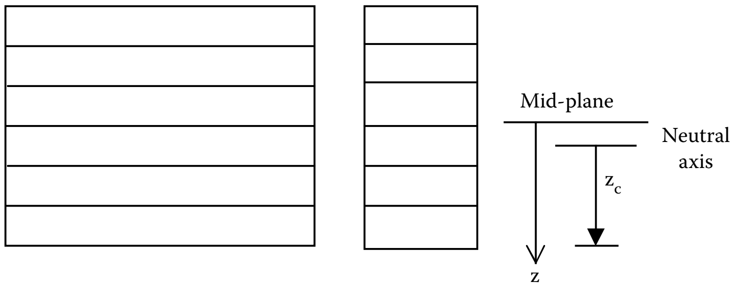

FIGURE 6.3
Laminated beam showing the midplane and the neutral axis.

giving

$$EI\kappa_x = M \ . \tag{6.4}$$

The formula for the bending stress is only valid for an isotropic material because it assumes that the elastic moduli is uniform in the beam. In the case of laminated materials, elastic moduli vary from layer to layer.

6.2 Symmetric Beams

To keep the introduction simple, we will discuss beams that are symmetric and have a rectangular cross-section[3] (Figure 6.3). Because the beam is symmetric, the loads and moments are decoupled in Equation (4.29) to give

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \left[D\right] \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{6.5}$$

or

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \left[D\right]^{-1} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} . \tag{6.6}$$

Now, if bending is only taking place in the x-direction, then

$$M_y = 0 \; , \; M_{xy} = 0$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \left[D\right]^{-1} \begin{bmatrix} M_x \\ 0 \\ 0 \end{bmatrix}, \tag{6.7}$$

that is,

$$\kappa_x = D_{11}^* M_x \tag{6.8a}$$

$$\kappa_y = D_{12}^* M_x \tag{6.8b}$$

$$\kappa_{xy} = D_{16}^* M_x \; , \tag{6.8c}$$

where D_{ij}^* are the elements of the $[D]^{-1}$ matrix as given in Equation (4.28c).

Because defining midplane curvatures (Equation 4.15),

$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2} \; ,$$

$$\kappa_y = -\frac{\partial^2 w_0}{\partial y^2} \; , \tag{6.9}$$

$$\kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} \; ,$$

the midplane deflection w_0 is not independent of y. However, if we have a narrow beam — that is, the length to width ratio (L/b) is sufficiently high, we can assume that $w_0 = w_0(x)$ only.

$$\kappa_x = -\frac{d^2 w_0}{dx^2} = D_{11}^* M_x \; . \tag{6.10}$$

Writing in the form similar to Equation (6.2) for isotropic beams,

$$\frac{d^2 w_0}{dx^2} = -\frac{M_x b}{E_x I} , \tag{6.11}$$

where

b = width of beam
E_x = effective bending modulus of beam
I = second moment of area with respect to the x–y-plane

From Equation (6.8a) and (6.11), we get

$$E_x = \frac{12}{h^3 D_{11}^*} . \tag{6.12}$$

Also,

$$I = \frac{bh^3}{12} \tag{6.13}$$

$$M = M_x b . \tag{6.14}$$

To find the strains, we have, from Equation (4.16),

$$\in_x = z\kappa_x \tag{6.15a}$$

$$\in_y = z\kappa_y \tag{6.15b}$$

$$\gamma_{xy} = z\kappa_{xy} . \tag{6.15c}$$

These global strains can be transformed to the local strains in each ply using Equation (2.95):

$$\begin{bmatrix} \in_1 \\ \in_2 \\ \gamma_{12} \end{bmatrix}_k = \left[R\right]\left[T\right]\left[R\right]^{-1} \begin{bmatrix} \in_x \\ \in_y \\ \gamma_{xy} \end{bmatrix}_k . \tag{6.16}$$

The local stresses in each ply are obtained using Equation (2.73) as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = \left[Q \right] \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix}_k . \quad (6.17)$$

The global stresses in each ply are then obtained using Equation (2.89) as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \left[T \right]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k . \quad (6.18)$$

Example 6.1

A simply supported laminated composite beam of length 0.1 m and width 5 mm (Figure 6.4) made of graphite/epoxy has the following layup of $[0/90/-30/30]_s$. A uniform load of 200 N/m is applied on the beam. What is the maximum deflection of the beam? Find the local stresses at the top of the third ply (–30°) from the top. Assume that each ply is 0.125 mm thick and the properties of unidirectional graphite/epoxy are as given in Table 2.1.

