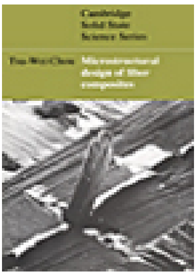


Microstructural Design of Fiber Composites

Tsu-Wei Chou

Book DOI: http://dx.doi.org/10.1017/CBO9780511600272

Online ISBN: 9780511600272

Hardback ISBN: 9780521354820

Paperback ISBN: 9780521019651

Chapter

2 - Thermoelastic behavior of laminated composites pp. 29-79

Chapter DOI: http://dx.doi.org/10.1017/CBO9780511600272.003

Cambridge University Press

2 Thermoelastic behavior of laminated composites

2.1 Introduction

Laminated composites are made by bonding unidirectional laminae together in predetermined orientations. The basis for analysis of thin laminated composites is the classical plate theory. When the thickness direction properties significantly contribute to the response of the laminate to an externally applied elastic field, the classical plate theory breaks down.

Fundamental to the treatment of thin laminates is the knowledge of the thermoelastic properties of a unidirectional lamina. These properties are predictable from the corresponding properties of constituent fiber and matrix materials as well as the fiber volume fraction. Having established the elastic response of a unidirectional lamina, the behavior of laminated composites is then analyzed from the strain and curvature of the mid-plane of the laminate as well as the force and moment resultants acting on its boundary edges. Because of the complexity of the constitutive equations for a general anisotropic laminated plate, simplifications of the stress–strain relations are accomplished through the manipulation of the geometric arrangement of the laminae. The lamination theory is a relatively mature subject; its treatment can be found in text books of, for instance, Ashton, Halpin and Petit (1969), Jones (1975), Vinson and Chou (1975), Christensen (1979), Tsai and Hahn (1980), Carlsson and Pipes (1987), and Chawla (1987), and in the review articles of Chou (1989a and b). A modification of the classical plate theory is in the inclusion of higher order terms in the displacement field expansion to account for the transverse shear deformation. An outline of such modifications adopted by various researchers is presented.

The classical thin laminated theory has been extended to take into consideration the effects of thermal and moisture diffusions, with particular emphasis on the transient behavior. Because of the large differences in the magnitudes of the thermal conductivity and moisture diffusion coefficients, the thermal and hygroscopic problems can be solved separately and their linear elastic fields can be superposed. Stress concentrations due to transient thermal effects are of particular interest in the study of laminate thermal shock resistance.

The mechanics of the thermoelastic behavior of laminated composites is fundamental to the understanding of the strength, fracture and fatigue behavior of all continuous-fiber composites including those reinforced with textile preforms.

2.2 Elastic behavior of a composite lamina

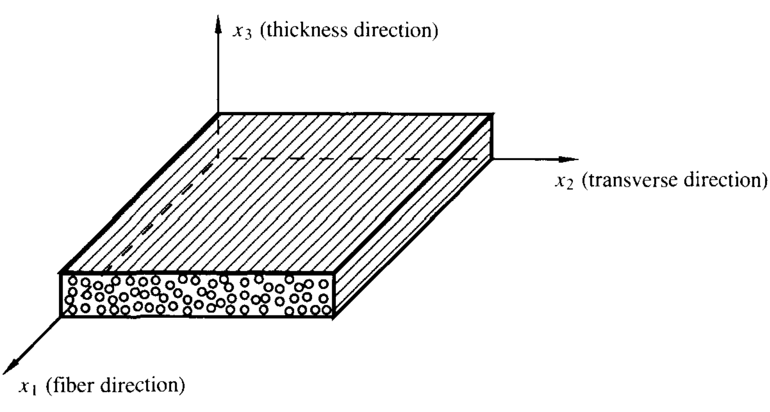
2.2.1 *Elastic constants*

It is well known that for a homogeneous isotropic material (i.e. the material properties are independent of the location and direction), two independent material elastic constants are sufficient to specify the constitutive relations. These could be any two of the five constants commonly used: E (Young's modulus), ν (Poisson's ratio), G (shear modulus), K (bulk modulus), and k (plane strain bulk modulus). The relations among these constants are

$$\begin{aligned} G &= E/2(1+\nu) \\ K &= E/3(1-2\nu) \\ k &= E/2(1-\nu-2\nu^2) \end{aligned} \tag{2.1}$$

Twenty-one independent constants are necessary to describe the elastic stress–strain relation of a generally anisotropic material (i.e. the material properties are different in different directions). However, due to the material symmetries, the number of the independent constants can be greatly reduced. Consider a lamina (Fig. 2.1) composed of unidirectional straight fibers in a matrix. Assume that

Fig. 2.1. A unidirectional fiber composite lamina.



it is homogeneous on a scale much larger than that of the inter-fiber spacing. Then, the unidirectional lamina can be treated as a homogeneous orthotropic continuum (i.e. having three mutually perpendicular planes of symmetry). The coordinates x_1–x_2–x_3 shown in Fig. 2.1 are known as the material principal coordinates, where x_1 is parallel to the fibers and x_2 lies in the plane of the lamina. For circular cross-section fibers randomly distributed in a unidirectional lamina, the lamina can be further assumed macroscopically as transversely isotropic, namely the material properties in planes transverse to the fiber direction are isotropic. Then, there are only five independent constants. The commonly used engineering elastic constants for the transversely isotropic lamina, referring to the fiber (x_1) and in-plane transverse (x_2) directions, are denoted by E_{11} (longitudinal Young's modulus), E_{22} (transverse Young's modulus), ν_{12} (Poisson's ratio due to loading in the x_1 direction and contraction in the x_2 direction), and G_{12} (in-plane shear modulus). These four independent elastic constants can be determined experimentally by three simple tensile tests of composite specimens with fiber orientations of 0°, 90° and $[\pm 45°]_{2s}$; the relevant testing standards are ASTM D3039-76 and ASTM D3518-76. The fifth independent constant, representing the transverse isotropic properties, could be either ν_{23} (transverse Poisson's ratio) or G_{23} (transverse shear modulus); the two are related by

$$G_{23} = \frac{E_{22}}{2(1 + \nu_{23})} \tag{2.2}$$

The other engineering constants are:

$$\begin{aligned}
\nu_{21} &= \frac{E_{22}}{E_{11}} \nu_{12} \\
E_{33} &= E_{22} \\
G_{13} &= G_{12} \\
\nu_{32} &= \nu_{23} \\
\nu_{13} &= \nu_{12} \\
\nu_{31} &= \nu_{21}
\end{aligned} \tag{2.3}$$

Various micromechanical models are available for predicting the elastic properties of unidirectional laminae from their constituent properties. Most of the matrices and some of the fibers used in composites can be considered as isotropic. Let the elastic constants of Eq. (2.1) for the isotropic fiber and matrix materials be denoted by the subscripts f and m, respectively. Also, the fiber volume fraction of the composite is indicated by V_f. Assuming no void in the composite, the volume fraction of matrix is

$$V_m = 1 - V_f \tag{2.4}$$

The following relations due to Hashin and Rosen (see Rosen 1973) are quoted for their concise forms and, hence, ease in application.

$$\begin{aligned}
E_{11} &= E_f V_f + E_m V_m + \frac{4V_f V_m (\nu_f - \nu_m)^2}{\dfrac{V_m}{k_f} + \dfrac{V_f}{k_m} + \dfrac{1}{G_m}} \\
E_{22} &= \frac{4k_t^* G_t^*}{k_t^* + G_t^*\left(1 + \dfrac{4k_t^* \nu_{12}^2}{E_{11}}\right)} \\
\nu_{12} &= \nu_f V_f + \nu_m V_m + \frac{V_f V_m (\nu_f - \nu_m)\left(\dfrac{1}{k_m} - \dfrac{1}{k_f}\right)}{\dfrac{V_m}{k_f} + \dfrac{V_f}{k_m} + \dfrac{1}{G_m}} \\
G_{12} &= G_m \frac{V_m G_m + (1 + V_f) G_f}{(1 + V_f) G_m + V_m G_f} \\
\nu_{23} &= \frac{E_{22}}{2G_t^*} - 1
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
k_f &= E_f / 2(1 - \nu_f - \nu_f^2) \\
k_m &= E_m / 2(1 - \nu_m - \nu_m^2) \\
k_t^* &= \frac{k_m k_f + (V_f k_f + V_m k_m) G_m}{V_m k_f + V_f k_m + G_m} \\
G_t^* &= G_m \frac{(\alpha + \beta_m V_f)(1 + \rho V_f^3) - 3V_f V_m^2 \beta_m^2}{(\alpha - V_f)(1 + \rho V_f^3) - 3V_f V_m^2 \beta_m^2} \\
\alpha &= (\gamma + \beta_m)/(\gamma - 1) \\
\beta_m &= \frac{1}{3 - 4\nu_m}, \qquad \beta_f = \frac{1}{3 - \nu_f} \\
\rho &= (\beta_m - \gamma\beta_f)/(1 + \gamma\beta_f) \\
\gamma &= G_f / G_m
\end{aligned} \tag{2.6}$$

Fibers such as carbon and Kevlar exhibit anisotropic behavior; their thermoelastic properties along and transverse to the fiber axis are significantly different. These fibers are considered to be transversely isotropic, and thus five independent constants are needed to describe their elastic properties, namely, E_{1f}, E_{2f}, G_{12f}, ν_{12f} and G_{23f}. The following expressions, due to Chamis (1983), describe the elastic properties of a unidirectional lamina composed of anisotropic fibers in an isotropic matrix:

$$
\begin{aligned}
E_{11} &= E_{1f}V_f + E_m V_m \\
E_{22} = E_{33} &= \frac{E_m}{1 - V_f(1 - E_m/E_{2f})} \\
G_{12} = G_{13} &= \frac{G_m}{1 - V_f(1 - G_m/G_{12f})} \\
G_{23} &= \frac{G_m}{1 - V_f(1 - G_m/G_{23f})} \\
\nu_{12} = \nu_{13} &= \nu_{12f}V_f + \nu_m V_m \\
\nu_{23} &= \frac{E_{22}}{2G_{23}} - 1
\end{aligned}
\tag{2.7}
$$

Halpin and Tsai (1967) have developed some semi-empirical relations for the laminar elastic properties. These expressions contain certain parameters which are influenced by the geometry of the reinforcing phases, their packing in the composite, and the loading conditions. Estimates of the values of these parameters can be obtained by comparing the Halpin–Tsai equation predictions with the numerical solutions employing formal elasticity theory (Halpin 1984). The effect of interfacial debonding on elastic properties has been discussed by Takahashi and Chou (1988).

