CHAPTER 8 STATISTICAL METHODS

8.1 INTRODUCTION

Variability in composite material property data may result from a number of sources including run-to-run variability in fabrication, batch-to-batch variability of raw materials, testing variability, and variability intrinsic to the material. It is important to acknowledge this variability when designing with composites and to incorporate it in design values of material properties. Procedures for calculating *statistically-based* material properties are provided in this chapter. With a properly designed test program (Chapter 2), these statistical procedures can account for some, but not all, of these sources for variability. A fundamental assumption is that one is measuring the desired properties. If this is not the case, then no statistical procedure is sufficient to account for other technical inadequacies.

Section 8.2 provides introductory material and guidance for the methods used in the remainder of the chapter. Readers unfamiliar with the statistical methods in the chapter should read Section 8.2 before the remainder of the chapter; more experienced readers may find it useful as a reference. Section 8.3 provides methods for evaluating data and calculating statistically-based properties. Section 8.4 contains other statistical methods, including methods for confidence intervals for a coefficient of variation, stress-strain curves, quality control, and alternate material evaluation. Section 8.5 contains statistical tables and approximate formulas.

8.1.1 Overview of methods for calculating statistically-based properties

Section 8.3 describes computational methods for obtaining A- and B-basis values from composite material data. Different approaches are used depending on whether the data can be grouped in a natural way (for example, because of batches or differences in environmental conditions). Data sets which either cannot be grouped, or for which there are negligible differences among such groups, are called *unstructured*. Otherwise, the data are said to be *structured*. The statistical methods in Section 8.3.2, which examine if the differences among groups of data are negligible, are useful for determining whether the data should be treated as structured or unstructured. Unstructured data are modeled using a Weibull, normal, or lognormal distribution, using the methods in Section 8.3.4. If none of these are acceptable, nonparametric basis values are determined. Structured data are modeled using *linear statistical models*, including *regression* and the *analysis of variance* (ANOVA), using the methods in Section 8.3.5.

8.1.2 Computer software

Non-proprietary computer software useful for analyzing material property data is available. `STAT17`, available from the MIL-HDBK-17 Secretariat upon request (see page ii), performs the calculations in the flowchart in Figure 8.3.1 with the exception of linear regression. `RECIPE` (REgression Confidence Intervals on PErcentiles), available from the National Institute of Standards and Technology, performs calculations that find material basis values from linear models including regression and analysis of variance. `RECIPE` can be obtained by anonymous ftp from '`ftp.nist.gov`', directory '`recipe`'. A non-proprietary general statistical analysis and graphics package `DATAPLOT` is also available from NIST by anonymous ftp from 'scf.nist.gov', directory 'pubs/dataplot[1].

8.1.3 Symbols

The symbols that are used in Chapter 8 and not commonly used throughout the remainder of this handbook are listed below, each with its definition and the section in which it is first used.

[1] Contact Stefan Leigh, Statistical Engineering Division, NIST, Gaithersburg, MD, 20899-0001, email: stefan.leigh@nist.gov.

SYMBOL	DEFINITION	SECTION
A	A-basis value	-
a	distribution limit	8.1.4
ADC	critical value of ADK	8.3.2.2
ADK	k-sample Anderson-Darling statistic	8.3.2.2
B	B-basis value	8.2.5.1
b	distribution limit	8.1.4
C	critical value	8.3.3.1
CV	coefficient of variation	8.2.5.2
e	error, residual	8.3.5.1
F	F-statistic	8.3.5.2.2
$F(x)$	cumulative distribution function	8.1.4
$f(x)$	probability density function	8.1.4
F_0	standard normal distribution function	8.3.4.3.2
IQ	informative quantile function	8.3.6.2
J	number of specimens per batch	8.2.5.3
k	number of batches	8.2.3
k_A	(1) one-sided tolerance limit factor, A-basis	8.3.4.3.3
	(2) Hanson-Koopmans coefficient, A-basis	8.3.4.5.2
k_B	(1) one-sided tolerance limit factor, B-basis	8.3.4.3.3
	(2) Hanson-Koopmans coefficient, B-basis	8.3.4.5.2
MNR	maximum normed residual test statistic	8.3.3.1
MSB	between-batch/group mean square	8.3.5.2.5
MSE	within-batch/group mean square	8.3.5.2.5
n	number of observations in a data set	8.1.4
n'	effective sample size	8.3.5.2.6
$\tilde{n}$	number of specimens required for comparable reproducibility	8.2.5.3
n^*	see Equation 8.3.5.2.6(b)	8.3.5.2.6
n_i	number of observations in batch/group i	8.3.2.1
OSL	observed significance level	8.3.1
$p(s)$	fixed condition	8.3.5.1
Q	quantile function	8.3.6.1
$\hat{Q}$	quantile function estimate	8.3.6.1
r	rank of observation	8.3.4.5.1
RME	relative magnitude of error	8.5
s	sample standard deviation	8.1.4
s^2	sample variance	8.1.4
s_L	standard deviation of log values	8.3.4.4
s_y	estimated standard deviation of errors from the regression line	8.3.5.3
SSB	between-batch/group sum of squares	8.3.5.2.3
SSE	within-batch/group sum of squares	8.3.5.2.3
SST	total sum of squares	8.3.5.2.3
T	tolerance limit factor	8.3.5.2.7
t	quantile of the t-distribution	8.3.3.1
T_i	temperature at condition i	8.3.5.1

SYMBOL	DEFINITION	SECTION
$t_{\gamma,0.95}(\delta)$	0.95 quantile of the non-central t-distribution with non-centrality parameter δ and degrees of freedom γ	8.3.5.3
TIQ	truncated informative quantile function	8.3.6.2
u	(1) ratio of mean squares	8.3.5.2.7
	(2) batch	8.3.5.1
V_A	one-sided tolerance limit factor for the Weibull distribution, A-basis	8.3.4.2.3
V_B	one-sided tolerance limit factor for the Weibull distribution, B-basis	8.3.4.2.3
w_{ij}	transformed data	8.3.5.2.1
$\bar{x}$	sample mean, overall mean	8.1.4
x_i	observation i in a sample	8.1.4
$\tilde{x}_i$	median of x values	8.3.5.2.1
x_{ij}	j^{th} observation in batch/group i	8.3.2.1
x_{ijk}	k^{th} observation in batch j at condition i	8.2.3
x_L	mean of log values	8.3.4.4
$x_{(r)}$	r^{th} observation, sorted in ascending order; observation of rank r	8.3.4.5.1
$z_{0.10}$	tenth percentile of the underlying population distribution	8.2.2
$z_{(i)}$	ranked independent values	8.3.2.1
$z_{p(s),u}$	regression constants	8.3.5.1
α	(1) significance level	8.3.3.1
	(2) scale parameter of Weibull distribution	8.1.4
$\hat{\alpha}$	estimate of α	8.3.4.2.1
β	shape parameter of Weibull distribution	8.1.4
$\hat{\beta}$	estimate of β	8.3.4.2.1
β_i	regression parameters	8.3.5.3
$\hat{\beta}_i$	least squares estimate of β_i	8.3.5.3
γ	degrees of freedom	8.3.5.3
δ	noncentrality parameter	8.3.5.3
θ_i	regression parameters	8.3.5.1
μ	population mean	8.1.4
μ_i	mean at condition i	8.2.3
ρ	correlation between any two measurements in the same batch	8.2.5.3
σ	population standard deviation	8.1.4
σ^2	population variance	8.1.4
σ_b^2	population between-batch variance	8.2.3
σ_e^2	population within-batch variance	8.2.3

8.1.4 Statistical terms

Definitions of the most often used statistical terms in this handbook are provided in this section. This list is certainly not complete; the user of this document with little or no background in statistical methods should also consult an elementary text on statistical methods such as Reference 8.1.4. Definitions for additional statistical terms are included in Section 1.7.

` *Population* -- The set of measurements about which inferences are to be made or the totality of possible measurements which might be obtained in a given testing situation. For example, "all possible ultimate tensile strength measurements for Composite Material A, conditioned at 95% relative humidity and room temperature". In order to make inferences about a population, it is often necessary to make assumptions about its distributional form. The assumed distributional form may also be referred to as the population.

Sample -- The collection of measurements (sometimes referred to as observations) taken from a specified population.

Sample size -- The number of measurements in a sample.

A-basis Value -- A statistically-based material property; a 95% lower confidence bound on the first percentile of a specified population of measurements. Also a 95% lower tolerance bound for the upper 99% of a specified population.

B-basis Value -- A statistically-based material property; a 95% lower confidence bound on the tenth percentile of a specified population of measurements. Also a 95% lower tolerance bound for the upper 90% of a specified population.

Compatible -- Descriptive term referring to different groups or subpopulations which may be treated as coming from the same population.

Structured data -- Data for which natural groupings exist, or for which responses of interest could vary systematically with respect to known factors. For example, measurements made from each of several batches could reasonably be grouped according to batch, and measurements made at various known temperatures could be modeled using linear regression (Section 8.3.5.2); hence both can be regarded as structured data.

Unstructured data -- Data for which all relevant information is contained in the response measurements themselves. This could be because these measurements are all that is known, or else because one is able to ignore potential structure in the data. For example, data measurements that have been grouped by batch and demonstrated to have negligible batch-to-batch variability (using the subsample compatibility methods of Section 8.3.2) may be considered unstructured.

Location parameters and statistics:

Population mean -- The average of all potential measurements in a given population weighted by their relative frequencies in the population. The population mean is the limit of the sample mean as the sample size increases.

Sample mean -- The average of all observations in a sample and an estimate of the population mean. If the notation $x_1, x_2, ..., x_n$ is used to denote the n observations in a sample, then the sample mean is defined by:

$$\bar{x} = \frac{x_1 + x_2 + ... + x_n}{n} \qquad 8.1.4(a)$$

or

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad 8.1.4(b)$$

Sample median -- After ordering the observations in a sample from least to greatest, the sample median is the value of the middle-most observation if the sample size is odd and the average of the two middle-most observations if the sample size is even. If the population is symmetric about its mean, the sample median is also a satisfactory estimator of the population mean.

Dispersion statistics:

Sample variance -- The sum of the squared deviations from the sample mean, divided by $n-1$, where n denotes the sample size. The sample variance is defined by:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad 8.1.4(c)$$

or

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} x_i^2 - \frac{n}{n-1}\bar{x}^2 \qquad 8.1.4(d)$$

Sample standard deviation -- The square root of the sample variance. The sample standard deviation is denoted by s.

Probability distribution terms:

Probability distribution -- A formula which gives the probability that a value will fall within prescribed limits. When the word *distribution* is used in this chapter, it should be interpreted to mean *probability distribution*.

Normal Distribution -- A two parameter (μ,σ) family of probability distributions for which the probability that an observation will fall between a and b is given by the area under the curve

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad 8.1.4(e)$$

between a and b. A normal distribution with parameters (μ,σ) has population mean μ and variance σ^2.

Lognormal Distribution -- A probability distribution for which the probability that an observation selected at random from this population falls between a and b ($0 < a < b < \infty$) is given by the area under the normal distribution between $\ln(a)$ and $\ln(b)$.

Two-Parameter Weibull Distribution -- A probability distribution for which the probability that a randomly selected observation from this population lies between a and b ($0 < a < b < \infty$) is given by

$$e^{-(a/\alpha)^\beta} - e^{-(b/\alpha)^\beta} \qquad 8.1.4(f)$$

where α is called the scale parameter and β is called the shape parameter.

Probability function terms:

Cumulative Distribution Function -- A function, usually denoted by $F(x)$, which gives the probability that a random variable lies between any prescribed pair of numbers, that is

$$\Pr(a < x \le b) = F(b) - F(a) \qquad 8.1.4(g)$$

Such functions are non-decreasing and satisfy

$$\lim_{x \to +\infty} F(x) = 1 \qquad 8.1.4(h)$$

The cumulative distribution function, F, is related to the probability density function, f, by

$$f(x) = \frac{d}{dx}F(x) \qquad 8.1.4(i)$$

provided that $F(x)$ is differentiable.

F-distribution -- A probability distribution which is employed in the analysis of variance, regression analysis, and tests for equality of variance. Tables of this distribution are readily available.

Probability Density Function -- A function $f(x) \geq 0$ for all x with

$$\int_{-\infty}^{\infty} f(x)dx = 1 \qquad 8.1.4(j)$$

The probability density function determines the cumulative distribution function $F(x)$ by

$$F(x) = \int_{-\infty}^{x} f(t)dt \qquad 8.1.4(k)$$

Note that the limits $(-\infty,\infty)$ may be conventional; for example, the exponential distribution satisfies the definition by defining its probability density function as

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \text{ and} \\ e^{-x} & \text{for } x > 0 \end{cases} \qquad 8.1.4(l)$$

The probability density function is used to calculate probabilities as follows:

$$\Pr(a < x \leq b) = \int_{a}^{b} f(x)dx \qquad 8.1.4(m)$$

Error and Variability:

Fixed Effect -- A systematic shift in a measured quantity due to a particular level change of a treatment or condition. The change in level for a treatment or condition is often under the control of the experimenter. A measured quantity could be compressive strength or tensile modulus. A treatment or condition could be test temperature, fabricator, and so on. For a fixed effect, the shift in the measured quantity is to be interpreted as a consistent change not only in the context of the observed data but also with respect to future data under the same treatment or condition.

Random Effect -- A shift in a measured quantity due to a particular level change of an external, usually uncontrollable, factor. The level of this factor is regarded as a random draw from an infinite population. The specific level of a random effect is never under the control of the experimenter, however it may remain fixed within a limited subgroup of observed data. A measured quantity could be compressive strength or tensile modulus. An external factor could be batch production leading to batch-to-batch differences. Fabricator-to-fabricator differences may be considered a random effect if the number of fabricators involved are considered to be a small sample of all present and future fabricators. For a random effect, the shift in the measured quantities is viewed as a random variable having mean zero and a nonzero variance. Within a subgroup experiencing a fixed level of an external factor, the measured quantities are correlated (shifting as a cluster around a population average with the magnitude of the shift depending on the level of the factor). Therefore, to obtain the most independent information concerning the

population of response values, it is better to have more subgroups than to have more measurements per subgroup.

Random Error -- That part of the data variation that is due to unknown or uncontrolled external factors and that affects each observation independently and unpredictably. It is the residual error in a model under analysis, the variability remaining after the variability due to fixed and random effects has been removed. Random error is a special case of a random effect. In both cases, the level of the random effect or error is uncontrollable but random errors vary independently from measurement to measurement (i.e., there are no random error shifts shared in common by several measurements). An important example of random error is the specimen-to-specimen variability occurring within a subgroup experiencing constant levels of treatment, condition, batch, and other external factors (fixed and random effects).

Material Variability -- A source of variability due to the spatial and consistency variations of the material itself and due to variations in its processing (e.g., the inherent microstructure, defect population, cross-link density, etc.). Components of material variability can be any combination of fixed effects, random effects, and random error.

8.2 BACKGROUND

This section provides introductory material and guidance for the methods used in the remainder of the chapter. Readers unfamiliar with the statistical methods in the chapter should read this section before the remainder of the chapter. For more experienced readers, this section may be a useful reference for the approach and use of terminology.

8.2.1 Statistically-based design values

A *design value* for a material is the minimum value of a material property expected to be used in the fabrication of the structure. The value can be deterministic or statistically based. S-basis value is the usual designation of a deterministic value; this implies that any material when test-sampled is rejected if any of its properties fall below the established S-value. Statistically-based design values acknowledge the stochastic nature of the material properties and, in general, will reduce the amount of incoming material testing. Deterministic and statistically based material design values are used in the same way in the deterministic design of the structure. For structural integrity, actual (including appropriate safety factors) stresses or strains in the structure can not exceed the material design values. If the structure is designed using probabilistic methods (by making reliability estimates) only statistically-based design values can be used.

To understand the definitions of 'statistically-based' design values, it is necessary to regard the material property of interest, not as a constant, but as a *random variable*, a quantity that varies from specimen to specimen according to some probability distribution. A reasonable first attempt at definitions of B-basis and A-basis material properties are the 10th and 1st percentiles of a material property distribution. One expects the property to usually be above these values, so these definitions are reasonable statistically-based counterparts to the traditional deterministic notion of a design value. Of course, there is an obvious problem in practice; one doesn't know the probability distribution of a material property. So far only simple ideas of probability theory have been used in these definitions; it is in addressing uncertainty in these percentiles that statistical inference plays an essential role.

8.2.2 Basis values for unstructured data.

Before breaking n specimens, imagine them each to have a strength value which can be represented as belonging to a common probability distribution. After breaking the specimens, one observes n numbers, and if n is large enough, a histogram of these numbers will approximate the unknown distribution. This probability distribution is referred to as a *population*, and the n numbers are a realization of a *random sample* of this population. Conceptually, one can do this thought-experiment many times, obtaining different sets of n numbers. A statistically-based B-basis material property is a *statistic*, calculated from a

random sample n, such that if one were to repeatedly obtain random samples of n specimens and calculate many of these basis values, 95% of the time the calculated values would fall below the (unknown) 10th percentile. An A-basis value is defined similarly, replacing the 10th percentile with the 1st. In statistical parlance, basis values are 95% lower confidence limits on prescribed percentiles, which are also sometimes referred to as *tolerance limits*.

Note that the definitions of statistically-based material properties have been developed in two steps. First a deterministic property was modeled with a probability distribution in order to take into account observed scatter in the property, and tentative definitions of basis values in terms of percentiles of this distribution were made. This takes into account uncertainty that remains, however much data on the property one obtains. But there is additional uncertainty, since instead of unlimited data, one has only n specimens. So the percentiles of our tentative definitions are replaced with conservative 'under-estimates' of these percentiles, thereby taking into account the additional uncertainty in a random material property due to limited data.

An example will help fix ideas. Let the tensile strength of a material have a normal distribution with a mean of 1000 MPa and a standard deviation of 125 MPa. The 10th percentile of this population is

$$z_{0.10} \doteq 1000 - (1.282)125 \doteq 840 \text{ MPa}$$

This would be the B-basis value if one had unlimited data, and hence knew the population. Assume instead that only $n = 10$ specimens are available. A B-basis value can be calculated for these n specimens (see Section 8.3.4.3), and if one were to obtain many such sets of 10 specimens from the same population, this basis value would be less than 840 MPa for 95% of these repeated samples. Substantial scatter is characteristic of basis values determined from small data sets, due primarily to uncertainty in the population variance (see Section 8.2.5).

The present discussion provides a fairly complete description of material basis values, if one is willing to make two simplifying assumptions: first that between-batch material property variability is negligible, and second that all of the data are obtained from tests at identical conditions. In Section 8.3.2, such data are defined to be *unstructured*. However, composite material properties often do vary substantially from batch to batch, and data on properties are usually obtained, not for a single set of fixed conditions but over a test matrix of some combination of temperatures, humidities, and stacking sequences. Data that exhibit these additional complexities will be called *structured* (see Section 8.3.2), and are analyzed using *regression* and *analysis of variance*. Regression analysis in general is discussed in Section 8.3.5.

8.2.3 Basis values in the presence of batch-to-batch variability

Composite materials typically exhibit considerable variability in many properties from batch to batch. Because of this variability, one should not indiscriminately pool data over batches and apply the unstructured data procedures discussed above and in Section 8.3.4. Basis values should incorporate the variability to be expected between batches or panels of a material, particularly when one has data on only a few batches or panels, or when one has a particular reason for suspecting that this variability could be non-negligible. Pooling batches involves the implicit assumption that this source of variability is negligible, and in the event that this is not the case, the values which result from pooling can be too optimistic. Before pooling data, the subsample compatibility methods of Section 8.3.2 should be applied. The interpretation of material basis values in the presence of between-batch (or panel, and so on) variability is discussed below for the simplest case of a one-way ANOVA model (Section 8.3.5.2).

The data for the present discussion consist of n measurements, all of the same property, of the same material, and tested under the same conditions, The only structure apparent in the data under this hypothetical scenario is that each specimen has been fabricated from one of k batches of raw material. (Equivalently, one might imagine material made from the same batch, but for which several autoclave runs had been required, resulting in non-negligible variability in properties between *panels* of specimens.) Each data value can be regarded as a sum of three parts. The first part is the unknown mean, the second

part is a shift in the mean due to the batch from which the specimen was obtained, and the third part is a random perturbation due to the scatter in measurements made on different specimens from the same batch.

The unknown constant mean corresponds to a set of fixed conditions (for example, 8-ply unidirectional tensile strength for a specific material, tested according to a well-defined test method, and at prescribed test conditions). If one were to produce batches endlessly, preparing specimens from each batch according to these fixed conditions, breaking specimens from each batch, and obtaining measurements of the property of interest, then the average of all of these measurements would approach this unknown constant in the limit of infinitely many batches. This unknown mean can be parameterized as a function of the conditions under which the specimens were prepared and tested, where the form of this function is known except for some constants; this is related to the notion of a *regression model*, which will be discussed in some detail in Section 8.3.5.1.

Imagine, however, that one were to test many specimens from a single batch. The average strength approaches a constant in this situation as well, but this constant will not be the same as in the case where each specimen comes from a different batch. In the situation discussed in the previous paragraph, the average converges to an overall population mean (a 'grand mean'), while the average converges to the population means for *a particular batch* in the present case. The difference between the overall population mean and the population mean for a particular batch is the second component of a material property measurement. This difference is a random quantity; it will vary from batch to batch in an unsystematic way. This random 'batch effect' is assumed to follow a normal probability distribution with a mean of zero, and some unknown variance called the *between-batch component of variance*, and denoted by σ_b^2.

Even when specimens are made from the same batch and tested under identical conditions, one will not get the same value every time. In addition to the population mean and the random 'batch effect' there is a third component to any measurement, which is also random, but which differs from specimen to specimen within a batch. This random quantity is called the within-batch variability, and it is modeled as a normally distributed random variable with a mean of zero and a variance σ_w^2, referred to as the *within-batch component of variance*.

To summarize, a measurement made on data on a particular specimen from a specific batch is modeled as a sum of three parts:

$$x_{ijk} = \mu_i + b_j + e_{ijk} \qquad 8.2.3$$

where x_{ijk} is the k^{th} measurement on data from batch j at a set of fixed conditions labeled by i. The random variables b_j and e_{ijk} have normal distributions with mean zero and variances σ_b^2 and σ_w^2, respectively. For the present discussion, there is only one set of fixed conditions, hence the subscript 'i' can be omitted. For the general regression and analysis of variance models discussed in Sections 8.3.5.1 and 8.3.5.2 there can be many combinations of fixed factors; there the 'i' subscript in Equation 8.2.3 must be retained.

If data from more than one batch are available, then RECIPE (Section 8.1.2) will use the data to determine basis values which with 95% confidence are less than the appropriate percentile of a randomly chosen observation from a randomly chosen *future* batch, for a particular set of fixed conditions. Such values protect against the possibility of batch-to-batch variability resulting in future batches which have lower mean properties than those batches for which data are available.

8.2.4 Batches, panels, and confounding

The model described in Equation 8.2.3 and Section 8.3.5 is based on the assumption of at most two sources of variability; these are referred to as 'between-batch variability' and within-batch variability'. In

the manufacturing of composites, however, there are typically at least three sources of variability. For composites made from prepreg, the additional source is due to the fact that several specimens are typically manufactured together as a ‘panel’, consequently a third source can be referred to as ‘between-panel’ variability.

When one has data on a material from several batches, but at only one set of fixed conditions, one cannot estimate batch and panel variabilities separately. Whenever data are obtained from a new batch, that data also comes from a different panel. (In statistical terminology, the batch and panel variances are confounded.) So what we call 'between-batch variability' in such cases is actually the sum of the between-batch and between-panel variances. Unless the between-panel variability is negligible, the between-batch variance will be over-estimated in such cases. This can result in material basis properties that are lower than they should be.

Next consider the situation where data are available from several batches at more than one set of fixed conditions (see Section 8.3.7.8). If one assumes also that data at different conditions from the same batch are from different panels, then one is able, in principle, to estimate the between-batch and between-panel variances separately. However, the regression models in this chapter and the RECIPE software include only one source of such variability. Consequently, the between-panel variance is confounded, not with the between-batch variance as above, but with the within-batch variance. This can result in material basis values that are somewhat higher than they should be. This is likely to be a less serious problem than the case where panel and batch variances are confounded for several reasons. Perhaps the most important of these is that of the sources of variability, that due to batches is the primary concern, and is being treated appropriately. Another reason is that there is typically considerable variability within panels, and if the between-panel variance is small with respect to the within-panel variability, then the material basis properties will not be substantially higher than they should be.

8.2.5 Sample size guidelines for determining basis values.

Material basis values are often regarded as material properties, that is, these values are interpreted as constants which can be used to help characterize the material and processing. Since basis values will *always* vary from one set of data to the next, even if the material, conditioning, and test remain unchanged, treating them as material constants is always an approximation.

However, if the calculations are based on 'enough' data, the basis values should be reproducible, to within engineering accuracy, across comparable data sets. The objective of this section is to illustrate the small-sample reproducibility problem and to provide guidance on how many data are necessary in basis value calculations in order for these values to be approximately reproducible.

How many data are 'enough' depends on many factors, including

1. The statistical model which is used to approximate the population from which the data is sampled,
2. The degree of reproducibility which is desired,
3. The variability in the property being measured, and
4. Variability in measurements of the property due to the test method

Because of this, it is impossible to give firm recommendations. The discussion in this section has another purpose. It is intended to provide background information and guidelines to assist the user of this handbook in making a sample size decision. We emphasize that this section deals *only* with the stability of basis values with respect to sample size. Another important issue relevant to the choice of a sample size, which deserves separate consideration, is the effect on basis values of statistical model assumptions - since there is considerable uncertainty in model selection from small samples. Additional discussion of the effect of sample size selection is found in Section 2.2.5.

8.2.5.1 Example

Table 8.2.5.1 presents tensile strength data (in ksi) for a unidirectional composite material, tested under room temperature dry conditions.

TABLE 8.2.5.1 *Room temperature dry tensile strength for a unidirectional composite material.*

226	227	226	232	252

The mean and standard deviation for these data are $\bar{x} = 232.6$ and $s = 11.13$. Using the normal model (Section 8.3.4.3), a B-basis value for these data is

$$B = \bar{x} - k_B s = 232.6 - 3.407(11.13) = 195 \qquad 8.2.5.1$$

The first point to be made is that a B-basis value determined from as few as five specimens is not likely to be sufficiently reproducible for it to be regarded as a material constant for most applications. For the present discussion, the plausible assumption is made that the above data are a sample from a normal distribution with a mean of 230 and a standard deviation of 10.

The theoretical population of B-basis values which corresponds to this assumed normal population of strength measurements can be calculated, and is displayed in Figure 8.2.5.1. Note that the observed basis value is near the mean of this population of basis values. This is to be expected since the parameters of the hypothetical normal distribution have been based on the same set of data from which the basis value was determined. However, note also that values within ±20 ksi of the basis value are also likely to be observed. Based on this analysis, one cannot rule out the possibility of the B-basis value of the next sample of five being as low as 180 ksi or as high as 220 ksi.

8.2.5.2 Mean and standard deviations of normal basis values

Basis values calculated from small samples exhibit high variability. One way of quantifying this is to calculate the theoretical mean, standard deviation, and coefficient of variation of basis values from hypothetical populations as functions of the number of specimens. Of course, these calculations are going to depend on the statistical model chosen and the parameters selected for this model. However, the objective of these calculations is not to provide rigid criteria, but rather to inform the user of the qualitative behavior of basis values.

A normal population with a mean of 100 and a standard deviation of 10 will be considered for the discussion in this subsection. The 10% coefficient of variation is typical of what is observed for many material properties, and the mean of 100 is within an order of magnitude of most strength measurements (in ksi) for unidirectional composite materials. The choice of the normal population is made because the normal basis values procedures have broad appeal, and because the required calculations can be done in closed form. Sample sizes for basis values from Weibull populations should as a rule be larger than those for normal populations in order to achieve the same degree of reproducibility. Only basis values for a simple random sample are considered here; ANOVA basis values are discussed in the next subsection.

The mean and one standard deviation limits for B-basis values from a normal population with a mean of 100 and a standard deviation of 10 is displayed in Figure 8.2.5.2(a) as a function of the number of specimens. Note the extremely high variability for sample sizes of ten or less.

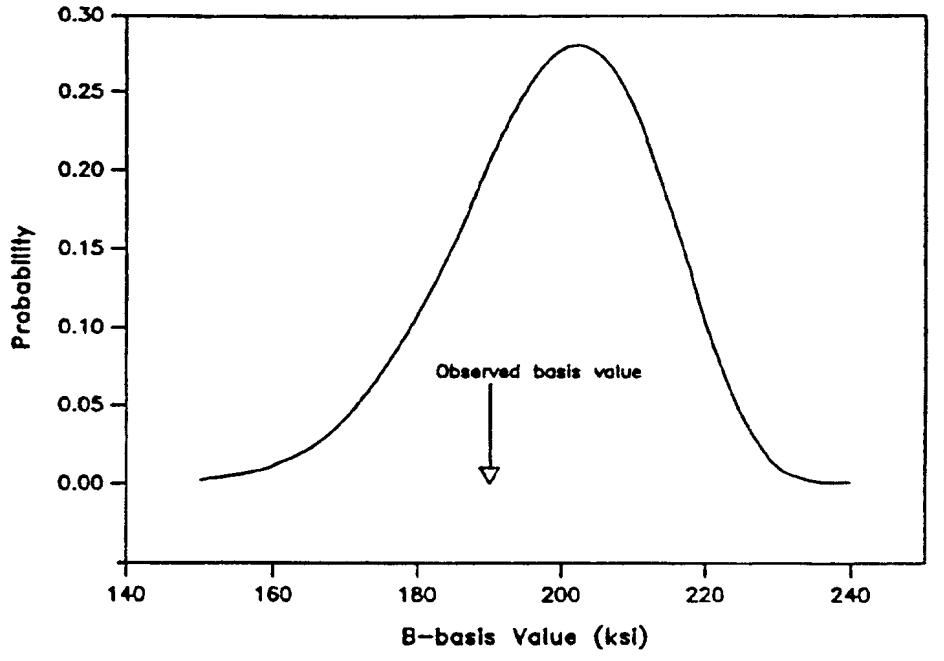

FIGURE 8.2.5.1 *B-basis value population for a sample of size five.*

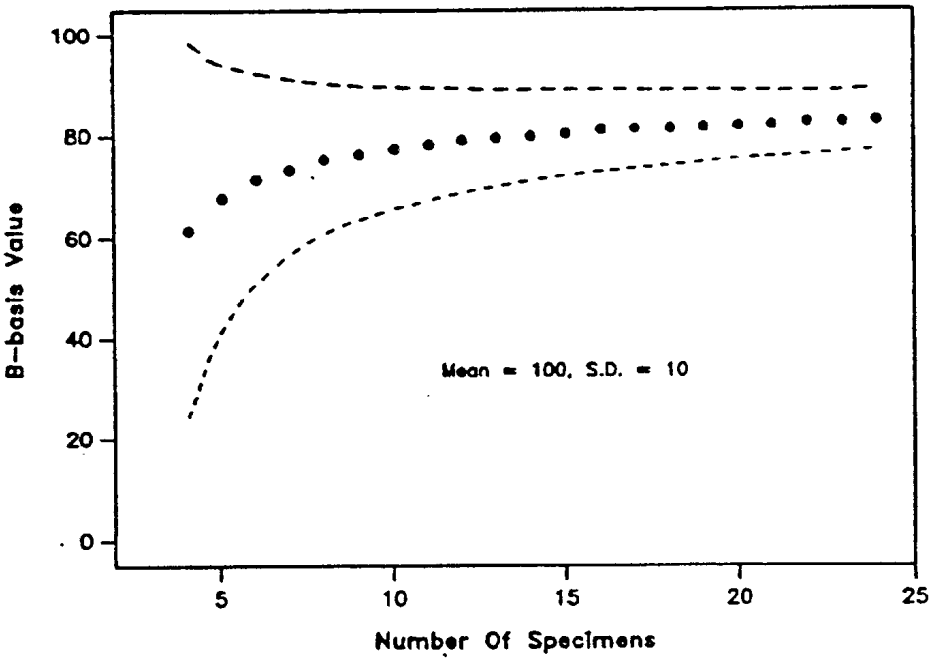

FIGURE 8.2.5.2(a) *Normal B-basis values with one-sigma limits.*

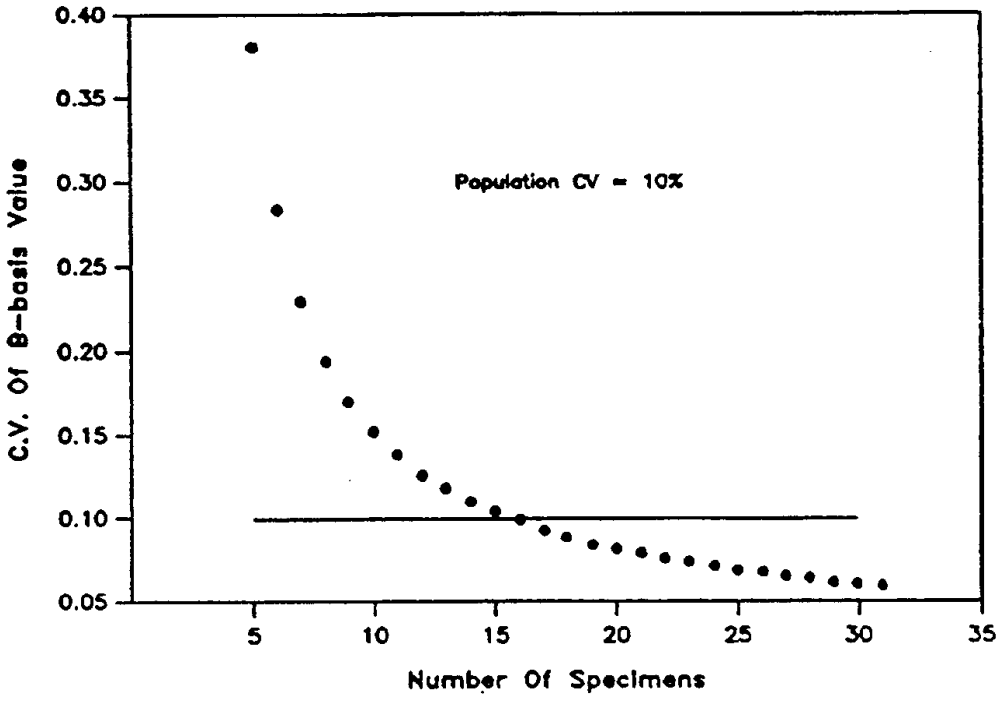

FIGURE 8.2.5.2(b) *C.V. of B-basis values: normal model.*

The coefficient of variation (CV) is the ratio of the standard deviation to the mean. It is, therefore, easy to obtain the CV as a function of sample size from the information in Figure 8.2.5.2(a). Figure 8.2.5.2(b) displays these CV values, with a horizontal line at 10% provided for reference.

Since an A-basis value is a 95% lower confidence limit on the *first* population percentile, while a B-basis value is a 95% lower confidence limit on the *tenth* percentile, it is obvious that, for a given amount of reproducibility in the basis values, substantially more data is required for A-basis than for B-basis. If one assumes that the measurements are a sample from a normal distribution, then it is reasonable to decide on the number of specimens as for B-basis and then multiply the resulting n by three to get an A-basis sample size. This is based on the assumption that the population coefficient of variation is less than 15%.