Solution

The shear and bending moment diagrams for the beam are given in Figure 6.5. The bending moment is maximum at the center of the beam and is given by

$$M = \frac{qL^2}{8} , \quad (6.19)$$

where

q = load intensity (N/m)
L = length of the beam (m)

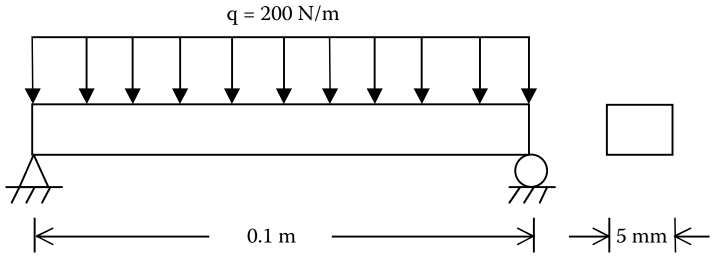

FIGURE 6.4
Uniformly loaded simply supported beam.

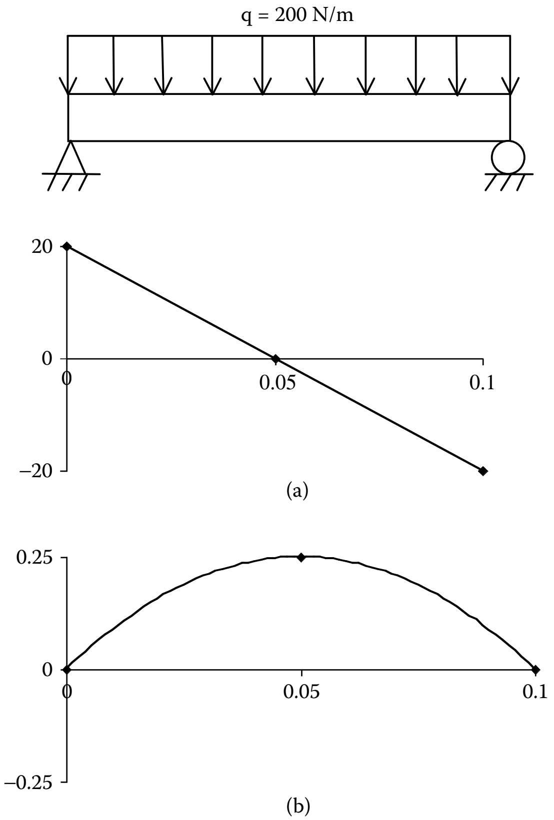

FIGURE 6.5
Shear (a) and bending moment (b) diagrams of a simply supported beam.

The maximum bending moment then is

$$M = \frac{200 \times 0.1^2}{8}$$

$$= 0.25 \text{ N-m.}$$

Without showing the calculations because they are shown in detail in Chapter 4 (see Example 4.2), we get

$$\left[D\right] = \begin{bmatrix} 1.015 \times 10^1 & 5.494 \times 10^{-1} & -4.234 \times 10^{-1} \\ 5.494 \times 10^{-1} & 5.243 \times 10^0 & -1.567 \times 10^{-1} \\ -4.234 \times 10^{-1} & -1.567 \times 10^{-1} & 9.055 \times 10^{-1} \end{bmatrix} Pa\text{-}m^3$$

$$[D]^{-1} = \begin{bmatrix} 1.009\times10^{1} & -9.209\times10^{-3} & 4.557\times10^{-2} \\ -9.209\times10^{-3} & 1.926\times10^{-1} & 2.901\times10^{-2} \\ 4.557\times10^{-2} & 2.901\times10^{-2} & 1.131\times10^{0} \end{bmatrix} \frac{1}{Pa\text{-}m^3} .$$

To find the maximum deflection of the beam, δ, we use the isotropic beam formula:

$$\delta = \frac{5qL^4}{384 E_x I} . \tag{6.20}$$

Now, in Equation (6.12),

$$h = (8)(0.125\times10^{-3})$$

$$= 0.001m$$

$$D_{11}^{*} = 1.009\times10^{-1} \frac{1}{Pa-m^3} .$$

Thus,

$$E_x = \frac{12}{h^3 D_{11}^{*}}$$

$$= \frac{12}{(0.001)^3 (1.009\times10^{-1})}$$

$$= 1.189\times10^{11}\,Pa$$

From Equation (6.13),

$$I = \frac{bh^3}{12}$$

$$= \frac{(5\times10^{-3})(0.001)^3}{12}$$

$$= 4.167\times10^{-13} m^4.$$

Therefore, from Equation (6.20),

$$\delta = \frac{(5)(200)(0.1)^4}{(384)(1.189\times 10^{11})(4.167\times 10^{-13})}$$