2.2.2 *Constitutive relations*

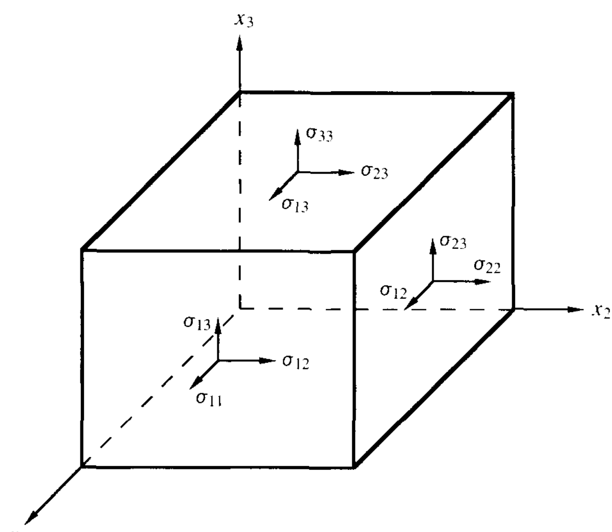
Consider a unidirectional lamina exhibiting orthotropic symmetry. The constitutive relations, referring to the material principal coordinates x_1–x_2–x_3, assume the general form (Vinson and Chou 1975):

$$
\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix}
=
\begin{pmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66}
\end{pmatrix}
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}
\tag{2.8}
$$

Here σ_{ij}, the stress tensors, are defined in Fig. 2.2. ε_{ij} are the strain tensors defined in a manner analogous to the stress components; it should be noted that the engineering shear strain $\gamma_{ij} = 2\varepsilon_{ij}$ $(i \neq j)$. S_{ij} denote the components of the compliance matrix. For the case of a transversely isotropic lamina with the x_2–x_3 plane being isotropic, the compliance constants are related to the engineering elastic constants as:

$$
\begin{aligned}
S_{11} &= \frac{1}{E_{11}} \\
S_{22} = S_{33} &= \frac{1}{E_{22}} \\
S_{12} = S_{13} &= -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \\
S_{23} &= -\frac{\nu_{23}}{E_{22}} \\
S_{44} &= \frac{1}{G_{23}} \\
S_{55} = S_{66} &= \frac{1}{G_{12}}
\end{aligned}
\tag{2.9}
$$

Fig. 2.2. Stress tensor components.



Equation (2.8) can be inverted to obtain the following stress–strain relations

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{pmatrix} \tag{2.10}$$

where C_{ij} are the components of the stiffness matrix. Again, for the case of transverse isotropy in the x_2–x_3 plane, the following relations hold:

$$\begin{aligned} C_{11} &= E_{11}(1 - \nu_{23}^2)/\Delta \\ C_{22} &= C_{33} = E_{22}(1 - \nu_{12}\nu_{21})/\Delta \\ C_{44} &= G_{23} \\ C_{55} &= C_{66} = G_{12} \\ C_{12} &= C_{13} = (\nu_{21} + \nu_{21}\nu_{23})E_{11}/\Delta = (\nu_{12} + \nu_{12}\nu_{23})E_{22}/\Delta \\ C_{23} &= (\nu_{23} + \nu_{12}\nu_{21})E_{22}/\Delta \\ \Delta &= 1 - 2\nu_{12}\nu_{21} - \nu_{23}^2 - 2\nu_{12}\nu_{21}\nu_{23} \end{aligned} \tag{2.11}$$

For a unidirectional composite lamina where the thickness is much smaller than the in-plane (x_1–x_2) dimensions, it is sufficient to consider the two-dimensional constitutive relations. Following the convention used in the composites literature, the following contracted notations, σ_i and ε_i, are introduced for the stress and strain components, respectively. Their relations to the tensorial stress and strain components are:

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}(=\tau_{23}), \quad \sigma_5 = \sigma_{13}(=\tau_{13}), \quad \text{and} \quad \sigma_6 = \sigma_{12}(=\tau_{12})$$

$$\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}(=\gamma_{23}), \quad \varepsilon_5 = 2\varepsilon_{13}(=\gamma_{13}), \quad \text{and} \quad \varepsilon_6 = 2\varepsilon_{12}(=\gamma_{12})$$

Under plane stress condition (i.e. $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$), and using the contracted notations, Eq. (2.8) can be reduced to

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} \tag{2.12}$$

where the compliance constants S_{ij} are given in Eq. (2.9). Also $\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2$ and $\varepsilon_4 = \varepsilon_5 = 0$. By inverting Eq. (2.12), the following two-dimensional stress–strain relations are obtained:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{pmatrix} \tag{2.13}$$

Here, the lamina exhibits orthotropic symmetry. The Q_{ij} in Eq. (2.13) are known as the reduced stiffness constants, and are related to the engineering constants as follows:

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{1 - \nu_{12}\nu_{21}} \\ Q_{12} &= \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}} \\ Q_{22} &= \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \\ Q_{66} &= G_{12} \end{aligned} \tag{2.14}$$

It should be noted that the Q_{ij} so obtained by assuming the plane stress condition of the unidirectional lamina are not identical to the C_{ij} given in Eq. (2.11). In fact, the difference between C_{ij} and Q_{ij} increases as the lamina becomes more isotropic. The inter-relations

Table 2.1. *Inter-relations among the different forms of elastic constants. After Chou* (1989b)

Engineering constant	E_{11}	E_{22}	ν_{12}	ν_{21}	G_{12}
Compliance	$1/S_{11}$	$1/S_{22}$	$-S_{12}/S_{11}$	$-S_{12}/S_{22}$	$1/S_{66}$
Reduced stiffness	$(Q_{11}Q_{22} - Q_{12}^2)/Q_{22}$	$(Q_{11}Q_{22} - Q_{12}^2)/Q_{11}$	Q_{12}/Q_{22}	Q_{12}/Q_{11}	Q_{66}
Compliance	S_{11}	S_{22}	S_{12}		S_{66}
Reduced stiffness	$Q_{22}/Q_{11}Q_{22} - Q_{12}^2)$	$Q_{11}/(Q_{11}Q_{22} - Q_{12}^2)$	$Q_{12}/(Q_{11}Q_{22} - Q_{12}^2)$		$1/Q_{66}$
Engineering constant	$1/E_{11}$	$1/E_{22}$	$-\nu_{12}/E_{11}$		$1/G_{12}$
Reduced stiffness	Q_{11}	Q_{22}	Q_{12}		Q_{66}
Engineering constant	$E_{11}/(1 - \nu_{12}\nu_{21})$	$E_{22}(1 - \nu_{12}\nu_{21})$	$\nu_{12}E_{22}/(1 - \nu_{12}\nu_{21})$		G_{12}
Compliance	$S_{22}/(S_{11}S_{22} - S_{12}^2)$	$S_{11}/(S_{11}S_{22} - S_{12}^2)$	$-S_{12}/(S_{11}S_{22} - S_{12}^2)$		$1/S_{66}$

among the engineering constants, compliance constants and reduced stiffness constants are summarized in Table 2.1.

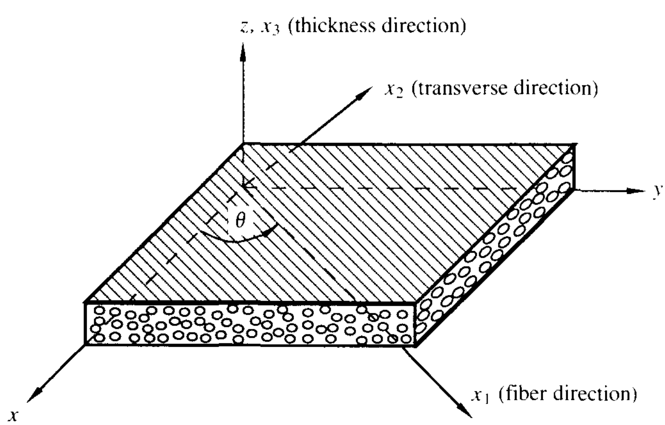
For a unidirectional lamina oriented at an angle θ with respect to the reference axes x–y (Fig. 2.3), the stress–strain relations in the x–y coordinates are

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} \tag{2.15}$$

where $\bar{Q}_{ij}$, the transformed reduced stiffness, are given by

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta \\
&\quad + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta \\
&\quad + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta \\
&\quad + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta \\
&\quad + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned} \tag{2.16}$$

Fig. 2.3. Fiber axis at an angle θ from the lamina reference axis x.



Note that in the x–y coordinate system the notations of τ_{xy} and γ_{xy} are introduced for the shear stress and strain, respectively. The unidirectional lamina referred to the x–y axes is termed generally orthotropic.

Equation (2.15) can be inverted to obtain the strain–stress relations in the following general form:

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} \tag{2.17}$$

in which the $\bar{S}_{ij}$ are the transformed compliance constants and their relations to S_{ij} and θ are

$$\begin{aligned}
\bar{S}_{11} &= S_{11}\cos^4\theta + (2S_{12}+S_{66})\sin^2\theta\cos^2\theta + S_{22}\sin^4\theta \\
\bar{S}_{12} &= S_{12}(\sin^4\theta+\cos^4\theta) + (S_{11}+S_{22}-S_{66})\sin^2\theta\cos^2\theta \\
\bar{S}_{22} &= S_{11}\sin^4\theta + (2S_{12}+S_{66})\sin^2\theta\cos^2\theta + S_{22}\cos^4\theta \\
\bar{S}_{16} &= (2S_{11}-2S_{12}-S_{66})\sin\theta\cos^3\theta \\
&\quad - (2S_{22}-2S_{12}-S_{66})\sin^3\theta\cos\theta \\
\bar{S}_{26} &= (2S_{11}-2S_{12}-S_{66})\sin^3\theta\cos\theta \\
&\quad - (2S_{22}-2S_{12}-S_{66})\sin\theta\cos^3\theta \\
\bar{S}_{66} &= 2(2S_{11}+2S_{22}-4S_{12}-S_{66})\sin^2\theta\cos^2\theta \\
&\quad + S_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{2.18}$$

The engineering constants of the unidirectional lamina referring to the x–y axes, which are not aligned with the material principal directions, can be expressed as functions of the off-axis angle, θ, by using Eqs. (2.9) and (2.18)

$$\begin{aligned}
\frac{1}{E_{xx}} &= \frac{1}{E_{11}}\cos^4\theta + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\sin^2\theta\cos^2\theta + \frac{1}{E_{22}}\sin^4\theta \\
\nu_{xy} &= E_{xx}\left(\frac{\nu_{12}}{E_{11}}(\sin^4\theta+\cos^4\theta)\right. \\
&\quad \left. - \left(\frac{1}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta\right) \\
\frac{1}{E_{yy}} &= \frac{1}{E_{11}}\sin^4\theta + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right)\sin^2\theta\cos^2\theta + \frac{1}{E_{22}}\cos^4\theta \\
\frac{1}{G_{xy}} &= 2\left(\frac{2}{E_{11}} + \frac{2}{E_{22}} + \frac{4\nu_{12}}{E_{11}} - \frac{1}{G_{12}}\right)\sin^2\theta\cos^2\theta \\
&\quad + \frac{1}{G_{12}}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{2.19}$$

The variations of E_{xx}, G_{xy}, and ν_{xy}, with fiber orientation angle, θ, for a Kevlar-49/epoxy composite are shown in Fig. 2.4.