8.2.5.3 Basis values using the ANOVA method

When the data come from several batches, and the between-batch variability is substantial, the flowchart (Figure 8.3.1) might indicate that the ANOVA method of Section 8.3.5.2 should be used. To decide how many specimens are required when the data are to come from several batches, begin by acting as if the data were from a single batch, and selecting a sample size, say n, based on the discussion of the previous subsection. If J is the number of specimens per batch (assumed equal for all batches) and ρ is the correlation between any two measurements taken on specimens from the same batch, then the number of specimens required for comparable reproducibility in the multi-batch case is approximately

$$\tilde{n} = \left[J\rho + 1 - \rho\right] n \qquad 8.2.5.3$$

If $\rho = 0$, there is no between-batch variability; hence $\tilde{n} = n$. At the other extreme, if $\rho = 1$, there is perfect correlation within each batch (that is, each batch consists of J copies of a single value), and $\tilde{n} = Jn$, one needs n batches to have the same degree of reproducibility as n specimens in the uncorrelated ($\rho = 0$) case. In practice, ρ is unknown. For sample size guidelines, letting $\rho = 1/2$ in Equation 8.2.5.3 is adequate for most applications. This suggests that $(n(J+1)/(2J))$ batches of size J are necessary for the same degree of reproducibility as a single sample of size n. It is usually preferable to divide a fixed number of specimens among as many batches as is possible. However, testing a new batch is much more expensive than testing several more specimens within a single batch. It is sometimes the case that the variability between two panels from the same batch, processed and tested separately, is comparable to the variability between two panels from different batches. When this is the case, it is reasonable to substitute multiple panels within a batch for multiple batches.

Suppose that an A-basis ANOVA value is desired which has the same degree of reproducibility as a B-basis value would have for a single sample of size $n = 5$. First, make the adjustment to an A-basis sample size: $n_A = 3 \bullet 5 = 15$, as described in Section 8.2.5.2. Next, assuming moderate between-batch variability and a batch size of (say) $J = 3$, calculate that $n_A[(J+1)/(2J)] = 10$ batches are required for the desired degree of reproducibility, for a total of 30 specimens.

8.3 CALCULATION OF STATISTICALLY-BASED MATERIAL PROPERTIES

Section 8.3 contains computational methods for obtaining B- and A-basis values from composite material test data.

8.3.1 Guide to computational procedures

The procedure used to determine a basis value depends on the characteristics of the data. The step-by-step procedure for selecting the appropriate computational method is illustrated by the flowchart in Figure 8.3.1. Details for the specific computational methods are provided in later sections.

Two approaches are used, with the selection dependent on whether the data are structured or not. The k-sample Anderson-Darling test in Section 8.3.2 examines the differences among groups of data to determine if they are significant or negligible, which also determines whether the data should be treated as structured or unstructured. The difference between structured and unstructured data is considered in Section 8.3.2. Briefly, data sets which either cannot be grouped, or for which there are negligible differences among such groups, are called *unstructured*. Otherwise, the data are said to be *structured*. All data should be examined for outliers, using the test in Section 8.3.3. From this point, different approaches are used for analysis depending on whether the data are unstructured or structured.

The approach for unstructured data is described first. If unstructured data were grouped and the differences among the groups found to be negligible, the groups are combined. The test for outliers should be performed again on the combined data. Tests for goodness-of-fit (Section 8.3.4.1) are performed for the Weibull, normal, and lognormal distributions in succession. If the observed significance level (OSL) for the Weibull distribution is greater than 0.05, indicating an adequate fit for the data to the Weibull distribution, then a Weibull basis value is recommended (Section 8.3.4.2). If the OSL for the Weibull distribution is less than 0.05 and the OSL for the normal distribution is greater than 0.05, then the normal basis value should be used (Section 8.3.4.3). If the OSL's from both the Weibull and normal goodness-of-fit tests are less than 0.05, and the OSL for the lognormal distribution is greater than 0.05, then a lognormal basis value is recommended (Section 8.3.4.4). If none of the three OSL's are greater than 0.05, then the nonparametric basis value procedures are recommended (Section 8.3.4.5). Section 8.3.4 provides the rationale for the order of the distribution selection. An alternative approach is to use the basis values corresponding to the best-fitting model. Exploratory data analysis (EDA) techniques, described in Section 8.3.6, can provide graphical illustrations of the data distribution in support of the goodness-of-fit tests.

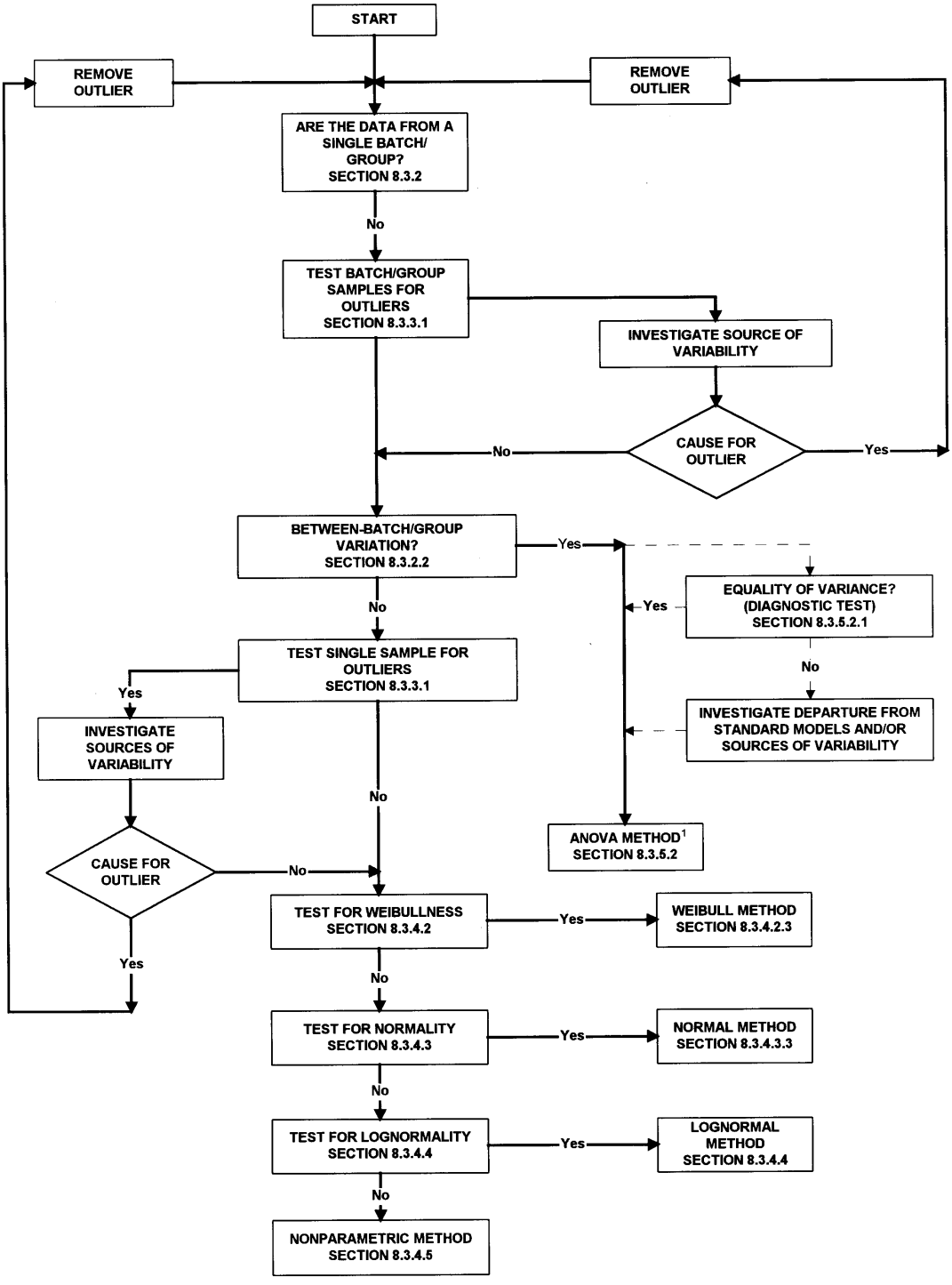

FIGURE 8.3.1 *Flowchart illustrating computational procedures for B-basis material property values.*[1]

[1]The ANOVA method applies to the simple multiple-batch case. Other scenarios may be addressed by linear regression (RECIPE). The acceptance of data analyzed by linear regression for inclusion in MIL-HDBK-17 is under consideration.

The approach for structured data divides the grouping of data according to fixed and random effects. A fixed effect is where an independent variable is set or measured. An example of a fixed effect is data obtained, by design or by chance, at different measured test temperatures. A random effect is the result of variability where the cause in unknown or unmeasurable. An example of a random effect is data obtained from several batches with significant batch-to-batch variability. (See definitions in Section 8.1.4.) Data sets with random effects, fixed effects or combinations of fixed and random effects require a basic understanding of linear models for regression and the analysis of variance. While a detailed exposition of this topic is beyond the scope of the handbook, an introduction with elementary references is provided in Section 8.3.5.1. The simplest case of structured data is where the only grouping is by a random effect, such as batches or panels. For this situation, basis values should be calculated by the analysis of variance (ANOVA) procedure (Section 8.3.5.2). Before basis values are calculated, a diagnostic test for equality of variances should be applied. Note that there is a special approach for determining basis values when the data consist of only two groups.

The case of one fixed effect and no random effects is linear regression (Section 8.3.5.3). For cases with no or one random effect and an arbitrary number of fixed effects, basis values from regression models can be calculated using the computer program RECIPE. A method for pooling small data sets from multiple environmental conditions is described in Section 8.3.5.4.

8.3.2 Subpopulation compatibility - structured or unstructured

Expected and unexpected behavior should be considered in determining whether there are natural or logical groupings of the data. Data for which natural groupings exist, or for which responses of interest could vary systematically with respect to known factors, are *structured* data. For example measurements made from each of several batches could reasonably be grouped according to batch, and measurements made at various known temperatures could be modeled using linear regression (Section 8.3.5); hence both can be regarded as structured data. In many ways, it is easier to analyze data which are *unstructured*; hence, it is often desirable to be able to show that a natural grouping of data has no significant effect. Data are considered unstructured if all relevant information is contained in the response measurements themselves. This could be because these measurements are all that is known, or else because one is able to ignore potential structure in the data. For example, data measurements that have been grouped by batch and demonstrated to have negligible batch-to-batch variability may be considered unstructured. An unstructured data set is a *simple random sample.*

The following section describes the k-sample Anderson-Darling test for showing the subpopulations are *compatible*, that is, the natural groupings have no significant effect. Compatible groups may be treated as part of the same population. Thus, a structured data set, with a natural grouping identified, can become an unstructured data set by showing that the natural grouping has no significant effect using the k-sample Anderson-Darling test.

For composite materials, it is recommended that batches (and panels where possible) be treated as natural groupings and tested for compatibility. Other groupings may result from expected behavior. Ply count might have a significant effect on ±45 shear test; thus specimens with different ply counts naturally fall into groupings for this test.The decision regarding grouping the data may also be affected by the purpose of the test program. As an example, consider the influence of strain rate on material properties. A test program may be designed to evaluate the effects of strain rate on a given property. That program would obtain data at selected and controlled values of strain rate. These would provide the natural grouping for the data. A subpopulation compatibility test could be used to determine if there was a significant effect; or a structured data approach, such as linear regression, could be used.

8.3.2.1 Notation for grouped data

For structured data, each data value belongs to a particular group, and there will generally be more than one value within each group. Therefore, double subscripts will be used to identify the observations. Let the data be denoted by x_{ij} for $i = 1, ..., k$ and $j = 1, ..., n_i$, where i is the group and j is the observation within that group. There are n_i data values in the ith of k groups. Then the total number of observations is

$n = n_1 + n_2 + ... + n_k$. The distinct values in the combined data set, ordered from smallest to largest, is denoted $z_{(1)}, z_{(2)}, ..., z_{(L)}$, where L will be less than n if there are tied observations.

8.3.2.2 The k-sample Anderson-Darling test

The k-sample Anderson-Darling test is a nonparametric statistical procedure that tests the hypothesis that the populations from which two or more groups of data were drawn are identical. The test requires that each group be an independent random sample from a population. For more information on this procedure, see Reference 8.3.2.2.

The k-sample Anderson-Darling statistic is

$$ADK = \frac{n-1}{n^2(k-1)} \sum_{i=1}^{k} \left[\frac{1}{n_i} \sum_{j=1}^{L} h_j \frac{(nF_{ij} - n_i H_j)^2}{H_j(n - H_j) - nh_j/4} \right] \qquad 8.3.2.2(a)$$

where

h_j = the number of values in the combined samples equal to $z_{(j)}$

H_j = the number of values in the combined samples less than $z_{(j)}$ plus one half the number of values in the combined samples equal to $z_{(j)}$, and

F_{ij} = the number of values in the ith group which are less than $z_{(j)}$ plus one half the number of values in this group which are equal to $z_{(j)}$.

Under the hypothesis of no difference in the populations, the mean and variance of ADK are approximately 1 and

$$\sigma_n^2 = Var(ADK) = \frac{an^3 + bn^2 + cn + d}{(n-1)(n-2)(n-3)(k-1)^2} \qquad 8.3.2.2(b)$$

with

$$a = (4g-6)(k-1) + (10-6g)S \qquad 8.3.2.2(c)$$

$$b = (2g-4)k^2 + 8Tk + (2g-14T-4)S - 8T + 4g - 6 \qquad 8.3.2.2(d)$$

$$c = (6T+2g-2)k^2 + (4T-4g+6)k + (2T-6)S + 4T \qquad 8.3.2.2(e)$$

$$d = (2T+6)k^2 - 4Tk \qquad 8.3.2.2(f)$$

where

$$S = \sum_{i=1}^{k} \frac{1}{n_i} \qquad 8.3.2.2(g)$$

$$T = \sum_{i=1}^{n-1} \frac{1}{i} \qquad 8.3.2.2(h)$$

and

$$g = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \frac{1}{(n-i)\,j} \qquad 8.3.2.2(i)$$

If the critical value

$$ADC = 1 + \sigma_n \left[1.645 + \frac{0.678}{\sqrt{k-1}} - \frac{0.362}{k-1} \right] \qquad 8.3.2.2(j)$$

is less than the test statistic in Equation 8.3.2.2(a), then one can conclude (with a five percent risk of being in error) that the groups were drawn from different populations. Otherwise, the hypothesis that the groups were selected from identical populations is not rejected, and the data may be considered unstructured with respect to the random or fixed effect in question. Table 8.5.6 contains the critical values (Equation 8.3.2.2(j)) for the case of where all of the n_i are equal. The example problem in Section 8.3.7.1, Step 2 demonstrates this procedure.

8.3.3 Detecting outliers

An *outlier* is an observation that is much lower or much higher than most other observations in a data set. Often outliers are erroneous values, perhaps due to clerical error, to the incorrect setting of environmental conditions during testing, or to a defective test specimen. Data should routinely be screened for outliers, since these values can have a substantial influence on the statistical analysis. In addition to the quantitative screening for outliers (Section 8.3.3.1), the data should also be examined visually, since no statistical procedure can be completely reliable for outlier detection.

The Maximum Normed Residual (MNR) method is used for quantitative screening for outliers. This test screens for outliers in an unstructured data set. If the data can be grouped naturally into subgroups (due to batches, manufacturers, temperatures, and so on), then one should form the smallest subgroups possible and screen each of these separately. Data from compatible subgroups, based on the previous section, should be combined and the screening test performed on the larger group. Of course, data should only be pooled when it makes sense to do so. For example, batches of data for the same property and environmental condition can be combined, but tension and compression data should never be pooled.

All values identified as outliers should be investigated. Those values for which a cause can be determined should be corrected if possible, and otherwise discarded. When error in data collection or recording are discovered, all data should be examined to determine whether similar errors occurred; these values should also be corrected or discarded. If no cause can be found for an outlier, it should be retained in the data set. If an outlier is clearly erroneous, it can be removed after careful consideration provided that the subjective decision to remove a value is documented as part of the data analysis. If any observations are corrected or discarded, both the statistical outlier test and the visual inspection should be repeated.

8.3.3.1 The maximum normed residual

The maximum normed residual (MNR) test is a screening procedure for identifying an outlier in an unstructured set of data. A value is declared to be an outlier by this method if it has an absolute deviation from the sample mean which, when compared to the sample standard deviation, is too large to be due to chance. This procedure assumes that observations which are not outliers can be regarded as a random sample from a normal population. The MNR method can only detect one outlier at a time, hence the significance level pertains to a single decision. Additional information on this procedure can be found in References 8.3.3.1(a) and (b).

Let $x_1, x_2, \ldots x_n$ denote the data values in the sample of size n, and let $\bar{x}$ and s be the sample mean and sample deviation, defined in Section 8.1.4. The MNR statistic is the maximum absolute deviation, from the sample mean, divided by the sample standard deviation:

$$MNR = \max_i \frac{|x_i - \bar{x}|}{s}, \qquad i = 1,2,\ldots,n \tag{8.3.3.1(a)}$$

The value of Equation 8.3.3.1(a) is compared to the critical value for the sample size n from Table 8.5.7. These critical values are computed from the following formula

$$C = \frac{n-1}{\sqrt{n}} \sqrt{\frac{t^2}{n-2+t^2}} \tag{8.3.3.1(b)}$$

where t is the $[1 - \alpha/(2n)]$ quantile of the t-distribution with $n - 2$ degrees of freedom and α is the significance level. The recommended significance level for this test is $\alpha = 0.05$.

If MNR is smaller than the critical value, then no outliers are detected in the sample; otherwise the data value associated with the largest value of $|x_i - \bar{x}|$ is declared to be a outlier.

If an outlier is detected, this value is omitted form the calculations and the MNR procedure is applied again. This process is repeated until no outliers are detected. Note that the jth time that a sample is screened for an outlier, the mean, standard deviation, and critical value are computed using a sample size of $n - j - 1$. It should be noted that for small samples, for example a batch containing five or six data, this procedure may identify most of the data as outliers, particularly if two or more of the values are identical. The example problem in Section 8.3.7.1, Step 1 demonstrates this procedure.

8.3.4 Basis values for unstructured data

The method employed in calculating basis values for unstructured data depends on the distributional form which is assumed. Section 8.3.4 contains procedures for performing a goodness-of-fit test for the Weibull, normal, and lognormal distributions.

As shown in Figure 8.3.1, it is recommended that the Weibull model be used if it adequately fits the data, even if other models apparently fit the data better. This preference for the Weibull distribution is based on two factors:

1. Theory suggests that the Weibull distribution is appropriate for the strength distribution of brittle materials such as composite fibers (see, for example, Reference 8.3.4(a).

2. The "Chain-of-Bundles" model for the strength of two- and three-dimensional unidirectional composites suggests that the Weibull model is appropriate for the strength distribution of such composites. This result is stated in References 8.3.4(b) and (c).

If the Weibull model cannot be shown to adequately fit the data, then the normal and lognormal tests are performed in succession. If none of these three population models can be demonstrated to adequately fit the data, then nonparametric procedures should be used to compute basis values.

The exploratory data analysis (EDA) techniques of Section 8.3.6 should also be used to graphically display the data, highlighting potential difficulties and providing graphical evidence of goodness-of-fit to support the quantitative conclusions of the tests in this section.

8.3.4.1 Goodness-of-fit tests

Each distribution is considered using the Anderson-Darling test statistic which is sensitive to discrepancies in the tail regions. The Anderson-Darling test compares the cumulative distribution function for the distribution of interest with the cumulative distribution function of the data. The data are first converted to a common representation for the distribution under consideration. For example, for a normal distribution, the data are normalized to a mean of 0 and a standard deviation of 1. An observed significance level (OSL) based on the Anderson-Darling test statistic is computed for each test. The OSL measures the probability of observing an Anderson-Darling test statistics as least as extreme as the value calculated if the distribution under consideration is in fact the underlying distribution of the data. The OSL is the probability of obtaining a value of the test statistic at least as large as that obtained if the hypothesis that the data are actually from the distribution being tested is true. If the OSL is less than or equal to 0.05, the hypothesis is rejected (with at most a five percent risk of being in error) and one proceeds as if the data are not from the distribution being tested.

In what follows, unless otherwise noted, the sample size is denoted by n, the sample observations by $x_1, ..., x_n$, and the sample observations ordered from least to greatest by $x_{(1)}, ..., x_{(n)}$.

8.3.4.2 Two-parameter Weibull distribution

In order to compute a basis value for a two-parameter Weibull population, it is first necessary to obtain estimates of the population shape and scale parameters. Section 8.3.4.2.1 contains a step-by-step procedure for calculating maximum likelihood estimates of these parameters. Calculations specific to the goodness-of-fit test for the Weibull distribution are provided in Section 8.3.4.2.2. The computational procedure for calculating basis values using these estimates is outlined in Section 8.3.4.2.3. The example problem in Section 8.3.7.1 demonstrates these procedures. For further information on these procedures, see Reference 8.3.4.2.

8.3.4.2.1 Estimating the shape and scale parameters of a Weibull distribution

The section describes the *maximum likelihood* method for estimating the parameters of the two-parameter Weibull distribution. The maximum-likelihood estimates of the shape and scale parameters are denoted $\hat{\beta}$ and $\hat{\alpha}$. The estimates are the solution to the pair of equations:

$$\frac{n}{\hat{\beta}} - n\ln\hat{\alpha} + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\left[\frac{x_i}{\hat{\alpha}}\right]^{\hat{\beta}}(\ln x_i - \ln\hat{\alpha}) = 0 \qquad 8.3.4.2.1(b)$$

and

$$\hat{\alpha}\hat{\beta}n - \frac{\hat{\beta}}{\hat{\alpha}^{\hat{\beta}-1}}\sum_{i=1}^{n}x_i^{\hat{\beta}} = 0 \qquad 8.3.4.2.1(a)$$

Equation 8.3.4.2.1(a) can be rewritten as

$$\hat{\alpha} = \left(\frac{\sum_{i=1}^{n}x_i^{\hat{\beta}}}{n}\right)^{\frac{1}{\hat{\beta}}} \qquad 8.3.4.2.1(c)$$

By substituting Equation 8.3.4.2.1(c) into Equation 8.3.4.2.1(b), the following equation is obtained.

$$\frac{n}{\hat{\beta}} + \sum_{i=1}^{n}\ln x_i - \frac{n}{\sum_{i=1}^{n}x_i^{\hat{\beta}}}\sum_{i=1}^{n}x_i^{\hat{\beta}}\ln x_i = 0 \qquad 8.3.4.2.1(d)$$

Equation 8.3.4.2.1(d) can be solved numerically for $\hat{\beta}$, which can then be substituted into Equation 8.3.4.2.1(c) to obtain $\hat{\alpha}$.

Figure 8.3.4.2.1 shows FORTRAN source code for three routines which compute the estimates of $\hat{\alpha}$ and $\hat{\beta}$ by the method described above. WBLEST is a subroutine which returns the estimates of the parameters, $\hat{\beta}$ and $\hat{\alpha}$. FNALPH is a function which calculates the estimate of the scale parameter, $\hat{\alpha}$. GFUNCT is a function which evaluates Equation 8.3.4.2.1(d). Arguments to WBLEST are

X	=	a vector of length NOBS containing the data (input),
NOBS	=	the number of data values, n (input),
BETA	=	estimate of the shape parameter (output),
ALPHA	=	estimate of the scale parameter (output).

The algorithm by which the FORTRAN code computes the estimates is described in the following paragraph.

```
C-----------------------------------------------------------------------------------------------
                        SUBROUTINE WBLEST(X,NOBS,ALPHA,BETA)
C
C               COMPUTE MLES FOR SHAPE PARAMETER (BETA) AND SCALE PARAMETER
C               (ALPHA) BY SOLVING THE EQUATION G(BETA) = 0, WHERE G IS
C               A MONOTONICALLY INCREASING FUNCTION OF BETA.
C               THE INITIAL ESTIMATE IS: RI=(1.28)/(STD. DEV. OF LOG(X)'S)
C               AND THE TOLERANCE IS  :  2*RI/(10**6).
C
                        DIMENSION X(NOBS)
C
                        RN = FLOAT(NOBS)
                        SUMY = 0.0
                        SUMYSQ = O.O
                        DO 2 I = 1, NOBS
                                Y = ALOG(X(I))
                                SUMY = SUMY + Y
                                SUMYSQ = SUMYSQ + (Y**2)
                2 CONTINUE
                        YSTD = SQRT((SUMYSQ - (SUMY**2)/RN)/(RN - 1.0))
                        XGM = EXP(SUMY/RN)
                        RI = 1.28/YSTD
                        TOL = 2.0*.000001*RI
                        BETAM = Rl
                        GFM = GFUNCT(X,NOBS,BETAM,XGM)
C
C               IF G(BETAM) .GE. 0, DIVIDE THE INITIAL ESTIMATE BY 2 UNTIL
C               THE ROOT IS BRACKETED BY BETAL ND BETAH.
C
                        IF(GFM .GE. 0.0) THEN
                                DO 3 J = 1, 20
                                        BETAH = BETAM
                                        BETAM = BETAM/2.0
                                        GFM = GFUNCT(X,NOBS,BETAM,XGM)
                                        IF (GFM .LE. O.0) GO TO 4
                3                       CONTINUE
                                        STOP 'GFM NEVER LE 0'
                4                       CONTINUE
                                        BETAL - BETAM
                        ENDIF
C
C                       IF G(BETAM) .LT. 0, MULTIPLY THE INITIAL ESTIMATE BY 2
C                       UNTIL THE ROOT IS BRACKETED BY BETAL AND BETAH
C
                        IF(GFM .LT. 0.0) THEN
                                DO 7 J = 1, 20
                                        BETAL=BETAM
                                        BETAM=BETAM*2.O
                                        GFM=GFUNCT(X,NOBS,BETAM,XGM)
                                        IF(GFM .GE. 0.0) GO TO 8
                7                       CONTINUE
                                        STOP 'GFM NEVER GE 0'
                8                       CONTINUE
                                        BETAH = BETAM
                        ENDIF
C
C               SOLVE THE EQUATION G(BETA) = O FOR BETA BY BISECTING THE
C               INTERVAL (BETAL,BETAH) UNTIL THE TOLERANCE IS MET
C
                10 CONTINUE
                        BETAM = (BETAL + BETAH) / 2.0
                        GFM = GFUNCT(X,NOBS,BETAM,XGM)
                        IF(GFM .GE. 0.0) THEN
                        BETAH = BETAM
                        ENDIF
                        IF(GFM .LT. 0.0) THEN
                        BETAL = BETAM
                        ENDIF
                        IF((BETAH - BETAL) .GT. TOL) GO TO 10
```

FIGURE 8.3.4.2.1 *FORTRAN routines for calculating two-parameter Weibull shape and shale parameter, estimates, continued on next page.*

```
C
                        BETA = (BETAL + BETAH) / 2.0
                        ALPHA = FNALPH(X,NOBS,BETA,XGM)
                        RETURN
                        END
C-------------------------------------------------------------------------------
                FUNCTION FNALPH(X,NOBS,BETA,XGM)
C
C       COMPUTE MLE FOR TWO-PARAMETER WEIBULL SCALE PARAMETER (ALPHA)
C       XGM IS THE GEOMETRIC MEAN OF THE X'S
C
                DIMENSION X(NOBS)
                RN = FLOAT(NOBS)
C
                SUMZ = 0.0
                DO 20 I = 1, NOBS
                        SUMZ = SUMZ I (X(I)/XGM)**BETA
        20 CONTINUE
C
                FNALPH = XGM*(SUMZ/RN)**(1./BETA)
C
                RETURN
                END
C-------------------------------------------------------------------------------



C-------------------------------------------------------------------------------
                Function GFUNCT(X,NOBS,BETA,XGM)
C
C       COMPUTE G FUNCTION USED IN ESTIMATING THE TWO-PARAMETER WEIBULL
C       SHAPE PARAMETER (BETA).
C       XGC IS THE GEOMETRIC MEAN OF THE X'S USED IN ESTIMATING ALPHA.
C
                DIMENSION X(NOBS)
                RN = FLOAT(NOBS)
C
                ALPHA = FNALPH(X,NOBS,BETA,XGM)
                SUMYZ = 0.0
                DO 10 I = 1, NOBS
                        SUMYZ = SUMYZ + ALOG(X(I))*((X(I)/ALPHA)**BETA - 1.)
        10 CONTINUE
C
                GFUNCT = (SUMYZ/RN) - 1.0/BETA
C
                RETURN
                END
C-------------------------------------------------------------------------------
```

FIGURE 8.3.4.2.1 *FORTRAN routines for calculating two-parameter Weibull shape and shale parameter, estimates,* concluded.

Equation 8.3.4.2.1(d) is a monotonically decreasing continuous function of $\hat{\beta}$. Designate the left-hand side of Equation 8.3.4.2.1(d) divided by n as $G(\hat{\beta})$ and obtain a solution for $\hat{\beta}$ by the following iterative procedure. Let S_y denote the standard deviation of $y_1,..., y_n$ where $y_1 = \ln(x_i)$ for $i = 1,...,n$. Calculate $I = 1.28/S_y$ as an initial guess at the solution and calculate $G(I)$. If $G(I) > 0$, then find the smallest positive integer k such that $G(1/2^k) < 0$ and let $L = I/2^2$ and $H = I/2^{k-1}$. If $G(I) < 0$, then find the smallest positive integer k such that $G(2^k I) > 0$ and let $L = 2^{k-1} I$ and $H = 2^k I$. In either case, the interval (L,H) contains the solution to $G(\hat{\beta}) = 0$. Now calculate $G(M)$ where $M = (L + H)/2$. If $G(M) = 0$, then the solution is $\hat{\beta} = M$. If $G(M) > 0$, then let $H = M$. If $G(M) < 0$ then let $L = M$. The new interval (L,H) still contains the solution to $G(\hat{\beta}) = 0$ but is only half as long as the old interval. Calculate a new M-value and begin the process of interval halving again. The process is repeated until $H\text{-}L < 2I/10^6$. The solution to $G(\hat{\beta}) = 0$ is then taken to be $M = (L + H)/2$. The solution is in error by at most $I/10^6$.

8.3.4.2.2 Goodness-of-fit test for the two-parameter Weibull distribution

The two-parameter Weibull distribution is considered by comparing the cumulative Weibull distribution function (Section 8.1.4) that best fits the data with the cumulative distribution function of the data. Using the shape and scale parameter estimates from Section 8.3.4.2.1, let

$$z_{(i)} = \left[x_{(i)}/\hat{\alpha}\right]^{\hat{\beta}}, \quad \text{for } i = 1,\ldots,n \qquad 8.3.4.2.2(a)$$

The Anderson-Darling test statistic is

$$AD = \sum_{i=1}^{n} \frac{1-2i}{n}\left[\ell n\left[1-\exp(-z_{(i)})\right] - z_{(n+1-i)}\right] - n \qquad 8.3.4.2.2(b)$$

and the observed significance level is

$$OSL = 1/\left\{1+\exp[-0.10+1.24\ln(AD^{*})+4.48AD^{*}]\right\} \qquad 8.3.4.2.2(c)$$

where

$$AD^{*} = \left(1+\frac{0.2}{\sqrt{n}}\right)AD \qquad 8.3.4.2.2(d)$$

This OSL measures the probability of observing an Anderson-Darling statistic at least as extreme as the value calculated if in fact the data are a sample from a two-parameter Weibull distribution. If OSL $\leq$ 0.05, one may conclude (at a five percent risk of being in error) that the population does not have a two-parameter Weibull distribution. Otherwise, the hypothesis that the population has a two-parameter Weibull distribution is not rejected. For further information on this procedure, see Reference 8.3.4.2.

8.3.4.2.3 Basis values for the two-parameter Weibull distribution

If the unstructured data set is from a population with a two-parameter Weibull distribution, the B-basis value is

$$B = \hat{q}\exp\left\{\frac{-V}{\hat{\beta}\sqrt{n}}\right\} \qquad 8.3.4.2.3(a)$$

where

$$\hat{q} = \hat{\alpha}(0.10536)^{1/\hat{\beta}} \qquad 8.3.4.2.3(b)$$

and V is the value in Table 8.5.8 corresponding to a sample of size n. A numerical approximation to the V values is given in Equation 8.5.8(h).

To calculate the A-basis value, use the appropriate V value from Table 8.5.9 substituting 8.3.4.2.3(c) for 8.3.4.2.3(b).

$$\hat{q} = \hat{\alpha}(0.01005)^{1/\beta} \qquad 8.3.4.2.3(c)$$

8.3.4.3 Normal distribution

In order to compute a basis value for a normally distributed population, it is necessary to obtain estimates of the population mean and standard deviation. Section 8.3.4.3.1 gives the equations for calculating these parameters. Section 8.3.4.3.2 provides the procedure for goodness-of-fit for the normal distri-

bution, and Section 8.3.4.3.3 gives the procedure for calculating basis values. The example problem in Section 8.3.7.2 demonstrates these procedures.

8.3.4.3.1 Estimating the mean and standard deviation parameters for the normal distribution

The population mean and standard deviation are estimated using the sample mean $\bar{x}$ and sample standard deviation s.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

8.3.4.3.2 Goodness-of-fit test for the normal distribution

The normal distribution is considered by comparing the cumulative normal distribution function (Section 8.1.4) that best fits the data with the cumulative distribution function of the data. Let

$$z_{(i)} = \frac{\bar{x}_{(i)} - \bar{x}}{s}, \quad \text{for } i = 1,\ldots,n \qquad 8.3.4.3.2(a)$$

where $x_{(i)}$ is the ith smallest sample observation, $\bar{x}$ is the sample average, and s is the sample standard deviation.

The Anderson-Darling test statistic is

$$AD = \sum_{i=1}^{n} \frac{1-2i}{n} \left\{ \ln\left[F_0(z_{(i)})\right] + \ln\left[1 - f_0(z_{(n+1-i)})\right] \right\} - n \qquad 8.3.4.3.2(b)$$

where F_0 is the standard normal distribution function (Equation 8.1.4(e)). The observed significance level is

$$OSL = 1/\left\{1 + \exp[-0.48 + 0.78\ln(AD^*) + 4.58\,AD^*]\right\} \qquad 8.3.4.3.2(c)$$

where

$$AD^* = \left(1 + \frac{0.2}{\sqrt{n}}\right) AD \qquad 8.3.4.3.2(d)$$

This OSL measures the probability of observing an Anderson-Darling statistic at least as extreme as the value calculated if in fact the data are a sample from a normal distribution. If $OSL \leq 0.05$, one may conclude (at a five percent risk of being in error) that the population is not normally distributed. Otherwise, the hypothesis that the population is normally distributed is not rejected. For further information on this procedure, see Reference 8.3.4.2.

8.3.4.3.3 Basis values for the normal distribution

If the unstructured data set is from a population with a normal distribution, the B-basis value is

$$B = \bar{x} - k_B s \qquad 8.3.4.3.3(a)$$

where k_B is the appropriate one-sided tolerance-limit factor from Table 8.5.10. A numerical approximation to the k_B values is given in Equation 8.5.10.