$$= 5.256\times 10^{-3}\, m$$

$$= 5.256\ mm\ .$$

The maximum curvature is at the middle of the beam and is given by

$$\begin{bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{bmatrix} = \begin{bmatrix} D_{11}^* \\ D_{12}^* \\ D_{16}^* \end{bmatrix} \frac{qL^2}{8b}$$

$$= \begin{bmatrix} 1.009\times 10^{-1} \\ -9.209\times 10^{-3} \\ 4.557\times 10^{-2} \end{bmatrix} \frac{200\times 0.1^2}{8\times 0.005}$$

$$= \begin{bmatrix} 1.009\times 10^{-1} \\ -9.209\times 10^{-3} \\ 4.557\times 10^{-2} \end{bmatrix} 50$$

$$= \begin{bmatrix} 5.045 \\ -0.4605 \\ 2.279 \end{bmatrix} \frac{1}{m}\ .$$

The global strains (Equation 6.15) at the top of the third ply (–30°) are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$= (-0.00025) \begin{bmatrix} 5.045 \\ -0.4605 \\ 2.279 \end{bmatrix}$$

$$= \begin{bmatrix} -1.261 \times 10^{-3} \\ 1.151 \times 10^{-4} \\ -5.696 \times 10^{-4} \end{bmatrix} \frac{m}{m} .$$

The global stresses (Equation 6.18) at the top of the third ply (–30°) then are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \left[\bar{Q}\right] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 1.094 \times 10^{11} & 3.246 \times 10^{10} & -5.419 \times 10^{10} \\ 3.246 \times 10^{10} & 2.365 \times 10^{10} & -2.005 \times 10^{10} \\ -5.419 \times 10^{10} & -2.005 \times 10^{10} & 3.674 \times 10^{10} \end{bmatrix} \begin{bmatrix} -1.261 \times 10^{-3} \\ 1.151 \times 10^{-4} \\ -5.698 \times 10^{-4} \end{bmatrix}$$

$$= \begin{bmatrix} -1.034 \times 10^{8} \\ -2.680 \times 10^{7} \\ 4.511 \times 10^{7} \end{bmatrix} Pa .$$

Example 6.2

In Example 6.1, the width-to-height ratio in the cross-section of the beam is $b/h = 5/1 = 5$. This may be considered as a narrow-beam cross-section. If the b/h ratio were large, the cross-section may be considered to be wide beam. What are the results of Example 6.1 if one considers the beam to be a wide beam?

Solution

In the case of wide beams, we consider

$$\kappa_y = \kappa_{xy} = 0 .$$

Then, from Equation (6.5),

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ 0 \\ 0 \end{bmatrix} ,$$

giving

$$M_x = D_{11}\kappa_x \tag{6.21}$$

$$M = M_x b = D_{11}\kappa_x b \ . \tag{6.22}$$

Thus, from Equation (6.9a), Equation (6.11), and Equation (6.21),

$$E_x = \frac{12D_{11}}{h^3}$$

$$= \frac{(12)(1.015\times 10^1)}{(0.001)^3}$$

$$= 1.218\times 10^{11}\,Pa$$

and, from Equation (6.20),

$$\delta = \frac{(5)(200)(0.1)^4}{(384)(1.218\times 10^{11})(4.167\times 10^{-13})}$$

$$= 5.131\times 10^{-3}\,m$$

$$= 5.131\ mm.$$

The relative difference in the value of deflection between the assumption of a wide and narrow beam is

$$|\epsilon_a| = \left|\frac{\delta_{narrow} - \delta_{wide}}{\delta_{narrow}}\right| \times 100$$

$$= \left|\frac{5.256 - 5.131}{5.256}\right| \times 100$$

$$= 2.357\% \ .$$

Because there is only 2.357% difference in the maximum deflection, does this mean that the assumption of wide beams influences the stresses only by a similar amount?

From Equation (6.21),

$$\kappa_x = \frac{M_x}{D_{11}}$$

$$= \frac{50}{1.015 \times 10^1}$$

$$= 4.926 \frac{1}{m}.$$

Because $\kappa_y = 0$, $\kappa_{xy} = 0$,

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} 4.926 \\ 0 \\ 0 \end{bmatrix} \frac{1}{m}.$$

The global strains (Equation 6.15) at the top of the third ply (–30°) are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$= (-0.00025) \begin{bmatrix} 4.926 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.232 \times 10^{-3} \\ 0 \\ 0 \end{bmatrix} \frac{m}{m}.$$