Jones (1975) discussed the extremum (largest or smallest) values of composite elastic properties, which do not necessarily occur in the principal material directions. It can be shown that E_{xx} is greater than both E_{11} and E_{22} for some values of θ if

$$G_{12} > \frac{E_{11}}{2(1+\nu_{12})} \tag{2.20}$$

and that E_{xx} is less than both E_{11} and E_{22} for some values of θ if

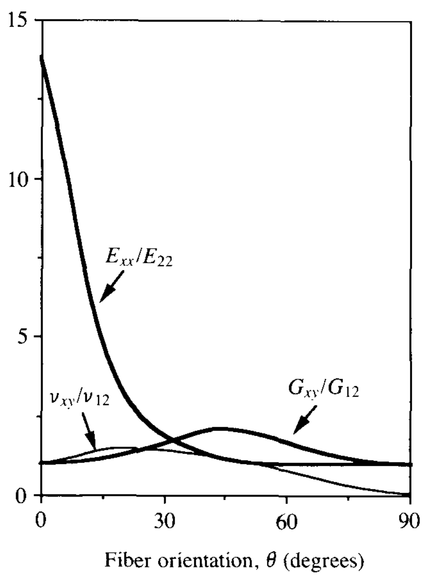
$$G_{12} > \frac{E_{11}}{2(E_{11}/E_{22}+\nu_{12})} \tag{2.21}$$

2.3 Elastic behavior of a composite laminate

2.3.1 *Classical composite lamination theory*

Based upon the constitutive relations for a lamina composed of a generally orthotropic material, Eq. (2.15), the constitutive relations for a laminate formed by bonding several laminae

Fig. 2.4. Variations of engineering elastic constants with fiber orientation angle, θ, for a Kevlar-49/epoxy composite with $V_f = 0.6$, $E_{11} = 76$ GPa, $E_{22} = 5.5$ GPa, $G_{12} = 2.3$ GPa and $\nu_{12} = 0.34$.



together is presented in this section. The orientation and material system of each lamina are general. Figure 2.5 depicts the geometry of an n-layered laminate of thickness h; the x–y plane coincides with the laminate geometric middle plane. Following the approach of the classical, linear, thin plate theory, the following assumptions are made (see Vinson and Chou 1975).

(1) A lineal element of the plate extending through the plate thickness, normal to the middle surface (x–y plane) in the unstressed state, upon the application of load: (a) undergoes at most a translation and a rotation with respect to the original coordinate system, and (b) remains normal to the deformed middle surface.

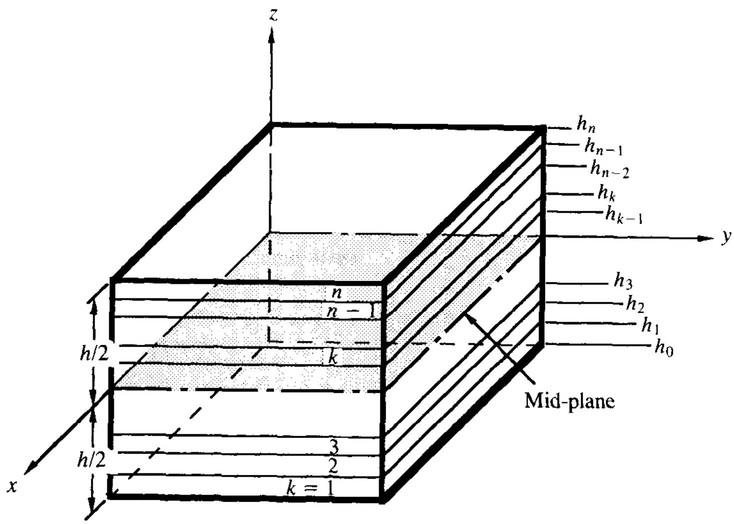
This assumption implies that the lineal element does not elongate or contract, and remains straight upon load applications.

(2) The plate resists lateral and in-plane loads by bending, transverse shear stress, and in-plane action, not through block-like compression or tension in the plate in the thickness direction.

Based upon the foregoing assumptions, also known as the Kirchhoff hypothesis for plates, the strain components can be derived

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{pmatrix} + z \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix} \tag{2.22}$$

Fig. 2.5. An n-layered laminate.



Here, ε_{xx}^{o}, ε_{yy}^{o} and γ_{xy}^{o} are the laminate mid-plane strain, which are expressed in terms of the mid-plane displacements u^{o} and v^{o} in the x and y directions, respectively:

$$\varepsilon_{xx}^{o} = \frac{\partial u^{o}}{\partial x}, \qquad \varepsilon_{yy}^{o} = \frac{\partial v^{o}}{\partial y}, \qquad \gamma_{xy}^{o} = \frac{\partial u^{o}}{\partial y} + \frac{\partial v^{o}}{\partial x} \tag{2.23}$$

The mid-plane curvatures are related to the z direction mid-plane displacement w^{o}

$$\kappa_{xx} = -\frac{\partial^2 w^{o}}{\partial x^2}, \qquad \kappa_{yy} = -\frac{\partial^2 w^{o}}{\partial y^2}, \qquad \kappa_{xy} = -\frac{\partial^2 w^{o}}{\partial x\, \partial y} \tag{2.24}$$

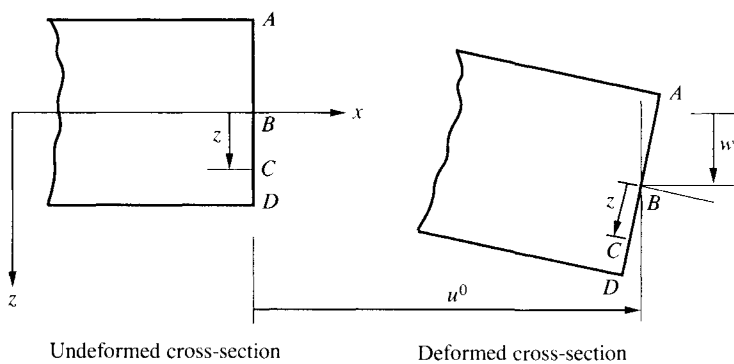
Note that κ_{xy} represents the twist curvature of the mid-plane. Figure 2.6 depicts the deformation associated with a typical cross-sectional element in a thin plate.

Also, following the approach of the classical plate theory, the resultant forces and moments, instead of the stresses, are utilized in the constitutive relations. Referring to Figs. 2.7 (a) and (b), the force and moment resultants of the laminate are obtained by integrating the stresses of each lamina, through the laminate thickness, h:

$$(N_x, N_y, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \tau_{xy})\, dz \tag{2.25}$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) z\, dz \tag{2.26}$$

Fig. 2.6. Deformation of a typical cross-sectional element in a thin laminated plate.

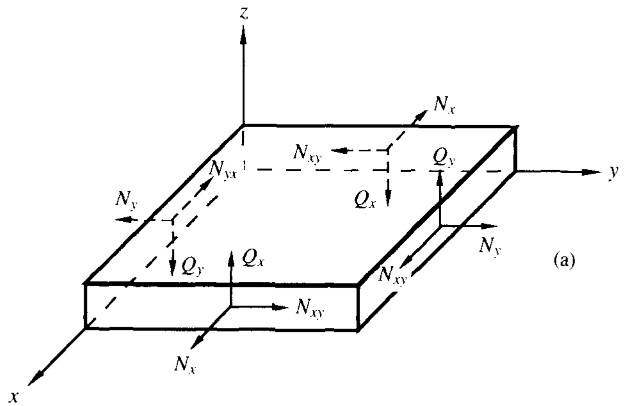


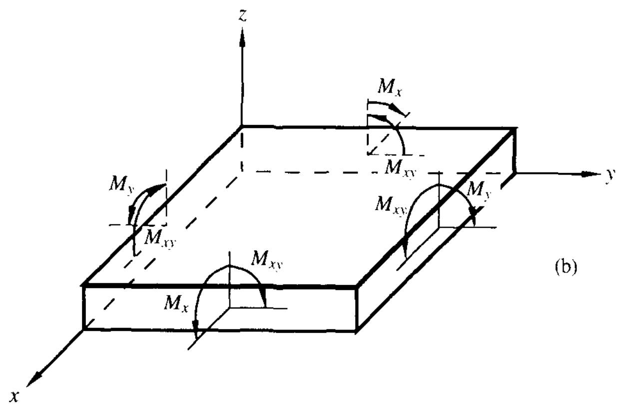
Substitution of Eqs. (2.15) and (2.16) into Eqs. (2.25) and (2.26) results in the following:

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx}^{\text{o}} \\ \varepsilon_{yy}^{\text{o}} \\ \gamma_{xy}^{\text{o}} \end{pmatrix}$$

$$+ \begin{pmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{pmatrix} \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix} \tag{2.27}$$

Fig. 2.7. (a) In-plane force resultants. (b) In-plane moment resultants.





$$\begin{pmatrix} M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{pmatrix} \begin{pmatrix} \varepsilon^{o}_{xx} \\ \varepsilon^{o}_{yy} \\ \gamma^{o}_{xy} \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{pmatrix} \begin{pmatrix} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix} \tag{2.28}$$

where

$$A_{ij} = \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k - h_{k-1})$$

$$B_{ij} = \tfrac{1}{2} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^2 - h_{k-1}^2) \tag{2.29}$$

$$D_{ij} = \tfrac{1}{3} \sum_{k=1}^{n} (\bar{Q}_{ij})_k (h_k^3 - h_{k-1}^3)$$

In Eqs. (2.27)–(2.29), A_{ij}, B_{ij}, and D_{ij} are called extensional stiffness, extension-bending coupling stiffness, and bending stiffness, respectively. The summation in Eqs. (2.29) is carried out over all the laminae; $(\bar{Q}_{ij})_k$ refers to the reduced stiffness of the kth layer. Eqs. (2.27) and (2.28) are often expressed in the condensed form as

$$\left(\frac{N}{M}\right) = \left(\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right) \left(\frac{\varepsilon^{o}}{\kappa}\right) \tag{2.30}$$

where $[\kappa]$ is composed of κ_{xx}, κ_{yy} and $2\kappa_{xy}$.

The constitutive relations of Eqs. (2.27) and (2.28) can be rearranged into other useful forms by partially or totally inverting them. The totally inverted forms of Eqs. (2.27) and (2.28) are given in the following condensed matrix expressions:

$$\begin{aligned} [\varepsilon^{o}] &= [A'][N] + [B'][M] \\ [\kappa] &= [B'][N] + [D'][M] \end{aligned} \tag{2.31}$$

where

$$[A'] = [A^*] - [B^*][D^{*-1}][C^*]$$

$$[B'] = [B^*][D^{*-1}] = -[D^{*-1}][C^*]$$

$$[D'] = [D^{*-1}]$$

and (2.32)

$$[A^*] = [A^{-1}]$$
$$[B^*] = -[A^{-1}][B]$$
$$[C^*] = [B][A^{-1}]$$
$$[D^*] = [D] - [B][A^{-1}][B]$$

An application of Eqs. (2.31) is found, for instance, when the stress and moment resultants acting on a laminated plate are specified. Then, with the knowledge of the elastic constants, the mid-plane strain and curvature of the laminate can be determined. The strain components of a specific lamina in terms of the plate reference axes can be derived from Eq. (2.22) and the corresponding stresses from Eq. (2.15). The existing criteria for laminar failure, due to combined in-plane stresses or strains, require the knowledge of stresses and strains along the fiber as well as the transverse directions. This information can be readily obtained by transformation of the stress and strain components to the principal material directions. Thus, the correlation between external loading on the laminated plate and the failure of an individual lamina can be established.

2.3.2 *Geometrical arrangements of laminae*

It has been established in Eqs. (2.29) that the elastic behavior of a composite laminate composed of unidirectional laminae is determined by the constituent material properties as well as the orientation and location of the individual laminae. These geometric aspects of the laminae are indicated by following the convention of the composites literature. For example, $[0°/45°_2/-45°_4/45°_2/0°]$ indicates the stacking sequence of a laminate with one layer at 0°, two layers at 45°, four layers at −45°, two layers at 45°, and one layer at 0°. Because of the mid-plane symmetry, this stacking sequence can also be expressed as $[0°/45°_2/-45°_2]_s$. Following this convention, the basic arrangements of laminae can be expressed as [0°] for unidirectional, [0°/90°] for cross-ply, and $[+\theta/-\theta]$ for angle-ply. The implications of the laminar geometrical arrangements on the laminar elastic behavior, namely, the $[A]$, $[B]$, and $[D]$ matrices, are discussed below.