To calculate the A-basis value, replace k_B with the appropriate value of k_A from Table 8.5.11 or the numerical approximation in Equation 8.5.11.

8.3.4.4 Lognormal distribution

The lognormal distribution is a positively skewed distribution that is simply related to the normal distribution. If something is lognormally distributed, then its logarithm is normally distributed. The natural (base e) logarithm is used in MIL-HDBK-17. See Section 8.1.4 for the definition of the lognormal distribution. The example problem in Section 8.3.7.3 demonstrates the application of the procedures in Section 8.3.4.3 for a lognormal distribution.

In order to fit test the goodness-of fit of the lognormal distribution, take the logarithm of the data and perform the Anderson-Darling test for normality from Section 8.3.4.3. Using the natural logarithm, let

$$z_{(i)} = \frac{\ln(\overline{x}_{(i)}) - \overline{x}_L}{s_L}, \quad \text{for } i = 1,\ldots,n \qquad 8.3.4.4(a)$$

where $x_{(i)}$ is the ith smallest sample observation, $\overline{x}_L$ and s_L are the mean and standard deviation of the $\ln(x_i)$ values.

The Anderson-Darling statistics is computed using Equation 8.3.4.3(b) and the observed significance level (OSL) is computed using Equation 8.3.4.3(c). This OSL measures the probability of observing an Anderson-Darling statistic at least as extreme as the value calculated if in fact the data are a sample from a lognormal distribution. If $OSL \le 0.05$, one may conclude (at a five percent risk of being in error) that the population is not lognormally distributed. Otherwise, the hypothesis that the population is lognormally distributed is not rejected. For further information on this procedure, see Reference 8.3.4.2.

The following procedure should be used to calculate basis values for unstructured data that is assumed to be a sample from a lognormal population. The equations presented in Section 8.3.4.3 are used to calculate the basis values. However, the calculations are performed using the logarithms of the data rather than the original observations. The computed B-basis value must then be transformed back to the original units by applying the inverse of the log transformation which was used.

8.3.4.5 Nonparametric basis values

These procedures should be used to compute basis values for unstructured data when one is unwilling to assume a particular population model, usually because the Weibull, normal, and lognormal models all provide inadequate fits to the data. One of two methods should be used, depending on the sample size.

8.3.4.5.1 Nonparametric basis values for large samples

To calculate a B-basis value for $n > 28$, determine the value r corresponding to the sample size n from Table 8.5.12. For sample sizes between tabulated values, select the r value associated with the largest tabulated sample size that is smaller than the actual n. The B-basis value is the rth lowest observation in the data set. For example, in a sample of size $n = 30$, the lowest ($r = 1$) observation is the B-basis value. A numerical approximation to the tabulated r values as a function of n is given in Section 8.5.12. The example problem in Section 8.3.7.4 demonstrates this procedure. Further information on this procedure may be found in Reference 8.3.4.5.1.

For $n > 298$, an A-basis value can calculated using the sample procedure, with the r value selected from Table 8.5.13.

8.3.4.5.2 The Hanson-Koopmans method

The following procedure (References 8.3.4.5.2(a) and (b)) can be a useful method for obtaining a B-basis value for sample sizes not exceeding 28. This procedure requires the assumption that the observations are a random sample from a population for which the logarithm of the cumulative distribution function is concave, an assumption satisfied by a large class of probability distributions. There is substantial empirical evidence that suggests the composite strength data satisfies this assumption, consequently this procedure can usually be recommended for use when n is less than 29. However, in view of the required assumption, this is not an unconditional recommendation.

The Hanson-Koopmans B-basis value is

$$B = x_{(r)}\left[\frac{x_{(1)}}{x_{(r)}}\right]^{k} \qquad 8.3.4.5.2(a)$$

where $x_{(1)}$ is the smallest and $x_{(r)}$ is the rth largest data value. The values of r and k depend on n and are tabulated in Table 8.5.14. This equation for the B-basis value should not be employed if $x_{(r)} = x_{(1)}$. The example problem in Section 8.3.7.5 demonstrates these procedures.

The Hanson-Koopmans method can be used to calculate A-basis values for n less than 299. Find the value k_A corresponding to the sample size n in Table 8.5.15. Let $x_{(n)}$ and $x_{(1)}$ be the largest and smallest data values. The A-basis value is

$$A = x_{(n)}\left[\frac{x_{(1)}}{x_{(n)}}\right]^{k} \qquad 8.3.4.5.2(b)$$

8.3.5 Basis values for structured data

Where possible, it is advantageous to reduce structured data to unstructured cases as discussed in Section 8.3.2. The analysis of unstructured data is possible for distributions other than a normal probability model, which is assumed by the procedures for structured data. Where the data are structured and cannot be combined according to the test in Section 8.3.2.2, the procedures in this section should be used. These procedures for basis value calculations for structured data assume a normal probability model. All of these procedures can be considered in terms of regression analysis. A general description of regression analysis of linear statistical models is provided in Section 8.3.5.1. Included in this section is a discussion of checking the required assumptions. Analysis of variance is a special case with one random effect and no fixed effects (Section 8.3.5.2). A case of one fixed effect and no random effects is *simple linear regression* (Section 8.3.5.3).

8.3.5.1 Regression analysis of linear statistical models

The objective of a regression analysis for material basis properties is to obtain basis values for a particular response (for example, tensile strength) as functions of fixed factors (such as temperature, lay-up, and humidity). The measured response values will be called *observations*, and the values which describe the conditions corresponding to these observations will be referred to as *covariates*. For example, if a linear relationship is assumed between tensile strength and temperature, then the mean strength at a temperature T_i is, in the limit of infinitely many observations at this temperature, equal to $\theta_0 + \theta_1 T_i$. The constants θ_0 and θ_1 are generally unknown and must be estimated from the data. The values that these constants multiply, here 1 and T_i, are covariates; together they describe the fixed conditions under which the ith strength observation was made. Linear regression refers to a method for the analysis of relationships which are linear functions of *unknown parameters* (here θ_0 and θ_1). These relationships need not be linear in *covariates*. For example, a quadratic model in which squared temperature (T^2) is introduced as an additional covariate can be analyzed using linear regression.

Assume that the data being analyzed consist of n observations at ℓ fixed conditions (or levels), and number these conditions $1, 2, .., \ell$. In the example of linear regression on temperature, there are ℓ temperatures, and ℓ corresponding sets of covariates: $(1, T_1)$, $(1, T_2)$, ..., $(1, T_\ell)$. It is necessary to indicate which fixed condition corresponds to each observation (recall the subscript i in Equation 8.2.3, so let the fixed conditions for observation s be $p(s)$. Also each observation is made on a specimen from one of m batches. These batches are numbered $1, 2, ..., m$, and $q(s)$ indicates the batch corresponding the sth observation. Denote the observations by x_s, for $s = 1, 2, ..., n$, where the sth value comes from fixed level $p(s)$ and from batch $q(s)$.

Assume that the $\{x_s\}$ represents a sample from a normal distribution with mean

$$\mu_{p(s)} = \theta_1 z_{p(s),1} + \theta_2 z_{p(s),2} + \ldots + \theta_r z_{p(s),r} \qquad 8.3.5.1(a)$$

where the $\{z_{p(s),u}\}$, for $1 \le p(s) \le \ell$ and $u = 1, ..., r$, are known constants and the $\{\theta_u\}$ are parameters to be estimated. For example, if mean strength is assumed to vary linearly with temperature, and if condition $p(s) = 1$ corresponds to 75 degrees, then

$$\mu_1 = \theta_1 + \theta_2 75 \qquad 8.3.5.1(b)$$

so $r = 2$, $z_{11} = 1$, and $z_{12} = 75$. Recall that the covariates $z_{p(s),u}$ are not required to be linear. For example, a quadratic relationship between strength and temperature would have covariates, 1, T_i, and T_i^2.

The means $\mu_{p(s)}$ can never be observed, but must be estimated from limited data. Each data value consists of the sum of $\mu_{p(s)}$ plus a random quantity $b_{q(s)} + e_s$, where $b_{q(s)}$ takes on a different value for each batch $q(s)$ and e_s takes on a different value for each observation. The random variables $\{b_{q(s)}\}$ and $\{e_s\}$ are assumed to be random samples from normal populations with means zero and variances σ_b^2 and σ_e^2. The variance σ_b^2 is the *between-batch variance*, and σ_e^2 is referred to as the *within-batch* (or *error*) *variance*. (For a more elementary discussion of these ideas, see Section 8.2.3.)

The model for the data can now be written as

$$x_s = \mu_{p(s)} + b_{q(s)} + e_s = \theta_1 z_{p(s),1} + \ldots + \theta_r z_{p(s),r} + b_{q(s)} + e_s \qquad 8.3.5.1(c)$$

where the $\{z_{p(s),u}\}$ are known, the $\{\theta_u\}$ are unknown fixed quantities, and the $\{b_{q(s)}\}$ and $\{e_s\}$ are random quantities with unknown variances. Equation 8.3.5.1(c) is called a *regression model*. Every regression analysis begins with the choice of a regression model.

Special cases of Equation 8.3.5.1(c) are frequently useful. If the levels correspond to data groups, with the covariates indicating which group is associated with each observation, then the regression model is an analysis of variance (ANOVA) (Section 8.3.5.2). This case is most frequently used to calculate basis values when there is significant batch-to-batch variability. When there is one continuous covariate, the case is called the simple linear regression model (Section 8.3.5.3). Details of the analysis are provided for these special cases in the following sections. The analysis of the more general case is beyond the scope of this handbook; however, the RECIPE software is available to perform the analysis and examples are shown in Sections 8.3.7.6 - 8.3.7.9.

The power gained by using regression models for basis values is obtained at the expense of additional assumptions. A *residual* is defined to be the difference between a data point and its fitted value. Using the residuals, the following assumptions need to be checked:

1. Check the validity of the assumed curvilinear relation between property and predictor variables, for example, straight line, quadratic, or other assumed relationship;

2. Check homogeneity of variance (variances are assumed constant over the range of predictor variables);

3. Check normality of regression residuals; and

4. Check for independence of residuals.

Also, one should not extrapolate beyond the range of the predictor variables without good cause.

A detailed discussion of the validation of a regression model is beyond the scope of this handbook; however it is discussed at length in most elementary texts, including References 8.3.5.1(a) - (d). Some elaboration at this point, though, might be helpful.

If a model fits well, then the residuals should be as likely to be positive as negative, and so they will alternate in sign every few values. They will have no apparent structure, and ideally will look like 'white noise'. If a model fits poorly, then there will often be long sequences of residuals that have the same sign, and curved patterns will typically be apparent in the residuals.

If the variance is high for a group of residuals, then these values will appear more scattered, and conversely for the case of low variability. This behavior can often be detected by examining residual plots. For example, if a simple linear regression has been performed of strength of specimens as a function of temperature, and if strength becomes more variable as temperature increases, then a plot of residuals against temperature might have a ‘megaphone’ shape.

There are also graphical procedures for checking the normality assumption for residuals. These can be found in most textbooks. It is also possible to apply the Anderson-Darling goodness-of-fit test for normality (Section 8.3.4.3) to the ratio of residuals to the standard deviation about the regression line (that is, e_i/s_y). A justification for this procedure can be found in Reference 8.3.5.1(e).

It is difficult to test for independence graphically. One possibility is to plot the odd-numbered residuals against the even-numbered ones, and to see if a trend is apparent. Further discussion can be found in the referenced textbooks. One form of lack of independence, ‘clustering’ due to batch effects, is addressed in the example in Section 8.3.7.9.

8.3.5.2 Analysis of variance

This section contains a discussion of one-way analysis of variance (ANOVA) procedures. Although these models can be written using the general notation of Equation 8.3.5.1(c), for the present discussion it is simpler to write the one-way ANOVA model as

$$x_{ij} = \mu + b_i + e_{ij}, \quad \begin{matrix} i = 1,\ldots,k \\ j = 1,\ldots,n_i \end{matrix} \qquad 8.3.5.2$$

where n_i is the number of values in the ith group, and x_{ij} represents the jth observation in the ith of k groups. The overall average of the population is μ, b_i is the effect attributed to the ith group, and e_{ij} is a random error term representing unexplained sources of variation. The error terms, e_{ij}, are assumed to be independently distributed normal random variables with mean zero and variance σ_e^2 (the within-group variance). The b_i may be regarded as fixed (unknown) constants, or else they may be modeled as realizations of a random variable, which is generally taken to be normally distributed with mean zero and variance σ_b^2 (the between-group variance).

The case of fixed b_i is called a *fixed-effects* analysis of variance, and it is appropriate for situations where the group means $\mu + b_i$ are *not* to be considered as samples from a population of means. For example, the groups might consist of strength measurements on composite material specimens having different numbers of plies. If the groups differ substantially in mean strength, one might consider determining basis values for the various numbers of plies. However, it clearly makes no sense to consider hypothetical random populations of specimens with different number of plies, and to regard the k groups which appear in the data as a random sample from such a population.

If the group means $\mu + b_i$ are considered to be a sample from a population of group means, then the model is a *random-effects* analysis of variance. For example, the data might come from k batches. In this case, one would typically be concerned as much with future batches as with those represented in the data. If one intends to use future batches in fabrication, then it does not make much sense to calculate basis values for each of the k observed batches. Rather, one might choose to determine basis values based on the populations of a random observation from an as yet unobtained batch. In this way, protection against batch-to-batch variability can be incorporated into design values. Reference 8.3.5.2(a) provides more information on analysis of variance procedures. The effect of sample size on an analysis of this type should be considered in test program design (Section 2.2.5.2).

The following calculations address batch-to-batch variability. In other words, the only grouping is due to batches and the compatibility test (Section 8.3.2) indicate that unstructured data methods should not be used. The method is based on the one-way analysis of variance (ANOVA) random-effects model and the procedure is documented in Reference 8.3.5.2(b).

The assumptions are that

1. The data from each batch are normally distributed,

2. The within-batch variance is the same from batch to batch, and

3. The batch means are normally distributed.

There is no test available for the first assumption. Simulation studies, however, suggest that moderate violation of this assumption does not have an adverse effect on the properties of the ANOVA method. The second assumption should be validated by performing the test described in Section 8.3.5.2.1. This test is currently recommended as a diagnostic, since extensive simulation suggests that violation of this assumption will likely result in conservatism, although non-conservatism can arise in some situations. There is no useful test for the third assumption unless data from many (twenty or more) batches are available.

In this analysis, all batches are treated the same (for example, no distinction is made between batches from different fabricators). If the batches are not from a single fabricator, then the approach shown in Section 8.3.7.9 should be used.

The organization of this subsection is as follows. The test for equality of variance is documented in the first two subsections. The next three subsections present computational procedures for statistics used in the ANOVA procedures. Next, a method for three or more batches, which should cover most cases of practical importance, is presented. The case of two batches is discussed separately.

8.3.5.2.1 Levene's test for equality of variances

The ANOVA method is derived under the assumption that the variances within each batch are equal. This section describes a widely-used test suggested by Levene (References 8.3.5.2.1(a) - (c)) for determining whether the sample variances for k groups differ significantly. This test is nonparametric; that is, it does not require strong assumptions about the form of the underlying populations.

To perform this test, form the transformed data

$$w_{ij} = |x_{ij} - \tilde{x}_i| \qquad 8.3.5.2.1$$

where $\tilde{x}_i$ is the *median* of the n_i values in the ith group. Then perform an F-test on these *transformed* data (Section 8.3.5.2.2). If the test statistic is greater than or equal to the tabulated F-distribution quantile, then the variances are declared to be significantly different. If the statistic is less than the tabulated value, then the hypothesis of equality of variance is not rejected.

If the test does reject the hypothesis that the variances are equal, it is recommended that an investigation of the reason for the unequal variances be carried out. This may reveal problems in the generation of the data or in the fabrication of the material. Basis values calculated using the ANOVA method are likely to be conservative if the variances differ substantially.

8.3.5.2.2 The F-test for equality of means

To test the assumption that the populations from which the k samples were drawn have the same mean, calculate the following F statistic:

$$F = \frac{\sum_{i=1}^{k} n_i(\bar{x}_i - \bar{x})^2 / (k-1)}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2/(n-k)} \qquad 8.3.5.2.2$$

where $\bar{x}_i$ is the average of the n_i values in the ith group, and $\bar{x}$ is the average of all n observations. If Equation 8.3.5.2.2 is greater than the $1 - \alpha$ quantile of the F-distribution having $k - 1$ numerator and $n - k$ denominator degrees of freedom, then one concludes (with a five percent risk of making an error) that the k population means are not all equal. For $\alpha = 0.05$, the required F quantiles are tabulated in Table 8.5.1.

This test is based on the assumption that the data are normally distributed; however, it is well known to be relatively insensitive to departures from this assumption.

8.3.5.2.3 One-way ANOVA computations based on individual measurements

When all of the observations in a sample are available, the first step is to compute the means.

$$\bar{x} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} x_{ij} / n \qquad 8.3.5.2.3(a)$$

and

$$\bar{x}_i = \sum_{j=1}^{n_i} x_{ij} / n_i, \quad \text{for } i = 1,\ldots,k \qquad 8.3.5.2.3(b)$$

where

$$n = \sum_{i=1}^{k} n_i \qquad 8.3.5.2.3(c)$$

is the total sample size. The required sums of squares can now be computed. The between-batch of squares is computed as

$$SSB = \sum_{i=1}^{k} n_i \bar{x}_i^2 - n\bar{x}^2 \qquad 8.3.5.2.3(d)$$

and the total sum of squares is

$$SST = \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}^2 - n\bar{x}^2 \qquad 8.3.5.2.3(e)$$

The within-batch, or error, sum of squares is computed by subtraction

$$SSE = SST - SSB \qquad 8.3.5.2.3(f)$$

8.3.5.2.4 One-way ANOVA computations based on summary statistics

It is often the case that only summary statistics are available for each group. If these summary statistics contain the sample averages $\bar{x}_i$, the standard deviations of the data from each group (s_i) and the group sizes (n_i), the sums of squares can be computed as follows. First, compute the overall mean,

$$\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i / n \qquad 8.3.5.2.4(a)$$

The between-batch sum of squares is computed using Equation 8.3.5.2.3(d). In terms of the s_i^2, the within-batch sum of squares is

$$SSE = \sum_{i=1}^{k} (n_i - 1) s_i^2 \qquad 8.3.5.2.4(b)$$

The total sum of squares, SST, is the sum of SSB and SSE.

8.3.5.2.5 The ANOVA table for a one-way model

An ANOVA table displays the information about sources of variation that is contained in the sums of squares. A typical ANOVA table, which is used for both the fixed effects and random effects models, is shown below. The first column identifies the source of variation. The degrees of freedom and the computed sums of squares are listed in the second and third columns. The fourth column contains mean squares which are defined as the sum of squares divided by its degrees of freedom. The final column contains an F statistic which is equal to the ratio of the mean squares. This statistic is used to test the hypothesis that there is significant sample-to-sample variation (Section 8.3.5.2.2). The statistic is compared to the upper 0.95th quantile of an F distribution with $k - 1$ numerator degrees of freedom and $n - k$ denominator degrees of freedom. Table 8.5.1 contains these critical F values. If the computed statistic is greater than the tabulated F value, this indicates that there is statistically significant sample-to-sample variation. If the computed statistic is less than the tabulated value, then the variation between samples is not statistically significant at the chosen significance level.

Source	Degrees of Freedom	Sum of Squares	Mean Squares	F Test
Samples	k-1	SSB	MSB = SSB/(k-1)	F =MSB/MSE
Error	n-k	SSE	MSE = SSE/(n-k)	
Total	n-1	SST		

8.3.5.2.6 Calculation of summary statistics for one-way ANOVA basis values

The first step in computing an ANOVA basis value is to compute summary statistics, including the batch averages, an estimate of the overall population mean, and estimates of the between-batch and within-batch variances. Since the batches need not have equal numbers of specimens, an 'effective batch size, is defined as

$$n' = \frac{n - n^*}{k-1} \qquad 8.3.5.2.6(a)$$

where

$$n^* = \sum_{i=1}^{k} \frac{n_i^2}{n} \qquad 8.3.5.2.6(b)$$

and

$$n = \sum_{i=1}^{k} n_i \qquad 8.3.5.2.6(c)$$

is the total sample size.

Next, the batch means ($\bar{x}_i$), overall mean ($\bar{x}$), and between- and within-batch sums of squares should be calculated as in Section 8.3.5.2.3 or 8.3.5.2.4) The between-batch mean square (MSB) and the within-batch mean square (MSE) are then obtained by dividing these sums of squares by the appropriated degrees of freedom, as in Section 8.3.5.2.5.

Using these two mean squares, an estimate of the population standard deviation is

$$S = \sqrt{\frac{MSB}{n'} + \left(\frac{n'-1}{n'}\right) MSE} \qquad 8.3.5.2.6(d)$$

8.3.5.2.7 Calculations for three or more batches

Let the tolerance limit factor for a simple random sample from a normal distribution with sample size n be denoted k_0, and let the tolerance limit factor for a simple random sample from a normal distribution of size k be denoted k_1. These tolerance limit factors can be obtained from Table 8.5.10 (for B-basis values) or 8.5.11 (for A-basis values). Denote the ratio of mean squares by

$$u = \frac{MSB}{MSE} \qquad 8.3.5.2.7(a)$$

If u is less than one, set u equal to one. The tolerance limit factor is

$$T = \frac{k_0 - k_1/\sqrt{n'} + (k_1 - k_0)w}{1 - \frac{1}{\sqrt{n'}}} \qquad 8.3.5.2.7(b)$$

where

$$w = \sqrt{\frac{u}{u + n' - 1}} \qquad 8.3.5.2.7(c)$$

The basis value is

$$B = \bar{x} - TS \qquad 8.3.5.2.7(d)$$

Whether this value is an A- or B-basis value depends only on whether k_0 and k_1 are taken from Table 8.5.10 or Table 8.5.11.

8.3.5.2.8 Calculations for two batches

If data on only two batches are available, then the ANOVA method is not useful. One has two alternatives:

1. Obtain more batches, or

2. Pool the two batches and use unstructured-data methods.

In order to decide which of these actions to take, look at the data from the two batches. If the difference between the two batch means is large when compared to the standard deviation of $\bar{x}$

$$s_{\bar{x}} = \sqrt{\frac{MSB}{n}} \qquad 8.3.5.2.8$$

and if this difference in means is also large enough to be of practical importance, then pooling cannot be advised. However, if the batches overlap substantially, or if the difference in batch means is too small to be of engineering importance, then one might be able to justify pooling and using the methods of Section 8.3.4. However, since the compatibility test (Section 8.3.2) has already indicated that the batches are not from the same population, it is probable that this visual inspection will not provide convincing evidence for combining the data and using the methods of Section 8.3.4. In this case, whenever possible, data from new batches should be obtained before proceeding. If this is not possible, then calculate the basis values for each batch separately, according to the methods in Section 8.3.4, and choose the lower of these numbers as an interim basis value, ideally to be replaced when more data can be obtained.

8.3.5.3 Simple linear regression

Simple linear regression is the special case of the general regression model (Equation 8.3.5.1(c), in which the covariates are 1 and z, and there is no random effect, such as batch-to-batch variability:

$$x_s = \mu_{p(s)} + e_s = \theta_1 + \theta_2 z_{p(s),2} + e_s \qquad 8.3.5.3(a)$$

Putting this in more familiar notation and assuming that β_0 and β_1 are fixed unknown parameters,

$$Y = \beta_0 + \beta_1 X + \varepsilon \qquad 8.3.5.3(b)$$

Assume that the experimenter chooses n values of x, x_1, x_2, ..., x_n which need not be distinct, and observes the corresponding y values; thus the data consist of the n pairs

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

In order for the statistical analysis to be valid we must have $n \geq 3$ and at least two distinct x values. Let $\hat{\beta}_0$ and $\hat{\beta}_1$ denote estimates of β_0 and β_1. Then for any x, which need not be one of the experimental values x_1, x_2, ..., x_n), a predicted or fitted value denoted $\hat{y}$ is obtained, that is

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x \qquad 8.3.5.3(c)$$

It is customary to estimate β_0 and β_1 using the principle of least squares, which may be defined as follows. Let β_0^* and β_1^* be any estimates of β_0 and β_1. Let

$$Q(\beta_0^*, \beta_1^*) = \sum_{i=1}^{n} (y_i - \hat{y}_i^*)^2 \qquad 8.3.5.3(d)$$

where $\hat{y}_i^* = \beta_0^* + \beta_1^* x_i$.

The least squares estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ are the values of β_0^* and β_1^* which minimize $Q(\beta_0^*, \beta_1^*)$. They are given by

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad 8.3.5.3(e)$$

and

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \qquad 8.3.5.3(f)$$

where

$$\bar{y} = \sum_{i=1}^{n} y_i / n \qquad 8.3.5.3(g)$$

and

$$\bar{x} = \sum_{i=1}^{n} x_i / n \qquad 8.3.5.3(h)$$

It is sometimes more convenient to calculate $\hat{\beta}_1$ by the following equivalent formula

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\overline{xy}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} \qquad 8.3.5.3(i)$$

Statistical significance (at level α) of this regression means that there is evidence the $\beta_1 \neq 0$ (with a probability of $\leq \alpha$ of reaching this conclusion when $\beta_1 = 0$). If $\beta_1 \neq 0$, then X is of value as a linear predictor of Y. In order for the usual test of significance to be valid, the following additional assumption is required; the Y's are independently normally distributed random variables with common variance σ^2 and means $\beta_0 + \beta_1 x_i$, for $i = 1, 2, ..., n$.

To test whether the regression is significant at level α, let

$$s_Y^2 = \frac{\sum_{i=1}^{n} (y_1 - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{n-2} \qquad 8.3.5.3(j)$$

and define

$$SSE = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 \qquad 8.3.5.3(k)$$

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 \qquad 8.3.5.3(l)$$

and

$$SSR = SST - SSE \qquad 8.3.5.3(m)$$

Then define

$$F = \frac{SSR}{s_Y^2} \qquad 8.3.5.3(n)$$

which has the F-distribution with 1 and n - 2 degrees of freedom. The regression is considered significant if the value in Equation 8.3.5.3(n) exceeds the $1 - \alpha$ quantile of the F-distribution with $\gamma_1 = 1$ and $\gamma_2 = n - 2$ degrees of freedom. Table 8.5.1 provides these values for $\alpha = 0.05$.

For given x_0, the B-basis value satisfies the condition that $B(x_o)$ is a B-basis value for the normal population with mean $f(x_0) = \beta_0 + \beta_1 x_0$ and variance σ^2. A B-basis, value, in the case of simple linear regression, can be determined as follows. For $x = x_0$, compute B as

$$B = (\hat{\beta}_0 + \hat{\beta}_1 x_0) - k_B s_y \qquad 8.3.5.3(o)$$

where s_y is the square root of $s_{\hat{y}}^2$ in Equation 8.3.5.3(j),

$$k_B = t_{\gamma,0.95}(\delta)\sqrt{\frac{1+\Delta}{n}} \qquad 8.3.5.3(p)$$

and $t_{\gamma,0.95}(\delta)$ is the 95th percentile of the non-central t-distribution with $\gamma = n - 2$ degrees of freedom and non-centrality parameter

$$\delta = \frac{1.282}{\sqrt{\frac{1+\Delta}{n}}} \qquad 8.3.5.3(q)$$

with

$$\Delta = \frac{n(x_0 - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad 8.3.5.3(r)$$

The following approximation to k_B can be used when n is greater than or equal to 10 and $0 \le \Delta \le 10$:

$$k_B = 1.282 + \exp\left[0.595 - 0.508\ln(n) + \frac{4.62}{n} + \left(0.488 - \frac{0.988}{n}\right)\ln(1.82 + \Delta)\right] \qquad 8.3.5.3(s)$$

To adapt Equation 8.3.5.3(o) to A-basis values, replace 1.282 by 2.326 in Equation 8.3.5.3(q). For A-basis values, k_a can be approximated by

$$k_A = 2.326 + \exp\left[0.659 - 0.514\ln(n) + \frac{6.58}{n} + \left(0.481 - \frac{1.42}{n}\right)\ln(3.71 + \Delta)\right] \qquad 8.3.5.3(t)$$

The example problem in Section 8.3.7.7 demonstrates the simple linear regression procedures. This case is expanded to linear regression with batch effects in Section 8.3.7.8.

8.3.5.4 Basis values using pooling of structured data

For small data sets (less than eighteen (18) per environmental condition) the utility of the approach described in Figure 8.3.1 can be increased by pooling data from tests at different environmental conditions. When using pooling procedures to expand the utility of small data sets to obtain higher basis values, the associated assumptions and limits must be validated in the data analysis (see References 8.3.5.4(a) and (b)). If the statistical checks and engineering data analysis described in these references

indicate that pooling is invalid, the other standard statistical methods described in this handbook should be applied. Unfortunately, some penalty is inherent when applying the standard methods with small data sets.

The general methodology is described in Reference 8.3.5.4(a) if the Weibull distribution fits the data and in (Reference 8.3.5.4(b)) if the use of normal distribution is desired. Either statistical distribution can be used, depending on the best fit to the data, although, in general, the use of the Weibull distribution will result in more conservative material basis values. In this data reduction method, data from multiple environments, batches, and panels are pooled to obtain population variability factors for each test type and failure mode, i.e., tension, compression, shear. Essentially the method uses the larger pooled data set to estimate variability but uses only the test sample at each environment to estimate the mean property value at that environment. As the large sample size is used to calculate the probabilities and confidence, the resulting basis values are generally less conservative than those obtained without resorting to pooling.

However, the pooling data reduction methodology requires the validation of several underlying assumptions in order to generate a valid material basis value (see References 8.3.5.4(a) and (b)). In order to pool the data sets, the variability across environments must be statistically equivalent and the failure modes for each environment should not significantly change.

8.3.6 Exploratory data analysis

Exploratory Data Analysis (EDA) techniques are simple, visual, qualitative procedures which often point out important features of data early in the analysis. Where possible, conclusions based on EDA should be used to supplement quantitative statistical methods. Two EDA techniques are described below; the *quantile box plot* and the *informative quantile functions*. A more complete treatment of this subject can be found in Reference 8.3.6.

8.3.6.1 The quantile box plot

The quantile box plot provides a graphical summary of the sample values. This procedure depicts the symmetry, tail sizes, and median value of the sample as well as indicating the possible existence of outliers and inhomogeneous data.

Let $F(x)$ be the underlying distribution function. The uth quantile of $F(x)$, q_u, is the solution to the equation $F(q_u) = u$. The quantile function, $Q(u)$, is defined by

$$Q(u) = F^{-1}(u) \quad 0 < u < 1 \qquad 8.3.6.1(a)$$

(see Figure 8.3.6.1(a)). Letting $x_{(1)} \leq x_{(2)} \leq \ldots x_{(n)}$ denote the ordered measurements for a sample of size n, $Q(u)$ is estimated by the piecewise linear function

$$\hat{Q}(u) = (nu - j + \frac{1}{2})\, x_{(j+1)} + (j + \frac{1}{2} - nu)\, x_{(j)} \qquad 8.3.6.1(b)$$

for

$$\frac{2j-1}{2n} \leq u < \frac{2j+1}{2n} \qquad 8.3.6.1(c)$$

Figure 8.3.6.1(b) is an example of a quantile box plot. The boxes are used to examine the symmetry and tail sizes of the underlying distribution. Flat spots in $Q(u)$ indicate modal values. Sharp rises in $Q(u)$ for u in the vicinity of 0 or 1 indicate the possible presence of outliers in the data. Sharp rises in $Q(u)$ within the boxes indicate the possible existence of two (or more) populations or gaps in the data. A thorough treatment of the use of the Quantile Box plot can be found in Reference 8.3.6.1.

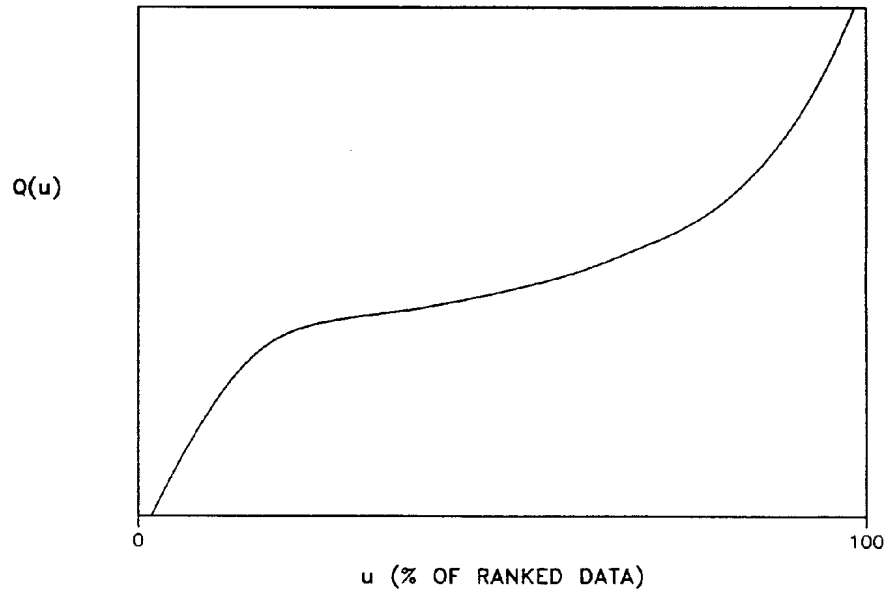

FIGURE 8.3.6.1(a) *The quantile function.*

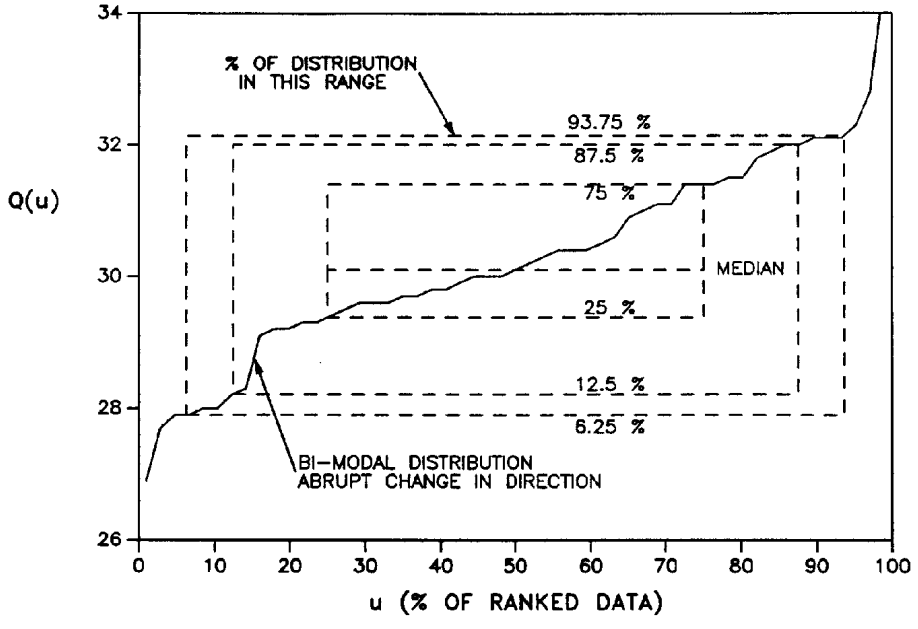

FIGURE 8.3.6.1(b) *Example of a quantile box plot.*

8.3.6.2 The informative quantile function

Techniques for obtaining B-basis values for unimodal data can be divided into two main categories: techniques for specific parametric families, and nonparametric techniques. The *Informative Quantile* (IQ) function can be used as an aid in identifying a parametric model which provides a satisfactory fit to the data. Parametric techniques have been most thoroughly discussed for the normal, lognormal, and two-parameter Weibull parametric families; thus only these techniques will be considered here. Henceforth in this section, any reference to the Weibull parametric family should be interpreted as a reference to the two-parameter Weibull parametric family.