The global stresses (Equation 6.18) at the top of the third ply (–30°) then are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \left[\bar{Q}\right] \begin{bmatrix} \in_x \\ \in_y \\ \gamma_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 1.094\times10^{11} & 3.246\times10^{10} & -5.419\times10^{10} \\ 3.246\times10^{10} & 2.365\times10^{10} & -2.005\times10^{10} \\ -5.419\times10^{10} & -2.005\times10^{10} & 3.674\times10^{10} \end{bmatrix} \begin{bmatrix} -1.232\times10^{-3} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.348\times10^{8} \\ -3.999\times10^{7} \\ 6.676\times10^{7} \end{bmatrix} Pa \ .$$

The relative differences in the stresses obtained using wide and narrow beam assumptions are

$$\left|\in_a\right|_{\sigma_x} = \left| \frac{\sigma_{x|narrow} - \sigma_{x|wide}}{\sigma_{x|narrow}} \right| \times 100$$

$$= \left| \frac{-1.034\times10^{8} - \left(-1.348\times10^{8}\right)}{-1.034\times10^{8}} \right|$$

$$= 30.37\%$$

$$\left|\in_a\right|_{\sigma_y} = \left| \frac{\sigma_{y|narrow} - \sigma_{y|wide}}{\sigma_{y|narrow}} \right| \times 100$$

$$= \left| \frac{-2.680\times10^{7} - \left(-3.999\times10^{7}\right)}{2.680\times10^{7}} \right| \times 100$$

$$= 49.22\% \ .$$

$$\left|\in_a\right|_{\tau_{xy}} = \left|\frac{\tau_{xy|narrow} - \tau_{xy|wide}}{\tau_{xy|narrow}}\right| \times 100$$

$$= \left|\frac{4.511\times10^7 - 6.676\times10^7}{4.511\times10^7}\right| \times 100$$

$$= 48.00\% \; .$$

6.3 Nonsymmetric Beams

In the case of nonsymmetric beams, the loads and moment are not decoupled. The relationship given by Equation (4.29) is

$$\begin{bmatrix} N \\ \hline M \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{bmatrix} \in_0 \\ \hline \kappa \end{bmatrix}$$

or

$$\begin{bmatrix} \in_0 \\ \hline \kappa \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]^{-1} \begin{bmatrix} N \\ \hline M \end{bmatrix} .$$

Assuming that the preceding 6×6 inverse matrix is denoted by $[J]$ — that is,

$$\left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right]^{-1} = [J] , \tag{6.23}$$

then

$$\begin{bmatrix} \in_x^0 \\ \in_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} . \tag{6.24}$$

If bending is taking place only in the x-direction, then M_x is the only nonzero component, giving

$$\in_x^0 = J_{14} M_x$$

$$\in_y^0 = J_{24} M_x$$

$$\gamma_{xy}^0 = J_{34} M_x$$

$$\kappa_x = J_{44} M_x$$

$$\kappa_y = J_{54} M_y$$

$$\kappa_{xy} = J_{64} M_{xy} \ . \tag{6.25}$$

The strain distribution in the beam, then, from Equation (4.16) is

$$\in_x = \in_x^0 + z\kappa_x \tag{6.26a}$$

$$\in_y = \in_y^0 + z\kappa_y \tag{6.26b}$$

$$\gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \ . \tag{6.26c}$$

Because the beam is unsymmetric, the neutral axis does not coincide with the midplane. The location of the neutral axis, z_n, is where $\in_x = 0$. From Equation (6.26a),

$$0 = \in_x^0 + z_n \kappa_x$$

$$= J_{14} M_x + z_n J_{44} M_x \ ,$$

giving

$$z_n = -\frac{J_{14}}{J_{44}} \ . \tag{6.27}$$

Because, from Equation (4.15),

$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2}$$

$$\kappa_y = -\frac{\partial^2 w_0}{\partial y^2}$$

$$\kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} ,$$

the deflection w_0 is not independent of y. However, if we have a narrow beam — that is, the length-to-width ratio (L/b) is sufficiently high, we can assume that $w_0 = w_0(x)$ only.

$$\kappa_x = \frac{d^2 w_0}{dx^2} = -J_{44} M_x , \tag{6.28}$$

writing in the form

$$\frac{d^2 w_0}{dx^2} = -\frac{M_x b}{E_x I} , \tag{6.29}$$

where

b = width of beam
E_x = effective bending modulus of beam
I = second moment of area with respect to the x–y-plane