The $[A]$ matrix relates the stress resultants with the mid-plane strains. The couplings between normal stress resultants and mid-plane shear strains, as well as shear stress resultants and mid-plane normal strains, are due to the components A_{16} and A_{26}. There is also the coupling between mid-plane stress resultants and the bending and twisting of the laminate through the $[B]$ matrix. In particular, the components B_{16} and B_{26} relate normal stress resultants with the twisting of the laminate. The $[B]$ matrix also plays a role in the coupling between the moment resultants and in-plane strains. Finally, the D_{16} and D_{26} terms are responsible for the interaction between the bending moment and twisting.

The various coupling effects in laminated composites can be minimized or eliminated through suitable choices of the laminae stacking sequence. As can be seen from Eqs. (2.29), the B_{ij} terms involve the squares of the z coordinates of the top and bottom faces of each lamina. Each term of B_{ij} vanishes if for every lamina above the mid-plane there is a lamina, identical in properties and orientation, located at the same distance below the mid-plane. Such mid-plane symmetry arrangements eliminate the bending–stretching coupling. The terms A_{16} and A_{26} both vanish under either of the following conditions: (a) all of the laminae assume 0°, 90° or cross-ply [0°/90°] configuration; (b) for every lamina of $+\theta$ orientation there is another lamina of the same property and thickness with a $-\theta$ orientation. The terms D_{16} and D_{26} are zero for the cases: (a) all of the lamina assume 0°, 90° or cross-ply configuration; and (b) for every lamina oriented at $+\theta$ at a given distance above the mid-plane there is an identical layer at the same distance below the mid-plane oriented at $-\theta$. It is obvious that the D_{16} and D_{26} terms are not zero for any mid-plane symmetric laminate, except for the cases of all 0°, all 90° and cross-ply. However, the magnitude of these terms can be made small by increasing the number of layers in the angle-ply configuration. Table 2.2 shows the effect of stacking sequence on the $[A]$, $[B]$ and

Table 2.2. *Effect of stacking sequence on* $[A]$, $[B]$ *and* $[D]$ *matrices. After Chou* (1989b)

	$\theta = 0°, 90°$	0°/90°	$\ldots +\theta_2/-\theta_1/$ $+\theta_1/-\theta_2 \ldots$ (anti-symmetry)	$\ldots +\theta_2/-\theta_1$ $-\theta_1/+\theta_2 \ldots$ (symmetry)	Same number of $+\theta$ and $-\theta$ layers
A_{16}, A_{26}	zero	zero	zero	–	zero
B_{11}, B_{22}, B_{12}, B_{66}	zero	–	zero	zero	–
B_{16}, B_{26}	zero	zero	–	zero	–
D_{16}, D_{26}	zero	zero	zero	–	–

$[D]$ matrices. The optimization of laminate design for strength has been discussed by Fukunaga and Chou (1988a and b).

2.4 Thick laminates

The term 'thick laminates' here is used to describe composite plates of which the thickness direction properties significantly contribute to the response of the material. Exact elasticity solutions of thick plates have demonstrated that the classical lamination theory of Section 2.3 is not applicable to the thick laminates. Experimental results (for example, Whitney 1972, and Stein and Jegley 1987) have shown significant departure from lamination theory predictions, for such properties as maximum deflections and natural frequencies, when (a) the plate thickness-to-width ratio and (b) the in-plane Young's modulus to interlaminar shear modulus ratio become high.

One reason for the departure of thick plate behavior from classical thin plate theory prediction is the presence of transverse shear deformation. The effect of transverse shear deformation is pronounced in anisotropic materials with high ratios of in-plane Young's moduli to interlaminar shear moduli; this is typical in laminated composites. Other assumptions of the classical plate theory (see Section 2.3) such as negligible transverse normal strains ($\varepsilon_z = 0$), and the linear in-plane strain variation with the z coordinate all contribute to the limitations of the theory. Furthermore, the strong interlaminar shear existing in thick laminates is responsible for delamination, particularly near the free edges. Thus, it is imperative to determine the magnitude and distribution of interlaminar shear in thick laminates.

In the following, the three-dimensional constitutive relations of a thick composite lamina are introduced first. Then, the classical and higher order theory for thick laminated composites is discussed.

2.4.1 *Three-dimensional constitutive relations of a composite lamina*

The three-dimensional constitutive equations of a composite lamina referring to the principal material coordinate system x_1–x_2–x_3 (Fig. 2.1) have been introduced in Eqs. (2.8) and (2.10), for the case of orthotropic symmetry. The relations between the stiffness constants and engineering elastic constants are:

$$
\begin{aligned}
&C_{11} = E_{11}(1 - \nu_{23}\nu_{32})/\Delta \\
&C_{22} = E_{22}(1 - \nu_{13}\nu_{31})/\Delta \\
&C_{33} = E_{33}(1 - \nu_{12}\nu_{21})/\Delta \\
&C_{44} = G_{23} \\
&C_{55} = G_{13} \\
&C_{66} = G_{12} \\
&C_{12} = (\nu_{21} + \nu_{23}\nu_{31})E_{11}/\Delta = (\nu_{12} + \nu_{13}\nu_{32})E_{22}/\Delta \\
&C_{13} = (\nu_{31} + \nu_{21}\nu_{32})E_{11}/\Delta = (\nu_{13} + \nu_{12}\nu_{23})E_{33}/\Delta \\
&C_{23} = (\nu_{32} + \nu_{12}\nu_{31})E_{11}/\Delta = (\nu_{23} + \nu_{13}\nu_{21})E_{33}/\Delta \\
&\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{13}\nu_{21}\nu_{32}
\end{aligned}
\tag{2.33}
$$

The general three-dimensional constitutive relation of a composite lamina referring to the reference coordinate x–y–z (Fig. 2.3) can be obtained from Eq. (2.10) by tensor transformation:

$$
\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{pmatrix} \tag{2.34}
$$

Here, the x–y plane coincides with the x_1–x_2 plane and the angle between the x_1 and x axes is θ. The stress and strain tensors in these two coordinate systems are related by

$$
\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = [T]^{-1} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} \qquad \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{pmatrix} = [T]^{-1} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix} \tag{2.35}
$$

The transformation matrix is

$$
[T] = \begin{pmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & -2\cos\theta\sin\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{pmatrix} \tag{2.36}
$$

$[T]^{-1}$ is obtained by changing θ to $-\theta$ in $[T]$. The stiffness matrix is derived from

$$[\bar{C}] = [T]^{-1}[C][T]^{-t} \tag{2.37}$$

with t indicating the matrix transpose and the explicit expressions of $[\bar{C}]$ are

$$\begin{aligned}
\bar{C}_{11} &= C_{11}\cos^4\theta + 2(C_{12}+2C_{66})\sin^2\theta\cos^2\theta + C_{22}\sin^4\theta \\
\bar{C}_{12} &= (C_{11}+C_{22}-4C_{66})\sin^2\theta\cos^2\theta + C_{12}(\sin^4\theta+\cos^4\theta) \\
\bar{C}_{13} &= C_{13}\cos^2\theta + C_{23}\sin^2\theta \\
\bar{C}_{16} &= (C_{11}-C_{12}-2C_{66})\sin\theta\cos^3\theta \\
&\quad + (C_{12}-C_{22}+2C_{66})\sin^3\theta\cos\theta \\
\bar{C}_{22} &= C_{11}\sin^4\theta + 2(C_{12}+2C_{66})\sin^2\theta\cos^2\theta + C_{22}\cos^4\theta \\
\bar{C}_{23} &= C_{13}\sin^2\theta + C_{23}\cos^2\theta \\
\bar{C}_{26} &= (C_{11}-C_{12}-2C_{66})\sin^3\theta\cos\theta \\
&\quad + (C_{12}-C_{22}+2C_{66})\sin\theta\cos^3\theta \\
\bar{C}_{33} &= C_{33} \\
\bar{C}_{36} &= (C_{13}-C_{23})\sin\theta\cos\theta \\
\bar{C}_{44} &= C_{44}\cos^2\theta + C_{55}\sin^2\theta \\
\bar{C}_{45} &= (C_{55}-C_{44})\sin\theta\cos\theta \\
\bar{C}_{55} &= C_{55}\cos^2\theta + C_{44}\sin^2\theta \\
\bar{C}_{66} &= (C_{11}+C_{22}-2C_{12}-2C_{66})\sin^2\theta\cos^2\theta \\
&\quad + C_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{2.38}$$

2.4.2 *Constitutive relations of thick laminated composites*

The classical laminated plate theory does not take into account the effect of transverse shear stress and strain. The inclusion of transverse shear deformation in the classical thin plate theory is achieved by allowing the transverse shear strains, ε_{xz} and ε_{yz}, to be non-zero. This gives rise to definitions of the shear force resultants:

$$(Q_x, Q_y) = \int (\sigma_{xz}, \sigma_{yz})\,\mathrm{d}z \tag{2.39}$$

These shear force resultants can be related to the transverse shear strains through the appropriate constitutive relations, Eq. (2.34) (see Vinson and Chou 1975).

Several higher order plate theories have been proposed to account for the transverse shear deformation. This is achieved by retaining higher order terms in the displacement field expansions, which are assumed in the form of power series of the z coordinate. The accuracy of these theories is generally greater for a greater number of terms retained in the series, but the complexity of the governing equations places severe limits on the number of terms for which solutions are realistically attainable.