The IQ function was developed to identify which univariate location-scale parametric distribution best describes an ordered group of data. A univariate location-scale parametric distribution is one whose distribution function F(x) can be expressed as

$$F(x) = F_o[(x-a)/b] \qquad 8.3.6.2(a)$$

where a and b are the location and scale parameters respectively, and $F_o(x)$ is the "standard" distribution with $a = 0$ and $b = 1$. The IQ function identifies the standard distributional form and is thus independent of the values of the location and scale parameters.

The Weibull and lognormal parametric families are not location-scale parametric families. However, these distributions are simply related to two location-scale families: the normal and the extreme value families.

The estimated IQ function is defined as

$$\hat{I}Q(u) = \frac{\hat{Q}(u) - \hat{Q}(0.5)}{2[\hat{Q}(0.75) - \hat{Q}(0.25)]} \qquad 8.3.6.2(b)$$

where $Q(u)$ is the estimated quantile function defined in Equation 8.3.6.1(b). The corresponding exact IQ function is denoted $IQ(u)$ and defined by Equation 8.3.6.2(a) with $\hat{Q}(u)$ replaced by $Q(u)$. In order to determine whether the data can be adequately modeled by either the normal or extreme value distribution, a plot of the estimated truncated IQ function, defined by

$$\hat{T}IQ(u) \begin{cases} -1 & \text{if } \hat{I}Q(u) \le -1 \\ \hat{I}Q(u) & \text{if } -1 < \hat{I}Q(u) \le 1 \\ 1 & \text{if } \hat{I}Q(u) > 1 \end{cases} \qquad 8.3.6.2(c)$$

is compared to the graph of the exact TIQ plots for these distributions (see Figures 8.3.6.2(a) and (b)). Though the TIQ plots for the data will be considerably less smooth than the exact TIQ plots, they may be compared for general shape and tail behavior.

In order to determine the adequacy of either the lognormal or the Weibull distribution, use the natural logarithms of the data to define the quantile function. Thus, Equation 8.3.6.1(b) becomes

$$\hat{Q}(u) = (nu - j + \frac{1}{2})\ln(x_{(j+1)}) + (j + \frac{1}{2} - nu)\ln(x_{(j)}) \qquad 8.3.6.2(d)$$

for

$$\frac{2j-1}{2n} \le u < \frac{2j+1}{2n} \qquad 8.3.6.2(d)$$

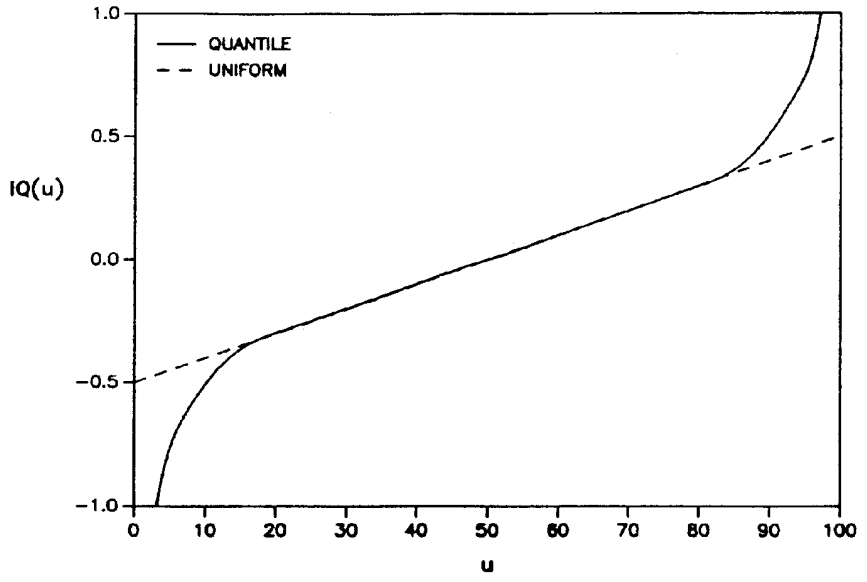

FIGURE 8.3.6.2(a) *TIQ plot of the normal distribution parametric family.*

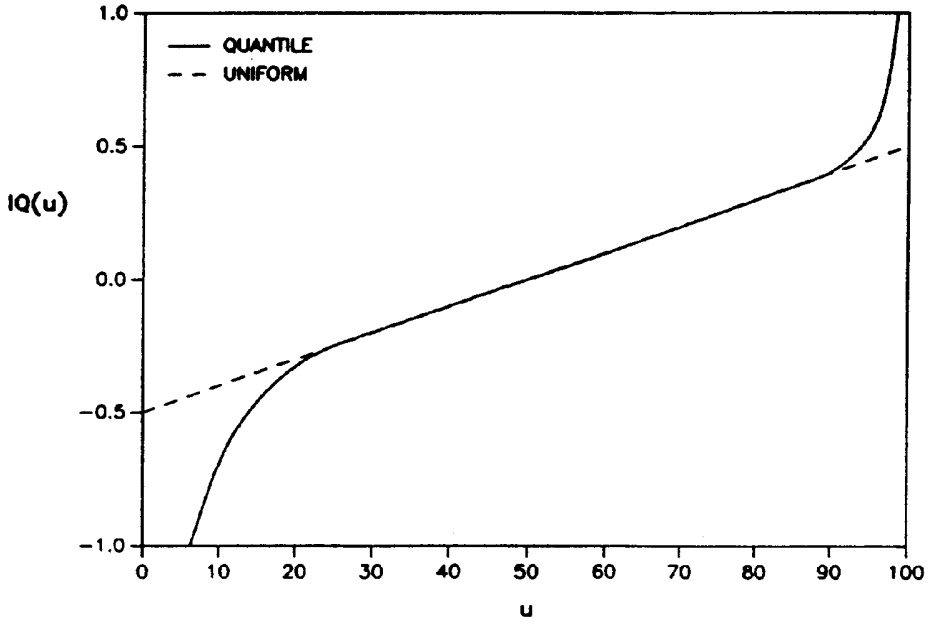

FIGURE 8.3.6.2(b) *TIQ plot of the extreme value distribution parametric family.*

The IQ and TIQ functions in Equations 8.3.6.2(b) and 8.3.6.2(c) are defined using this quantile function.

Thus, to determine whether the data can be adequately modeled by the normal distribution, compare the $\hat{T}IQ$ plot for the original data to the exact TIQ plot for the normal distribution in Figure 8.3.6.2(a). To determine whether the data can be adequately modeled by the lognormal distribution, compare the $\hat{T}IQ$ plot for the log data to the exact TIQ plot for the normal distribution in Figure 8.3.6.2(a). The adequacy of the two-parameter Weibull distribution is determined by comparing the $\hat{T}IQ$ plot for the log data to the exact TIQ plot for the extreme value distribution in Figure 8.3.6.2(b). For further information concerning the quantile function and the informative quantile function, the reader is referred to References 8.3.6.2(a) and 8.3.6.2(b).

8.3.7 Examples of computational procedures

This section illustrates the computational procedures using mechanical property data sets. In progressing through the example problems, the flowchart in Figure 8.3.1 are followed, and appropriate references to specific sections are made. Each example notes which software, STAT17 or RECIPE or both, provides the calculation for each step (see Section 8.1.2). All example data sets are listed in Table 8.3.7. Data files provided with the software are identified for each example.[1]

8.3.7.1 Problem 1 - Outlier detection, multiple-sample tests, and the Weibull distribution

The data set for this problem consists of compressive strength measurements from ten batches of material. This problem illustrates the outlier detection procedure, the k-sample Anderson-Darling test, the two-parameter Weibull goodness-of-fit test and the calculation of B-basis values by the Weibull method. Calculations for all steps may be performed by STAT17 and may be demonstrated using example data set, `example.d01`.

Problem 1 - Step 1. The first step is to screen the data for outliers using the MNR procedure as described in Section 8.3.3.1. The screening procedure is performed separately on each batch. The relevant calculations for the first batch, with a sample mean of 568.8 and a sample standard deviation of 757.9, are shown in the table below.

x_i	$\lvert r_i \rvert = \left\lvert \frac{x_i - \bar{x}}{s} \right\rvert = \left\lvert \frac{x_i 568.8}{757.9} \right\rvert$
125.9	0.584
136.6	0.570
1444	1.155

The MNR statistic is the largest absolute residual, or 1.155. Since this is greater than the $n = 3$ critical value of 1.154 from Table 8.5.7, the third observation is identified as an outlier. An examination of the laboratory record shows a measured value of 144.4. The data point was corrected and the MNR test repeated. The batch mean was recalculated as 135.7 and the batch standard deviation as 9.31. No outliers were detected. Similar calculations for the remaining batches identify no other outliers in this set of data. Visual inspection of the data also does not identify any outliers.

[1]Note that the example data sets identified for STAT17 correspond to those distributed with Version 5.0.

TABLE 8.3.7 *Example data sets for Section 8.3.7,* continued on next page.

Problem 1		Problem 3		Problem 4		Problem 4	
Batch	Data	Batch	Data	Batch	Data	Batch	Data
1	136.64	1	85.39	1	7300.000	3	6400.000
1	125.91	1	97.12	1	6100.000	3	8700.000
1	1444.5	1	92.66	1	7300.000	3	7400.000
2	107.79	1	96.43	1	7900.000	3	8600.000
2	114.58	1	90.72	1	6800.000	3	7900.000
2	110.70	1	95.84	1	6900.000	3	8500.000
3	125.50	2	97.30	1	1300.000	3	8400.000
3	118.79	2	109.47	1	7000.000	3	7200.000
3	131.24	2	101.35	1	8200.000	3	6500.000
4	125.91	2	98.01	1	7800.000	3	8100.000
4	127.86	2	86.18	1	7300.000	3	7900.000
4	125.91	2	100.91	1	6900.000	3	6200.000
5	134.41	3	96.05	1	8400.000	3	5900.000
5	124.60	3	92.20	1	8800.000	3	6400.000
5	127.54	3	90.86	2	7300.000	3	6800.000
6	139.35	3	101.27	2	8000.000	3	6700.000
6	119.03	3	101.23	2	8000.000	3	6400.000
6	125.81	3	93.15	2	8500.000	3	8100.000
7	120.00	4	114.32	2	6400.000	3	6700.000
7	121.94	4	100.14	2	7000.000	3	6900.000
7	132.58	4	91.24	2	6300.000	3	7500.000
8	119.28	4	86.11	2	6700.000	3	7600.000
8	118.30	4	93.42	2	8500.000	3	8200.000
8	126.12	4	92.65	2	6800.000	3	7000.000
9	109.50	5	97.58	2	9500.000		
9	121.23	5	97.75	2	7500.000		
9	130.03	5	97.95	2	7600.000		
10	118.71	5	112.49	2	9500.000		
10	126.56	5	95.75	2	6900.000		
10	124.60	5	110.53	2	6900.000		
				2	6400.000		
Problem 2		Problem 4		2	6100.000	Problem 5	
Batch	Data	Batch	Data	2	7300.000	Batch	Data
1	106.5	1	5700.000	2	7700.000	1	118.58
1	94.0	1	6300.000	2	7300.000	1	121.77
1	116.1	1	6700.000	2	6200.000	1	137.54
1	98.8	1	6300.000	2	6900.000	1	140.39
1	114.2	1	6300.000	2	6200.000	1	134.03
2	113.8	1	8600.000	2	6400.000	2	133.44
2	98.1	1	8300.000	2	6300.000	2	114.56
2	102.0	1	8000.000	2	5500.000	2	123.28
2	106.0	1	7300.000	2	6400.000	2	130.33
2	98.1	1	7600.000	2	6300.000	2	138.00
3	105.2	1	6100.000	2	5500.000	3	122.69
3	103.3	1	9100.000	3	8500.000	3	137.10
3	103.3	1	7800.000	3	7500.000	3	137.49
3	101.3	1	7100.000	3	6900.000	3	122.87
3	100.4	1	7400.000	3	8200.000	3	135.82
4	94.8	1	7000.000	3	7500.000		
4	105.4	1	6700.000	3	8200.000		
4	101.5	1	6300.000	3	7400.000		
4	95.8	1	6800.000	3	8100.000		
4	102.5	1	7300.000	3	7400.000		

TABLE 8.3.7 *Example data sets for Section 8.3.7,* concluded.

Problem 6	
Batch	Data
1	328.1174
1	334.7674
1	347.7833
1	346.2661
1	338.7314
2	297.0387
2	293.4595
2	308.0419
2	326.4864
2	318.1297
2	309.0487
3	337.0930
3	317.7319
3	321.4292
3	317.2652
3	291.8881
4	297.6943
4	327.3973
4	303.8629
4	313.0984
4	323.2769
5	312.9743
5	324.5192
5	334.5965
5	314.9458
5	322.7194
6	291.1215
6	309.7852
6	304.8499
6	288.0184
6	294.1995

Problem 7	
Temperature	Data
75	328.1174
75	334.7674
75	347.7833
75	346.2661
75	338.7314
75	340.8146
-67	343.5855
-67	334.1746
-67	348.6610
-67	356.3232
-67	344.1524

Problem 8		
Temperature	Batch	Data
75	1	328.1174
75	1	334.7674
75	1	347.7833
75	1	346.2661
75	1	338.7314
75	2	297.0387
75	2	293.4595
75	2	308.0419
75	2	326.4864
75	2	318.1297
75	2	309.0487
75	3	337.0930
75	3	317.7319
75	3	321.4292
75	3	317.2652
75	3	291.8881
75	4	297.6943
75	4	327.3973
75	4	303.8629
75	4	313.0984
75	4	323.2769
75	5	312.9743
75	5	324.5192
75	5	334.5965
75	5	314.9458
75	5	322.7194
75	6	291.1215
75	6	309.7852
75	6	304.8499
75	6	288.0184
75	6	294.1995
-67	1	340.8146
-67	1	343.5855
-67	1	334.1746
-67	1	348.6610
-67	1	356.3232
-67	1	344.1524
-67	2	308.6256
-67	2	315.1819
-67	2	317.6867
-67	2	313.9832
-67	2	309.3132
-67	2	275.1758
-67	3	321.4128
-67	3	316.4652
-67	3	331.3724
-67	3	304.8643
-67	3	309.6249
-67	3	347.8449
-67	4	331.5487
-67	4	316.5891
-67	4	303.7171
-67	4	320.3625

Problem 8		
Temperature	Batch	Data
-67	4	315.2963
-67	4	322.8280
-67	5	340.0990
-67	5	348.9354
-67	5	331.2500
-67	5	330.0000
-67	5	340.9836
-67	5	329.4393
-67	7	330.9309
-67	7	328.4553
-67	7	344.1026
-67	7	343.3584
-67	7	344.4717
-67	7	351.2776
-67	8	331.0259
-67	8	322.4052
-67	8	327.6699
-67	8	296.8215
-67	8	338.1995

Problem 9		
Source	Batch	Data
1	1	75.8
1	1	78.4
1	1	82.0
1	2	68.8
1	2	70.9
1	2	73.5
1	3	74.5
1	3	74.8
1	3	78.8
2	4	81.3
2	4	87.7
2	4	89.0
2	5	88.2
2	5	91.2
2	5	94.2

Problem 1 - Step 2. The k-sample Anderson-Darling test described in Section 8.3.2.2 will be employed next to determine whether or not the data from the ten batches should be combined. The first step is to order the pooled sample. Table 8.3.7.1 lists the 27 sorted, distinct values in the column labeled $z_{(j)}$. The

remaining columns show the h_j, H_j, and F_{1j} values used in calculating the terms in the statistic arising from the first batch $(i=1)$. The column labeled f_{1j} shows the number of times that $z_{(j)}$ is represented in the first batch and is used in calculating F_{1j}. From these numbers, it follows that

$$\frac{1}{n_i}\sum_{j=1}^{l} h_j \frac{\left(nF_{ij}n_iH_j\right)^2}{H_j\left(n\ H_j\right)\frac{nh_j}{4}} = \frac{1}{3}\sum_{j=1}^{27} h_j \frac{\left(30F_{1j}3H_j\right)^2}{H_j\left(30H_j\right)30\frac{h_j}{4}} = 363.33$$

When these calculations are repeated for the remaining nine batches, the k-sample Anderson-Darling statistic is computed as

TABLE 8.3.7.1 *Illustration of k-sample Anderson-Darling statistic calculations for the first batch.*

j	$z_{(j)}$	h_j	H_j	f_{1j}	F_{1j}
1	107.8	1	0.5	0	0.0
2	109.5	1	1.5	0	0.0
3	110.7	1	2.5	0	0.0
4	114.6	1	3.5	0	0.0
5	118.3	1	4.5	0	0.0
6	118.7	1	5.5	0	0.0
7	118.8	1	6.5	0	0.0
8	119.0	1	7.5	0	0.0
9	119.3	1	8.5	0	0.0
10	120.0	1	9.5	0	0.0
11	121.2	1	10.5	0	0.0
12	121.9	1	11.5	0	0.0
13	124.6	2	13.0	0	0.0
14	125.5	1	14.5	0	0.0
15	125.8	1	15.5	0	0.0
16	125.9	3	17.5	1	0.5
17	126.1	1	19.5	0	1.0
18	126.6	1	20.5	0	1.0
19	127.5	1	21.5	0	1.0
20	127.9	1	22.5	0	1.0
21	130.0	1	23.5	0	1.0
22	131.2	1	24.5	0	1.0
23	132.6	1	25.5	0	1.0
24	134.4	1	26.5	0	1.0
25	136.6	1	27.5	1	1.5
26	139.4	1	28.5	0	2.0
27	144.4	1	29.5	1	2.5

$$ADK = \frac{n-1}{n^2(k-1)}\sum_{i=1}^{k}\left\{\frac{1}{n_i}\sum_{j=1}^{L} h_j \frac{\left(nF_{ij}-n_iH_j\right)^2}{H_j\left(n-H_j\right)-nh_j/4}\right\}$$

$$= \frac{30-1}{30^2(10-1}x \quad \text{the sum in } \sum$$

$$= 1.24$$

The computed value of the statistic is compared to the critical value from Equation 8.3.2.2(j), which is 1.37. Since the computed value of 1.24 is less than the critical value of 1.37, the hypothesis that the populations from which these groups were drawn are identical is not rejected. Conclude that the data from these batches may be combined into a single sample.

Problem 1 - Step 3. The maximum normed residual (MNR) test is performed on the pooled data. No potential outliers are detected in the pooled data. (see Problem 1 - Step 1 for details of the outlier detection procedure.)

Problem 1 - Step 4. In order to perform the two-parameter Weibull goodness-of-fit test described in Section 8.3.4.2.2, it is necessary to compute estimates of the scale and shape parameters, $\hat{\alpha}$ and $\hat{\beta}$. A procedure for doing this is described in Section 8.3.4.2.1. The geometric mean of the data is computed as

$$\bar{x}_G = \exp\left[\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)\right] = \exp\left[\frac{1}{53}\sum_{i=1}^{53}\ln(x_i)\right] = 67.501$$

For a given value of $\hat{\beta}$, $\hat{\alpha}$ is calculated as

$$\hat{\alpha} = \bar{x}_G\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\bar{x}_G}\right)^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}$$

$$\hat{\alpha} = 67.501\left[\frac{1}{31}\sum_{i=1}^{31}\left(\frac{x_i}{67.501}\right)^{\hat{\beta}}\right]^{\frac{1}{\hat{\beta}}}$$

In order to calculate $\hat{\beta}$, define the function $G(\hat{\beta})$ by

$$G(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i)\left(\left[\frac{x_i}{\hat{\alpha}}\right]^{\hat{\beta}}-1\right)-\frac{1}{\hat{\beta}}$$

$$= \frac{1}{30}\sum_{i=1}^{30}\ln(x_i)\left(\left[\frac{x_i}{\hat{\alpha}}\right]^{\hat{\beta}}-1\right)-\frac{1}{\hat{\beta}}$$

where $\hat{\alpha}$ is calculated as above. The estimate, $\hat{\beta}$, is the solution to the equation $G\left(\hat{\beta}\right)=0$. An iterative technique for solving this equation is given in Section 8.3.4.2.1, and begins by setting

$$\hat{\beta} = \frac{1.28}{S_y} = \frac{1.28}{0.0673} = 19.02$$

The solution is $\hat{\beta}=15.35$, which in turn gives $\hat{\alpha}=128.39$.

The first five ordered observations are listed below with the transformations necessary to compute the goodness-of-fit test statistic.

x_i	$z_{(i)} = \left(\frac{x_{(i)}}{\hat{\alpha}}\right)^{\hat{\beta}} = \left(\frac{x_{(i)}}{128.39}\right)^{15.35}$
107.8	0.0684
109.5	0.0869
110.7	0.1027
114.6	0.1748
118.3	0.2847

The Anderson-Darling goodness-of-fit statistic and observed significance level are calculated according to Section 8.3.4.2.2 as follows.

$$\begin{aligned} AD &= \sum_{i=1}^{n} \frac{1-2i}{n}\left\{\ln[1-\exp(-z_{(i)})] - z_{(n+1-i)}\right\} - n \\ &= \sum_{i=1}^{30} \frac{1-2i}{30}\left\{\ln[1-\exp(-z_{(i)})] - z_{(31-i)}\right\} - 30 \\ &= 0.699 \end{aligned}$$

$$AD^{*} = \left(1 + 0.2/\sqrt{n}\right) AD = \left(1 + 0.2/\sqrt{30}\right) 0.699 = 0.7245$$

$$\begin{aligned} OSL &= 1/\left\{1+\exp\left[-0.10+1.24\ln(AD^{*})+4.48AD^{*}\right]\right\} \\ &= 1/\left\{1+\exp\left[-0.10+1.24\ln(0.7245)+4.48(0.7245)\right]\right\} \\ &= 0.0576 \end{aligned}$$

Since the Weibull goodness-of-fit test yields an OSL value greater than 0.05, there is insufficient evidence to contradict the assumption that the data follow a two-parameter Weibull distribution. Hence, the two parameter Weibull method in Section 8.3.4.2.3 should be used to compute the B-basis value.

Problem 1 - Step 5. The parameter estimates $\hat{\alpha}$ and $\hat{\beta}$ calculated in the previous step are used to compute the B-basis value for the sample as described in Section 8.3.4.2.3. The quantities necessary to compute the B-basis value are:

$$V_B = 5.057 \text{ (from Table 8.5.8)}$$

$$\hat{\alpha} = 128.39$$

$$\hat{\beta} = 15.35$$

$$\hat{Q} = \hat{\alpha}(0.10536)^{1/\hat{\beta}} = (128.39)(0.10536)^{1/15.35} = 110.88$$

The B-basis value is calculated as

$$B = \hat{Q}\exp\left(\frac{-V_B}{\hat{\beta}\sqrt{n}}\right) = 110.88\exp\left(\frac{-5.057}{15.35\sqrt{30}}\right) = 104.41$$

For presentation in MIL-HDBK-17, this B-basis value would be rounded to 104.

8.3.7.2 Problem 2 - Normal distribution

The data set for this problem consists of compressive test measurements from four batches of material. This problem illustrates the normal goodness-of-fit test and the calculation of B-basis values by the

normal method. Calculations for all steps may be performed by STAT17 and may be demonstrated using example data set, example.d02. Calculations for Step 6 may be performed by RECIPE and may be demonstrated using example data set, pr2.dat. This also provides an example of the use of RECIPE for a simple random sample.

Problem 2 - Step 1. There are no detected outliers in this set of data. (See Problem 1 for details of the outlier detection calculations.)

Problem 2 - Step 2. The k-sample Anderson-Darling test statistic is $\mathrm{ADK} = 1.01$ (see Problem 1 for a detailed computation of the k-sample statistic). Since this is less than the critical value of 1.73, conclude that the data from the batches may be combined and treated as a single sample. The next step is to investigate the form of the distribution.

Problem 2 - Step 3. The maximum normed residual (MNR) test is performed on the pooled data. No potential outliers are detected in the pooled data. (see Problem 1 - Step 1 for details of the outlier detection procedure.)

Problem 2 - Step 4. The Weibull goodness-of-fit test yields an observed significance level of 0.008. (See Problem 3 for details of the computation for the Weibull goodness-of-fit test.) Since this is less than 0.05, the normal goodness-of-fit test described in Section 8.3.4.3.2 is performed.

Problem 2 - Step 5. The mean and standard deviation of the sample are 103.1 and 6.175, respectively. The first five ordered observations are listed below with the z-values and the values of the standard normal distribution necessary for calculation of the normal Anderson-Darling statistic.

$x_{(i)}$	$z_{(i)} = \frac{x_{(i)} - \bar{x}}{s} = \frac{x_{(i)} - 106.5}{6.436}$	$F_0(z_{(i)})$
98.0	-1.313	0.0951
99.0	-1.158	0.1235
100.0	-1.002	0.1582
100.0	-1.002	0.1582
100.0	-1.002	0.1582
..	..	..

$$\begin{aligned}
\mathrm{AD} &= \sum_{i=1}^{n} \frac{1-2i}{n}\left\{\ln[F_o(z_{(i)})] + \ln[1 - F_o(z_{(n+1-i)})]\right\} - n \\
&= \sum_{i=1}^{20} \frac{1-2i}{20}\left\{\ln[F_o(-z_{(i)})] + \ln[1 - F_o(z_{(21-i)})]\right\} - 20 \\
&= 0.570
\end{aligned}$$

$$\mathrm{AD}^* = \left[1 + \frac{4}{n} - \frac{25}{n^2}\right]\mathrm{AD} = \left[1 + \frac{4}{20} - \frac{25}{20^2}\right](0.570) = 0.648$$

$$\begin{aligned}
\mathrm{OSL} &= 1/\left\{1 + \exp\left[-0.48 + 0.78\ln(\mathrm{AD}^*) + 4.58\mathrm{AD}^*\right]\right\} \\
&= 1/\left\{1 + \exp\left[-0.48 + 0.78\ln(0.648) + 4.58(0.648)\right]\right\} \\
&= 0.163
\end{aligned}$$

Since the normal goodness-of-fit test yields an OSL value (0.163) greater than 0.05, there is insufficient evidence to contradict the assumption that the data are normally distributed. Hence, the normal method in Section 8.3.4.3.3 is used to compute a B-basis value.

Problem 2 - Step 6. From Table 8.5.10, the one-sided tolerance limit factor, k_B, is 1.93. The B-basis value for a normally distributed sample is computed as

$$B = \bar{x} - k_B s = 103.1 - (1.927)(6.175) = 91.2$$

For presentation in MIL-HDBK-17, this B-basis value would be rounded to 91, since this corresponds to the number of significant figures obvious in the data.

RECIPE can also be used to calculate this basis value Since it has been shown that the batches may be pooled, this problem represents an unstructured (that is, simple random) sample of n observations from a single batch at a fixed set of conditions. For this case, there $\ell = 1$ is condition, and $m = 1$ batch, so $p(s) = q(s) = 1$ for each s. This model can be written

$$y_s = \theta_1 + e_s$$

Note that $b_{q(s)}$ does not appear in this equation since the between-batch variability has been shown to be negligible (Step 3).

```
#
#  RECIPE Problem #2: Random sample of 4 batches with no
#                     batch-to-batch variability
#
#  -- For this example, we have 20 observations: all at the same
#     fixed level and from one population.  RECIPE is a very
#     general program which is here used for a very simple
#     example. This example might seem confusing because it
#     is so special.  If so, consider the more complicated
#     examples, particularly Example #4. Ironically, the
#     simpler examples may then be easier to understand.
#
#  --  ntot, nlvl, nbch, npar, npts, prob, conf
#
   20 1 1 1 1 .9d0 .95d0
#
#  --  Fixed levels. Here nlvl=1 and npar=1; that is there is only
#     one fixed level and one regression parameter (a constant mean),
#     so this part of the input consists of one row and one column,
#     containing just the number '1'.
#
  1
#
#  --  Fixed level, batch number, response value.  Note that there
#     is only one level (nlvl=1) and one batch (nbch=1).
#
  1    1  99.
  1    1  100.
  1    1  106.
# (this just shows that comments can be put anywhere: even among
#  the data values. This is useful, for example, if a data value
#  is to be removed from the analysis. Simply put a '#' at the
#  beginning of the appropriate line, and decrease 'ntot' by 1
#  in the first noncomment line)
  1    1  107.
  1    1  110.
  1    1  98.
  1    1  103.
  1    1  111.
```

```
  1   1  119.
  1   1  121.
  1   1  100.
  1   1  100.
  1   1  104.
  1   1  108.
  1   1  116.
  1   1  103.
  1   1  104.
  1   1  106.
  1   1  106.
  1   1  108.
#
# --  Points at which to evaluate tolerance limit. Here the only fixed
#     effect is a constant mean, so this part of the input is trivial.
 1
```

Lines which begin with a '#' are *comment lines* which are ignored by the program. Comment lines can be inserted anywhere and are intended to make RECIPE data files self-documenting. The input to this program is free-format, so it doesn't matter which column values are in, so long as they are in the correct order and separated by spaces. The sole exception to this is that comment lines must have a '#' in column 1.

The first non-comment line of any RECIPE files has seven constants:

RECIPE mnemonic	Symbol	Definition
ntot	n	total number of observations
nlevel	l	number of fixed levels
nbch	m	number of batches
npar	r	number of fixed parameters
npts	-	number of basis values to be calculated
prob	-	content
conf	-	confidence

It is necessary to specify the number of points at which the basis values will be determined. For example, if a linear regression model relates strength to temperature, then a basis value can be calculated at any number of temperatures, that is, the temperatures at which basis values are determined need not correspond to values for which data are available. The fifth number npts specifies the number of basis values which are to be calculated. The sixth and seventh values, prob and conf, give the *content* and *confidence* which are to be used. For purpose of basis calculations, one need only remember that prob should be 0.99d0 for A-basis values and 0.90d0 for B-basis values, and that conf should be 0.95d0.

In this example, note that there are $n = 20$ observations, at $l = 1$ fixed level, from $m = 1$ batch, with $r = 1$ fixed parameter, and that a single B-basis value is to be calculated. (Since this corresponds to a simple random sample, it only makes sense to calculate one B-basis value.)

The next $l = 1$ noncomment lines specify the fixed levels; for this example there is only one fixed level, and it is just the mean, so this part of the file has only one line with a '1' in it. The following $n = 20$ noncomment lines each gives, from left to right, a fixed level $p(s)$ (here $p(s) = 1$), batch $q(s)$ (here $q(s) = 1$), and observation (strength y_s for $s = 1,...,20$). The next npts = 1 noncomment lines give the z's

corresponding to each point at which a basis value is to be calculated. Again, because this example is a simple random sample, this part of the file consists of only a single line with a '1'.

RECIPE is executed as follows:

```
recipe
 Filename (without .dat extension) ?
ex2

 RECIPE : One-Sided Random-Effect Regression Tolerance Limits
 (Version 1.0, April 1995)

 *** Simulated pivot critical value file ex2.crt not found.
   Satterthwaite approximation will be used.

     Probability   Confidence   Regression  Tolerance Limit
       0.90          0.95     104.400000     88.312898
```

The first two columns of the output indicate a B-basis value has been calculated. The third column gives the value of a point on the least squares regression line (here just the sample mean) and the fourth column gives the corresponding basis value (here the usual normal B-basis value for a single sample of five specimens). These results provide a mean of 104 and a B-basis value of 88 (Note the number of significant figures). This approach can also be used for data from one batch. A warning is provided as a reminder that one cannot estimate between-batch variability with data from a single batch, and consequently a basis value has been calculated under the assumption that there is no between-batch variability.

There are two methods that RECIPE can use to calculate allowable. One involves the use of a Satterthwaite approximation (Reference 8.3.7.2(a)) and the other requires using an auxiliary program SIMPVT to obtain a quantile of a pivotal random variable for which the probability distribution cannot be determined in analytical form. Usually, these two methods will give very nearly the same answers, at least for material basis value calculations. The simpler Satterthwaite approximation is therefore recommended for general use. Auxiliary programs SIMPVT and SIMCOV, which use simulation to approximate the appropriate pivoted quantile and to assess the quality of the Satterthwaite approximate, respectively, are available with RECIPE (Section 8.1.2). For more information see References 8.3.7.2(a) and (b).

8.3.7.3 Problem 3 - Lognormal distribution

The data set for this problem consists of transverse tension test measurements from five batches of material. This problem illustrates the lognormal goodness-of-fit test and the calculation of B-basis values by the lognormal method. Calculations for all steps may be performed by STAT17 and may be demonstrated using example data set, example.d03.

Problem 3 - Step 1. There are no detected outliers in this set of data. (See Problem 1 for details of the outlier detection calculations.)

Problem 3 - Step 2. The k-sample Anderson-Darling test statistic is $ADK = 1.27$. (See Problem 1 for details of the computation of the k-sample statistic.) Since this is less than the critical value of 1.64, conclude that the data from the batches may be combined into a single sample.

Problem 3 - Step 3. The maximum normed residual (MNR) test is performed on the pooled data. No potential outliers are detected in the pooled data. (see Problem 1 - Step 1 for details of the outlier detection procedure.)

Problem 3 - Step 4. The observed significance levels (OSL) for the two-parameter Weibull and the normal goodness-of-fit tests are given below:

Distribution	OSL
Two-parameter Weibull	0.001
Normal	0.042

(See Problems 1 and 2 for details of the computations for these tests.) Since the OSL 's are both less than 0.05, neither of the distributions adequately describe the data. Thus, the lognormal goodness-of-fit test is performed.

Problem 3 - Step 5. In order to perform the lognormal goodness-of-fit test described in Section 8.3.4.4, the natural logarithms of the data are used. The average and standard deviation of the transformed data are

$$\bar{x}_L = 4.57 \qquad s_L = 1.6050$$

The first five ordered observations are listed below with the transformations necessary to compute the goodness-of-fit statistic. The goodness-of-fit statistic and observed significance level are calculated as:

$x_{(i)}$	$\ln(x_{(i)})S$	$z_{(i)} = \frac{\ln(x_{(i)}) - \bar{x}_L}{s_L} = \frac{\ln(x_{(i)}) - 4.57}{1.6050}$	$F_o[z_{(i)}]$
85.39	4.447228998	-1.727771002	0.042014598
86.11	4.455625549	-1.719374451	0.042773061
86.18	4.456438132	-1.718561868	0.042847046
90.72	4.50777784	-1.66722216	0.047735087
90.86	4.50931986	-1.66568014	0.047888539
..	..	..	..

$$\begin{aligned} AD &= \sum_{i=1}^{n} \frac{1-2i}{n}\left\{\ln[F_o(z_{(i)})] + \ln[1 - F_o(z_{(n+1-i)})]\right\} - n \\ &= \sum_{i=1}^{30} \frac{1-2i}{30}\left\{\ln[F_o(-z_{(i)})] + \ln[1 - F_o(z_{(31-i)})\right\} - 31 \\ &= 0.597 \end{aligned}$$

$$AD^* = \left[1 + \frac{4}{n} - \frac{25}{n^2}\right]AD = \left[1 + \frac{4}{30} - \frac{25}{30^2}\right](0.597) = 0.177S$$

$$\begin{aligned} OSL &= 1/\left\{1 + \exp\left[-0.48 + 0.78\ln(AD^*) + 4.58AD^*\right]\right\} \\ &= 1/\left\{1 + \exp\left[-0.48 + 0.78\ln(0.177) + 4.58(0.177)\right]\right\} \\ &= 0.098 \end{aligned}$$

Since the lognormal goodness-of-fit test results in an OSL value greater than 0.05, there is insufficient evidence to contradict the assumption that the data are lognormally distributed. Hence, the lognormal method in Section 8.3.4.5.1 is used to compute a B-basis value.