From Equation (6.28) and Equation (6.29), we get

$$E_x = \frac{12}{h^3 J_{44}} . \tag{6.30}$$

Also,

$$I = \frac{bh^3}{12}$$

$$M = M_x b .$$

To find the strains, we have, from Equation (4.16),

$$\in_x = \in_x^o + z\kappa_x \tag{6.31a}$$

$$\in_y = \in_y^o + z\kappa_y \tag{6.31b}$$

$$\gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \,. \tag{6.31c}$$

These global strains can be transformed to the local strains in each ply using Equation (2.95):

$$\begin{bmatrix} \in_1 \\ \in_2 \\ \gamma_{12} \end{bmatrix}_k = [R][T][R]^{-1} \begin{bmatrix} \in_x \\ \in_y \\ \gamma_{xy} \end{bmatrix}_k . \tag{6.32}$$

The local stresses in each ply are obtained using Equation (2.73) as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = [Q] \begin{bmatrix} \in_1 \\ \in_2 \\ \gamma_{12} \end{bmatrix}_k . \tag{6.33}$$

The global stresses in each ply are then obtained using Equation (2.89) as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k . \tag{6.34}$$

Example 6.3

A simply supported laminated composite beam (Figure 6.4) of length 0.1 m and width 5 mm made of graphite/epoxy has the following layup: $[0/90/-30/30]_2$. A uniform load of 200 N/m is applied on the beam. What is the maximum deflection of the beam? Find the local stresses at the top of the third ply (–30°) from the top. Assume that each ply is 0.125 mm thick and the properties of unidirectional graphite/epoxy are as given in Table 2.1.

Solution

The stiffness matrix found by using Equation (4.28) and Equation (4.29) is

$$\begin{bmatrix} 1.027\times10^{8} & 1.768\times10^{7} & 3.497\times10^{-10} & -1.848\times10^{3} & 1.848\times10^{3} & 1.694\times10^{3} \\ 1.768\times10^{7} & 5.986\times10^{7} & 2.608\times10^{-9} & 1.848\times10^{3} & -1.848\times10^{3} & 6.267\times10^{2} \\ 3.497\times10^{-10} & 2.608\times10^{-9} & 2.195\times10^{7} & 1.694\times10^{3} & 6.267\times10^{2} & 1.848\times10^{3} \\ -1.848\times10^{3} & 1.848\times10^{3} & 1.694\times10^{3} & 9.231 & 1.473 & 4.234\times10^{-1} \\ 1.848\times10^{3} & -1.848\times10^{3} & 6.267\times10^{2} & 1.473 & 4.319 & 1.567\times10^{-1} \\ 1.694\times10^{3} & 6.267\times10^{2} & 1.848\times10^{3} & 4.234\times10^{-1} & 1.567\times10^{-1} & 1.829 \end{bmatrix}.$$

The inverse of the matrix is

$$\begin{bmatrix} 1.068\times10^{-8} & -3.409\times10^{-9} & 7.009\times10^{-10} & 4.298\times10^{-6} & -7.241\times10^{-6} & -9.809\times10^{-6} \\ -3.409\times10^{-9} & 1.829\times10^{-8} & 4.042\times10^{-10} & -6.097\times10^{-6} & 1.142\times10^{-5} & -3.083\times10^{-6} \\ 7.009\times10^{-10} & 4.042\times10^{-10} & 5.035\times10^{-8} & -6.339\times10^{-6} & -3.460\times10^{-6} & -4.989\times10^{-5} \\ 4.298\times10^{-6} & -6.097\times10^{-6} & -6.339\times10^{-6} & 1.194\times10^{-1} & -4.335\times10^{-2} & -1.940\times10^{-2} \\ -7.241\times10^{-6} & 1.142\times10^{-5} & -3.460\times10^{-6} & -4.355\times10^{-2} & 2.551\times10^{-1} & -5.480\times10^{-3} \\ -9.809\times10^{-6} & -3.083\times10^{-6} & -4.989\times10^{-5} & -1.940\times10^{-2} & -5.480\times10^{-3} & 6.123\times10^{-1} \end{bmatrix}$$

$$h = 8\times\left(0.125\times10^{-3}\right)$$

$$= 0.001\ m$$

$$J_{44} = 1.194\times10^{-1}\,\frac{1}{Pa\text{-}m^{3}}\ .$$

Now, in Equation (6.30),

$$E_x = \frac{12}{h^3 J_{44}}$$

$$= \frac{12}{\left(0.001\right)^3\left(1.194\times10^{-1}\right)}$$

$$= 1.005\times10^{11}\ Pa.$$

From Equation (6.13),

$$I = \frac{bh^3}{12}$$

$$= \frac{\left(5 \times 10^{-3}\right)\left(0.001\right)^3}{12}$$

$$= 4.167 \times 10^{-13}\, m^4 \,.$$

Thus, from Equation (6.20),

$$\delta = \frac{\left(5\right)\left(200\right)\left(0.1\right)^4}{\left(384\right)\left(1.005 \times 10^{11}\right)\left(4.167 \times 10^{-13}\right)}$$