Among the various proposed displacement field expansions, the simplest one includes the linear term in z; it has been adopted by many workers (for example, Reissner 1945, Whitney and Pagano 1970),

$$\begin{aligned} u(x, y, z) &= u^{\mathrm{o}}(x, y) + z\psi_x(x, y) \\ v(x, y, z) &= v^{\mathrm{o}}(x, y) + z\psi_y(x, y) \\ w(x, y, z) &= w^{\mathrm{o}}(x, y) \end{aligned} \tag{2.40}$$

where u, v and w are the displacement components in the x, y and z coordinates (Fig. 2.2), respectively; u^{o}, v^{o} and w^{o} denote the mid-plane displacements of a point (x, y); and ψ_x and ψ_y are the rotations of the normal to the mid-plane about the y and x axes, respectively. It is noted that, unlike the classical plate theory, due to the existence of transverse shear deformation,

$$\begin{aligned} \psi_x &\neq -\frac{\partial w^{\mathrm{o}}}{\partial x} \\ \psi_y &\neq -\frac{\partial w^{\mathrm{o}}}{\partial y} \end{aligned} \tag{2.41}$$

The new curvatures expressions, which are different from Eq. (2.24) are given by

$$\begin{aligned} \kappa_{xx} &= \frac{\partial \psi_x}{\partial x} \neq -\frac{\partial^2 w^{\mathrm{o}}}{\partial x^2} \\ \kappa_{yy} &= \frac{\partial \psi_y}{\partial y} \neq -\frac{\partial^2 w^{\mathrm{o}}}{\partial y^2} \\ \kappa_{xy} &= \frac{1}{2}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \neq -\frac{\partial^2 w^{\mathrm{o}}}{\partial x\, \partial y} \end{aligned} \tag{2.42}$$

Then, the strain–displacement relations of linear elasticity are

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u^{\mathrm{o}}}{\partial x} + z\,\frac{\partial \psi_x}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v^{\mathrm{o}}}{\partial y} + z\,\frac{\partial \psi_y}{\partial y} \\
\varepsilon_{zz} &= 0 \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\psi_y + \frac{\partial w^{\mathrm{o}}}{\partial y}\right) \\
\varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\left(\psi_x + \frac{\partial w^{\mathrm{o}}}{\partial x}\right) \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left[\frac{\partial u^{\mathrm{o}}}{\partial y} + \frac{\partial v^{\mathrm{o}}}{\partial x} + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)\right]
\end{aligned}
\tag{2.43}
$$

By substituting Eqs. (2.34) and (2.43) into Eqs. (2.25), (2.26) and (2.39), the constitutive relations of the laminated plate in terms of stress resultants and displacement variables can be obtained as

$$
\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix}
=
\begin{pmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{pmatrix}
\begin{pmatrix}
\dfrac{\partial u^{\mathrm{o}}}{\partial x} \\
\dfrac{\partial v^{\mathrm{o}}}{\partial y} \\
\dfrac{\partial u^{\mathrm{o}}}{\partial y} + \dfrac{\partial v^{\mathrm{o}}}{\partial x} \\
\dfrac{\partial \psi_x}{\partial x} \\
\dfrac{\partial \psi_y}{\partial y} \\
\dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x}
\end{pmatrix}
\tag{2.44}
$$

and

$$\begin{pmatrix} Q_y \\ Q_x \end{pmatrix} = c_k \begin{pmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{pmatrix} \begin{pmatrix} \dfrac{\partial w^{\text{o}}}{\partial y} + \psi_y \\ \dfrac{\partial w^{\text{o}}}{\partial x} + \psi_x \end{pmatrix} \tag{2.45}$$

where

$$A_{ij} = \int \bar{C}_{ij}\,\mathrm{d}z \qquad (i, j = 1, 2, 4, 5, 6)$$

$$(B_{ij}, D_{ij}) = \int \bar{C}_{ij}(z, z^2)\,\mathrm{d}z \qquad (i, j = 1, 2, 6)$$

and c_k in Eq. (2.45) is a correction factor for the kth lamina which, according to Lo, Christensen and Wu (1977a), is determined by matching the approximated solution with the exact elasticity solution in order to satisfy appropriately the requirements of vanishing transverse shear stress on the top and bottom surfaces of the thick plate.

Having obtained the constitutive relations, the problem of thick laminated plates can be solved by substituting Eqs. (2.44) and (2.45) into the plate equation of motion. Then, a set of partial differential equations in terms of the displacement variables u^{o}, v^{o}, w^{o}, ψ_x and ψ_y can be derived. These unknowns can be solved with the appropriate initial and boundary conditions, which are determined from the total energy of the system (Whitney and Pagano 1970).

The approach outlined above demonstrates an example of the high order laminated plate theories, where only the in-plane displacement terms linear in z are included in Eqs (2.40); and it differs from the classical plate theory only by the terms ψ_x and ψ_y as shown in Eqs (2.41). As pointed out by Lo, Christensen and Wu (1977a&b), despite the increased generality of the shear deformation theory, the related flexural stress distributions show little improvement over the classical laminated plate theory. Thus, it is apparent that higher order terms are needed in the power series expansion of the assumed displacement field to properly model the behavior of thick laminates.

Among the various higher order displacement fields proposed, Lo, Christensen and Wu (1977a&b) suggested the following displacement forms:

$$
\begin{aligned}
u &= u^{\mathrm{o}} + z\psi_x + z^2\xi_x + z^3\phi_x \\
v &= v^{\mathrm{o}} + z\psi_y + z^2\xi_y + z^3\phi_y \\
w &= w^{\mathrm{o}} + z\psi_z + z^2\xi_z
\end{aligned}
\tag{2.46}
$$

where the cubic terms in z for the in-plane displacement field and the square terms in z for the out-of-plane deformations are used; a total of 11 displacement functions (u^{o}, v^{o}, w^{o}, ψ_x, ψ_y, ψ_z, ξ_x, ξ_y, ξ_z, ϕ_x and ϕ_y) are involved. Much improvement over the classical theory predictions is observed; however, the complexity of the analysis has increased tremendously.

The format of solution to higher order systems generally involves the application of the principle of potential energy to derive the pertinent governing equations of equilibrium. Using the strain–displacement relations and the assumed displacement field, in conjunction with the equations of equilibrium, a set of partial differential equations in terms of the displacements used is derived. The number of equations is determined by the number of terms retained in the assumed displacement form. With the appropriate initial and boundary conditions, the solution of these equations describes the elastic behavior of the plate. The details of such approaches can be found, for example, in the work of Whitney and Pagano (1970), Whitney and Sun (1973), Lo, Christensen and Wu (1977a&b), and Reddy (1984).

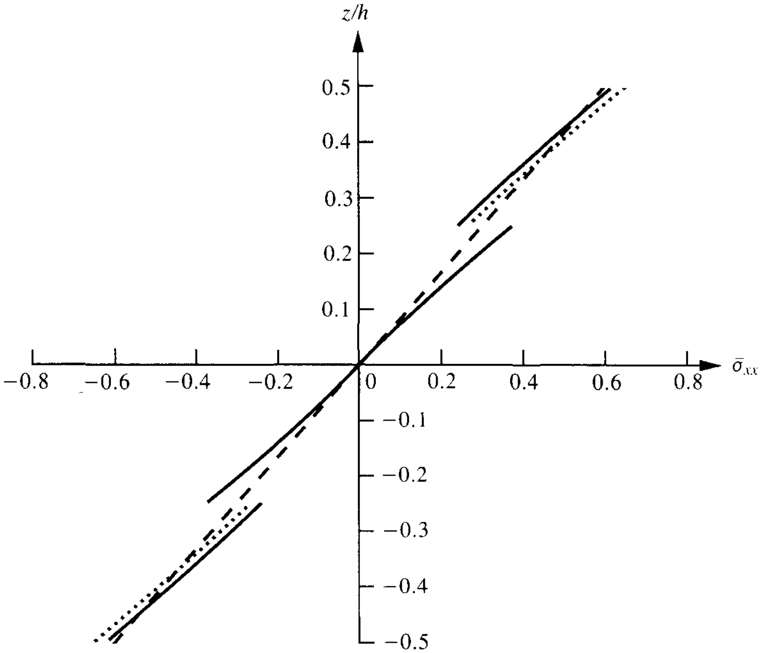
Although accounting for the higher order plate deformation in thick laminates involves a great deal more complexity than the classical thin plate approach, it is evident that the extra effort to accurately describe their fundamentally different elastic behavior is required. The numerical results of the flexural stress distribution in an infinite $[+30, -30]_s$ laminate of carbon/epoxy composite, subjected to a pressure q, on the top surface ($z = h/2$) of the form

$$
q = q_0 \sin \frac{\pi x}{L} \tag{2.47}
$$

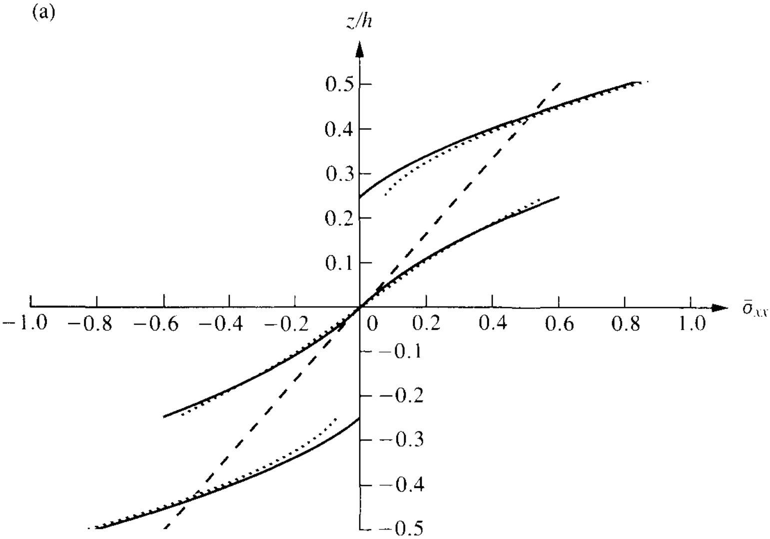
are shown in Fig. 2.8 (a) and (b) (see Lo, Christensen and Wu 1977b). Here the length L characterizes the load distribution. The in-plane stress σ_{xx} is normalized as $\bar{\sigma}_{xx} = \sigma_{xx}/q_0S^2$, $S = L/h$. The results indicate that the higher order theory is necessary for determining the deformation of plates with small L/h ratio.

Sun and Li (1988) and Luo and Sun (1989) have adopted a global–local method for the analysis of thermoelastic fields of thick

Fig. 2.8. (a) Flexural stress distributions for a $[+30, -30]_s$ angle-ply laminate for $L/h = 10$. (b) Flexural stress distributions for a $[+30, -30]_s$ angle-ply laminate for $L/h = 4$. —— exact elastic solution; .... higher order laminated plate theory; ---- classical laminated plate theory. (After Lo, Christensen and Wu 1977b.)



(a)



(b)

laminated composites consisting of a repeating sublaminate (the typical cell). The effective moduli and thermal expansion coefficients are obtained from the sublaminate and used to obtain the global (average) stress and strain solutions. A refining procedure is then introduced in which the global solution is used directly to recover the stresses in the lamina or used as boundary conditions in a sublaminate to perform the exact thermoelastic analysis.

2.5 Thermal and hygroscopic behavior

Besides externally applied load, deformations in laminated composites can also occur due to changes in temperature and absorption of moisture. This is known as the hygrothermal effect. As polymers undergo both dimensional and property changes in a hygrothermal environment, so do composites utilizing polymers as the matrix. Since fibers are fairly insensitive to environmental changes, the environmental susceptibility of composites is mainly through the matrix. Consequently, in a unidirectional composite the temperature–moisture environment has a much greater effect on the transverse and shear properties than the longitudinal properties.

The thermal diffusivity and moisture diffusion coefficient are used as measures of the rates at which the temperature and moisture concentrations change within the material. In general, these parameters depend on the temperature and moisture concentration. However, over the range of temperature and moisture concentration that prevails in typical applications of composites, the thermal diffusivity is usually several orders of magnitude greater than the moisture diffusion coefficient. Consequently, thermal diffusion takes place at a rate much faster than moisture diffusion, and the temperature will reach equilibrium long before the moisture concentration does. This allows one to solve the heat-conduction and moisture-diffusion problems and the resulting elastic fields separately.

The knowledge of anisotropic heat conduction is basic to the solution of thermal stresses in laminated composites. Investigations of such problems have been performed by Poon and Chang (1978), and Chu, Weng and Chen (1983) using transformation theory, by Chang (1977), Huang and Chang (1980), and Nomura and Chou (1986) using Green's function method, by Tauchert and Aköz (1974) using a complex variable method, and by Katayama, Saito and Kobayashi (1974) using a finite difference technique. The solution of the steady-state thermoelastic problem of anisotropic material appears to be initiated by Mossakowska and Nowacki

(1958), Sharma (1958), and Singh (1960). Then Takeuti and Noda (1978), Sugano (1979), and Noda (1983) have examined the transient temperature and thermal stress fields of transversely isotropic elastic medium.