Problem 3 - Step 6. The B-basis value for lognormally distributed data is computed as

$$B = \exp[\bar{x}_L - k_B s_L] = \exp[4.57 - 1.78(1.6050)] = 85.09$$

For presentation in MIL-HDBK-17, this B-basis value would be rounded to 85.1.

8.3.7.4 Problem 4 - Nonparametric method

The data set for this problem consists of transverse tensile strain-to-failure measurements for three batches of material. This problem illustrates the calculation of B-basis values by the nonparametric method. Calculations for all steps may be performed by STAT17 and may be demonstrated using example data set, `example.d04`.

Problem 4 - Step 1. There was one detected outlier, 1300, in this set of data. No reason could be found so it was retained in the data set (See Problem 1 for details of the outlier detection calculations.)

Problem 4 - Step 2. The k-sample Anderson-Darling test statistic is $ADK = 1.44$. (See Problem 1 for details of the computation of the k-sample statistic.) Since this is less than the critical value of 1.86, conclude that the data from the batches may be combined into a single sample.

Problem 4 - Step 3. The maximum normed residual (MNR) test is performed on the pooled data. The outlier detected in Step 1 was again identified as an outlier in the pooled data. (see Problem 1 - Step 1 for details of the outlier detection procedure.)

Problem 4 - Step 4. The results of the goodness-of-fit tests for the three distributions are:

Distribution	OSL
Two-parameter Weibull	0.003
Normal	0.011
Lognormal	0.000

(See problems 1, 2, and 3 for details of the computations for each of these tests.)

Since all of the observed significance levels are less than 0.05, it is concluded that the data do not follow any of the three distributions. Thus, the nonparametric method described in Section 8.3.4.5.1 must be used to calculate the B-basis value.

Problem 4 - Step 5. The first step in computing a B-basis value by the nonparametric method is to order the data values from smallest to largest. The five smallest values are 1300, 5500, 5500, 5700, and 5900. The next step is to obtain the appropriate rank from Table 8.5.12 corresponding to the sample of size n. With an n of 97, the rank of the observation to be used as a B-basis value is $r = 5$. Thus, the fifth observation, or 5900, is the B-basis value for this sample.

8.3.7.5 Problem 5 - Hanson-Koopmans method

The data set for this problem consists of compressive strength measurements for three batches of material. This problem illustrates the situation where none of the standard distributions adequately fit the data, and there is insufficient data to perform the nonparametric method. Calculations for all steps may be performed by STAT17 and may be demonstrated using example data set, `example.d05`.

Problem 5 - Step 1. There are no detected outliers in this set of data. (See Problem 2 for details of the outlier detection calculations.)

Problem 5 - Step 2. The k-sample Anderson-Darling test statistic is $ADK = 0.60$. (See Problem 1 for details of the computation of the k-sample statistic.) Since this is less than the critical value of 1.89, conclude that the data from the batches may be combined into a single sample.

Problem 5 - Step 3. The maximum normed residual (MNR) test is performed on the pooled data. No potential outliers are detected in the pooled data. (see Problem 1 - Step 1 for details of the outlier detection procedure.)

Problem 5 - Step 4. The results of the goodness-of-fit tests for the three distributions are:

Distribution	OSL
Two-parameter Weibull	0.047
Normal	0.039
Lognormal	0.035

(See problems 1, 2, and 3 for details of the computations for each of these tests.)

Since all of the observed significance levels are less than 0.05, it is concluded that the data do not follow any of the three distributions. The Hanson-Koopmans method should be used to calculate a B-basis value for these data, since there are only 15 data values.

Problem 5 - Step 5. Following the procedure described in Section 8.3.4.5.2, a B-basis value can be estimated. For $n = 15$, from Table 8.5.14 it is determined that $r = 8$ and $k = 1.54$. After ranking the data in ascending order, the first and eighth values are found.

$$x_{(1)} = 114.6 \qquad x_{(8)} = 133.4$$

The B-basis value is calculated as

$$B = x_{(r)}\left[\frac{x_{(1)}}{x_{(r)}}\right]^{k} = 133.4\left[\frac{114.6}{133.4}\right]^{1.54} = 104.365$$

These data can be included in MIL-HDBK-17 as interim data, but the B-value would not be reported in the handbook.

8.3.7.6 Problem 6 - Analysis of variance (ANOVA) method

The data set for this problem consists of tensile strength measurements from six batches of material. This problem illustrates the test for normality of multiple samples, the equality of variance test, and the calculation of basis values by the analysis of variance (ANOVA) method. Calculations for all steps may be performed by `STAT17` and may be demonstrated using example data set, `example.d06`. Calculations for Step 4 may be performed by `RECIPE` and may be demonstrated using example data set, `ex2.dat`.

Problem 6 - Step 1. There are no detected outliers in this set of data. (See Problem 1 for details of the outlier detection computations.)

Problem 6 - Step 2. The k-sample Anderson-Darling test statistic is $ADK = 2.45$. (See Problem 1 for details of the computation of the k-sample statistic.) Since ADK is greater than the critical value of 1.56, the hypothesis that the populations from which these groups are drawn are identical is rejected.

Problem 6 - Step 3. The equality of variance test described in Section 8.3.5.2.1 is used to determine if the within-batch variances are significantly different. The sample sizes (n_i), group medians ($\tilde{x}_i$), and group averages of $w_{ij} = \left|x_{ij} - \tilde{x}_i\right|$ are tabulated below.

Batch	n_i	$\tilde{x}_i$	$\bar{w}_i$
1	5	338.7	6.233
2	6	308.5	9.187
3	5	317.7	9.874
4	5	313.1	9.823
5	5	322.7	6.239
6	5	294.2	7.099

The transformed data are $w_{11} = |328.1 - 338.7| = 10.6$, $w_{12} = |334.8 - 338.7| = 3.9$, ..., $w_{65} = |294.2 - 294.2| = 0$. The test statistic is

$$F = \frac{\sum_{i=1}^{k} n_i(\bar{w}_i - \bar{w})^2/(k-1)}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(w_{ij} - \bar{w}_i)^2/(n-k)}$$

$$= \frac{\sum_{i=1}^{6} n_i(\bar{w}_i - \bar{w})^2/(6-1)}{\sum_{i=1}^{6}\sum_{j=1}^{n_i}(w_{ij} - \bar{w}_i)^2/(31-6)}$$

$$= 0.29$$

95th percentile of an F random variable with $\gamma_1 = k - 1 = 5$ and $\gamma_2 = n - k = 25$ degrees of freedom from Table 8.5.1 is 2.60. Since 0.29 is less than 2.60, the hypothesis that within-group variances are equal is not rejected.

Since the equality of variance test is a diagnostic test, a B-basis value may still be calculated, even when the hypothesis that higher group variances are equal is rejected. However, a nonconservative B-basis value can result in some instances when the variances are unequal. Unequal variances suggest potential problems with consistency in fabrication or processing of the different batches. The B-basis value calculated in such cases should be used with caution.

Problem 6 - Step 4. Summary statistics for the data are given in the table below.

Batch	n_i	$\bar{x}_i$	s_i
1	5	339.133	8.159
2	6	308.701	12.443
3	5	317.081	16.236
4	5	313.066	12.556
5	5	321.951	8.614
6	5	297.595	9.307

Preliminary ANOVA calculations covered in Section 8.3.5.2.6 are:

$$n^* = \sum_{i=1}^{k} n_i^2 / n = (5^2 + \ldots + 5^2)/31 = 5.19$$

$$n' = (n - n^*)/(k-1) = (31 - 5.19)/(6-1) = 5.16$$

$$\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i / n = [5(339.133)+\ldots+5(297.595)]/31 = 316$$

$$MSB = \sum_{i=1}^{k} \frac{n_i(\bar{x}_i - \bar{x})^2}{k-1} = \frac{1}{6-1}\left[5(339-316)^2+\ldots+5(298-316)^2\right] = 983.0$$

$$MSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{(x_{ij}-\bar{x}_i)^2}{n-k} = \frac{1}{n-k}\sum_{i=1}^{k}(n_i-1)s_i^2 = \frac{1}{31-6}\left[4(8.159)^2+\ldots+4(9.307)^2\right] = 134.8$$

The following tolerance limit factors are obtained from Table 8.5.10 (for B-basis values).

$$k_0 = 1.768 \qquad k_1 = 3.007$$

Note that the approximation to Table 8.5.10 is not used for small degrees of freedom. The tolerance limit factor is calculated as follows. Denote the ratio of mean squares by

$$\begin{aligned} S &= \sqrt{\frac{MSB}{n'} + \left(\frac{n'-1}{n'}\right)MSE} \\ &= \sqrt{\frac{983.0}{5.16} + \left(\frac{5.16-1}{5.16}\right)134.8} \\ &= 17.297 \end{aligned}$$

$$u = \frac{MSB}{MSE} = \frac{983.0}{134.8} = 7.292$$

(If u is less than one, set u equal to one.)

$$w = \sqrt{\frac{u}{u+n'1}} = \sqrt{\frac{7.292}{7.292+5.16-1}} = 0.7980$$

The tolerance limit factor is

$$\begin{aligned} t &= \frac{k_0 - k_1/\sqrt{n'} + (k_1-k_0)W}{1-\frac{1}{\sqrt{n'}}} \\ &= \frac{1.768 - 3.007/\sqrt{5.16} + (3.007-1.768)0.798}{1-\frac{1}{\sqrt{5.16}}} \\ &= 2.560 \end{aligned}$$

Thus, a B-basis value is calculated as

$$B = \bar{x} - ts = 316 - 2.560(17.297) = 271.72$$

For presentation in MIL-HDBK-17, this B-basis value would be rounded to 272.

The calculations for Step 4 can be performed using RECIPE, when batch-to-batch variability is significant or the ANOVA approach is desired. In this example, there are data on several batches, each tested

under the same set of fixed conditions. Since there is only one set of fixed conditions, the model for this example has a constant mean, but now there are both between-batch and within-batch components of variance. So $l = 1$, and

$$y_s = \theta_1 + b_{q(s)} + e_s$$

This is the usual random-effects ANOVA (or simply 'ANOVA') model of Section 8.3.5.2.

```
#
#  RECIPE Example #2: Basis value from a one-way ANOVA model
#  This corresponds to MIL-HDBK-17, Problem #6
#
#  -- This example has 31 observations in 6 batches, for which
#     an ANOVA B-basis value is to be determined
#
#   --  ntot, nlvl, nbch, npar, npts, prob, conf
   31 1 6 1 1 .9d0 .95d0
#
#  -- Fixed levels. Here we are fitting a one-way ANOVA model, so there
#     is only one fixed level, and only one fixed parameter (the mean)
#     to estimate.
   1
#
#  -- Fixed level number, batch number, strength. Since we have
#     only one fixed level, the first column is all ones. The
#     second column gives the batch number, and the third column
#     gives the strength values.
    1   1  328.1174
    1   1  334.7674
    1   1  347.7833
    1   1  346.2661
    1   1  338.7314
    1   2  297.0387
    1   2  293.4595
    1   2  308.0419
    1   2  326.4864
    1   2  318.1297
    1   2  309.0487
    1   3  337.0930
    1   3  317.7319
    1   3  321.4292
    1   3  317.2652
    1   3  291.8881
    1   4  297.6943
    1   4  327.3973
    1   4  303.8629
    1   4  313.0984
    1   4  323.2769
    1   5  312.9743
    1   5  324.5192
    1   5  334.5965
    1   5  314.9458
    1   5  322.7194
    1   6  291.1215
    1   6  309.7852
    1   6  304.8499
    1   6  288.0184
    1   6  294.1995
#
#   -- Points at which to evaluate tolerance limit. For the one-way
#      ANOVA model used here, there is only one point at which the
#      evaluation can be done: it corresponds to the one fixed
#      level of the model.
    1
```

The output is similar in form to the example in Problem 2, Step 6.

```
recipe
  Filename (without .dat extension) ?
ex2

  RECIPE : One-Sided Random-Effect Regression Tolerance Limits
  (Version 1.0, April 1995)

  *** Simulated pivot critical value file ex2.crt not found.
     Satterthwaite approximation will be used.

        Probability    Confidence   Regression  Tolerance Limit
           0.90           0.95     316.010884     271.672860
```

The results include a mean of 316 and a B-basis value of 272. Note, however, that the warning message that was output for Problem 2 does not appear. The between-batch variability can be estimated since there are six batches. The fourth column gives the one-way random effects ANOVA basis value.

8.3.7.7 Problem 7 - Linear regression

The data set for this problem consists of tensile test measurements at two fixed temperatures. This problem illustrates the regression analysis procedures presented in Section 8.3.5.3. Calculations for Step 1 may be performed by STAT17 and may be demonstrated using example data set, `example.d07`. Calculations for Steps 2 through 5 may be performed by RECIPE and may be demonstrated using example data set, `ex3.dat`. Note that a linear relationship between strength and temperature is not appropriate for all temperature ranges.

Problem 7 - Step 1. In this example, x represents the temperature and y the tensile strength determined from a group of tension tests. Outlier detection is useful applied to each temperature or fixed condition. There are no detected outliers for either temperature in this set of data.

Problem 7 - Step 2. From the data in Table 8.3.7, the following quantities may be calculated:

$$n = 11 \qquad (\Sigma x)^2 = 13225$$
$$\Sigma x = 115 \qquad (\Sigma y)^2 = 14163006$$
$$\Sigma y = 3763 \qquad (\Sigma x)(\Sigma y) = 432788.3$$
$$\Sigma x^2 = 56195 \qquad \Sigma xy = 37033.94$$
$$\Sigma y^2 = 1288172$$

$$S_{xx} = \Sigma x^2 - (\Sigma x)^2 / n = 56195 - (13225)/20 = 54992$$
$$S_{xy} = \Sigma xy - (\Sigma x)(\Sigma y)/n = 37033.94 - (115)(3763)/11 = 2310.459$$
$$S_{yy} = \Sigma y^2 - (\Sigma y)^2 / n = 1288172 - (14163006)/11 = 626.5063$$

The slope of regression line is:

$$b = \frac{S_{xy}}{S_{xx}} = \frac{-2310.459}{54992} = 0.0420$$

The y-intercept of the regression line is:

$$a = \frac{\Sigma y - b\Sigma x}{n} = \frac{3763}{11} \frac{(-0.0420)(115)}{11} = 342.1 - (-0.438) = 342.5644$$

Thus, the final equation of the least squares regression line is:

$$y^* = a + b\bar{x} = 342.5644 - 0.0420\bar{x}$$

Using this equation, the values of y^* in the table below are computed for the values of x in the data set.

x	y	y^*	$e = y - y^*$
75	328.1174	339.4134	-11.2959667
75	334.7674	339.4134	-4.6459667
75	347.7833	339.4134	8.3699333
75	346.2661	339.4134	6.8527333
75	338.7314	339.4134	-0.6819667
75	340.8146	339.4134	1.4012333
-67	343.5855	345.3793	-1.7938400
-67	334.1746	345.3793	-11.2047400
-67	348.6610	345.3793	3.2816600
-67	356.3232	345.3793	10.9438600
-67	344.1524	345.3793	-1.2269400

The root mean square error is computed as follows:

$$s_Y = \sqrt{\frac{\Sigma(y - y^*)^2}{n-2}} = \sqrt{\frac{529.5}{9}} = 7.669818$$

and R^2 is computed as follows:

$$R^2 = \frac{b^2 S_{xx}}{S_{yy}} = \frac{(-0.0420)^2(54992)}{626.5063} = 0.1549$$

Thus, 15% of the variability in the y data about its average is explained by the linear relationship between y and x.

Problem 7 - Step 3. One of the assumptions made in linear regression analysis is that the residuals are normally distributed about the regression line. The validity of this assumption may be checked by performing a normal goodness-of-fit test on the residuals as discussed in Section 8.3.5.1. Note that the $z_{(i)}$ values used in the Anderson-Darling statistic are defined as $z_{(i)} = e_{(i)} / s_y$, where $e_{(i)}$ is the i^{th} ordered residual and S_y is the root-mean-square error from the regression. The eleven ordered residuals and the preliminary goodness-of-fit calculations are shown in the following table.

$e_{(i)}$	$z_{(i)} = \frac{e_{(i)}}{S_y} = \frac{e_{(i)}}{58.83}$
-11.2959667	-1.47278162
-4.6459667	-0.60574667
8.3699333	1.09128187
6.8527333	0.89346752
-0.6819667	-0.08891563
1.4012333	0.18269447
-1.7938400	-0.23388299
-11.2047400	-1.46088734
3.2816600	0.42786674
10.9438600	1.42687349
-1.2269400	-0.15996990

The normal goodness-of-fit test statistic is 0.222 with an OSL of 0.590. (See Problem 2 for details of the computation for the normal goodness-of-fit test.) Since the OSL is greater than 0.05, there is insufficient evidence to contradict the assumption that the residuals are normally distributed.

Problem 7 - Step 4. There are multiple y observations for several of the x values. Thus, it is possible to construct an analysis of variance table to test the adequacy of the regression as discussed in Section 8.3.5.3. The sums of squares for the three primary lines of the analysis of variance table are calculated as follows:

$$SSR = b^2 S_{xx} = (-0.0420)^2(54992) = 97.7138$$
$$SST = S_{yy} = 626.5063$$
$$SSE = SST - SSR = 626.5063 - 97.7138 = 529.4349$$

The mean squares are calculated as shown below.

$$MSR = SSR = 97.07138$$
$$MSE = SSE/n-2 = 529.4349/9 = 58.82611$$
$$F = MSR/MSE = 97.07138/58.82611 = 1.650141$$

The analysis of variance table is shown below.

Source of Variation	Degrees of Freedom	Sum of Squares, SS	Mean Squares, MS	F_{calc}
Regression	1	97.07	97.07	$F = 1.65$
Error	9	529.4	58.83	
Total	10	626.5		

The F value of 1.65 with 1 and $n-2=9$ degrees of freedom is less than the value of 5.12 from Table 8.5.1 corresponding to 1 and 9 degrees of freedom, so the regression may be negligible.

Problem 7 - Step 5. With the linear regression equation from step 1, lower tolerance limits may be calculated at any temperature (x value) by the procedure in Section 8.3.5.3. Details for computing a B-basis value at $x = 25$ are given below.

The average temperature value in the data set is:

$$\bar{x} = \sum x/n = 115/11 = 10.45$$

The Δ factor required to compute the tolerance limit factor, , is:

$$\Delta = \frac{(x_0 - \bar{x})^2}{\sum_{i=1}^{n} (x_0 - \bar{x})^2 / n} = \frac{(25 - 10.45)^2}{(54992)/11} = 0.0423$$

The approximation for the k' factor is:

$$k'_B = 1.282 + \exp\left[0.595 - 0.508\ln(n) + \frac{4.62}{n} + \left(0.486 - \frac{0.986}{n}\right)\ln(1.82 + \Delta)\right]$$

$$= 1.282 + \exp\left[0.595 - 0.508\ln(11) + \frac{4.62}{11} + \left(0.486 - \frac{0.986}{11}\right)\ln(1.82 + 0.0423)\right]$$

$$= 2.33$$

Thus, a B-basis value at $x = 25$ is computed as

$$B = (a + b\,x_0) - k_B s_y = \left[342.5 + (-0.0420)(20) - 2.33(7.669818)\right] = 323.64339$$

For presentation in MIL-HDBK-17, this value would be rounded to 324.

RECIPE provides the linear regression calculations for this problem. There are data from a single batch, so that $m = 1$; but the possibility of several conditions ($l > 1$) is included. To fix ideas, assume that there are several sets of unidirectional tensile strength data from a single batch, with each set being tested at a different temperature, and with all other conditions held constant. Assume further that the strength for this material is believed to vary linearly with temperature, at least for temperatures within the range of the data. With only one batch, the between-batch variability cannot be estimated. The regression model appropriate for this situation is

$$y_s = \theta_1 z_{p(s),1} + \theta_2 z_{p(s),2} + e_s$$

This is the simple linear regression model of Section 8.3.5.3.

The file ex3.dat, which corresponds to this problem, is:

```
#
#  RECIPE Example #3: Regression model with data from a single batch
#  This corresponds to MIL-HDBK-17, Problem #7
#
#  -- This dataset has 11 observations at two fixed levels. The
#     data come from 1 batch, there are two fixed parameters to
#     estimate (the slope and intercept of a straight line), and
#     a B-basis value is to be calculated at 7 points on this line.
#
#   --  ntot, nlvl, nbch, npar, npts, prob, conf
   11 2 1 2 7 .9d0 .95d0
#
#  -- We are fitting a model y=a+bT at two levels: T=75 degrees and
#     T=-67 degrees.  The first column corresponds to 'a' in this
#     linear equation; the second column corresponds to 'b'. Note
#     that these values need not be given in any special order,
#     for example (1, -67) need not come before (1, 75).  The
#     important thing is that the order of the rows given here
#     must correspond to the level indicator, p(s), given with each
#     response value.
```

```
   1 75
   1 -67
#
#  -- Now we have the 11 observations.  The first column is the
#     level (=1 for 75 degrees, =2 for -67 degrees), the second
#     column is the batch (always 1), and in the third column are
#     the strength observations.
#
   1    1   328.1174
   1    1   334.7674
   1    1   347.7833
   1    1   346.2661
   1    1   338.7314
   1    1   340.8146
   2    1   343.5855
   2    1   334.1746
   2    1   348.6610
   2    1   356.3232
   2    1   344.1524
#
#  -- Finally, we give the seven points at which basis
#     values are to be determined.  These correspond
#     to seven different temperatures -67,...,50. Note
#     that the first column of ones is required because
#     of the intercept in the regression model
  1 -67
  1 -50
  1 -25
  1  0
  1 25
  1 50
  1 75
```

Note that the first noncomment line of `ex3.dat` indicates (in order, from left to right) that there are 11 observations in all, that the data are at 2 fixed levels, that all of the data are from a single batch, that the fixed part of the model involves 2 unknown parameters (actually, a straight line is being fit to the data), that the basis value curve will be evaluated at 7 points, and that the tolerance limits to be calculated are B-basis values.

This example illustrates a simplification of the common situation where a material basis value is required as a function of temperature. One has data at two fixed levels, corresponding to the temperatures -67 and 75 °F, and one would like to determine basis values at the 7 temperatures -67, -50, -25, 0, 25, 50, and 75 °F. The intercept of the linear function is, of course, constant for all temperatures, so the first column equals 1 for the 2 rows that give the levels of the fixed effect, as well as the 7 rows that give the points at which the basis values are to be evaluated. The output from running `RECIPE` on these data is

```
recipe
  Filename (without .dat extension) ?
ex3

  RECIPE : One-Sided Random-Effect Regression Tolerance Limits
  (Version 1.0, April 1995)

  *** Simulated pivot critical value file ex3.crt not found.
     Satterthwaite approximation will be used.

  regini : Warning: between-batch variance cannot
           be estimated from these data. Results
           will be based on the assumption that the
           between-batch variability is negligible.

        Probability    Confidence   Regression  Tolerance Limit
```

```
0.90          0.95     345.379340     325.887099
0.90          0.95     344.665104     325.747683
0.90          0.95     343.614756     325.338699
0.90          0.95     342.564409     324.619436
0.90          0.95     341.514062     323.538853
0.90          0.95     340.463714     322.102027
0.90          0.95     339.413367     320.366619
```

Each of the last seven lines gives a point on the regression line, and the corresponding point on the B-basis curve for each of the seven sets of covariates (temperatures) in the file `ex3.dat`. Note that there is a warning message, since one cannot estimate between-batch variability using data from a single batch. The basis values calculated are valid under the assumption that the between-batch variability is zero (or at least negligible).

8.3.7.8 Problem 8 - Simple linear regression with a random effect

The data set for this problem consists of compression test measurements at two temperatures with several batches represented at each temperature. This problem illustrates the same situation as Problem 7 except data are available for more than one batch. Calculations for Step 1 can be performed by `STAT17` and can be demonstrated using example data set, `example.d08`. Calculations for Step 2 are demonstrated by example data set, `ex4.dat`. and `RECIPE`. Note that a linear relationship between strength and temperature is not appropriate for all temperature ranges.

Problem 8 - Step 1. In this example, x represents the temperature and y the tensile strength determined from a group of tension tests. Outlier detection is useful applied to each temperature or fixed condition. There are no detected outliers for either temperature in this set of data.

Problem 8 - Step 2. The random batch effect $b_{q(s)}$ can now be introduced into the model, leading to

$$y_s = \theta_1 z_{p(s),1} + \theta_2 z_{p(s),2} + b_{q(s)} + e_s$$

where $z_{p(s),1} = 1$, $z_{p(s),2} = T_i$, the i^{th} test temperature, and $b_{q(s)}$ is the batch mean for the $q(s)^{th}$ batch. The file `ex4.dat`, which corresponds to this problem, is

```
#
#     RECIPE Example #4: Regression model with data from several
#                        batches
#     This corresponds to MIL-HDBK-17, Problem #8
#
#  -- In this example, we have 72 strength observations on data
#     from 8 batches.  A straight-line regression is fit with
#     two fixed levels (temperatures).  B-basis values are calculated
#     for 7 points along this curve.
#
#  -- ntot, nlvl, nbch, npar, npts, prob, conf
 72 2 8 2 7 .9d0 .95d0
#
#  -- There are two fixed levels, corresponding to
#     75 and -67 degrees.
 1  75
 1 -67
#
#  -- The following 72  rows give the fixed level in the
#     first column, the batch in the second column, and the
#     strength observation in the third column.
   1   1  328.1174
   1   1  334.7674
   1   1  347.7833
```

1	1	346.2661
1	1	338.7314
1	2	297.0387
1	2	293.4595
1	2	308.0419
1	2	326.4864
1	2	318.1297
1	2	309.0487
1	3	337.0930
1	3	317.7319
1	3	321.4292
1	3	317.2652
1	3	291.8881
1	4	297.6943
1	4	327.3973
1	4	303.8629
1	4	313.0984
1	4	323.2769
1	5	312.9743
1	5	324.5192
1	5	334.5965
1	5	314.9458
1	5	322.7194
1	6	291.1215
1	6	309.7852
1	6	304.8499
1	6	288.0184
1	6	294.1995
2	1	340.8146
2	1	343.5855
2	1	334.1746
2	1	348.6610
2	1	356.3232
2	1	344.1524
2	2	308.6256
2	2	315.1819
2	2	317.6867
2	2	313.9832
2	2	309.3132
2	2	275.1758
2	3	321.4128
2	3	316.4652
2	3	331.3724
2	3	304.8643
2	3	309.6249
2	3	347.8449
2	4	331.5487
2	4	316.5891
2	4	303.7171
2	4	320.3625
2	4	315.2963
2	4	322.8280
2	5	340.0990
2	5	348.9354
2	5	331.2500
2	5	330.0000
2	5	340.9836
2	5	329.4393
2	7	330.9309
2	7	328.4553
2	7	344.1026
2	7	343.3584
2	7	344.4717
2	7	351.2776

```
   2   8  331.0259
   2   8  322.4052
   2   8  327.6699
   2   8  296.8215
   2   8  338.1995
#
#    -- The following 7 rows give the points at which
#       the B-basis value is to be calculated: these
#       correspond to 7 temperatures -67,-50,...,75.
  1 -67
  1 -50
  1 -25
  1  0
  1 25
  1 50
  1 75
```

A run of RECIPE produces the output:

```
recipe
 Filename (without .dat extension) ?
ex4

  RECIPE : One-Sided Random-Effect Regression Tolerance Limits
  (Version 1.0, April 1995)

  *** Simulated pivot critical value file ex4.crt not found.
     Satterthwaite approximation will be used.

        Probability    Confidence   Regression  Tolerance Limit
           0.90           0.95     327.537310     286.895095
           0.90           0.95     326.157386     285.580736
           0.90           0.95     324.128085     283.557672
           0.90           0.95     322.098785     281.470595
           0.90           0.95     320.069485     279.335972
           0.90           0.95     318.040184     277.119935
           0.90           0.95     316.010884     274.783636
```

The input and output files have the same form as Problem 7. The important distinction between Problem 7 and Problem 8 is that the basis values in Problem 8 account for between-batch variability, while in Problem 7 the calculated basis values are strictly valid for a specific batch. Note also that the warning message that appeared in Problem 7 does not appear here, since there are data from several batches.

8.3.7.9 Problem 9 - One-way mixed-model ANOVA: basis values with data from multiple sources

The data set for this problem consists of tensile test measurements for several batches each from more than one manufacturer. Calculations for Steps 1 and 2 may be performed by STAT17 and may be demonstrated using example data set, `example.d09`. Calculations for Step 3 are demonstrated by example data set, `ex5.dat`. and RECIPE.

Suppose that one has several batches of data from each of several manufacturers, and that these manufacturers wish to combine their resources to determine basis values. If one is absolutely certain that the manufacturing and testing are identical for all of the data, then one can ignore the fact that the data came from multiple sources. Often, however, there will be slight differences among the manufacturers in the way that the material was fabricated, tested, or both. In such cases, if one is unwilling to assume that the variability between and within batches are close to being the same for all manufacturers, there is no alternative to applying the usual ANOVA method (as in Section 8.3.5.2) separately to each manufacturer's data. However, if one is willing to assume that each set of data exhibits the same variability (with a possible different mean for each manufacturer), then *all* of the batches can be used to deter-

mine a basis value for *each* manufacturer. These basis values will often be substantially higher, and closer together, than if each manufacturer had acted alone.

Problem 9 - Step 1. As in Problem 6, each batch should be examined for outliers. No outliers were identified for these batches.

Problem 9 - Step 2. For this case, it may also be worthwhile to group the data by manufacturer, and evaluate each group for outliers. The outlier detection procedure is demonstrated in Problem 1.

Problem 9 - Step 3. To develop a regression model for this example, let the mean for the i^{th} manufacturer by μ_i. If there are l manufacturers, we have $r = l$ unknown fixed parameters, $\mu_1, \mu_2, \ldots, \mu_l$ in addition to the components of variance σ_b^2 and σ_e^2. Hence, the regression model is of the form

$$\begin{aligned} y_s &= \theta_1 z_{p(s),1} + \theta_2 z_{p(s),2} + \theta_3 z_{p(s),3} + \ldots + \theta_\ell z_{p(s),\ell^2} + b_{q(s)} + e_s \\ &= \mu_{p(s)} + b_{q(s)} + e_s \end{aligned}$$

The z's are taken to be $z_{p(s),u} = \delta_{p(s),u}$, where $\delta_{p(s),u}$ (the Kronecker δ) equals one where $p(s) = u$, and zero otherwise. The fixed parameters are $\theta_i = \mu_i$.

The example data set `ex5.dat`, which corresponds to this problem, contains data on several batches of the same material from each of two manufacturers. For this example, assume that the variability is the same for each manufacturer. The number of fixed levels $l = r = 2$.

```
#
#  RECIPE Example #5: Basis values using data from multiple sources
#  This corresponds to MIL-HDBK-17, Problem #9
#
#   -- In this example, we have five batches of data: three from
#      one source, and two from a second source.  We would like
#      to use all five batches of data to get a tolerance limit
#      for each source.
#
#   --  ntot, nlvl, nbch, npar, npts, prob, conf
#
  15 2 5 2 2 .9d0 .95d0
#
#   -- The fixed part of this model is a different mean for
#      each of the two sources
   1 0
   0 1
#
#   -- Here are the 15 data values.  Column 1 indicates the
#      fixed level (data source), and column 2 indicates the
#      number of the batch. The third column gives the strength
#      values.
  1  1  75.8
  1  1  78.4
  1  1  82.0
  1  2  68.8
  1  2  70.9
  1  2  73.5
  1  3  74.5
  1  3  74.8
  1  3  78.8
  2  4  81.3
  2  4  87.7
  2  4  89.0
```

```
  2  5  88.2
  2  5  91.2
  2  5  94.2
#
#    -- The tolerance limit are to be calculated at two
#       points, which correspond to the two sources. So
#       we just repeat the two lines for the fixed part
#       of the model here.
  1 0
  0 1
```

From the file `ex5.dat` one can see that there are 15 data values, and that a regression model is being used with $r = 2$ parameters. The first column of the 15 rows of `ex5.dat` that contain data indicates the fixed level, the second column for these rows indicates the batches, and the third column gives the strength values. The fixed part of the model has two means, one for each data source. So the rows that give the fixed levels, and the rows that give the points at which basis values are to be evaluated have a '1' in one column and a '2' in the other. Contrast this with Problems 2 and 6 where there is only one fixed level and so the corresponding rows have just one column having a single value, 1.

The RECIPE output for this example is:

```
recipe
 Filename (without .dat extension) ?
ex5

  RECIPE : One-Sided Random-Effect Regression Tolerance Limits
  (Version 1.0, April 1995)

  *** Simulated pivot critical value file ex5crt not found.
     Satterthwaite approximation will be used.

        Probability    Confidence   Regression  Tolerance Limit
            0.90           0.95
                                    75.27778       59.401536
            0.90           0.95
                                    88.600000      71.902179
```

The B-basis values are therefore 59.4 and 71.9 for the two manufacturers. As a simple exercise in using `RECIPE`, one can show (following Problem 6, using the data from this problem) that if each manufacturer had used only their own data, then the B-basis values would be 52.8 and 34.6, respectively. Note that the mixed model gives basis values which are higher and closer together. In particular, the very low value 34.6 is due to the second manufacturer having data from only two batches.

8.4 STATISTICAL METHODS

8.4.1 Tests for determining equivalency between an existing database and a new dataset for the same material

There are several situations where it is required to determine whether a sample of test data is equivalent to a baseline data set for the same raw material:

- For material batch certification and acceptance, it must be shown that the properties of the batch are "equivalent" to the qualification database; i.e., the batch data meet the material specification acceptance limits.
- A material supplier wishes to modify the production process for the raw material.
- A part manufacturer wants to design using a common database of material properties and basis values that was developed by another organization. Whether they are using the exact same fabrication process as was used for the laminates used to obtain the shared database, or are using a

modified fabrication process, the manufacturer must demonstrate the "equivalency" of its production methods in obtaining the same material properties.

- A part manufacturer who has established a database of material properties, specification values and basis values, and who wants to modify its fabrication process without regenerating the property database.