$$= 6.219 \times 10^{-3} \;\; m$$

$$= 6.219 \;\; mm.$$

The maximum bending moment occurs at the middle of the beam and is given by

$$M_{\max} = \frac{qL^2}{8}$$

$$= \frac{200 \times 0.1^2}{8}$$

$$= 0.25 \;\; N\text{-}m$$

$$M_{x|max} = \frac{M_{max}}{b}$$

$$= \frac{0.25}{0.005}$$

$$= 50 \frac{N\text{-}m}{m}.$$

Calculating the midplane strains and curvature from Equation (6.24) gives

$$\begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} =$$

$$\begin{bmatrix} 1.068\times10^{-8} & -3.409\times10^{-9} & 7.009\times10^{-10} & 4.298\times10^{-6} & -7.241\times10^{-6} & -9.809\times10^{-6} \\ -3.409\times10^{-9} & 1.829\times10^{-8} & 4.042\times10^{-10} & -6.097\times10^{-6} & 1.142\times10^{-5} & -3.083\times10^{-6} \\ 7.009\times10^{-10} & 4.042\times10^{-10} & 5.035\times10^{-8} & -6.339\times10^{-6} & -3.460\times10^{-6} & -4.989\times10^{-5} \\ 4.298\times10^{-6} & -6.097\times10^{-6} & -6.339\times10^{-6} & 1.194\times10^{-1} & -4.335\times10^{-2} & -1.940\times10^{-2} \\ -7.241\times10^{-6} & 1.142\times10^{-5} & -3.460\times10^{-6} & -4.355\times10^{-2} & 2.551\times10^{-1} & -5.480\times10^{-3} \\ -9.809\times10^{-6} & -3.083\times10^{-6} & -4.989\times10^{-5} & -1.940\times10^{-2} & -5.480\times10^{-3} & 6.123\times10^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \end{bmatrix},$$

giving

$$\begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} 2.149\times10^{-4} \\ -3.048\times10^{-4} \\ -3.169\times10^{-4} \end{bmatrix}$$

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} 5.970 \\ -2.178 \\ -9.700\times10^{-1} \end{bmatrix}.$$

The global strains (Equation 6.31) at the top of the third ply (–30°) are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 2.149\times10^{-4} \\ -3.048\times10^{-4} \\ -3.169\times10^{-4} \end{bmatrix} + \left(-0.00025\right) \begin{bmatrix} 5.970 \\ -2.178 \\ -9.700\times10^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -1.278\times10^{-3} \\ 2.397\times10^{-4} \\ -7.431\times10^{-5} \end{bmatrix} \frac{m}{m}.$$

The global stresses (Equation 6.34) at the top of the third ply (–30°) are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \left[\bar{Q}\right] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 1.094 \times 10^{11} & 3.246 \times 10^{10} & -5.419 \times 10^{10} \\ 3.246 \times 10^{10} & 2.365 \times 10^{10} & -2.005 \times 10^{10} \\ -5.419 \times 10^{10} & -2.005 \times 10^{10} & 3.674 \times 10^{10} \end{bmatrix} \begin{bmatrix} -1.278 \times 10^{-3} \\ 2.397 \times 10^{-4} \\ 7.431 \times 10^{-5} \end{bmatrix}$$

$$= \begin{bmatrix} -1.280 \times 10^{8} \\ -3.431 \times 10^{7} \\ 6.170 \times 10^{7} \end{bmatrix} Pa \ .$$

Example 6.4

In Example 6.3, the width-to-height ratio in the cross-section of the beam is $b/h = 5/1 = 5$. This may be considered as a narrow-beam cross-section. If the b/h ratio were large, the cross-section may be considered to be wide beam. What are the results of Example 6.3 if one considers the beam to be a wide beam?