In the category of thermally and elastically orthotropic media, the steady-state temperature and thermal stress field have been investigated for problems of semi-infinite domain (Aköz and Tauchert 1972), a slab bounded by two parallel infinite planes (Tauchert and Aköz 1974) and a rectangular slab (Aköz and Tauchert 1978). The transient thermal stress analysis of thermally and elastically orthotropic laminae has been performed by H. Wang and Chou (1985, 1986), Wang, Pipes and Chou (1986), and Y. Wang and Chou (1988, 1989); their approaches are recapitulated in the following.

In Section 2.5.1, the thermoelastic constitutive equations for a three-dimensional orthotropic material are introduced. These equations are then simplified to the two-dimensional case of unidirectional laminae, and the classical lamination theory is generalized to take into account the thermal and hygroscopic effects. Then, three transient thermal and hygroscopic problems are discussed to illustrate the formulation of the boundary value problems and the solution techniques. The first problem is for the diffusion of moisture through the thickness of a laminated composite (Section 2.5.2). It is assumed that the diffusion equation is one-dimensional (z direction), while the elastic field is two-dimensional (x–y plane). The second problem focuses on the effect of heat conduction on interlaminar thermal stresses (Section 2.5.3). It is assumed, in this case, that heat flows across the width of a laminated plate (one-dimensional heat conduction) and the resulting thermal stress field is three dimensional. Finally, a two-dimensional heat conduction problem is formulated for a rectangular-shaped unidirectional lamina subjected to thermal boundary conditions at its four edges. The two-dimensional thermal elastic field is obtained. In all three problems, the thermal transient effects on stress distribution are demonstrated.

2.5.1 *Basic equations*

2.5.1.1 *Constitutive relations*

Deformations of a unidirectional lamina resulting from hygrothermal effects can be described by a modified set of linear constitutive equations: i.e., the total strain minus the non-mechanical strain is linearly related to the stress.

The non-mechanical strain is measured from a stress-free reference state, and the elastic moduli used in the calculation are taken at the final environmental conditions. For example, in the fabrication of polymer matrix composite laminates, the curing of an individual ply results in different deformations along the fiber and transverse directions. The constraint of deformation of a single ply due to the presence of other plies in a multi-directional laminate gives rise to residual stresses. Since most of the cross-linking in the polymer occurs at the highest curing temperature, the polymer matrix can be considered as still viscous enough to allow complete relaxation of the residual stress. Thus, the highest curing temperature can be regarded as the stress-free temperature.

By taking into account the non-mechanical strain in Eq. (2.10) for hygrothermally induced deformation, the laminated plate analysis developed in Section 2.3 can be modified to determine the overall elastic response. The stresses due to moisture absorption and temperature change are identically analogous, in that they are dilatational and self-equilibrating when the whole laminate is considered. In general, the longitudinal properties of polymer matrix composites are far less sensitive to temperature and moisture than the transverse and shear properties of unidirectional composites, because of the excellent retention of mechanical properties by the fibers. The greatest reduction in properties occurs when temperature and moisture are combined, such as in hot and humid environments. However, the combination of temperature and moisture could render a laminate free of residual stresses. This can be understood by considering, for example, a [0°/90°] cross-ply based upon a resin matrix. The thermal stress induced from fabrication is tensile in the transverse direction of a ply, while the residual stresses induced by moisture absorption are compressive. Some details of analysis of such phenomena are developed in the following.

Referring to the principal material coordinate axes of a unidirectional lamina, the three-dimensional orthotropic stress–strain relations of Eq. (2.10) can be written as

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{pmatrix}=\begin{pmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}-\alpha_{11}T-\beta_{11}m\\ \varepsilon_{22}-\alpha_{22}T-\beta_{22}m\\ \varepsilon_{33}-\alpha_{33}T-\beta_{33}m\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{pmatrix} \tag{2.48}$$

where α_{ii} are the coefficients of thermal expansion and β_{ii} are the coefficients of hygroscopic expansion; the subscripts of these coefficients indicate the principal material axes x_i ($i = 1$–3). Also, T denotes a small uniform temperature change from the 'stress-free' temperature; m is the change in moisture concentration referring to a 'moisture-free' environment. Both $\alpha_{ii}T$ and $\beta_{ii}m$ indicate non-mechanical strains.

Referring to the reference axes x–y and following Eq. (2.34), Eq. (2.48) can be rewritten as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{pmatrix}$$

$$\times \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}T - \beta_{xx}m \\ \varepsilon_{yy} - \alpha_{yy}T - \beta_{yy}m \\ \varepsilon_{zz} - \alpha_{zz}T - \beta_{zz}m \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} - \alpha_{xy}T - \beta_{xy}m \end{pmatrix} \qquad (2.49)$$

where

$$\begin{aligned} \alpha_{xx} &= \alpha_{11}\cos^2\theta + \alpha_{22}\sin^2\theta & \beta_{xx} &= \beta_{11}\cos^2\theta + \beta_{22}\sin^2\theta \\ \alpha_{yy} &= \alpha_{22}\cos^2\theta + \alpha_{11}\sin^2\theta & \beta_{yy} &= \beta_{22}\cos^2\theta + \beta_{11}\sin^2\theta \\ \alpha_{zz} &= \alpha_{33} & \beta_{zz} &= \beta_{33} \\ \alpha_{xy} &= (\alpha_{11} - \alpha_{22})\sin\theta\cos\theta & \beta_{xy} &= (\beta_{11} - \beta_{22})\sin\theta\cos\theta \end{aligned} \qquad (2.50)$$

and θ is defined in Fig. 2.3.

The relations given in Eq. (2.49) require that the thermoelastic deformations of the medium are accurately described by linear coefficients of thermal expansion over the range of temperatures of interest, an often used assumption. Similarly, the deformations induced by the hygroscopic nature of the medium are characterized by linear coefficients of hygroscopic expansion, an assumption which follows from existing experimental data.

The elastic constitutive relations for a laminate subjected to both thermal and hygroscopic environments have been formulated by Pipes, Vinson and Chou (1976). For the purpose of laminar analysis, Eq. (2.49) is reduced to

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} - \alpha_{xx}T - \beta_{xx}m \\ \varepsilon_{yy} - \alpha_{yy}T - \beta_{yy}m \\ 2\varepsilon_{xy} - \alpha_{xy}T - \beta_{xy}m \end{pmatrix} \tag{2.51}$$

Substituting Eq. (2.51) into Eq. (2.25) and following the notation of Eq. (2.30), the constitutive equation is expressed in the following condensed form:

$$[N] = [A][\varepsilon^{o}] + [B][\kappa] - [N]^{T} - [N]^{m} \tag{2.52}$$

In Eq. (2.52), the effective thermal force resultants, $[N]^{\mathrm{T}}$, and effective hygroscopic force resultants $[N]^{\mathrm{m}}$ are introduced with the following definitions:

$$\begin{aligned}
N_x^{\mathrm{T}} &= \int_{-h/2}^{h/2} (\bar{Q}_{11}\alpha_{xx} + \bar{Q}_{12}\alpha_{yy} + \bar{Q}_{16}\alpha_{xy})T(z, t)\,\mathrm{d}z \\
N_y^{\mathrm{T}} &= \int_{-h/2}^{h/2} (\bar{Q}_{12}\alpha_{xx} + \bar{Q}_{22}\alpha_{yy} + \bar{Q}_{26}\alpha_{xy})T(z, t)\,\mathrm{d}z \qquad (2.53\mathrm{a}) \\
N_{xy}^{\mathrm{T}} &= \int_{-h/2}^{h/2} (\bar{Q}_{16}\alpha_{xx} + \bar{Q}_{26}\alpha_{yy} + \bar{Q}_{66}\alpha_{xy})T(z, t)\,\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}
N_x^{\mathrm{m}} &= \int_{-h/2}^{h/2} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})m(z, t)\,\mathrm{d}z \\
N_y^{\mathrm{m}} &= \int_{-h/2}^{h/2} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})m(z, t)\,\mathrm{d}z \qquad (2.53\mathrm{b}) \\
N_{xy}^{\mathrm{m}} &= \int_{-h/2}^{h/2} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})m(z, t)\,\mathrm{d}z
\end{aligned}$$

where t denotes time. Consider the kth layer of the laminate; and define $\int^{z} T(\xi, t)\,\mathrm{d}\xi \equiv R(z, t)$ and $\int^{z} m(\xi, t)\,\mathrm{d}\xi \equiv H(z, t)$. Then, Eqs. (2.53) are written as summations

$$N_x^{\mathrm{T}} = \sum_{k=1}^{n} (\bar{Q}_{11}\alpha_{xx} + \bar{Q}_{12}\alpha_{yy} + \bar{Q}_{16}\alpha_{xy})_k [R(h_k, t) - R(h_{k-1}, t)]$$

$$N_y^{\mathrm{T}} = \sum_{k=1}^{n} (\bar{Q}_{12}\alpha_{xx} + \bar{Q}_{22}\alpha_{yy} + \bar{Q}_{26}\alpha_{xy})_k [R(h_k, t) - R(h_{k-1}, t)]$$

$$N_{xy}^{\mathrm{T}} = \sum_{k=1}^{n} (\bar{Q}_{16}\alpha_{xx} + \bar{Q}_{26}\alpha_{yy} + \bar{Q}_{66}\alpha_{xy})_k \times [R(h_k, t) - R(h_{k-1}, t)] \tag{2.54a}$$

$$N_x^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k [H(h_k, t) - H(h_{k-1}, t)]$$

$$N_y^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k [H(h_k, t) - H(h_{k-1}, t)]$$

$$N_{xy}^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k \times [(H(h_k, t) - H(h_{k-1}, t)] \tag{2.54b}$$

Parallel to the treatment of in-plane response, the flexural response of the laminate is obtained by substituting Eq. (2.51) into Eq. (2.26)

$$[M] = [B][\varepsilon^{\mathrm{o}}] + [D][\kappa] - [M]^{\mathrm{T}} - [M]^{\mathrm{m}} \tag{2.55}$$

Here, the effective thermal moment resultant, $[M]^{\mathrm{T}}$, and effective hygroscopic moment resultant, $[M]^{\mathrm{m}}$, are defined as

$$M_x^{\mathrm{T}} = \int_{-h/2}^{h/2} (\bar{Q}_{11}\alpha_{xx} + \bar{Q}_{12}\alpha_{yy} + \bar{Q}_{16}\alpha_{xy}) T(z, t) z \, \mathrm{d}z$$

$$M_y^{\mathrm{T}} = \int_{-h/2}^{h/2} (\bar{Q}_{12}\alpha_{xx} + \bar{Q}_{22}\alpha_{yy} + \bar{Q}_{26}\alpha_{xy}) T(z, t) z \, \mathrm{d}z \tag{2.56a}$$

$$M_{xy}^{\mathrm{T}} = \int_{-h/2}^{h/2} (\bar{Q}_{16}\alpha_{xx} + \bar{Q}_{26}\alpha_{yy} + \bar{Q}_{66}\alpha_{xy}) T(z, t) z \, \mathrm{d}z$$

$$M_x^{\mathrm{m}} = \int_{-h/2}^{h/2} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy}) m(z, t) z \, \mathrm{d}z$$

$$M_y^{\mathrm{m}} = \int_{-h/2}^{h/2} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy}) m(z, t) z \, \mathrm{d}z \tag{2.56b}$$

$$M_{xy}^{\mathrm{m}} = \int_{-h/2}^{h/2} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy}) m(z, t) z \, \mathrm{d}z$$