This section is intended to provide the statistical procedures for determining data "equivalency" in the above types of situations. These procedures are not intended for use in determining the acceptability for use of an alternate ("second source") material; for this situations refer to Section 8.4.2. In the following procedures, a probability level of rejecting a "good" material, α, must be selected. Other sections of this Volume recommend the appropriate value of α for particular situations; for instance, see Section 2.3.7 for requirements to substantiate the use of MIL-HDBK-17 Volume 2 data.

The mechanical and chemical properties of material specimens are subject to random variability. Hence, one must accept the possibility of making an error in declaring a "good" material property to have failed a statistical test. For a fixed number of test specimens in a sample, the probability of this undesirable event occurring (defined as α in the following statistical tests) can only be made small at the expense of decreased likelihood of detecting failures in material when failure should be declared. The selection of the value for the probability of failing a statistical test in error, α, is a compromise between the two types of errors. If the statistical tests are being used with test programs where retests of "failed" properties are allowed, then a slightly higher value for α can be used, as the α after one retest will be effectively α^2.

The criteria used to determine equivalency between a large database of a given composite material and a subsequent test sample of the same material are selected based on the material properties of interest.

The criterion for modulus or physical properties such as per ply thickness require that the mean value be within an acceptable range; neither a high nor a low mean is desirable. The criterion for these properties is designed to reject either a high or a low mean value. The appropriate statistical method is given below as "Test for Change in Mean".

The criterion for strength properties, on the other hand, must reject either a low mean or a low minimum individual value. The appropriate statistical method for strength properties is given below as "Test for Decrease in Mean or Minimum Individual". This test was developed to have equal probability of rejecting a "good" set of data with either the test on the mean or on the minimum individual property. This balance between the two conditions of the test gives the maximum "statistical power", and is an improvement over the ad-hoc methods used in industry to set material specification acceptance limits.

The criterion for certain chemical and physical properties, such as volatile content or porosity level, must reject a high mean value, as the desired property value is 0. The appropriate statistical method for these properties is given below as "Test for a High Mean".

Test for Decrease in Mean or Minimum Individual – The mean, and standard deviation are approximated by $\bar{x}$ and s, from the individual test condition (environment) of the original material qualification database. The pass/fail thresholds for mean properties, W_{mean}, are determined by equation 8.4.1(a). The k_n^{Mean} values are given in Table 8.5.17. The mean values from experimental tests must meet or exceed

$$W_{mean} = \bar{x} - k_n^{Mean} S \qquad 8.4.1(a)$$

The pass/fail thresholds for minimum individual properties, $W_{minimum\ individual}$, are determined by equation 8.4.1(b). The k_n^{Indv} values are given in Table 8.5.18. The minimum individual values from experimental tests must meet or exceed

$$W_{\text{Minimum Individual}} = \bar{x} - k_n^{\text{Indv}} S \qquad 8.4.1(b)$$

Test for Change in Mean - Since the sample sizes of the original database, n_1, and the new data sample, n_2, are different, a pooled standard deviation, S_p, is used as an estimator of common population standard deviation.

$$S_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \qquad 8.4.1(c)$$

Using the pooled standard deviation and the mean values of the original and new data sets, the test statistic, t_0, is calculated using:

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{S_p \bullet \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \qquad 8.4.1(d)$$

Since this is a two-sided t-test, the required t value is $t_{a,n} = t_{\alpha/2, n_1 \_ n_2 - 2}$. Note that $a = \alpha/2$ for the two–sided test. $t_{a,n}$ is obtained from Table 8.5.19.

For a material to pass the this test, the test statistic, t_0, must satisfy:

$$-t_{\alpha/2, n_1 + n_2 - 2} \le t_0 \le t_{\alpha/2, n_1 + n_2 - 2} \qquad 8.4.1(e)$$

Test for a High Mean - For this test, the test statistic, t_0, is obtained using equation 8.4.1(d). This test is designed to detect undesirably high mean values such as in the case of volatile content of prepreg. The mean of the "follow-on" property is said to be less than or equal to the mean of the "original" property if equation 8.4.1(f) is satisfied, an indication of an acceptable material and/or process. This is a one-sided t-test so $t_{a,n} = t_{\alpha, n_1 + n_2 - 2}$. Note that $a = \alpha$ for one-sided test. $t_{a,n}$ is obtained from Table 8.5.19. Thus, for a material to pass this test, the test statistic, t_0, must satisfy:

$$t_0 \le t_{\alpha, n_1 + n_2 - 2} \qquad 8.4.1(f)$$

Recommended Values for α

For determining batch acceptance limits for material specifications, setting the probability of rejecting a good property (α) to 0.01 (1%) is recommended for all test methods that utilize the test statistics. A minimum of five specimens for strength properties and three specimens for modulus properties is recommended for material batch acceptance testing.

For determining material equivalency (such as the second through fourth situations listed in the introduction to this section), setting the probability of rejecting a good property (α) to 0.05 (5%) is recommended for all test methods that utilize the test statistics. One retest is allowed for each property, reducing the actual probability to 0.0025 (0.25%). A minimum of eight specimens is recommended for strength property comparisons (typically four specimens from two separate panels and processing cycles). A minimum of four specimens is recommended for modulus comparisons (typically two specimens from two separate panels and processing cycles). In the case where one or more properties fail the appropriate criteria, one may choose to retest only those properties that failed the criteria.

8.4.2 Alternate material statistical procedures

Considerable data, including allowables, are often available on the in-house fabrication of a particular material system with raw material obtained from a particular supplier. A change in some aspect of the material system is contemplated, such as a switch to a new supplier. The additional testing which is required is specified in Sections 2.3.4. The present section describes statistical procedures which will help determine when the original and alternative materials differ to an extent that is too large to be plausibly attributed to chance. If the methods of this section indicate a statistically significant difference, and if the magnitude of this difference is meaningful from an engineering standpoint, then the alternative material probably should not be qualified without further testing.

This section assumes that data as required by Section 2.3.4 are available. Because differences in materials are usually observed to be differences in the mean of a property, the methods of this section focus on the comparison of means. It is important to note that, although no formal test for comparing variances is provided, differences in variability which are substantially larger than what is consistent with experience with similar materials should be investigated.

The means for each mechanical property for which data are available for both the original and the alternative material are compared using a two sample t-test which allows for a random batch effect (Section 8.4.2.1). This analysis will result in a set of observed significance levels and confidence intervals. Any of the mean differences which are statistically significant at the five percent level should be investigated.

To get a single number which measures the difference between the materials, let p_i be the OSL for the ith property as determined in Section 8.4.2.1, and let m be the number of properties compared. Calculate the following:

$$P = -2 \sum_{i=1}^{m} \ln(p_i) \qquad 8.4.2$$

The larger P is, the more evidence there is in the data for a difference between the materials. Compare P with the 95th percentile of a chi-square distribution with $2m$ degrees of freedom (Table 8.5.2) in order to determine if the differences in the means is significant at the five percent significance level.

This combined test is strictly valid only when the m sets of data are statistically independent of each other. However, since the tests are on the same material and the same batches, this independence will never exactly hold. In many situations, the tests will be approximately independent and the combined value P will provide a useful measure of the extent to which the two sets of tests differ. If it is apparent from examining the data that some batches are consistently high and others low, i.e. that independence does not hold, then the combined test should be interpreted with caution.

8.4.2.1 Comparing two groups of batches

This section considers the problem of testing whether a statistically significant difference exists between the means of two sets of measurements, where each set consists of several batches. The methods of this section might be applied, for example, to compare the mean room temperature tensile strength of specimens made from three batches at one site to another set of measurements on the same mechanical property consisting of five batches manufactured at another site.

The two sets of data are represented by x_{ij} and y_{ij} where the first subscript indicates the batch and the second subscript denotes the values within each batch. We assume here that both the x and y sets of data are sampled from one-way, balanced, random effects models (see Section 8.3.5.2):

$$x_{ij} = \mu^{(1)} + b_i^{(1)} + e_{ij(1)} \qquad 8.4.2.1(a)$$

where $i = 1,...,k_1$ and $j = 1,...,n_1$ and

$$y_{ij} = \mu^{(2)} + b_i^{(2)} + e_{ij(2)} \qquad 8.4.2.1(b)$$

where $i = 1,...,k_2$ and $j = 1,...,n_2$. The number of batches and batch size are k_1 and n_1 for the x's and k_2 and n_2 for the y's.

The ANOVA model represents each observation as the sum of three components; $\mu^{(\ell)}$ is the overall mean, $\mu^{(\ell)} + b_i^{(\ell)}$ is the population average for the ith batch, and $e_{ij(\ell)}$ represents the variation within each batch, where ℓ equals one for the x data and two for the y data. The error terms $e_{ij(\ell)}$ are assumed to be independently distributed normal random variables with mean zero and variance σ_e^2 (the within batch variance).

The batch means $b_i^{(\ell)}$ are assumed to be independent random variables following a normal distribution with zero and a variance of σ_b^2 (the between batch variance). The within-batch variance is assumed to be the same for all batches.

Denote the batch averages for the x's by $\bar{x}_i$, for $i = 1,...,k_1$, and the batch averages for the y's by $\bar{y}_i$, for $i = 1,...,k_2$. the test statistic uses the following four quantities:

$$\bar{x} = \frac{1}{k_1} \sum_{i=1}^{k_1} \bar{x}_i \qquad 8.4.2.1(c)$$

$$\bar{y} = \frac{1}{k_2} \sum_{i=1}^{k_2} \bar{y}_i \qquad 8.4.2.1(d)$$

$$s_x^2 = \frac{n_1}{k_1 - 1} \sum_{i=1}^{k_1} (\bar{x} - \bar{x}_i)^2 \qquad 8.4.2.1(e)$$

$$s_y^2 = \frac{n_2}{k_2 - 1} \sum_{i=1}^{k_2} (\bar{y} - \bar{y}_i)^2 \qquad 8.4.2.1(f)$$

If $k_1 = 1$, then let $s_x^2 = 0$; if $k_2 = 1$, let $s_y^2 = 0$; if $k_1 = k_2 = 1$, then the method of this subsection should not be used. In terms of the statistics in Equations 8.4.2.1(c) - (f), the test statistic is

$$T = \frac{\left|\bar{x} - \bar{y}\right|}{\left(\frac{s_x^2}{k_1 n_1} + \frac{s_y^2}{k_2 n_2} \right)^{0.5}} \qquad 8.4.2.1(g)$$

To test the hypothesis that $\mu^{(1)} = \mu^{(2)}$ at the α significance level, compare T with $t_{1-\alpha/2,\gamma}$, the $100(1 - \alpha/2)$ quantile of a central t random variable with $\gamma = k_1 + k_2 - 2$ degrees of freedom (Table 8.5.3). If T does not exceed this t quantile, then conclude that the data are consistent with the hypothesis that the population means are equal, otherwise conclude that there is a statistically significant difference in the population means (at the α level of significance).

A $100(1 - \alpha)$ confidence interval is

$$\left|\bar{x} - \bar{y}\right| \pm t_{1-a/2,\gamma} \left(\frac{s_x^2}{k_1 n_1} + \frac{s_y^2}{k_2 n_2} \right)^{0.5} \qquad 8.4.2.1(h)$$

The observed significance level, or OSL, is the probability of observing a value of T as large or larger than the T actually observed if indeed the hypothesis of equal means is true. An OSL which is less than the significance level α indicates that the null hypothesis can be rejected at the α level of significance. The OSL is a function of T and $\gamma = k_1 + k_2 - 2$. For γ greater than ten, the following approximation is usually adequate. Calculate

$$u = \frac{T\left(1 - \frac{1}{4\gamma}\right)}{\left(1 + \frac{T^2}{2\gamma}\right)} \qquad 8.4.2.1(i)$$

Determine the probability P that a standard normal random variable is less than u. This probability can be determined from a table of the normal distribution, such as Table 8.5.5. The OSL is equal to $2(1 - P)$. If γ is less than 10, then the above approximation is not sufficiently accurate and the OSL should be obtained from Table 8.5.4.

For example, consider the strength measurements in Table 8.4.2.1. The specimens tested to give these data were taken from a group of three consecutive batches and a group of five consecutive batches. The second group of batches was produced more than a year after the first group. Because of this time difference, one should not consider these data to be eight random batches from a single population without justification. A more prudent approach is to regard these test results as a random sample of three batches from one population and a random sample of five batches from a possible different population.

TABLE 8.4.2.1 *Strength measurements from two groups of consecutive batches*.

Set 1			Set 2		
Mean	Variance	n	Mean	Variance	n
402.2	138.7	5	408.4	40.8	5
387.8	1002.2	5	395.8	113.2	5
389.4	321.8	5	357.2	451.7	5
			376.2	119.7	5
			377.0	189.5	5

For the data in Table 8.4.2.1, Equations 8.4.2.1(c) - (g) give the following:

$$\bar{x} = 393 \qquad 8.4.2.1(j)$$

$$\bar{y} = 383 \qquad 8.4.2.1(k)$$

$$s_x^2 = 311 \qquad 8.4.2.1(l)$$

$$s_y^2 = 1946 \qquad 8.4.2.1(m)$$

$$T = \frac{\left|393 - 383\right|}{\left(\frac{311}{(3)(5)} + \frac{1946}{(5)(5)} \right)^{0.5}} = 1.007 \qquad 8.4.2.1(n)$$

From Table 8.5.3, the 97.5 percentile of the t distribution with 5 + 3 - 2 = 6 degrees of freedom is $t_{0.975,6}$ = 2.45. Since 1.007 is less than 2.45, one concludes that there is no statistically significant difference in the mean strength for the two sets of data at the five percent significance level. Because $k_1 + k_2 - 2 = 6$ is less than ten, use Table 8.5.4 to obtain an OSL of 0.0.36. A 95 percent confidence interval for the difference in the means is given by

$$\left|393 - 383\right| \pm (2.45)\left(\frac{311}{(3)(5)} + \frac{1946}{(5)(5)}\right)^{0.5} \qquad 8.4.2.1(o)$$

$$10 \pm 24.3$$

Note that the confidence interval must contain zero for this example since the difference in the means is not significant at the 95 percent level.

8.4.3 Confidence intervals for the coefficient of variation

The coefficient of variation is the ratio of the population standard deviation to the population mean. This section provides a method for calculating confidence intervals for a coefficient of variation, assuming that the underlying distribution is normal. The coefficient of variation of the population is estimated by the sample coefficient of variation

$$c = \frac{s}{\bar{x}} \qquad 8.4.3(a)$$

where s is the sample standard deviation and $\bar{x}$ is the sample mean.

An approximate 100γ % confidence interval for the coefficient of variation has lower limit

$$c_l = c\left[\left(\frac{u_1+2}{n}-1\right)c^2+\frac{u_1}{n+1}\right]^{-\frac{1}{2}} \qquad 8.4.3(b)$$

and upper limit

$$c_h = c\left[\left(\frac{u_2+2}{n}-1\right)c^2+\frac{u_2}{n+1}\right]^{-\frac{1}{2}} \qquad 8.4.3(c)$$

where u_1 and u_2 are $100\,(1+\gamma)/2$ and $100\,(1-\gamma)/2$ percentiles of the χ^2 distribution with $n-1$ degrees of freedom. Values of u_1 and u_2 are tabulated in Table 8.5.16 for γ equal to 0.9, 0.95, and 0.99.

8.4.3.1 Example of CV confidence interval calculation

A sample of five specimens has sample mean $\bar{x} = 103.8$, sample standard deviation $s = 4.161$, and sample coefficient of variation

$$c = \frac{4.161}{103.8} = 0.0400 \qquad 8.4.3.1$$

The constants u_1 and u_2 are, from Table 8.5.16, found to be $u_1 = 5(2.2287) = 11.1435$ and $u_2 = 5(0.0968883) = 0.48444$. By substituting in Equations 8.4.3(a) and (b), an interval, which contains the population coefficient of variation, with 95% confidence, is determined to have lower limit $c_l = 0.0240$ and upper limit, $c_h = 0.115$.

8.4.3.2 Comment on the approximation

This approximate method is adequate for situations where the population coefficient of variation is less than 35%. It is usually extremely accurate, and it is exact in the limit of large samples and also in the limit of small population coefficient of variation. For details of the derivation and properties of this approximation, see Reference 8.4.3.2. For measurements made on populations with coefficients of variation substantially larger than this, an exact (but somewhat complicated) method is available. However, if one is willing to consider the possibility of a population C.V. much larger than 35%, then in order for the normal model to make sense, one must also accept the possibility of negative values. Hence, if a quantity is necessarily positive, then a very large C.V. implies that the normal model does not make physical sense. Consequently for those cases where this approximation fails, one would usually not want to assume a normal model anyway, so one would seldom, if ever, need the complicated exact procedure.

8.4.4 Statistical procedures for process control

8.4.4.1 $\bar{x}$ bar chart including batch effect

Assume that the data are a sample from a one-way, balanced random effects analysis of variance model (see Section 8.3.5.2):

$$x_{ij} = \mu + b_i + e_{ij}, \quad i = 1,\ldots,k \quad j = 1,\ldots,k \qquad 8.4.4.1(a)$$

where k is the number of accepted batches, n is the number of specimens in each batch, and x_{ij} represents the jth specimen in the ith batch.

This ANOVA model represents each observation as the sum of three components; μ is the overall average of the population, b_i is the population average for the ith batch, and e_{ij} is a random error term which represents variation within each batch. The error terms, e_{ij}, are assumed to be independently distributed normal random variables with a mean of zero and a variance of σ_e^2 (the within-batch variance). The batch means, b_i, are assumed to be independent random variables following a normal distribution with a mean of zero and a variance of σ_b^2 (the between-batch variance).

The acceptability of a new batch is to be tested using the data in the k previously accepted batches. This new batch is referred to as the $(k+1)$th batch. Denote the grand mean on the basis of k batches as

$$\bar{x}^{(k)} = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij} / (kn) \qquad 8.4.4.1(b)$$

The batch means are computed as

$$\bar{x}_i = \sum_{j=1}^{n} x_{ij} / n \quad \text{for } i = 1,\ldots,k \qquad 8.4.4.1(c)$$

In this section, a superscript in parentheses indicates the number of batches of data used to calculate a statistic. For example, $\bar{x}^{(k)}$ is the grand mean based on all batches up to and including the kth batch. From these quantities, the required sums of squares can be computed. The between-batch mean square is computed as

$$MSB^{(k)} = \frac{1}{k-1} \sum_{i=1}^{k} n(\bar{x}_i - \bar{x}^{(k)})^2 \qquad 8.4.4.1(d)$$

Assume that the $(k+1^{th})$ batch is also described by the model in Equation 8.4.4.1(a). The mean of the $(k+1^{th})$ batch has a normal distribution with mean μ and a variance of

$$\frac{1}{n}\left(n\sigma_b^2 + \sigma_e^2\right)$$

It follows that the difference between the grand mean and the $(k+1^{th})$ mean,

$$\bar{x}^{(k)} - \bar{x}_{k+1}$$

has a normal distribution with a mean of zero and a variance of

$$\frac{k+1}{k}\ \frac{\left(n\ \sigma_b^2 + \sigma_e^2\right)}{n}$$

Also,

$$\frac{k+1}{kn}\ MSB^{(k)} \sim \frac{k+1}{k(k-1)}\ \frac{\left(n\ \sigma_b^2 + \sigma_e^2\right)}{n}\ \chi^2_{k-1} \qquad 8.4.4.1(e)$$

where the $\sim$ indicates "is distributed as" and χ^2_{k-1} denotes the χ^2 distribution with $k-1$ degrees of freedom. Dividing the difference between the grand mean and the $(k+1)^{th}$ batch mean by the left hand side of Equation 8.4.4.1(e) gives

$$V^{(k+1)} = \frac{\bar{x}^{(k)} - \bar{x}_{k+1}}{\left[\frac{k+1}{k(k-1)}\ \sum_{i=1}^{k}\ (\bar{x}_i - \bar{x}^{(k)})^2\right]^{1/2}} \sim t_{k-1} \qquad 8.4.4.1(f)$$

where t_{k-1} denotes the central t-distribution with $k-1$ degrees of freedom. This last relationship is the basis of the control chart. $V^{(k+1)}$, calculated from the new mean and all previously accepted batch means, is compared to the t-distribution limits. Specifically, $V^{(k+1)}$ is compared to the a quantile of the central t-distribution with $k-1$ degrees of freedom, $t_{k-1,\alpha}$, from Table 8.5.3. If the absolute value of $V^{(k+1)}$ exceeds $t_{k-1,\alpha}$, the $(k+1)^{th}$ batch is not accepted. These limits approach the normal distribution limits as the number of batches increases. Because of the variable control limits, it should be possible to start using this chart after very few batches. It may be reasonable to use it after data from four or five batches have been obtained.

If the $(k+1)$th batch is accepted, the grand mean and the between-batch mean square are updated as follows:

$$\bar{x}^{(k+1)} = \frac{k\ \bar{x}^{(k)} - \bar{x}_{k+1}}{k+1} \qquad 8.4.4.1(g)$$

$$MSB^{(k+1)} = \frac{k-1}{k}\ MSB^{(k)} + \frac{n}{k+1}\left(\bar{x}^{(k)} - \bar{x}_{k+1}\right)^2 \qquad 8.4.4.1(h)$$

Finally, note that this procedure can fail if there is a trend in the means. Such a trend would inflate the estimate of the variance and result in limits which are too wide. Because of this, the above procedure is used with a "runs" test for trends. Example charts are shown in Figure 8.4.4.1(a) and (b). These figures show the limits, $t_{k-1,\alpha}$ and $-t_{k-1,\alpha}$, and $V^{(k+1)}$ for each successive batch.

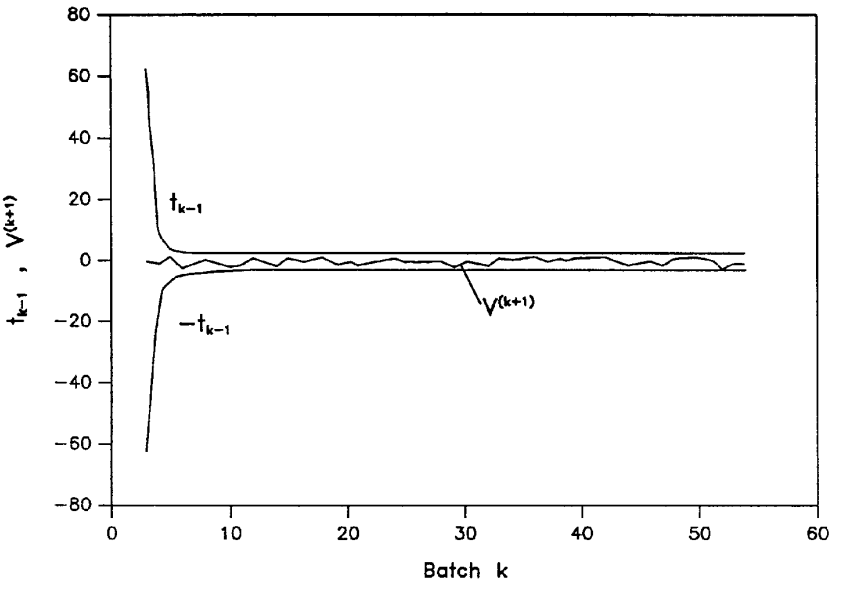

FIGURE 8.4.4.1(a) *Example control chart for means beginning with the third batch (0.01 level).*

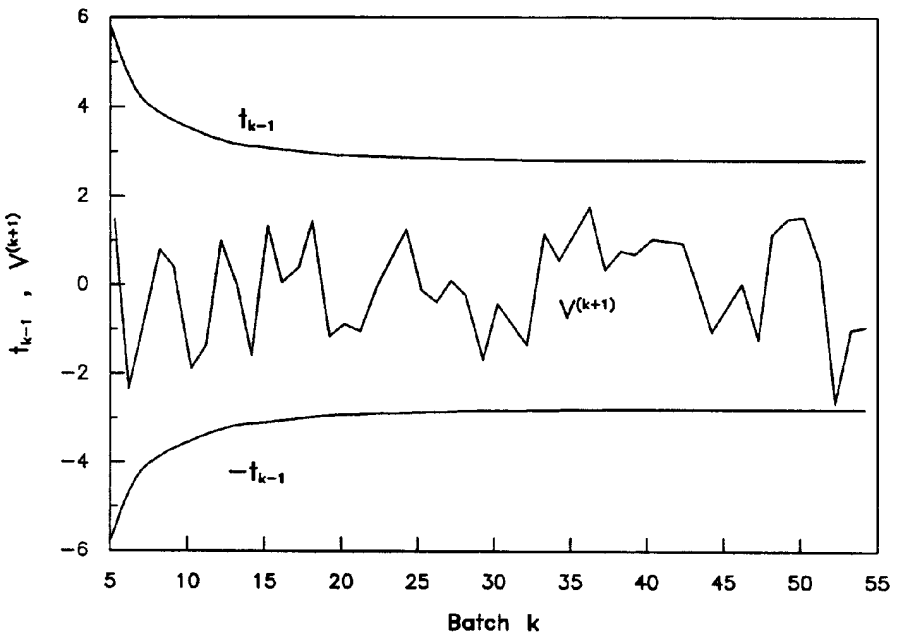

FIGURE 8.4.4.1(b) *Example control chart for means beginning with the fifth batch (0.01 level).*

8.4.4.2 s^2 chart for the within-batch component of variance

Let s^2 be the sample variance of the jth accepted batch. Test the (k+1)th batch variance using the following statistic:

$$U_{k+1} = \frac{s^2_{k+1}}{\sum_{j=1}^{k} s_j^2 / k} \sim F \qquad 8.4.4.2$$

where F denotes the standard F-distribution with n -1 numerator degrees of freedom and k(n-1) denominator degrees of freedom. Compare U_{k+1} to the a quantile of the F-distribution for n-1 numerator degrees of freedom and k(n-1) denominator degrees of freedom, F_α. These values are provided in Table 8.5.1 for the 0.95 level. If the statistic U_{k+1} exceeds F_α, the $(k+1)^{th}$ batch is not accepted. The control limits will approach constants as the denominator degrees of freedom for the F statistic become large. As with the means chart, this variance chart should be useful after data have been obtained from a few batches. An example S^2 chart is shown in Figure 8.4.4.2. This figure shows the limit, F_α, and U for each successive batch.

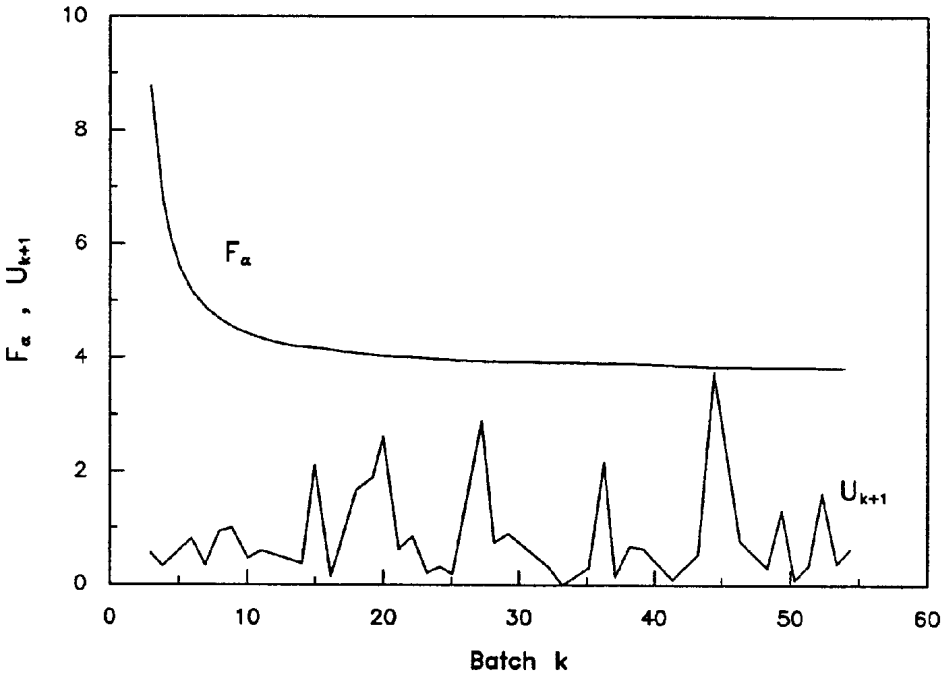

FIGURE 8.4.4.2 *Example control chart for variances beginning with the third batch (0.01 level).*

8.4.4.3 Test for trend in batch means

The $\bar{x}$ chart including batch effect of Section 8.4.4.1 may not perform properly if there is a systematic trend, either upward or downward, in the batch means for the initial batches received. Such a trend would make $V^{(k+1)}$ (Equation 8.4.4.1(f)) too small by inflating the denominator. This could result in batches being accepted which would have been rejected had there been no trend. If a trend is detected in the batch means before the control limits have leveled off (e.g., before the 25th batch; see Figure

8.4.4.1(b)) then caution should be used when accepting batches, especially if the trend can be seen to be downward. A diagnostic test is given in this subsection which should be performed on the first 25 batches received in order to determine whether a statistically significant trend exists. After 25 batches, this diagnostic should be discontinued, since the control limits will no longer be substantially effected by any trend which might be present, so the validity of the test in Section 8.4.4.1 need no longer be questioned. The idea behind this test is to fit a straight line through the batch means by least squares, and to determine if this line has a statistically significant slope.

Let the mean for the ith accepted batch be denoted $\bar{x}_i$, and let t_i be a time associated with the arrival of this batch. For example, the time of arrival of the first batch may be represented by $t_1 = 0$, and the remaining t_i may be the number of days which have elapsed since the first batch arrived. Assume that k batches have been accepted thus far, and that the following quantities have been calculated and therefore are available:

$$\bar{x}^{(k)} = \sum_{i=1}^{k} \bar{x}_i / k \qquad 8.4.4.3(a)$$

$$\bar{t}^{(k)} = \sum_{i=1}^{k} t_i / k \qquad 8.4.4.3(b)$$

$$S_{tt}^{(k)} = \sum_{i=1}^{k} \left(t_i - \bar{t}^{(k)}\right)^2 \qquad 8.4.4.3(c)$$

$$S_{xx}^{(k)} = \sum_{i=1}^{k} \left(\bar{x}_j - \bar{x}^{(k)}\right)^2 \qquad 8.4.4.3(d)$$

$$S_{tx}^{(k)} = \sum_{i=1}^{k} \left(t_i - \bar{t}^{(k)}\right)\left(\bar{x}_i - \bar{x}^{(k)}\right) \qquad 8.4.4.3(e)$$

$$b^{(k)} = \frac{S_{tx}^{(k)}}{S_{tt}^{(k)}} \qquad 8.4.4.3(f)$$

$$S_R^{(k)} = S_{xx}^{(k)} - 2b^{(k)}S_{tx}^{(k)} + \left[b^{(k)}\right]^2 S_{tt}^{(k)} \qquad 8.4.4.3(g)$$

The slope of the least squares line based on k batches is $b^{(k)}$, and the standard deviation about the least squares line is

$$S_D^{(k)} = \left[S_R^{(k)} / (k-2)\right]^{1/2} \qquad 8.4.4.3(h)$$

When the (k+1)st batch arrives, at time t_{k+1}, the following steps should be performed.

Step 1 Update $S_{tt}^{(k)}$, $S_{tt}^{(k)}$, and $S_{tx}^{(k)}$.

$$S_{tt}^{(k+1)} = S_{tt}^{(k)} + \frac{k}{k+1}\left(\bar{t}^{(k)} - t_{k+1}\right)^2 \qquad 8.4.4.3(i)$$

$$S_{xx}^{(k+1)} = S_{xx}^{(k)} + \frac{k}{k+1}\left(\bar{x}^{(k)} - \bar{x}_{k+1}\right)^2 \qquad 8.4.4.3(j)$$

$$S_{tx}^{(k+1)} = S_{tx}^{(k)} + \frac{k}{k+1}\left(\bar{t}^{(k)} - t_{k+1}\right)\bar{x}_{k+1} \qquad 8.4.4.3(k)$$

Step 2 Calculate $b^{(k+1)}$ and $S_R^{(k+1)}$.

$$b^{(k+1)} = \frac{S_{tx}^{(k+1)}}{S_{tt}^{(k+1)}} \qquad 8.4.4.3(l)$$

$$S_R^{(k+1)} = S_{xx}^{(k+1)} - 2b^{(k+1)} S_{tx}^{(k+1)} + b^{(k+1)^2} S_{tt}^{(k+1)} \qquad 8.4.4.3(m)$$

Step 3 Calculate the trend statistic.

$$U^{(k+1)} = b^{(k+1)} \sqrt{\frac{(k-1)S_{tt}^{(k+1)}}{S_R^{(k+1)}}} \qquad 8.4.4.3(n)$$

Step 4 Determine $t_{k-1,\alpha/2}$, the α quantile of the central t distribution with $k-1$ degrees of freedom (Table 8.5.3). If $|U^{(k+1)}|$ is greater than $t_{k-1,\alpha/2}$, then a statistically significant trend has been detected and should be investigated.

The level α of the test is somewhat arbitrary, but probably should be taken to be small (e.g., 0.001) in order to make small the probability of making an error by declaring a trend when no trend exists.

Step 5 Update the means.

$$\bar{x}^{(k+1)} = \frac{k\bar{x}^{(k)} + \bar{x}_{k+1}}{k+1} \qquad 8.4.4.3(o)$$

$$\bar{t}^{(k+1)} = \frac{k\bar{t}^{(k)} + t_{(k+1)}}{k+1} \qquad 8.4.4.3(p)$$

The following remarks should be made concerning this test:

1) The test should not be performed once the control limits in the $\bar{x}$ chart including batch effect (Section 8.4.4.1) have leveled off. This should occur at or before 25 batches have been accepted.

2) This trend test is only designed to detect situations where the validity of the test in Section 8.4.4.1 is called into question. It is not suitable as a general purpose trend test.

3) The updating Equations 8.4.4.3(i - k) and 8.4.4.3(o - p) make it unnecessary to use Equations 8.4.4.3(a - e) after each batch. Calculating quantities for a test on the $(k+1)^{th}$ batch based on the results from the test on the kth batch requires only a hand calculator.

4) A control chart analogous to Figure 8.4.4.1 may be prepared for the trend test as well, and it could provide useful information.

8.4.5 Average stress-strain curves and bearing load-deformation curves

It is highly desirable to have average stress-strain curves for tension, compression, and in-plane shear loading and to have average bearing load-deformation curves. However, the equations suggested below to represent the average curves are continuous and cannot represent discontinuities that may be caused by initiation of damage. Thus, it is also desirable to have graphs of actual data sets that are representative of individual tests.

Average curves will be determined using a best fit procedure with a minimum of two data sets from each batch. The best fit of each data set will be obtained to each of the seven algebraic functions below using the procedure described in References 8.4.5(a) and (b).

1) linear
2) parabolic
3) inverse parabolic
4) Ramberg-Osgood exponential
5) bilinear
6) parabolic-linear
7) parabolic-exponential

The equations for each function are given in the Section 8.4.5.1. The average error of fit is determined by the product of the root mean square (RMS) stress (load) error and the RMS percent stress (load) error. By using the product of these two values, sensitivity to errors for high stress (load) values and for the initial portion of the data set is provided. The function with the smallest average error of fit is chosen for each data set.