Solution

In the case of the wide beams, we consider

$$\kappa_y = 0 \ .$$

Then, from Equation (6.24),

$$\begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ 0 \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ M_x \\ M_y \\ 0 \end{bmatrix}, \qquad (6.35)$$

we get

$$\in_x^0 = J_{14} M_x + J_{15} M_y \tag{6.36a}$$

$$\in_y^0 = J_{24} M_x + J_{25} M_y \tag{6.36b}$$

$$\gamma_{xy}^0 = J_{34} M_x + J_{35} M_y \tag{6.36c}$$

$$\kappa_x = J_{44} M_x + J_{45} M_y \tag{6.36d}$$

$$0 = J_{54} M_x + J_{55} M_y \tag{6.36e}$$

$$\kappa_{xy} = J_{64} M_x + J_{65} M_y \ . \tag{6.36f}$$

To find the neutral axis, $\in_x = 0$, we use Equation (6.36a) and Equation (6.36e) to give

$$z_n = -\frac{J_{14} J_{55} - J_{15} J_{54}}{J_{44} J_{55} - J_{45} J_{54}} \tag{6.37}$$

$$M_{beam} = bM_x = b \frac{J_{55}}{J_{44} J_{55} - J_{54} J_{45}} \kappa_x \ . \tag{6.38}$$

From Equation (6.9a), Equation (6.11), and Equation (6.38),

$$E_x = \frac{12}{h^3} \frac{J_{55}}{\left(J_{44} J_{55} - J_{45} J_{54}\right)}$$

$$= \frac{12}{\left(0.001\right)^3} \frac{2.551 \times 10^{-1}}{\left(1.194 \times 10^{-1}\right)\left(2.551 \times 10^{-1}\right) - \left(-4.355 \times 10^{-2}\right)\left(-4.355 \times 10^{-2}\right)}$$

$$= 1.071 \times 10^{11} Pa.$$

Thus, from Equation (6.20), we get

$$\delta = \frac{\left(5\right)\left(200\right)\left(0.1\right)^4}{\left(384\right)\left(1.072 \times 10^{11}\right)\left(4.167 \times 10^{-13}\right)}$$

$$= 5.830 \times 10^{-3} m$$

$$= 5.830\ mm.$$

From Example 6.3, the maximum bendings' moment per unit width is

$$M_{x|\max} = 50 \frac{N\text{-}m}{m} .$$

From Equation (6.36e),

$$M_y \Big|_{\max} = -\frac{J_{54}}{J_{55}} M_x$$

$$= -\frac{-4.355 \times 10^{-2}}{2.551 \times 10^{-1}} (50)$$

$$= 8.497 \frac{N\text{-}m}{m} .$$

From Equation (6.35),

$$\begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ 0 \\ \kappa_y \end{bmatrix} =$$

$$\begin{bmatrix} 1.068 \times 10^{-8} & -3.409 \times 10^{-9} & 7.009 \times 10^{-10} & 4.298 \times 10^{-6} & -7.241 \times 10^{-6} & -9.809 \times 10^{-6} \\ -3.409 \times 10^{-9} & 1.829 \times 10^{-8} & 4.042 \times 10^{-10} & -6.097 \times 10^{-6} & 1.142 \times 10^{-5} & -3.083 \times 10^{-6} \\ 7.009 \times 10^{-10} & 4.042 \times 10^{-10} & 5.035 \times 10^{-8} & -6.339 \times 10^{-6} & -3.460 \times 10^{-6} & -4.989 \times 10^{-5} \\ 4.298 \times 10^{-6} & -6.097 \times 10^{-6} & -6.339 \times 10^{-6} & 1.194 \times 10^{-1} & -4.335 \times 10^{-2} & -1.940 \times 10^{-2} \\ -7.241 \times 10^{-6} & 1.142 \times 10^{-5} & -3.460 \times 10^{-6} & -4.355 \times 10^{-2} & 2.551 \times 10^{-1} & -5.480 \times 10^{-3} \\ -9.809 \times 10^{-6} & -3.083 \times 10^{-6} & -4.989 \times 10^{-5} & -1.940 \times 10^{-2} & -5.480 \times 10^{-3} & 6.123 \times 10^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 50 \\ 8.497 \\ 0 \end{bmatrix} .$$

$$= \begin{bmatrix} 1.534 \times 10^{-4} \\ -2.078 \times 10^{-4} \\ -3.463 \times 10^{-4} \\ 5.602 \\ 0 \\ -1.017 \end{bmatrix}$$

The global strains (Equation 6.15) at the top of the third ply (–30°) are

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 1.534 \times 10^{-4} \\ -2.078 \times 10^{-4} \\ -3.463 \times 10^{-4} \end{bmatrix} + \left(-0.00025\right) \begin{bmatrix} 5.602 \\ 0 \\ -1.017 \end{bmatrix}$$

$$= \begin{bmatrix} -1.247 \times 10^{-3} \\ -2.078 \times 10^{-4} \\ -9.221 \times 10^{-5} \end{bmatrix} \frac{m}{m}.$$