By introducing the integrals of $R(z, t)$ (i.e., $S(z, t) = \int^z R(\xi, t)\,d\xi$), and $H(z, t)$ (i.e., $J(z, t) = \int^z H(\xi, t)\,d\xi$), Eqs. (2.56) are also expressed as summations:

$$\begin{aligned}
M_x^{\mathrm{T}} &= \sum_{k=1}^{n} (\bar{Q}_{11}\alpha_{xx} + \bar{Q}_{12}\alpha_{yy} + \bar{Q}_{16}\alpha_{xy})_k[h_k R(h_k, t) \\
&\quad - h_{k-1}R(h_{k-1}, t) - S(h_k, t) + S(h_{k-1}, t)] \\
M_y^{\mathrm{T}} &= \sum_{k=1}^{n} (\bar{Q}_{12}\alpha_{xx} + \bar{Q}_{22}\alpha_{yy} + \bar{Q}_{26}\alpha_{xy})_k[h_k R(h_k, t) \\
&\quad - h_{k-1}R(h_{k-1}, t) - S(h_k, t) + S(h_{k-1}, t)] \qquad (2.57a) \\
M_{xy}^{\mathrm{T}} &= \sum_{k=1}^{n} (\bar{Q}_{16}\alpha_{xx} + \bar{Q}_{26}\alpha_{yy} + \bar{Q}_{66}\alpha_{xy})_k[h_k R(h_k, t) \\
&\quad - h_{k-1}R(h_{k-1}, t) - S(h_k, t) + S(h_{k-1}, t)]
\end{aligned}$$

$$\begin{aligned}
M_x^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k[h_k H(h_k, t) \\
&\quad - h_{k-1}H(h_{k-1}, t) - J(h_k, t) + J(h_{k-1}, t)] \\
M_y^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k[h_k H(h_k, t) \\
&\quad - h_{k-1}H(h_{k-1}, t) - J(h_k, t) + J(h_{k-1}, t)] \qquad (2.57b) \\
M_{xy}^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k[h_k H(h_k, t) \\
&\quad - h_{k-1}H(h_{k-1}, t) - J(h_k, t) + J(h_{k-1}, t)]
\end{aligned}$$

Finally, Eqs. (2.52) and (2.55) are combined as

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} + \begin{pmatrix} N_x^{\mathrm{T}} \\ N_y^{\mathrm{T}} \\ N_{xy}^{\mathrm{T}} \\ M_x^{\mathrm{T}} \\ M_y^{\mathrm{T}} \\ M_{xy}^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} N_x^{\mathrm{m}} \\ N_y^{\mathrm{m}} \\ N_{xy}^{\mathrm{m}} \\ M_x^{\mathrm{m}} \\ M_y^{\mathrm{m}} \\ M_{xy}^{\mathrm{m}} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx}^{\mathrm{o}} \\ \varepsilon_{yy}^{\mathrm{o}} \\ \gamma_{xy}^{\mathrm{o}} \\ \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{pmatrix} \qquad (2.58)$$

The response of a laminate subjected to known mechanical force and moment resultants, and both thermal and hygroscopic effects, can be determined by calculating the effective thermal and hygroscopic resultants and inverting Eq. (2.58). The inversion would yield laminate mid-plane strains, ε^{o}, and curvatures, κ. The strains of the laminae could then be calculated by Eq. (2.22). Given the strains, the stresses within each lamina could be determined according to Eq. (2.51).

2.5.1.2 *Thermal and moisture diffusion equations*

The three-dimensional heat conduction equation for a general anisotropic solid of constant conductivity coefficients is given by (Ozisik 1980)

$$K_{xx}^{\mathrm{T}}\frac{\partial^2 T}{\partial x^2}+K_{yy}^{\mathrm{T}}\frac{\partial^2 T}{\partial y^2}+K_{zz}^{\mathrm{T}}\frac{\partial^2 T}{\partial z^2}+2K_{xy}^{\mathrm{T}}\frac{\partial^2 T}{\partial x\,\partial y}+2K_{xz}^{\mathrm{T}}\frac{\partial^2 T}{\partial x\,\partial z}+2K_{yz}^{\mathrm{T}}\frac{\partial^2 T}{\partial y\,\partial z}=\rho C_{\mathrm{p}}\frac{\partial T}{\partial t} \tag{2.59}$$

Here, K_{ij}^{T} denote the coefficients of heat conduction, ρ is mass density, and C_{p} is the specific heat. The temperature of the elastic medium, T, is a function of location (x, y, z) and the time, t. It is understood that there is no internal heat generation of the elastic body.

For a thermally orthotropic material, with respect to the reference axes x–y–z, Eq. (2.59) is simplified as

$$K_{xx}^{\mathrm{T}}\frac{\partial^2 T}{\partial x^2}+K_{yy}^{\mathrm{T}}\frac{\partial^2 T}{\partial y^2}+K_{zz}^{\mathrm{T}}\frac{\partial^2 T}{\partial z^2}=\rho C_{\mathrm{p}}\frac{\partial T}{\partial t} \tag{2.60}$$

Here, the thermal conductivities K_{xx}^{T}, K_{yy}^{T} and K_{zz}^{T} are related to the conductivities along the material principal direction, i.e. K_{11}^{T}, K_{22}^{T} and K_{33}^{T} using transformation equations identical in form to those given in Eqs. (2.50).

An equation identical in form to Eq. (2.59) can be written for moisture diffusion. Consider, for instance, the diffusion of moisture along the laminate thickness (z) direction, the governing equation is reduced to

$$K_{zz}^{\mathrm{m}}\frac{\partial^2 m}{\partial z^2}=\frac{\partial m}{\partial t} \tag{2.61}$$

where K^{m}_{zz} is the moisture diffusion coefficient and $m = m(z, t)$ denotes the moisture concentration distribution. Equation (2.61) is further discussed in Section 2.5.2.

In Section 2.5.3, the transient interlaminar stress induced by heat conduction through the laminate width (y) direction is discussed. Then, $T = T(y, t)$, and Eq. (2.59) for each lamina is reduced to

$$K^{\mathrm{T}}_{yy} \frac{\partial^2 T}{\partial y^2} = \rho C_{\mathrm{p}} \frac{\partial T}{\partial t} \tag{2.62}$$

In Section 2.5.4, heat conduction in the plane of a unidirectional composite is considered. The governing equation for heat conduction becomes

$$K^{\mathrm{T}}_{xx} \frac{\partial^2 T}{\partial x^2} + K^{\mathrm{T}}_{yy} \frac{\partial^2 T}{\partial y^2} = \rho C_{\mathrm{p}} \frac{\partial T}{\partial t} \tag{2.63}$$

2.5.2 *Hygroscopic behavior*

2.5.2.1 *Moisture concentration functions*

Pipes, Vinson and Chou (1976) assume that the classical diffusion equation (see Jost 1960) governs the absorption and desorption of moisture by a hygroscopic material as given in Eq. (2.61). Consider first the case of moisture absorption. If the laminate is assumed to be initially moisture free, while its surfaces $z = \pm h/2$ are exposed to a moisture concentration M_{o}, then moisture concentration in the laminate at position z and time t is

$$m(z, t) = M_{\mathrm{o}}\left[1 - \sum_{n=0}^{\infty} m_n \cos(a_n z)\right] \tag{2.64}$$

where

$$a_n = \frac{(2n+1)\pi}{h}$$

and

$$m_n = \frac{4}{\pi}\left\{\frac{(-1)^n}{2n+1} \exp[-a_n^2 K^{\mathrm{m}}_{zz} t]\right\}$$

From Eq. (2.64), the effective hygroscopic force resultant can be readily determined by combining Eqs. (2.54b) and (2.64):

$$N_x^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k M_{\mathrm{o}} \times \left[h_k - h_{k-1} - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (\sin(a_n h_k) - \sin(a_n h_{k-1})) \right]$$

$$N_y^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k M_{\mathrm{o}} \times \left[h_k - h_{k-1} - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (\sin(a_n h_k) - \sin(a_n h_{k-1})) \right] \quad (2.65)$$

$$N_{xy}^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k M_{\mathrm{o}} \times \left[h_k - h_{k-1} - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (\sin(a_n h_k) - \sin(a_n h_{k-1})) \right]$$

The effective hygroscopic moment resultant is then determined from Eqs. (2.57b)

$$M_x^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k M_{\mathrm{o}} \times \left[\tfrac{1}{2}(h_k^2 - h_{k-1}^2) - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) - \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right]$$

$$M_y^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k M_{\mathrm{o}} \times \left[\tfrac{1}{2}(h_k^2 - h_{k-1}^2) - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) - \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right] \quad (2.66)$$

$$M_{xy}^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k M_{\mathrm{o}} \times \left[\tfrac{1}{2}(h_k^2 - h_{k-1}^2) - \sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) - \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right]$$

Next, consider the desorption of moisture. The laminate containing a uniformly distributed moisture concentration, M_{o}, is exposed to a moisture-free environment on its surfaces $z = \pm h/2$. The solution of the diffusion Eq. (2.61) corresponding to these boundary conditions is

$$m(z, t) = M_{\mathrm{o}} \sum_{n=0}^{\infty} m_n \cos(a_n z) \tag{2.67}$$

The corresponding effective hygroscopic force and moment resultants are

$$\begin{aligned}
N_x^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k M_{\mathrm{o}} \sum_{n=0}^{\infty} \frac{m_n}{a_n} \times (\sin(a_n h_k) - \sin(a_n h_{k-1})) \\
N_y^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k M_{\mathrm{o}} \sum_{n=0}^{\infty} \frac{m_n}{a_n} \times (\sin(a_n h_k) - \sin(a_n h_{k-1})) \qquad (2.68) \\
N_{xy}^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k M_{\mathrm{o}} \sum_{n=0}^{\infty} \frac{m_n}{a_n} \times (\sin(a_n h_k) - \sin(a_n h_{k-1)})) \\
M_x^{\mathrm{m}} &= \sum_{k=1}^{n} (\bar{Q}_{11}\beta_{xx} + \bar{Q}_{12}\beta_{yy} + \bar{Q}_{16}\beta_{xy})_k M_{\mathrm{o}} \times \left[\sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) + \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right]
\end{aligned}$$