An average curve for all the data sets is determined up to the minimum strength of all data sets using the best-fit functions with smallest average error and the following procedure. At increments of one hundredth of the minimum strength, the average strain (deformation) for each best-fit function is determined. These average strains (deformations) and stresses (loads) are again fit to the seven algebraic functions to obtain an average curve. The function with the smallest average error of fit and the constants for this function will be reported with the average curve in Volume 2.

All of the strengths and strain- and deformation-to-failure values, not just those used in determining the average curve will be included as a scatter plot at the top of the average curve. Example stress-strain curves which include the scatter plot are shown in Figures 8.4.5(a) and (b).

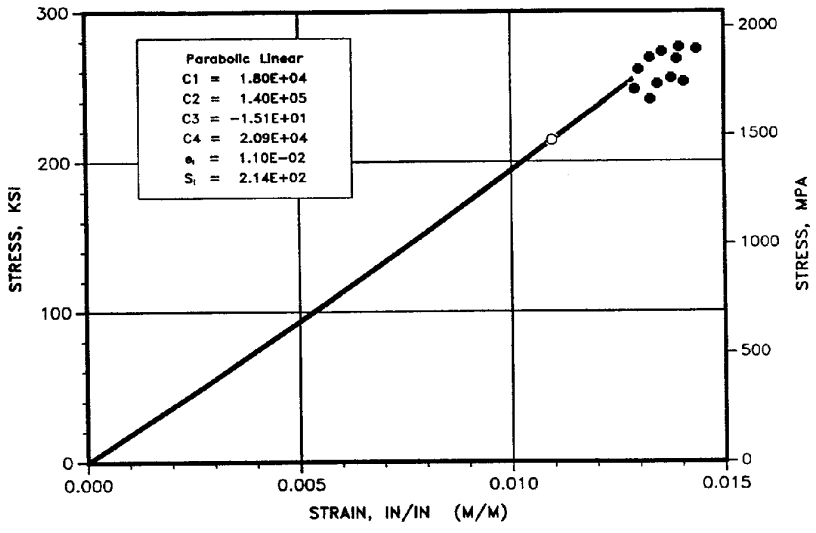

FIGURE 8.4.5(a) *Tensile stress-strain curve for AS4/3501-6 carbon/epoxy.*

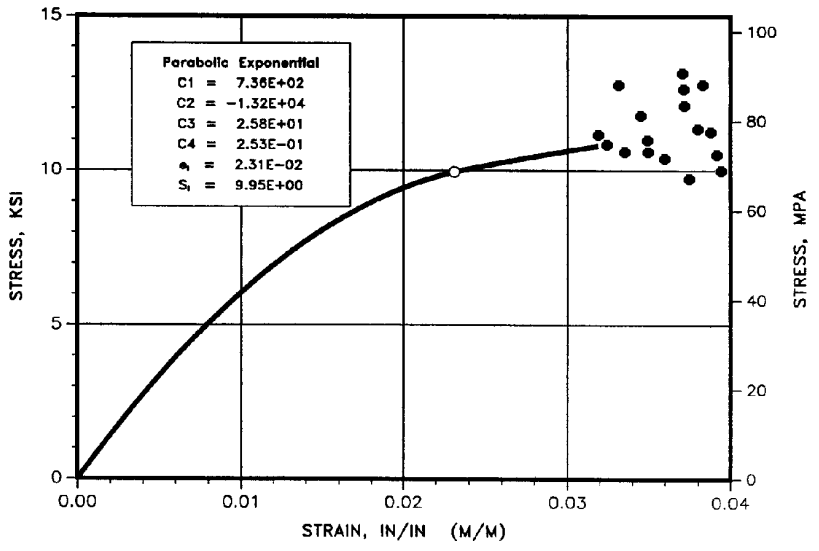

FIGURE 8.4.5(b) *Shear stress-strain curve for AS4/3501-6 carbon-epoxy.*

8.4.5.1 Fitting equations

The fitting equations for the stress-strain curves based on References 8.4.5(a) and (b) are presented below. Also included with each type of curve are the equations for the secant modulus and tangent modulus curves. These curves will be presented in MIL-HDBK-17 as discussed in Section 8.4.4. Each of the functions are based on the terms stress (s) and strain (e). The secant modulus functions are calculated as the secant modulus between the current value of strain and zero strain.

$$E_s = \frac{s(e) - s(0)}{e - 0} = \frac{s(e)}{e} \qquad 8.4.5.1(a)$$

The equation for the tangent modulus at any value of strain is:

$$E_t = \frac{ds}{de} \qquad 8.4.5.1(b)$$

Linear:

$$s = C_1 e \qquad 8.4.5.1(c)$$

$$E_t = E_s = C_1 \qquad 8.4.5.1(d)$$

Parabolic:

$$s = C_1 e + C_2 e^2 \qquad 8.4.5.1(e)$$

$$E_t = C_1 + 2C_2 e \qquad 8.4.5.1(f)$$

$$E_s = C_1 + C_2 e \qquad 8.4.5.1(g)$$

Inverse Parabolic:

$$e = C_2 s + C_3 s_2 \qquad 8.4.5.1(h)$$

$$E_t = \pm(C_2^2 + 4C_3 e)^{-1/2} \qquad 8.4.5.1(i)$$

$$E_s = \frac{C_2 \pm (C_2^2 + 4C_3 e)^{1/2}}{2C_3 e} \qquad 8.4.5.1(j)$$

The ± symbol has the same sign as the constant C_3.

Ramberg-Osgood Exponential:

$$e = \frac{s}{C_2} + 0.002 \left[\frac{s}{C_1}\right]^n, \quad n = C_3 \qquad 8.4.5.1(k)$$

Both secant modulus and tangent modulus values for the Ramberg-Osgood exponential function are found numerically.

The remaining functions have two fitted portions of the curves. The intersection of these two portions (e_i, s_i) is found as part of the fitting procedure.

Bilinear:
Below (e_i,s_i)

$$s = C_1 e \qquad 8.4.5.1(c)$$

$$E_t = E_s = C_1 \qquad 8.4.5.1(d)$$

Above (e_i,s_i)

$$s = C_2 + C_3 e \qquad 8.4.5.1(l)$$

$$E_t = C_3 \qquad 8.4.5.1(m)$$

$$E_s = \frac{C_2}{e} + C_3 \qquad 8.4.5.1(n)$$

Parabolic - Linear:
Below(e_i,s_i)

$$s = C_1 e + C_2 e^2 \qquad 8.4.5.1(e)$$

$$E_t = C_1 + 2C_2 e \qquad 8.4.5.1(f)$$

$$E_s = C_1 + C_2 e \qquad 8.4.5.1(g)$$

Above (e_i,s_i)

$$s = C_3 + C_4 e \qquad 8.4.5.1(o)$$

$$E_t = C_4 \qquad 8.4.5.1(p)$$

$$E_s = \frac{C_3}{e} + C_4 \qquad 8.4.5.1(q)$$

Parabolic - Exponential:
Below (e_i,s_i)

$$s = C_1 + C_2 e^2 \qquad 8.4.5.1(e)$$

$$E_t = C_1 + 2C_2 e \qquad 8.4.5.1(f)$$

$$E_s = C_1 + C_2 e \qquad 8.4.5.1(g)$$

Above (e_i,s_i)

$$s = C_3 e^n, \quad n = C_4 \qquad 8.4.5.1(r)$$

$$E_t = nC_3 e^{n-1} \qquad 8.4.5.1(s)$$

$$e_s = C_3 e^{n-1} \qquad 8.4.5.1(t)$$

In all cases, the type of curve and the values of the constants will be shown on the typical stress-strain curve figures. When there are two regions in a stress-strain curve, the value of the strain and stress and the intersection of the two regions is also shown on the figure.

8.5 STATISTICAL TABLES AND APPROXIMATIONS

This section contains a number of tables which are required for the analyses described in Section 8.3. Tables 8.5.1, 8.5.4 through 8.5.6, and 8.5.14 were generated specifically for MIL HDBK-17. The remaining tables were adapted from MIL-HDBK-5 (see Reference 8.3.4.5.1).

For some of the tabulated values, theoretical derivations and numerical approximations are provided below. The approximations are useful in computer applications when the software required to generate the tabulated values is not available. The accuracy of the approximations is measured by the relative magnitude of error (RME). The RME is defined as

$$\text{RME} = \frac{|\text{approximate} - \text{value}|}{\text{actual value}} \qquad 8.5$$

and measures the percentage error in the approximate value with respect to the actual value.

8.5.1 Quantiles of the F-distribution

An approximation to the $F_{0.95}$ values in Table 8.5.1 is

$$F_{0.95} = \exp\left(2\delta\left[1 + \frac{z^2 - 1}{3} - \frac{4\sigma^2}{3} \right] + 2\sigma z \sqrt{1 + \frac{\sigma^2 (z^2 - 3)}{6}} \right) \qquad 8.5.1(a)$$

where

$$\delta = 0.5\{1/(\gamma_2 - 1) - 1/(\gamma_1 - 1)\} \qquad 8.5.1(b)$$

$$\sigma^2 = 0.5\left\{1/(\gamma_2 - 1) + 1/(\gamma_1 - 1)\right\} \qquad 8.5.1(c)$$

z = 1.645

γ_1 = numerator degrees of freedom

γ_2 = denominator degrees of freedom.

Equations 8.5.1(a-c) are not valid when either γ_1 or γ_2 equals one. The following equations are to be used for these special cases:

For γ_1 = 1

$$F_{0.95} = \left[1.95996400 + \frac{2.37227200}{\gamma_2} + \frac{2.82250000}{\gamma_2^2} + \frac{2.555585200}{\gamma_2^3} + \frac{1.58953600}{\gamma_2^4} \right]^2 \qquad 8.5.1(e)$$

For γ_2 = 1

$$F_{0.95} = \left[0.06270671 + \frac{0.01573832}{\gamma_1} + \frac{0.00200073}{\gamma_1^2} - \frac{0.00243852}{\gamma_1^3} - \frac{0.00064811}{\gamma_1^4}\right]^{-2} \qquad 8.5.1(f)$$

8.5.2 Quantiles of the χ^2 distribution

An approximation to the chi-squared quantiles ($\chi^2_{0.95}$) in Table 8.5.2 is:

$$\chi^2_{0.95} = \gamma\left[1 - \frac{2}{9\gamma} + 1.645\left(\frac{2}{9\gamma}\right)^{\frac{1}{2}}\right]^3 + \frac{9}{100\gamma} \qquad 8.5.2$$

where γ is the degrees of freedom. This approximation is accurate to within 0.2% of the tabulated values. (See Reference 8.5.2.)

8.5.3 Upper-tail quantiles for the t-distribution

Table 8.5.3 was generated specifically for MIL-HDBK-17.

8.5.4 Two-tail probabilities for the t-distribution

Table 8.5.4 was generated specifically for MIL-HDBK-17.

8.5.5 Upper-tail probabilities for the standard normal distribution

Table 8.5.5 was generated specifically for MIL-HDBK-17.

8.5.6 Critical values for the k-sample Anderson-Darling test at the $\alpha = 0.05$ significance level

The k-sample Anderson-Darling test critical values in Table 8.5.6 were calculated using Equation 8.3.2.2(j) for the case of samples of equal size n.

8.5.7 Critical values for the MNR outlier test

The critical values in Table 8.5.7 are computed by the following formula:

$$V_c = \frac{n-1}{\sqrt{n}}\sqrt{\frac{t^2}{n-2+t^2}} \qquad 8.5.7$$

where t is the [1 - γ/(2n)] quantile of the t-distribution with n - 2 degrees of freedom, γ is the significance level of the test, and n is the sample size. Numbers in Table 8.5.7 are computed with a significance level of γ = 0.05. (See Reference 8.3.3.1(b).)

8.5.8 One-sided B-basis tolerance factors, V_B, for the Weibull distribution

The V values in Table 8.5.8 are calculated using the following statistical results. First, define the random variables

$$A_i = \frac{\ln(x_i) - \ln(\hat{\alpha})}{1/\hat{\beta}} \qquad i = 1,\ldots,n \qquad 8.5.8(a)$$

where x_i is a Weibull random variable with unknown shape and scale parameters β and α and $\hat{\alpha}$ and $\hat{\beta}$ are the maximum likelihood estimators (MLE's) of β and α given by Equations 8.3.4.2.1(a) and 8.3.4.2.1(c). For a particular n, the V_B value is the 0.95th quantile of the conditional distribution of the random variable

$$V_B = \frac{\sqrt{n}[\ln(\hat{Q}) - \ln(Q)]}{1/\hat{\beta}} \qquad 8.5.8(b)$$

given that

$$A_i = \frac{\ln(x_i') - \ln(\hat{\alpha}')}{1/\hat{\beta}'} \qquad 8.5.8(c)$$

where

$$x_i' = -\ln\left(1 - \frac{i - 0.5}{n + 0.25}\right) \quad i = 1, \ldots, n \qquad 8.5.8(d)$$

$$\hat{Q} = \hat{\alpha}(0.10536)^{1/\beta} \qquad 8.5.8(e)$$

$$Q = \alpha(0.10536)^{1/\beta} \qquad 8.5.8(f)$$

and $\hat{\alpha}'$ and $\hat{\beta}'$ are the MLE's of the two-parameter Weibull scale and shape parameters for the sample x_1', ..., x_n'. The conditional distribution of V_B is determined by the relationship

$$V_B = \sqrt{n}[Z + \ln(0.10536)] \qquad 8.5.8(g)$$

where the distribution of Z is given in Theorem 4.1.3 on page 150 of reference 8.3.4.2. Numerical integration was used to determine the V values in Table 8.5.8 based on these results.

An approximation to the V values in Table 8.5.8 is:

$$V_B \approx 3.803 + \exp\left\{1.79 - 0.516\ln(n) + \frac{5.1}{n-1}\right\} \qquad 8.5.8(h)$$

This approximation is accurate to within 0.5% of the tabulated values for n greater than or equal to 16.

8.5.9 One-sided A-basis tolerance factors, V_A, for the Weibull distribution

The V_A values in Table 8.5.9 are calculated as described in Section 8.5.8 (Reference 8.5.9). An approximation to the V_A values is:

$$V_A \approx 6.649 + \exp\left[2.55 - 0.526\ln(n) + \frac{4.76}{n}\right] \qquad 8.5.9$$

This approximation is accurate within 0.5% of the tabulated values for n greater than or equal to 16.

8.5.10 One-sided B-basis tolerance factors, k_B, for the normal distribution

The k_B values in Table 8.5.10 are calculated as $1/\sqrt{n}$ times the 0.95th quantile of the noncentral t-distribution with noncentrality parameter $1.282\sqrt{n}$ and $n - 1$ degrees of freedom. An approximation to the k_B values in Table 8.5.10 is:

$$k_B \approx 1.282 + \exp\{0.958 - 0.520\ln(n) + 3.19/n\} \qquad 8.5.10$$

This approximation is accurate to within 0.2% of the tabulated values for n greater than or equal to 16.

8.5.11 One-sided A-basis tolerance factors, k_A, for the normal distribution

The k_A values in Table 8.5.11 are calculated as $1/\sqrt{n}$ times the 0.95th quantile of the noncentral t-distribution with noncentrality parameter 2.326 $\sqrt{n}$ and $n - 1$ degrees of freedom (Reference 8.5.11). An approximation to the k_A values in Table 8.5.11 is:

$$k_A \approx 2.326 + \exp\left(1.34 - 0.522\ln(n) + \frac{3.87}{n}\right) \qquad 8.5.11$$

This approximation is accurate to within 0.2% of the tabulated values for n greater than or equal to 16.

8.5.12 Ranks, r_B, for determining nonparametric B-basis values

For $n > 29$, an approximation to the ranks for B-basis values in Table 8.5.12 is

$$r_B = \frac{n}{10} - 1.645\sqrt{\frac{9n}{100}} + 0.23 \qquad 8.5.12$$

rounded to the nearest integer. This approximation is exact for all but 12 values of n in the range of the table ($29 \le n \le 10499$). For this small percentage of n values (0.1%), the approximation errs by one rank on the conservative side.

8.5.13 Ranks, r_A, for determining nonparametric A-basis values

For $n \ge 299$, an approximation to the ranks for A-basis values in Table 8.5.13 is:

$$r_A \approx \frac{n}{100} - 1.645\sqrt{\frac{99n}{10{,}000}} + 0.29 + \frac{19.1}{n} \qquad 8.5.13$$

For n less than 299, an A-allowable cannot be computed. This approximation is exact for all but 23 values of n in the range of the table ($299 \le n \le 11691$). For this small percentage of n values (0.2%), the approximation errs by one rank on the conservative side (Reference 8.3.4.5.1).

8.5.14 Nonparametric B-basis values for small sample sizes

The values in Table 8.5.14 are based on Reference 8.3.4.5.2(a).

8.5.15 Non-parametric A-basis values for small sample sizes

The values in Table 8.5.15 are based on Reference 8.3.4.5.2(b).

8.5.16 Critical values for approximate confidence limits on the coefficient of variation

Values for u_1 and u_2, $100\,(1+\gamma)/2$ and $100\,(1\gamma)/2$ percentiles of the χ^2 distribution with $n1$ degrees of freedom, are tabulated in Table 8.5.16 for γ equal to 0.9, 0.95, and 0.99.

8.5.17 One-sided tolerance factors for acceptance limits on mean values, for normal distribution

The values listed in Table 8.5.17 are used for determining equivalency between two data sets and for the establishment of material specification requirements for minimum average values. These constants are used for equivalency tests that are designed to detect a decrease in the mean value. The values in the table were obtained from Reference 8.3.5.4(b) and reference 8.5.17.

8.5.18 One-sided tolerance factors for acceptance limits on individual values, for normal distribution

The values listed in Table 8.5.18 are used for determining equivalency between two data sets and for the establishment of material specification requirements for minimum individual values. These constants are used for equivalency tests that are designed to detect a decrease in an individual value. The values in the table were obtained from Reference 8.3.5.4(b) and Reference 8.5.17.

8.5.19 Upper and lower tail quantiles for two-sided t-distribution

The values listed in Table 8.5.19 are used for determining equivalency between two data sets and for the establishment of material specification requirements for minimum and/or maximum mean values. These constants are used for equivalency tests that are designed to detect an increase in a mean value, or to detect a change in a mean value (either up or down). The values in the table were obtained from Reference 8.3.5.4(b) and Reference 8.5.17.

TABLE 8.5.1 *Quantiles of the F-distribution, continued on next page.*

		γ_1 numerator degrees of freedom								
		1	2	3	4	5	6	7	8	9
	1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
	2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
γ_2	3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
	4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
	5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
d										
e	6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
n	7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
o	8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
m	9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
i	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
n										
a	11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
t	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
o	13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
r	14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
d	16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
e	17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
g	18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
r	19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
e	20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
e										
s	21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
	22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
o	24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
f	25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
	26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
f	27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
r	28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
e	29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
e	30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
d										
o	40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
m	60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
	120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96
	∞	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88

TABLE 8.5.1 *Quantiles of the F-distribution,* concluded.

		γ_1 numerator degrees of freedom									
		10	12	15	20	24	30	40	60	120	∞
	1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
	2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.51
γ_2	3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
d											
e	6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
n	7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
o	8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
m	9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
i	10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
n											
a	11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
t	12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
o	13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
r	14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
	15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
d	16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
e	17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
g	18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
r	19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
e	20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
e											
s	21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
	22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
	23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
o	24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
f	25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
	26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
f	27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
r	28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
e	29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
e	30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
d											
o	40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
m	60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
	∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

TABLE 8.5.2 *Quantiles of the χ^2 distribution.*

γ	$\chi^2_{0.95}$
1	3.84
2	5.99
3	7.82
4	9.49
5	11.07
6	12.60
7	14.07
8	15.51
9	16.93
10	18.31
11	19.68
12	21.03
13	22.37
14	23.69
15	25.00
16	26.30
17	27.59
18	28.88
19	30.15
20	31.42
21	32.68
22	33.93
23	35.18
24	36.42
25	37.66
26	38.89
27	40.12
28	41.34
29	42.56
30	43.78

TABLE 8.5.3 *Upper-tail quantiles for the t-distribution.*

γ	0.75	0.90	0.95	0.975	0.99	0.995
1	1.0000	3.0777	6.3137	12.7062	31.8205	63.6568
2	0.8165	1.8856	2.9200	4.3027	6.9646	9.9248
3	0.7649	1.6377	2.3534	3.1825	4.5407	5.8409
4	0.7407	1.5332	2.1318	2.7764	3.7470	4.6041
5	0.7267	1.4759	2.0150	2.5706	3.3649	4.0322
6	0.7176	1.4398	1.9432	2.4469	3.1427	3.7074
7	0.7111	1.4149	1.8946	2.3646	2.9980	3.4995
8	0.7064	1.3968	1.8595	2.3060	2.8965	3.3554
9	0.7027	1.3830	1.8331	2.2622	2.8214	3.2498
10	0.6998	1.3722	1.8125	2.2281	2.7638	3.1693
11	0.6974	1.3634	1.7959	2.2010	2.7181	3.1058
12	0.6955	1.3562	1.7823	2.1788	2.6810	3.0545
13	0.6938	1.3502	1.7709	2.1604	2.6503	3.0123
14	0.6924	1.3450	1.7613	2.1448	2.6245	2.9768
15	0.6912	1.3406	1.7530	2.1314	2.6025	2.9467
16	0.6901	1.3368	1.7459	2.1199	2.5835	2.9208
17	0.6892	1.3334	1.7396	2.1098	2.5669	2.8982
18	0.6884	1.3304	1.7341	2.1009	2.5524	2.8784
19	0.6876	1.3277	1.7291	2.0930	2.5395	2.8609
20	0.6870	1.3253	1.7247	2.0860	2.5280	2.8453
21	0.6864	1.3232	1.7207	2.0796	2.5176	2.8314
22	0.6858	1.3212	1.7171	2.0739	2.5083	2.8188
23	0.6853	1.3195	1.7139	2.0687	2.4999	2.8073
24	0.6848	1.3178	1.7109	2.0639	2.4922	2.7969
25	0.6844	1.3163	1.7081	2.0595	2.4851	2.7874
30	0.6828	1.3104	1.6973	2.0423	2.4573	2.7500
40	0.6807	1.3031	1.6839	2.0211	2.4233	2.7045
60	0.6786	1.2958	1.6706	2.0003	2.3901	2.6603
120	0.6765	1.2886	1.6577	1.9799	2.3578	2.6174
∞	0.6745	1.2816	1.6449	1.9600	2.3263	2.5758

TABLE 8.5.4 *Two-tail probabilities for t-distribution.*

T	degrees of freedom, γ									
	1	2	3	4	5	6	7	8	9	10
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.25	0.8440	0.8259	0.8187	0.8149	0.8125	0.8109	0.8098	0.8089	0.8082	0.8076
0.50	0.7048	0.6667	0.6514	0.6433	0.6383	0.6349	0.6324	0.6305	0.6291	0.6279
0.75	0.5903	0.5315	0.5077	0.4950	0.4870	0.4816	0.4777	0.4747	0.4724	0.4705
1.00	0.5000	0.4226	0.3910	0.3739	0.3632	0.3559	0.3506	0.3466	0.3434	0.3409
1.25	0.4296	0.3377	0.2999	0.2794	0.2666	0.2578	0.2515	0.2466	0.2428	0.2398
1.50	0.3743	0.2724	0.2306	0.2080	0.1939	0.1843	0.1773	0.1720	0.1679	0.1645
1.75	0.3305	0.2222	0.1784	0.1550	0.1405	0.1307	0.1236	0.1182	0.1140	0.1107
2.00	0.2952	0.1835	0.1393	0.1161	0.1019	0.0924	0.0856	0.0805	0.0766	0.0734
2.25	0.2662	0.1534	0.1099	0.0876	0.0743	0.0654	0.0592	0.0546	0.0510	0.0482
2.50	0.2422	0.1296	0.0877	0.0668	0.0545	0.0465	0.0410	0.0369	0.0339	0.0314
2.75	0.2220	0.1107	0.0707	0.0514	0.0403	0.0333	0.0285	0.0251	0.0225	0.0205
3.00	0.2048	0.0955	0.0577	0.0399	0.0301	0.0240	0.0199	0.0171	0.0150	0.0133
3.25	0.1900	0.0831	0.0475	0.0314	0.0227	0.0175	0.0141	0.0117	0.0100	0.0087
3.50	0.1772	0.0728	0.0395	0.0249	0.0173	0.0128	0.0100	0.0081	0.0067	0.0057
3.75	0.1659	0.0643	0.0331	0.0199	0.0133	0.0095	0.0072	0.0056	0.0046	0.0038
4.00	0.1560	0.0572	0.0280	0.0161	0.0103	0.0071	0.0052	0.0039	0.0031	0.0025
4.25	0.1471	0.0512	0.0239	0.0132	0.0081	0.0054	0.0038	0.0028	0.0021	0.0017
4.50	0.1392	0.0460	0.0205	0.0108	0.0064	0.0041	0.0028	0.0020	0.0015	0.0011
4.75	0.1321	0.0416	0.0177	0.0090	0.0051	0.0032	0.0021	0.0014	0.0010	0.0008
5.00	0.1257	0.0377	0.0154	0.0075	0.0041	0.0025	0.0016	0.0011	0.0007	0.0005
5.25	0.1198	0.0344	0.0135	0.0063	0.0033	0.0019	0.0012	0.0008	0.0005	0.0004
5.50	0.1145	0.0315	0.0118	0.0053	0.0027	0.0015	0.0009	0.0006	0.0004	0.0003
5.75	0.1096	0.0289	0.0104	0.0045	0.0022	0.0012	0.0007	0.0004	0.0003	0.0002
6.00	0.1051	0.0267	0.0093	0.0039	0.0018	0.0010	0.0005	0.0003	0.0002	0.0001
6.25	0.1010	0.0247	0.0083	0.0033	0.0015	0.0008	0.0004	0.0002	0.0001	0.0001
6.50	0.0972	0.0229	0.0074	0.0029	0.0013	0.0006	0.0003	0.0002	0.0001	0.0001
6.75	0.0936	0.0213	0.0066	0.0025	0.0011	0.0005	0.0003	0.0001	0.0001	0.0001
7.00	0.0903	0.0198	0.0060	0.0022	0.0009	0.0004	0.0002	0.0001	0.0001	0.0000
7.25	0.0873	0.0185	0.0054	0.0019	0.0008	0.0003	0.0002	0.0001	0.0000	0.0000
7.50	0.0844	0.0173	0.0049	0.0017	0.0007	0.0003	0.0001	0.0001	0.0000	0.0000
7.75	0.0817	0.0162	0.0045	0.0015	0.0006	0.0002	0.0001	0.0001	0.0000	0.0000
8.00	0.0792	0.0153	0.0041	0.0013	0.0005	0.0002	0.0001	0.0000	0.0000	0.0000

TABLE 8.5.5 *Upper tail probabilities for the standard normal distribution.*

x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
0.00	0.50000	0.59871	0.69146	0.77337	0.84134	0.89435	0.93319	0.95994	0.97725	0.98778
0.01	0.50399	0.60257	0.69497	0.77637	0.84375	0.89617	0.93448	0.96080	0.97778	0.98809
0.02	0.50798	0.60642	0.69847	0.77935	0.84614	0.89796	0.93574	0.96164	0.97831	0.98840
0.03	0.51197	0.61026	0.70194	0.78230	0.84850	0.89973	0.93699	0.96246	0.97882	0.98870
0.04	0.51595	0.61409	0.70540	0.78524	0.85083	0.90147	0.93822	0.96327	0.97932	0.98899
0.05	0.51994	0.61791	0.70884	0.78814	0.85314	0.90320	0.93943	0.96407	0.97982	0.98928
0.06	0.52392	0.62172	0.71226	0.79103	0.85543	0.90490	0.94062	0.96485	0.98030	0.98956
0.07	0.52790	0.62552	0.71566	0.79389	0.85769	0.90658	0.94179	0.96562	0.98077	0.98983
0.08	0.53188	0.62930	0.71904	0.79673	0.85993	0.90824	0.94295	0.96637	0.98124	0.99010
0.09	0.53586	0.63307	0.72240	0.79955	0.86214	0.90988	0.94408	0.96712	0.98169	0.99036
0.10	0.53983	0.63683	0.72575	0.80234	0.86433	0.91149	0.94520	0.96784	0.98214	0.99061
0.11	0.54380	0.64058	0.72907	0.80511	0.86650	0.91309	0.94630	0.96856	0.98257	0.99086
0.12	0.54776	0.64431	0.73237	0.80785	0.86864	0.91466	0.94738	0.96926	0.98300	0.99111
0.13	0.55172	0.64803	0.73565	0.81057	0.87076	0.91621	0.94845	0.96995	0.98341	0.99134
0.14	0.55567	0.65173	0.73891	0.81327	0.87286	0.91774	0.94950	0.97062	0.98382	0.99158
0.15	0.55962	0.65542	0.74215	0.81594	0.87493	0.91924	0.95053	0.97128	0.98422	0.99180
0.16	0.56356	0.65910	0.74537	0.81859	0.87698	0.92073	0.95154	0.97193	0.98461	0.99202
0.17	0.56749	0.66276	0.74857	0.82121	0.87900	0.92220	0.95254	0.97257	0.98500	0.99224
0.18	0.57142	0.66640	0.75175	0.82381	0.88100	0.92364	0.95352	0.97320	0.98537	0.99245
0.19	0.57535	0.67003	0.75490	0.82639	0.88298	0.92507	0.95449	0.97381	0.98574	0.99266
0.20	0.57926	0.67364	0.75804	0.82894	0.88493	0.92647	0.95543	0.97441	0.98610	0.99286
0.21	0.58317	0.67724	0.76115	0.83147	0.88686	0.92785	0.95637	0.97500	0.98645	0.99305
0.22	0.58706	0.68082	0.76424	0.83398	0.88877	0.92922	0.95728	0.97558	0.98679	0.99324
0.23	0.59095	0.68439	0.76730	0.83646	0.89065	0.93056	0.95818	0.97615	0.98713	0.99343
0.24	0.59483	0.68793	0.77035	0.83891	0.89251	0.93189	0.95907	0.97670	0.98745	0.99361

Note: To find the probability that a standard normal random variable is less than x, enter the table at the cell for which the sum of the row and column headings equals x (e.g., for $x = 0.73 = 0.5 + 0.23$, we have, from row 23 and column 2, $P = 0.76730$). If x is less than zero, use the absolute value of x to get a value P', and let the probability be $P = 1 - P'$ (e.g., for $x = -0.73$, $P = 1 - 0.76730 = 0.23270$)

TABLE 8.5.6 *Critical values for the k-sample Anderson-Darling test at the $\alpha = 0.05$ significance level.*

n^* \ k^*	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	2.11	1.80	1.65	1.56	1.50	1.46	1.42	1.39	1.37	1.35	1.33	1.32	1.31	1.29
4	2.20	1.86	1.70	1.60	1.54	1.49	1.45	1.42	1.39	1.37	1.36	1.34	1.33	1.31
5	2.25	1.89	1.73	1.63	1.56	1.51	1.47	1.43	1.41	1.39	1.37	1.35	1.34	1.32
6	2.29	1.92	1.74	1.64	1.57	1.52	1.48	1.45	1.42	1.40	1.38	1.36	1.34	1.33
7	2.32	1.94	1.76	1.65	1.58	1.53	1.49	1.45	1.43	1.40	1.38	1.36	1.35	1.34
8	2.34	1.95	1.77	1.66	1.59	1.53	1.49	1.46	1.43	1.41	1.39	1.37	1.35	1.34
9	2.35	1.96	1.78	1.67	1.59	1.54	1.50	1.46	1.43	1.41	1.39	1.37	1.36	1.34
10	2.37	1.97	1.78	1.67	1.60	1.54	1.50	1.47	1.44	1.41	1.39	1.37	1.36	1.35
11	2.38	1.97	1.79	1.68	1.60	1.55	1.50	1.47	1.44	1.42	1.39	1.38	1.36	1.35
12	2.39	1.98	1.79	1.68	1.60	1.55	1.51	1.47	1.44	1.42	1.40	1.38	1.36	1.35
13	2.39	1.98	1.80	1.68	1.61	1.55	1.51	1.47	1.44	1.42	1.40	1.38	1.36	1.35
14	2.40	1.99	1.80	1.69	1.61	1.55	1.51	1.47	1.44	1.42	1.40	1.38	1.37	1.35
15	2.41	1.99	1.80	1.69	1.61	1.55	1.51	1.48	1.45	1.42	1.40	1.38	1.37	1.35
16	2.41	2.00	1.80	1.69	1.61	1.56	1.51	1.48	1.45	1.42	1.40	1.38	1.37	1.35
17	2.42	2.00	1.81	1.69	1.61	1.56	1.51	1.48	1.45	1.42	1.40	1.38	1.37	1.35
18	2.42	2.00	1.81	1.69	1.62	1.56	1.51	1.48	1.45	1.42	1.40	1.39	1.37	1.35
19	2.42	2.00	1.81	1.70	1.62	1.56	1.52	1.48	1.45	1.43	1.40	1.39	1.37	1.36
20	2.43	2.01	1.81	1.70	1.62	1.56	1.52	1.48	1.45	1.43	1.40	1.39	1.37	1.36
	2.49	2.05	1.84	1.72	1.64	1.58	1.53	1.50	1.46	1.44	1.42	1.40	1.38	1.37

TABLE 8.5.7 *Critical values for the MNR outlier test.*

n	CV	n	CV	n	CV	n	CV	n	CV
-	-	41	3.047	81	3.311	121	3.448	161	3.539
-	-	42	3.057	82	3.315	122	3.451	162	3.541
3	1.154	43	3.067	83	3.319	123	3.453	163	3.543
4	1.481	44	3.076	84	3.323	124	3.456	164	3.545
5	1.715	45	3.085	85	3.328	125	3.459	165	3.547
6	1.887	46	3.094	86	3.332	126	3.461	166	3.549
7	2.020	47	3.103	87	3.336	127	3.464	167	3.551
8	2.127	48	3.112	88	3.340	128	3.466	168	3.552
9	2.215	49	3.120	89	3.344	129	3.469	169	3.554
10	2.290	50	3.128	90	3.348	130	3.471	170	3.556
11	2.355	51	3.136	91	3.352	131	3.474	171	3.558
12	2.412	52	3.144	92	3.355	132	3.476	172	3.560
13	2.462	53	3.151	93	3.359	133	3.479	173	3.561
14	2.507	54	3.159	94	3.363	134	3.481	174	3.563
15	2.548	55	3.166	95	3.366	135	3.483	175	3.565
16	2.586	56	3.173	96	3.370	136	3.486	176	3.567
17	2.620	57	3.180	97	3.374	137	3.488	177	3.568
18	2.652	58	3.187	98	3.377	138	3.491	178	3.570
19	2.681	59	3.193	99	3.381	139	3.493	179	3.572
20	2.708	60	3.200	100	3.384	140	3.495	180	3.574
21	2.734	61	3.206	101	3.387	141	3.497	181	3.575
22	2.758	62	3.212	102	3.391	142	3.500	182	3.577
23	2.780	63	3.218	103	3.394	143	3.502	183	3.579
24	2.802	64	3.224	104	3.397	144	3.504	184	3.580
25	2.822	65	3.230	105	3.401	145	3.506	185	3.582
26	2.841	66	3.236	106	3.404	146	3.508	186	3.584
27	2.859	67	3.241	107	3.407	147	3.511	187	3.585
28	2.876	68	3.247	108	3.410	148	3.513	188	3.587
29	2.893	69	3.252	109	3.413	149	3.515	189	3.588
30	2.908	70	3.258	110	3.416	150	3.517	190	3.590
31	2.924	71	3.263	111	3.419	151	3.519	191	3.592
32	2.938	72	3.268	112	3.422	152	3.521	192	3.593
33	2.952	73	3.273	113	3.425	153	3.523	193	3.595
34	2.965	74	3.278	114	3.428	154	3.525	194	3.596
35	2.978	75	3.283	115	3.431	155	3.527	195	3.598
36	2.991	76	3.288	116	3.434	156	3.529	196	3.599
37	3.003	77	3.292	117	3.437	157	3.531	197	3.601
38	3.014	78	3.297	118	3.440	158	3.533	198	3.603
39	3.025	79	3.302	119	3.442	159	3.535	199	3.604
40	3.036	80	3.306	120	3.445	160	3.537	200	3.606