The global stresses (Equation 6.18) at the top of the third ply (–30°) are

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \left[\bar{Q}\right] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$= \begin{bmatrix} 1.094 \times 10^{11} & 3.246 \times 10^{10} & -5.419 \times 10^{10} \\ 3.246 \times 10^{10} & 2.365 \times 10^{10} & -2.005 \times 10^{10} \\ -5.419 \times 10^{10} & -2.005 \times 10^{10} & 3.674 \times 10^{10} \end{bmatrix} \begin{bmatrix} -1.247 \times 10^{-3} \\ -2.078 \times 10^{-4} \\ -9.221 \times 10^{-3} \end{bmatrix}$$

$$= \begin{bmatrix} -1.382 \times 10^{8} \\ -4.354 \times 10^{7} \\ 6.833 \times 10^{7} \end{bmatrix}.$$

The relative differences $\left|\epsilon_a\right|$ in the stresses obtained using wide and narrow beam assumptions are

$$\left|\epsilon_a\right|_{\sigma_x} = \left| \frac{\sigma_{x|narrow} - \sigma_{x|wide}}{\sigma_{x|narrow}} \right| \times 100$$

$$= \left| \frac{-1.280 \times 10^8 - \left(-1.382 \times 10^8\right)}{-1.280 \times 10^8} \right| \times 100$$

$$= 7.97\%$$

$$\left|\in_a\right|_{\sigma_x} = \left| \frac{\sigma_{y|narrow} - \sigma_{y|wide}}{\sigma_{y|narrow}} \right| \times 100$$

$$= \left| \frac{-3.431 \times 10^7 - \left(-4.354 \times 10^7\right)}{-3.431 \times 10^7} \right| \times 100$$

$$= 26.90\%$$

$$\left|\in_a\right|_{\tau_{xy}} = \left| \frac{\tau_{xy|narrow} - \tau_{xy|wide}}{\tau_{xy|narrow}} \right| \times 100$$

$$= \left| \frac{6.170 \times 10^7 - 6.836 \times 10^7}{6.170 \times 10^7} \right| \times 100$$

$$= 10.79\%.$$

6.4 Summary

In this chapter, we reviewed the bending of isotropic beams and then extended the knowledge to study stresses and deflection in laminated composite beams. The beams could be symmetric or unsymmetric, and wide or narrow cross-sectioned. Differences in the deflection and stress are calculated between the results of a wide and a narrow beam.

Key Terms

Bending stress
Second moment of area

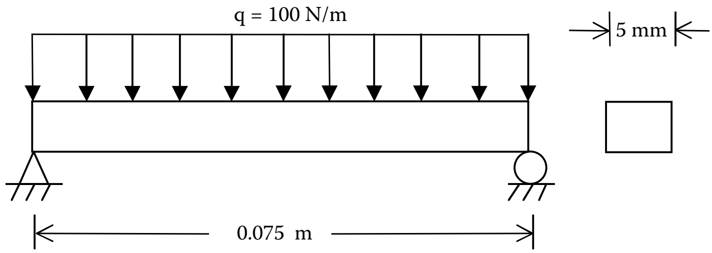

FIGURE 6.6
Uniformly loaded simply supported beam.

Symmetric beams
Wide beams
Narrow beams
Unsymmetric beams

Exercise Set

6.1 A simply supported laminated composite beam (Figure 6.6) made of glass/epoxy is 75 mm long and has the layup of $[\pm 30]_{2s}$. A uniform load is applied on the beam that is 5 mm in width. Assume each ply is 0.125 mm thick and the properties of glass/epoxy are from Table 2.1.

1. What is the maximum deflection of the beam?
2. Find the local stresses at the top of the laminate.

6.2 A simply supported laminated composite beam (Figure 6.6) made of glass/epoxy is 75 mm long and has the layup of $[\pm 30]_4$. A uniform load is applied on the beam that is 5 mm in width. Assume each ply is 0.125 mm thick and the properties of glass/epoxy are from Table 2.1.

1. What is the maximum deflection of the beam?
2. Find the local stresses at the top of the laminate.

6.3 Calculate the bending stiffness of a narrow beam cross-ply laminate $[0/90]_{2s}$. Now compare it by using the average modulus of the laminate. Assume that each ply is 0.125 mm thick and the properties of glass/epoxy are from Table 2.1.

References

1. Buchanan, G.R., *Mechanics of Materials*, HRW Inc., New York, 1988.
2. Ugural, A.C. and Fenster, S.K., *Advanced Strength and Applied Elasticity*, 3rd ed. Prentice Hall, Englewood Cliffs, NJ, 1995.
3. Swanson, S.R., *Introduction to Design and Analysis with Advanced Composite Materials,* Prentice Hall, Englewood Cliffs, NJ, 1997.