TABLE 8.5.8 One-sided B-basis tolerance factors, V_B , for the Weibull distribution, continued on next page

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n = 10 - 192							
n	V_B	n	V_B	n	V_B	n	V_B
10	6.711	45	4.764	80	4.477	130	4.309
11	6.477	46	4.751	81	4.471	132	4.305
12	6.286	47	4.738	82	4.466	134	4.301
13	6.127	48	4.725	83	4.462	136	4.296
14	5.992	49	4.713	84	4.457	138	4.292
15	5.875	50	4.702	85	4.452	140	4.288
16	5.774	51	4.691	86	4.448	142	4.284
17	5.684	52	4.680	87	4.443	144	4.280
18	5.605	53	4.670	88	4.439	146	4.277
19	5.533	54	4.659	89	4.435	148	4.273
20	5.469	55	4.650	90	4.431	150	4.269
21	5.412	56	4.640	91	4.427	152	4.266
22	5.359	57	4.631	92	4.423	154	4.262
23	5.310	58	4.622	93	4.419	156	4.259
24	5.265	59	4.631	94	4.415	158	4.256
25	5.224	60	4.605	95	4.411	160	4.253
26	5.186	61	4.597	96	4.407	162	4.249
27	5.150	62	4.589	97	4.404	164	4.246
28	5.117	63	4.582	98	4.400	166	4.243
29	5.086	64	4.574	99	4.396	168	4.240
30	5.057	65	4.567	100	4.393	170	4.237
31	5.030	66	4.560	102	4.386	172	4.234
32	5.003	67	4.553	104	4.380	174	4.232
33	4.979	68	4.546	106	4.373	176	4.229
34	4.956	69	4.539	108	4.367	178	4.226
35	4.934	70	4.533	110	4.361	180	4.224
36	4.913	71	4.527	112	4.355	182	4.221
37	4.893	72	4.521	114	4.349	184	4.218
38	4.875	73	4.515	116	4.344	186	4.216
39	4.857	74	4.509	118	4.339	188	4.213
40	4.840	75	4.503	120	4.334	190	4.211
41	4.823	76	4.498	122	4.328	192	4.208
42	4.807	77	4.492	124	4.323		
43	4.792	78	4.487	126	4.317		
44	4.778	79	4.482	128	4.314		

TABLE 8.5.8 *One-sided B-basis tolerance factors, V_B, for the Weibull distribution,* concluded.

n = 194 - ∞					
n	V_B	n	V_B	n	V_B
194	4.206	300	4.119	850	3.992
196	4.204	310	4.113	875	3.989
198	4.201	320	4.108	900	3.986
200	4.199	330	4.103	925	3.983
204	4.195	340	4.098	950	3.981
208	4.191	350	4.093	975	3.979
212	4.186	360	4.089	1000	3.976
216	4.182	370	4.085	1100	3.968
220	4.179	380	4.081	1200	3.960
224	4.175	390	4.077	1300	3.954
228	4.171	400	4.073	1400	3.948
232	4.168	425	4.076	1500	3.943
236	4.164	450	4.067	1600	3.939
240	4.161	475	4.060	1700	3.934
244	4.157	500	4.053	1800	3.931
248	4.154	525	4.047	1900	3.927
252	4.151	550	4.041	2000	3.924
256	4.148	575	4.035	3000	3.901
260	4.145	600	4.030	4000	3.887
264	4.142	625	4.025	5000	3.878
268	4.140	650	4.020	6000	3.872
272	4.137	675	4.016	7000	3.866
276	4.134	700	4.012	8000	3.862
280	4.131	725	4.008	9000	3.859
284	4.129	750	4.005	10000	3.856
288	4.126	775	4.001	15000	3.846
292	4.124	800	3.998	20000	3.840
296	4.121	825	3.995	∞	3.803

TABLE 8.5.9 *One-sided A-basis tolerance limit factors, V_A, for the Weibull distribution, (Reference 8.5.11), continued on next page.*

n	V_A	n	V_A	n	V_A	n	V_A
10	12.573	44	8.629	78	8.038	124	7.706
11	12.093	45	8.600	79	8.028	126	7.697
12	11.701	46	8.573	80	8.017	128	7.687
13	11.375	47	8.547	81	8.007	130	7.678
14	11.098	48	8.522	82	7.997	132	7.669
15	10.861	49	8.498	83	7.988	134	7.660
16	10.654	50	8.474	84	7.978	136	7.652
17	10.472	51	8.452	85	7.969	138	7.643
18	10.311	52	8.430	86	7.960	140	7.635
19	10.166	53	8.409	87	7.951	142	7.627
20	10.035	54	8.389	88	7.942	144	7.619
21	9.917	55	8.369	89	7.933	146	7.612
22	9.809	56	8.349	90	7.925	148	7.604
23	9.710	57	8.330	91	7.916	150	7.597
24	9.619	58	8.313	92	7.908	152	7.590
25	9.535	59	8.295	93	7.900	154	7.583
26	9.457	60	8.278	94	7.892	156	7.576
27	9.385	61	8.262	95	7.884	158	7.569
28	9.318	62	8.246	96	7.877	160	7.563
29	9.251	63	8.230	97	7.867	162	7.556
30	9.195	64	8.215	98	7.862	164	7.550
31	9.139	65	8.200	99	7.855	166	7.544
32	9.087	66	8.186	100	7.845	168	7.538
33	9.037	67	8.172	102	7.834	170	7.532
34	8.990	68	8.158	104	7.820	172	7.526
35	8.946	69	8.145	106	7.811	174	7.520
36	8.904	70	8.132	108	7.795	176	7.515
37	8.863	71	8.119	110	7.783	178	7.509
38	8.825	72	8.107	112	7.771	180	7.504
39	8.789	73	8.095	114	7.759	182	7.499
40	8.754	74	8.083	116	7.748	184	7.493
41	8.721	75	8.071	118	7.737	186	7.488
42	8.689	76	8.060	120	7.727	188	7.483
43	8.658	77	8.049	122	7.717	190	7.478

TABLE 8.5.9 *One-sided A-basis tolerance limit factors, V_A, for the Weibull distribution, (Reference 8.5.11), concluded.*

n	V_A	n	V_A	n	V_A	n	V_A
192	7.473	268	7.333	475	7.152	1000	6.989
194	7.469	272	7.328	500	7.138	1100	6.972
196	7.454	276	7.322	525	7.126	1200	6.958
198	7.459	280	7.317	550	7.114	1300	6.945
200	7.455	284	7.312	575	7.103	1400	6.934
204	7.446	288	7.307	600	7.093	1500	6.924
208	7.437	292	7.302	625	7.084	1600	6.915
212	7.429	296	7.297	650	7.075	1700	6.907
216	7.421	300	7.292	675	7.066	1800	6.899
220	7.413	310	7.280	700	7.058	1900	6.892
224	7.404	320	7.270	725	7.051	2000	6.886
228	7.397	330	7.259	750	7.044	3000	6.841
232	7.390	340	7.249	775	7.037		
236	7.383	350	7.240	800	7.031		
240	7.376	360	7.229	825	7.025		
244	7.370	370	7.222	850	7.019		
248	7.363	380	7.214	875	7.013		
252	7.357	390	7.206	900	7.008		
256	7.351	400	7.198	925	7.003		
260	7.345	425	7.183	950	6.998		
264	7.339	450	7.167	975	6.993		

TABLE 8.5.10 *One-sided B-basis tolerance limit factors, k_B, for the normal distribution, continued on next page.*

N = 2 - 137							
n	k_B	n	k_B	n	k_B	n	k_B
2	20.581	36	1.725	70	1.582	104	1.522
3	6.157	37	1.718	71	1.579	105	1.521
4	4.163	38	1.711	72	1.577	106	1.519
5	3.408	39	1.704	73	1.575	107	1.518
6	3.007	40	1.698	74	1.572	108	1.517
7	2.756	41	1.692	75	1.570	109	1.516
8	2.583	42	1.686	76	1.568	110	1.515
9	2.454	43	1.680	77	1.566	111	1.513
10	2.355	44	1.675	78	1.564	112	1.512
11	2.276	45	1.669	79	1.562	113	1.511
12	2.211	46	1.664	80	1.560	114	1.510
13	2.156	47	1.660	81	1.558	115	1.509
14	2.109	48	1.655	82	1.556	116	1.508
15	2.069	49	1.650	83	1.554	117	1.507
16	2.034	50	1.646	84	1.552	118	1.506
17	2.002	51	1.642	85	1.551	119	1.505
18	1.974	52	1.638	86	1.549	120	1.504
19	1.949	53	1.634	87	1.547	121	1.503
20	1.927	54	1.630	88	1.545	122	1.502
21	1.906	55	1.626	89	1.544	123	1.501
22	1.887	56	1.623	90	1.542	124	1.500
23	1.870	57	1.619	91	1.540	125	1.499
24	1.854	58	1.616	92	1.539	126	1.498
25	1.839	59	1.613	93	1.537	127	1.497
26	1.825	60	1.609	94	1.536	128	1.496
27	1.812	61	1.606	95	1.534	129	1.495
28	1.800	62	1.603	96	1.533	130	1.494
29	1.789	63	1.600	97	1.531	131	1.493
30	1.778	64	1.597	98	1.530	132	1.492
31	1.768	65	1.595	99	1.529	133	1.492
32	1.758	66	1.592	100	1.527	134	1.491
33	1.749	67	1.589	101	1.526	135	1.490
34	1.741	68	1.587	102	1.525	136	1.489
35	1.733	69	1.584	103	1.523	137	1.488

TABLE 8.5.10 *One-sided B-basis tolerance limit factors, k_B, for the normal distribution,* concluded.

n = 138 - ∞							
n	k_B	n	k_B	n	k_B	n	k_B
138	1.487	172	1.464	230	1.438	400	1.398
139	1.487	173	1.464	235	1.436	425	1.395
140	1.486	174	1.463	240	1.434	450	1.391
141	1.485	175	1.463	345	1.433	475	1.388
142	1.484	176	1.462	250	1.431	500	1.386
143	1.483	177	1.461	255	1.430	525	1.383
144	1.483	178	1.461	260	1.428	550	1.381
145	1.482	179	1.460	265	1.427	575	1.378
146	1.481	180	1.460	270	1.425	600	1.376
147	1.480	181	1.459	275	1.424	625	1.374
148	1.480	182	1.459	280	1.422	650	1.372
149	1.479	183	1.458	285	1.421	675	1.371
150	1.478	184	1.458	290	1.420	700	1.369
151	1.478	185	1.457	295	1.419	725	1.367
152	1.477	186	1.457	300	1.417	750	1.366
153	1.476	187	1.456	305	1.416	775	1.364
154	1.475	188	1.456	310	1.415	800	1.363
155	1.475	189	1.455	315	1.414	825	1.362
156	1.474	190	1.455	320	1.413	850	1.361
157	1.473	191	1.454	325	1.412	875	1.359
158	1.473	192	1.454	330	1.411	900	1.358
159	1.472	193	1.453	335	1.410	925	1.357
160	1.472	194	1.453	340	1.409	950	1.356
161	1.471	195	1.452	345	1.408	975	1.355
162	1.470	196	1.452	350	1.407	1000	1.354
163	1.470	197	1.451	355	1.406	1500	1.340
164	1.469	198	1.451	360	1.405	2000	1.332
165	1.468	199	1.450	365	1.404	3000	1.323
166	1.468	200	1.450	370	1.403	5000	1.313
167	1.467	205	1.448	375	1.402	10000	1.304
168	1.467	210	1.446	380	1.402	∞	1.282
169	1.466	215	1.444	385	1.401		
170	1.465	220	1.442	390	1.400		
171	1.465	225	1.440	395	1.399		

TABLE 8.5.11 *One-sided A-basis tolerance limit factors, k_A, for the normal distribution, Reference 8.5.11), continued on next page.*

n	k_A	n	k_A	n	k_A	n	k_A
2	37.094	36	2.983	70	2.765	104	2.676
3	10.553	37	2.972	71	2.762	105	2.674
4	7.042	38	2.961	72	2.758	106	2.672
5	5.741	39	2.951	73	2.755	107	2.671
6	5.062	40	2.941	74	2.751	108	2.669
7	4.642	41	2.932	75	2.748	109	2.667
8	4.354	42	2.923	76	2.745	110	2.665
9	4.143	43	2.914	77	2.742	111	2.663
10	3.981	44	2.906	78	2.739	112	2.662
11	3.852	45	2.898	79	2.736	113	2.660
12	3.747	46	2.890	80	2.733	114	2.658
13	3.659	47	2.883	81	2.730	115	2.657
14	3.585	48	2.876	82	2.727	116	2.655
15	3.520	49	2.869	83	2.724	117	2.654
16	3.464	50	2.862	84	2.721	118	2.652
17	3.414	51	2.856	85	2.719	119	2.651
18	3.370	52	2.850	86	2.716	120	2.649
19	3.331	53	2.844	87	2.714	121	2.648
20	3.295	54	2.838	88	2.711	122	2.646
21	3.263	55	2.833	89	2.709	123	2.645
22	3.233	56	2.827	90	2.706	124	2.643
23	3.206	57	2.822	91	2.704	125	2.642
24	3.181	58	2.817	92	2.701	126	2.640
25	3.158	59	2.812	93	2.699	127	2.639
26	3.136	60	2.807	94	2.697	128	2.638
27	3.116	61	2.802	95	2.695	129	2.636
28	3.098	62	2.798	96	2.692	130	2.635
29	3.080	63	2.793	97	2.690	131	2.634
30	3.064	64	2.789	98	2.688	132	2.632
31	3.048	65	2.785	99	2.686	133	2.631
32	3.034	66	2.781	100	2.684	134	2.630
33	3.020	67	2.777	101	2.682	135	2.628
34	3.007	68	2.773	102	2.680	136	2.627
35	2.995	69	2.769	103	2.678	137	2.626

TABLE 8.5.11 *One-sided A-basis tolerance limit factors, k_A, for the normal distribution, Reference 8.5.11), concluded.*

n	k_A	n	k_A	n	k_A	n	k_A
138	2.625	172	2.591	230	2.552	400	2.494
139	2.624	173	2.590	235	2.549	425	2.489
140	2.622	174	2.589	240	2.547	450	2.484
141	2.621	175	2.588	245	2.544	475	2.480
142	2.620	176	2.587	250	2.542	500	2.475
143	2.619	177	2.587	255	2.540	525	2.472
144	2.618	178	2.586	260	2.537	550	2.468
145	2.617	179	2.585	265	2.535	575	2.465
146	2.616	180	2.584	270	2.533	600	2.462
147	2.615	181	2.583	275	2.531	625	2.459
148	2.613	182	2.583	280	2.529	650	2.456
149	2.612	183	2.582	285	2.527	675	2.454
150	2.611	184	2.581	290	2.525	700	2.451
151	2.610	185	2.580	295	2.524	725	2.449
152	2.609	186	2.580	300	2.522	750	2.447
153	2.608	187	2.579	305	2.520	775	2.445
154	2.607	188	2.578	310	2.518	800	2.443
155	2.606	189	2.577	315	2.517	825	2.441
156	2.605	190	2.577	320	2.515	850	2.439
157	2.604	191	2.576	325	2.514	875	2.438
158	2.603	192	2.575	330	2.512	900	2.436
159	2.602	193	2.575	335	2.511	925	2.434
160	2.601	194	2.574	340	2.509	950	2.433
161	2.600	195	2.573	345	2.508	975	2.432
162	2.600	196	2.572	350	2.506	1000	2.430
163	2.599	197	2.572	355	2.505	1500	2.411
164	2.598	198	2.571	360	2.504	2000	2.399
165	2.597	199	2.570	365	2.502	3000	2.385
166	2.596	200	2.570	370	2.501	5000	2.372
167	2.595	205	2.566	375	2.500	10,000	2.358
168	2.594	210	2.563	380	2.499	∞	2.326
169	2.593	215	2.560	385	2.498		
170	2.592	220	2.557	390	2.496		
171	2.592	225	2.555	395	2.495		

TABLE 8.5.12 *Ranks, r_B, for determining nonparametric B-basis values.*

n	R_b	n	r_B	n	r_B
28	†	660	54	3901	360
29	1	682	56	4005	370
46	2	704	58	4109	380
61	3	726	60	4213	390
76	4	781	65	4317	400
89	5	836	70	4421	410
103	6	890	75	4525	420
116	7	945	80	4629	430
129	8	999	85	4733	440
142	9	1053	90	4836	450
154	10	1107	95	4940	460
167	11	1161	100	5044	470
179	12	1269	110	5147	480
191	13	1376	120	5251	490
203	14	1483	130	5354	500
215	15	1590	140	5613	525
227	16	1696	150	5871	550
239	17	1803	160	6130	575
251	18	1909	170	6388	600
263	19	2015	180	6645	625
275	20	2120	190	6903	650
298	22	2226	200	7161	675
321	24	2331	210	7418	700
345	26	2437	220	7727	730
368	28	2542	230	8036	760
391	30	2647	240	8344	790
413	32	2752	250	8652	820
436	34	2857	260	8960	850
459	36	2962	270	9268	880
481	38	3066	280	9576	910
504	40	3171	290	9884	940
526	42	3276	300	10191	970
549	44	3380	310	10499	1000[1]
571	46	3484	320		
593	48	3589	330		
615	50	3693	340		
638	52	3797	350		

†B-value does not exist for $n < 28$.
[1]For $n > 10499$, use Equation 8.5.12.

TABLE 8.5.13 *Ranks, r_A, for determining non-parametric A-basis values* (Reference (8.6.8 (c)).

n	r_A	n	r_A	n	r_A	n	r_A
≤298	†	3603	27	6657	54	9627	81
299	1	3719	28	6769	55	9736	82
473	2	3834	29	6879	56	9854	83
628	3	3949	30	6990	57	9954	84
773	4	4064	31	7100	58	10063	85
913	5	4179	32	7211	59	10172	86
1049	6	4293	33	7322	60	10281	87
1182	7	4407	34	7432	61	10390	88
1312	8	4521	35	7543	62	10498	89
1441	9	4635	36	7653	63	10607	90
1568	10	4749	37	7763	64	10716	91
1693	11	4862	38	7874	65	10824	92
1818	12	4975	39	7984	66	10933	93
1941	13	5088	40	8094	67	11041	94
2064	14	5201	41	8204	68	11150	95
2185	15	5314	42	8314	69	11258	96
2306	16	5427	43	8423	70	11366	97
2426	17	5539	44	8533	71	11475	98
2546	18	5651	45	8643	72	11583	99
2665	19	5764	46	8753	73	11691	100[1]
2784	20	5876	47	8862	74		
2902	21	5988	48	8972	75		
3020	22	6099	49	9081	76		
3137	23	6211	50	9190	77		
3254	24	6323	51	9300	78		
3371	25	6434	52	9409	79		
3487	26	6545	53	9518	80		

† A-value does not exist for $n < 299$[1]
For $N > 11691$, use Equation 8.5.13

TABLE 8.5.14 *Nonparametric B-basis factors for small sample sizes (Reference 8.3.4.5.2(a)).*

n	r_B	k_B
2	2	35.177
3	3	7.859
4	4	4.505
5	4	4.101
6	5	3.064
7	5	2.858
8	6	2.382
9	6	2.253
10	6	2.137
11	7	1.897
12	7	1.814
13	7	1.738
14	8	1.599
15	8	1.540
16	8	1.485
17	8	1.434
18	9	1.354
19	9	1.311
20	10	1.253
21	10	1.218
22	10	1.184
23	11	1.143
24	11	1.114
25	11	1.087
26	11	1.060
27	11	1.035
28	12	1.010

TABLE 8.5.15 *Nonparametric A-basis factors for small sample sizes (Reference 8.3.4.5.2(b)).*

n	k_A	n	k_A	n	k_A
2	80.00380	38	1.79301	98	1.31553
3	16.91220	39	1.77546	100	1.30806
4	9.49579	40	1.75868	105	1.29036
5	6.89049	41	1.74260	110	1.27392
6	5.57681	42	1.72718	115	1.25859
7	4.78352	43	1.71239	120	1.24425
8	4.25011	44	1.69817	125	1.23080
9	3.86502	45	1.68449	130	1.21814
10	3.57267	46	1.67132	135	1.20620
11	3.34227	47	1.65862	140	1.19491
12	3.15540	48	1.64638	145	1.18421
13	3.00033	49	1.63456	150	1.17406
14	2.86924	50	1.62313	155	1.16440
15	2.75672	52	1.60139	160	1.15519
16	2.65889	54	1.58101	165	1.14640
17	2.57290	56	1.56184	170	1.13801
18	2.49660	58	1.54377	175	1.12997
19	2.42833	60	1.52670	180	1.12226
20	2.36683	62	1.51053	185	1.11486
21	2.31106	64	1.49520	190	1.10776
22	2.26020	66	1.48063	195	1.10092
23	2.21359	68	1.46675	200	1.09434
24	2.17067	70	1.45352	205	1.08799
25	2.13100	72	1.44089	210	1.08187
26	2.09419	74	1.42881	215	1.07595
27	2.05991	76	1.41724	220	1.07024
28	2.02790	78	1.40614	225	1.06471
29	1.99791	80	1.39549	230	1.05935
30	1.96975	82	1.38525	235	1.05417
31	1.94324	84	1.37541	240	1.04914
32	1.91822	86	1.36592	245	1.04426
33	1.89457	88	1.35678	250	1.03952
34	1.87215	90	1.34796	275	1.01773
35	1.85088	92	1.33944	299	1.00000
36	1.83065	94	1.33120		
37	1.81139	96	1.32324		

TABLE 8.5.16 *Critical values for approximate confidence limits on the coefficient of variation.*

n	Confidence level					
	Lower limit C_1			Upper limit C_u		
	0.99	0.95	0.90	0.90	0.95	0.99
2	0.3562	0.4461	0.5101	15.989	31.999	160.051
3	0.4344	0.5207	0.5778	4.415	6.285	14.124
4	0.4834	0.5665	0.6196	2.920	3.729	6.467
5	0.5188	0.5991	0.6493	2.372	2.874	4.396
6	0.5464	0.6242	0.6720	2.089	2.453	3.485
7	0.5688	0.6444	0.6903	1.915	2.202	2.980
8	0.5875	0.6612	0.7054	1.797	2.035	2.660
9	0.6036	0.6755	0.7183	1.711	1.916	2.439
10	0.6177	0.6878	0.7293	1.645	1.826	2.278
20	0.7018	0.7604	0.7939	1.370	1.461	1.666
30	0.7444	0.7964	0.8255	1.280	1.344	1.487
40	0.7718	0.8191	0.8453	1.232	1.284	1.397
50	0.7914	0.8353	0.8594	1.202	1.246	1.341
60	0.8065	0.8476	0.8701	1.181	1.220	1.303
70	0.8185	0.8574	0.8785	1.165	1.200	1.274
80	0.8284	0.8654	0.8855	1.152	1.185	1.252
90	0.8368	0.8722	0.8913	1.142	1.172	1.235
100	0.8440	0.8780	0.8963	1.134	1.162	1.220
125	0.8583	0.8895	0.9062	1.118	1.142	1.193
150	0.8692	0.8982	0.9137	1.106	1.128	1.173
200	0.8849	0.9106	0.9243	1.090	1.109	1.147
250	0.8959	0.9193	0.9317	1.080	1.096	1.129
500	0.9243	0.9416	0.9507	1.055	1.066	1.088

TABLE 8.5.17 *One-sided tolerance factors for acceptance limits on mean values, for normal distribution.*

α	Number of Samples (n)								
	2	3	4	5	6	7	8	9	10
0.5	0.1472	0.1591	0.1539	0.1473	0.1410	0.1354	0.1303	0.1258	0.1217
0.25	0.6266	0.5421	0.4818	0.4382	0.4048	0.3782	0.3563	0.3379	0.3221
0.1	1.0539	0.8836	0.7744	0.6978	0.6403	0.5951	0.5583	0.5276	0.5016
0.05	1.3076	1.0868	0.9486	0.8525	0.7808	0.7246	0.6790	0.6411	0.6089
0.025	1.5266	1.2626	1.0995	0.9866	0.9026	0.8369	0.7838	0.7396	0.7022
0.01	1.7804	1.4666	1.2747	1.1425	1.0443	0.9678	0.9059	0.8545	0.8110
0.005	1.9528	1.6054	1.3941	1.2488	1.1411	1.0571	0.9893	0.9330	0.8854
0.0025	2.1123	1.7341	1.5049	1.3475	1.2309	1.1401	1.0668	1.0061	0.9546
0.001	2.3076	1.8919	1.6408	1.4687	1.3413	1.2422	1.1622	1.0959	1.0397
0.0005	2.4457	2.0035	1.7371	1.5546	1.4196	1.3145	1.2298	1.1596	1.1002
0.00025	2.5768	2.1097	1.8287	1.6363	1.4941	1.3835	1.2943	1.2203	1.1578
0.0001	2.7411	2.2429	1.9436	1.7390	1.5877	1.4701	1.3752	1.2966	1.2301
0.00005	2.8595	2.3389	2.0266	1.813	1.6553	1.5326	1.4337	1.3517	1.2824
0.000025	2.9734	2.4313	2.1065	1.8844	1.7204	1.5928	1.4900	1.4048	1.3327
0.00001	3.1179	2.5487	2.2079	1.9751	1.8031	1.6694	1.5616	1.4723	1.3968

TABLE 8.5.18 *One-sided tolerance factors for acceptance limits on individual values, for normal distribution.*

α	Number of Samples (n)								
	2	3	4	5	6	7	8	9	10
0.5	0.7166	1.0254	1.2142	1.3498	1.4548	1.5400	1.6113	1.6724	1.7258
0.25	1.2887	1.5407	1.6972	1.8106	1.8990	1.9711	2.0317	2.0838	2.1295
0.1	1.8167	2.0249	2.1561	2.2520	2.3272	2.3887	2.4407	2.4856	2.525
0.05	2.1385	2.3239	2.4420	2.5286	2.5967	2.6527	2.7000	2.7411	2.7772
0.025	2.4208	2.5888	2.6965	2.7758	2.8384	2.8900	2.9337	2.9717	3.0052
0.01	2.7526	2.9027	2.9997	3.0715	3.1283	3.1753	3.2153	3.25	3.2807
0.005	2.9805	3.1198	3.2103	3.2775	3.3309	3.3751	3.4127	3.4455	3.4745
0.0025	3.1930	3.3232	3.4082	3.4716	3.5220	3.5638	3.5995	3.6307	3.6582
0.001	3.4549	3.5751	3.6541	3.7132	3.7603	3.7995	3.8331	3.8623	3.8883
0.0005	3.6412	3.7550	3.8301	3.8864	3.9314	3.9690	4.0011	4.0292	4.0541
0.00025	3.8188	3.9270	3.9987	4.0526	4.0958	4.1319	4.1628	4.1898	4.2138
0.0001	4.0421	4.1439	4.2117	4.2629	4.304	4.3384	4.3678	4.3936	4.4166
0.00005	4.2035	4.3011	4.3664	4.4157	4.4554	4.4886	4.5172	4.5422	4.5644
0.000025	4.3592	4.4530	4.5160	4.5637	4.6022	4.6344	4.6620	4.6863	4.7079
0.00001	4.5573	4.6466	4.7069	4.7527	4.7897	4.8206	4.8473	4.8707	4.8915

TABLE 8.5.19 *Upper and lower tail quantiles for two-sided t-distribution.*

n	a									
	0.4	0.25	0.1	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

REFERENCES

8.1.4 Moore, D.S. and McCabe, G.P., *Introduction to the Practice of Statistics*, Second Edition, W.H. Freeman, New York, 1993.

8.3.2.2 Scholz, F.W. and Stephens, M.A., “K-Sample Anderson-Darling Tests of Fit,” *Journal of the American Statistical Association*, Vol 82, 1987, pp. 918-924.

8.3.3.1(a) Snedecor, G.W. and Cochran, W.G., *Statistical Methods*, 7th ed., The Iowa State University Press, 1980, pp. 252-253.

8.3.3.1(b) Stefansky, W., "Rejecting Outliers in Factorial Designs," *Technometrics*, Vol 14, 1972, pp. 469-479.

8.3.4(a) Bury, K.V., *Statistical Models in Applied Science*, John Wiley and Sons, 1975, reprinted by Krieger, 1989, pp. 585-589.

8.3.4(b) Smith, R.L., et al., "Lower Tail Approximations for the Probability of Failure of Three-Dimensional Fibrous Composites with Hexagonal Geometry," *Proc. Royal Society London*, Vol A388, 1983, pp. 353-391.

8.3.4(c) Phoenix, S.L. and Smith, R.L. "A Comparison of Probabilistic Techniques for the Strength of Fibrous Materials under Local Load Sharing Among Fibers," *International Journal of Solids and Structures*, Vol 19, 1983, pp. 479-496.

8.3.4.2 Lawless, J.F., *Statistical Models and Methods for Lifetime Data*, John Wiley & Sons, 1982, pp. 150, 452-460.

8.3.4.5.1 *Metallic Materials and Elements for Aerospace Vehicle Structures*, MIL-HDBK-5E, Naval Publications and Forms Center, Philadelphia, Pennsylvania, 1 June 1987, pp. 9-166,9-167.

8.3.4.5.2(a) Hanson, D.L. and Koopmans, L.H., "Tolerance Limits for the Class of Distribution with Increasing Hazard Rates," *Annals of Math. Stat.*, Vol 35, 1964, pp. 1561-1570.

8.3.4.5.2(b) Vangel, M.G., “One-Sided Nonparametric Tolerance Limits,” *Communications in Statistics: Simulation and Computation*, Vol. 23, 1994, p. 1137.

8.3.5.1(a) Draper, N.R. and Smith, H., *Applied Regression Analysis*, 2nd ed., John Wiley & Sons, 1981.

8.3.5.1(b) Hinkelmann, K and Kempthorne, O., *Design and Analysis of Experiments Volume 1: Introduction to Experimental Design*, New York, John Wiley & Sons, 1994.

8.3.5.1(c) Scheffé, H., *The Analysis of Variance*, John Wiley & Sons, 1959.

8.3.5.1(d) Box, G.E.P., Hunter, W.G., and Hunter, J.S., *Statistics for Experimenters*, John Wiley & Sons, 1981.

8.3.5.1(e) Pierce, D.A. and Kopecky, K.J., "Testing Goodness-of-Fit for the Distribution of Errors in Regression Models," *Biometrika*, Vol 66, 1979, pp. 1-5.

8.3.5.2(a) Searle, S.R., *Linear Models*, John Wiley & Sons, 1971, pp. 473-474.

8.3.5.2(b) Vangel, M.G., "New Methods for One-Sided Tolerance Limits for a One-Way Balanced Random Effects ANOVA Model," *Technometrics*, Vol 34, 1992, pp. 176-185.

8.3.5.2.1(a) Lehmann, E.L., *Testing Statistical Hypotheses*, John Wiley & Sons, 1959, pp. 274-275.

8.3.5.2.1(b) Levene, H., "Robust Tests for Equality of Variances," in *Contributions to Probability and Statistics*, ed. I. Olkin, Palo, Alto, CA: Stanford University Press, 1960.

8.3.5.2.1(c) Conover, W.J., Johnson, M.E., and Johnson, M.M., "A Comparative Study of Tests for Homogeneity of Variances, with Applications to the Outer Continental Shelf Bidding Data," *Technometrics*, Vol. 23, pp. 351-361, 1981

8.3.5.4(a) Shyprykevich, P., "The Role of Statistical Data Reduction in the Development of Design Allowables for Composites," *Test Methods for Design Allowables for Fibrous Composites: 2nd Volume,* ASTM STP 1003, C.C. Chamis, ed., American Society for Testing and Materials, Philadelphia, PA, 1989, pp. 111-135.

8.3.5.4(b) Tomblin, John S., Ng, Yeow C., and Raju, K. Suresh, "Material Qualification and Equivalency for Polymer Matrix Composite Material Systems," DOT/FAA/AR-00/47, March 2001.

8.3.6 Tukey, J.W., *Exploratory Data Analysis*, Addison-Wesley Publishing Company, 1977

8.3.6.1 Parzen, E., "A Density-Quantile Function Perspective on Robust Estimation," *Robustness in Statistics*, Academic Press, New York, 1979

8.3.6.2(a) Parzen, E., "Entropy Interpretation of Tests for Normality by Shapiro-Wilk Statistics," presented at the Twenty-Eighth Conference on the Design of Experiments in Army Research, Development, and Testing, 20-22 October 1982, Monterey, California

8.3.6.2(b) Parzen, E., "Informative Quantile Function and Identification of Probability Distribution Types," Technical Report A-26, Department of Statistics, Texas A&M University (August 1983)

8.3.7.2(a) Vangel, M.G., "One-Sided β-Content Tolerance Intervals for Mixed Models with Two Components of Variance," *Technometrics*, submitted

8.3.7.2(b) Vangel, M.G., "Design Allowables From Regression Models Using Data From Several Batches," in *12th Symposium on Composite Materials: Testing and Design*, American Society of Testing and Materials, to appear, 1996

8.4.3.2 Vangel, M.G., "Confidence Intervals for a Normal Coefficient of Variation," *The American Statistician*, Vol. 14, pp. 21-26, 1996

8.4.5(a) Papirno, R., "Average Stress-Strain Curves for Resin Matrix Composites," *Composite Technology Review*, 1986, pp. 107-116

8.4.5(b) Papirno, R., "Algebraic Approximations of Stress-Strain Curves for Kevlar-Reinforced Composites," *Journal of Testing and Evaluation*, Vol 13, 1985, pp. 115-122

8.5.2 Johnson, N.L. and Kotz, S., *Distributions in Statistics: Continuous Univariate Distributions - 1*, John Wiley & Sons, 1970, p. 176

8.5.9 Jones, R.A., Ossiander, M., Scholz, F.W., and Shorack, G.R., "Tolerance Bounds for Log-gamma Regression Model," *Technometrics,* Vol 27, May 1985, pp. 109-118

8.5.11 Odeh, R.E. and Owen, D.B., *Tables of Normal Tolerance Limits, Sampling Plans and Screening*, Marcel Dekker, 1980.

8.5.17 Vangel, Mark, "Acceptance and Compliance Testing Using the Sample Mean and an Extrema", submitted to Technometrics, 2001.