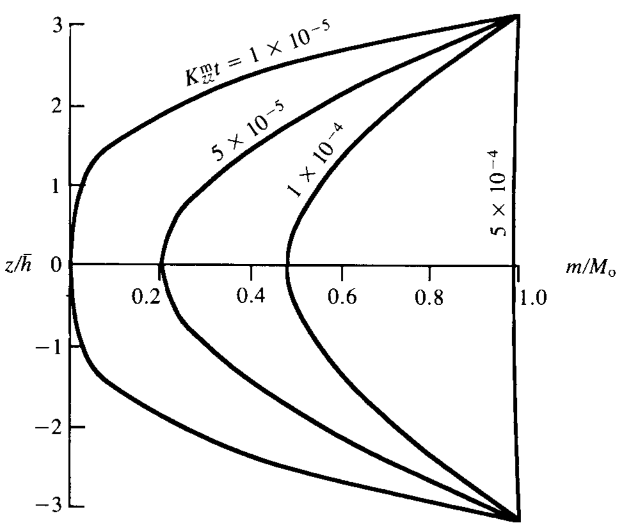
$$M_y^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{12}\beta_{xx} + \bar{Q}_{22}\beta_{yy} + \bar{Q}_{26}\beta_{xy})_k M_{\mathrm{o}}$$
$$\times \left[\sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) \right. \quad (2.69)$$
$$\left. + \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right]$$

$$M_{xy}^{\mathrm{m}} = \sum_{k=1}^{n} (\bar{Q}_{16}\beta_{xx} + \bar{Q}_{26}\beta_{yy} + \bar{Q}_{66}\beta_{xy})_k M_{\mathrm{o}}$$
$$\times \left[\sum_{n=0}^{\infty} \frac{m_n}{a_n} (h_k \sin(a_n h_k) - h_{k-1} \sin(a_n h_{k-1})) \right.$$
$$\left. + \sum_{n=0}^{\infty} \frac{m_n}{a_n^2} (\cos(a_n h_k) - \cos(a_n h_{k-1})) \right]$$

2.5.2.2 *Hygroscopic stress field*

Pipes, Vinson and Chou (1976) have illustrated the hygroscopic effects on a carbon–epoxy system (T300/5208) comprising a six-ply laminate of $[0°/+45°/-45]_s$, where each lamina is of the thickness $\bar{h}$. It is assumed that the diffusion coefficient, K_{zz}^{m}, and the coefficients of expansion, α and β, are constant over the ranges of

Fig. 2.9. Moisture distribution profiles during absorption. (After Pipes, Vinson and Chou 1976.)



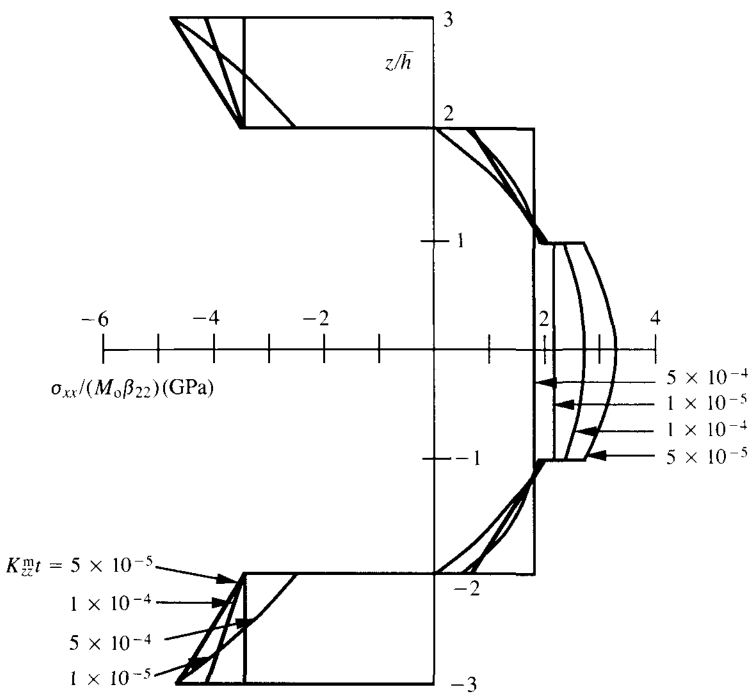
temperature and moisture concentration of interest. The material properties are $E_{11} = 143$ GPa, $E_{22} = 10.1$ GPa, $\nu_{12} = 0.31$, $G_{12} = 4.14$ GPa, $\beta_{11} = 0$, $\beta_{22} = 6.67 \times 10^{-3}$/wt% and $\bar{h} = 0.1397$ mm.

Figure 2.9 illustrates the moisture profiles across the laminate which is moisture free at time $t = 0$, and then exposed to an environment on both surfaces of moisture concentration M_o. The range of $K_{zz}^{m}t$ values is between 1×10^{-5} and 5×10^{-4}. It is seen that by $K_{zz}^{m}t = 5 \times 10^{-5}$ the moisture concentration at the mid-surface is 20% of that at the surface.

Figure 2.10 shows the profiles of σ_{xx}, which is compressive in the outer, 0°, laminae, because of the expansion caused by the moisture gradients of Fig. 2.9, and the inner four laminae at ±45° are all in tension. Stress values are maximum at the outer surfaces. The profiles of σ_{yy} follow the same trend as σ_{xx}, and $\sigma_{yy} > \sigma_{xx}$ at each time. In both cases the steady state is achieved at $K_{zz}^{m}t > 5 \times 10^{-4}$.

Figure 2.11 shows that $\tau_{xy} = 0$ in the outer two layers because they are at the orientation of $\theta = 0°$; the same would occur for any

Fig. 2.10. Distribution of stress, σ_{xx} during moisture absorption. (After Pipes, Vinson and Chou 1976.)



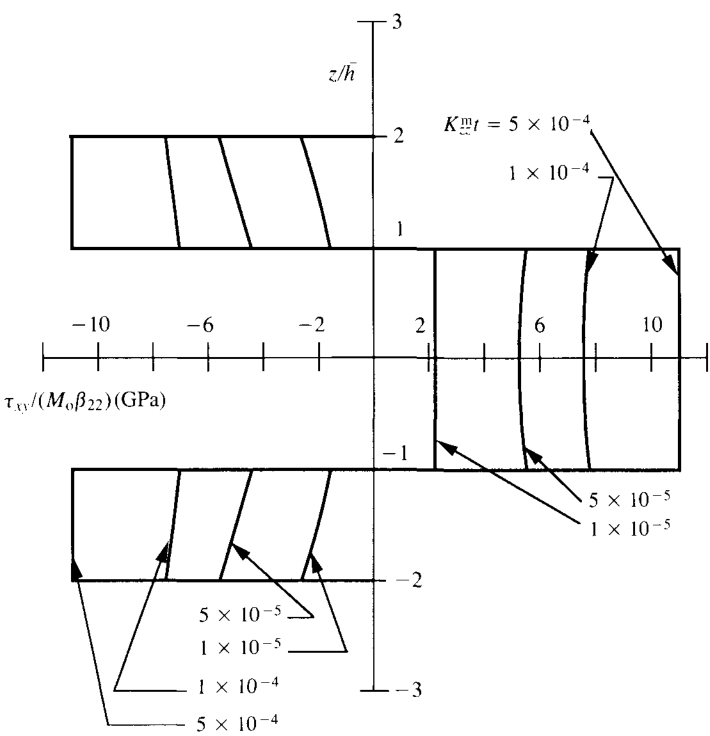
layers at $\theta = 90°$ in balanced laminates. The in-plane shear stresses increase with time, because of the increasing strains caused by increased moisture content; by the time of steady state ($K_{zz}^{m}t > 5 \times 10^{-4}$) the shear stresses are much larger than either the σ_{xx} or σ_{yy} stress. These large shear stresses imply large interlaminar shear stresses, σ_{zx} and σ_{zy}, near laminate discontinuities.

2.5.3 *Transient interlaminar thermal stresses*

2.5.3.1 *Transient temperature field*

Consider an x direction infinite laminated plate subjected to a temperature field $T = T_o$ on two edges ($y = \pm b$) at time $t = 0^+$ (Fig. 2.12). By assuming that the temperature field in each layer is independent of the thickness direction, i.e. $T = T(y, t)$, the heat conduction equation for each lamina follows Eq. (2.62).

Fig. 2.11. Distribution of stress, τ_{xy}, during moisture absorption. (After Pipes, Vinson and Chou 1976.)



The boundary and initial conditions are

$$T(\pm b, t) = T_o \qquad T(y, 0) = 0 \tag{2.70}$$

The solution of the governing equation Eq. (2.62) by the method of separation of variables is

$$T = T_0\left(1 + \sum_{n=0}^{\infty} a_n \cos(b_n Y) e^{-c_n t}\right) \tag{2.71}$$

where

$$Y = \frac{y}{b}$$

$$a_n = \frac{(-1)^n 4}{(2n-1)\pi}$$

$$b_n = (n - \tfrac{1}{2})\pi$$

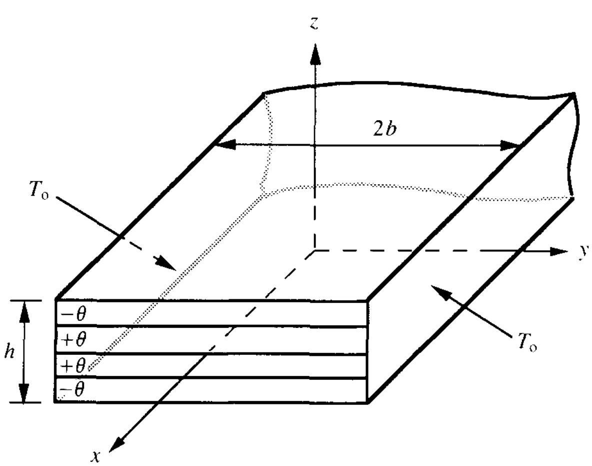
$$c_n = \left[(n - \tfrac{1}{2})\frac{\pi g}{b}\right]^2$$

$$g^2 = \frac{K_{yy}}{\rho C_p}$$

2.5.3.2 *Thermal stress field*

Y. Wang and Chou (1989) have considered the transient thermal stresses in an orthotropic composite laminate. Since the

Fig. 2.12. Geometry of an angle-ply laminate for analytical modeling.



thermal boundary conditions are uniform along the surfaces $y = \pm b$, the displacements are independent of the x axis and expressed as:

$$\begin{aligned} u &= u(y, z, t) \\ v &= v(y, z, t) \\ w &= w(y, z, t) \end{aligned} \tag{2.72}$$

The stress–strain relations for such an orthotropic laminate follow Eq. (2.49):

$$\begin{aligned} \alpha_{xx} &= \bar{C}_{11}\varepsilon_{xx} + \bar{C}_{12}\varepsilon_{yy} + \bar{C}_{13}\varepsilon_{zz} + 2\bar{C}_{16}\varepsilon_{xy} - \tilde{\alpha}_1 T \\ \sigma_{yy} &= \bar{C}_{12}\varepsilon_{xx} + \bar{C}_{22}\varepsilon_{yy} + \bar{C}_{23}\varepsilon_{zz} + 2\bar{C}_{26}\varepsilon_{xy} - \tilde{\alpha}_2 T \\ \sigma_{zz} &= \bar{C}_{13}\varepsilon_{xx} + \bar{C}_{23}\varepsilon_{yy} + \bar{C}_{33}\varepsilon_{zz} + 2\bar{C}_{36}\varepsilon_{xy} - \tilde{\alpha}_3 T \\ \sigma_{yz} &= 2\bar{C}_{44}\varepsilon_{yz} \\ \sigma_{xz} &= 2\bar{C}_{55}\varepsilon_{xz} \\ \sigma_{xy} &= \bar{C}_{16}\varepsilon_{xx} + \bar{C}_{26}\varepsilon_{yy} + \bar{C}_{36}\varepsilon_{zz} + 2\bar{C}_{66}\varepsilon_{xy} - \tilde{\alpha}_6 T \end{aligned} \tag{2.73}$$

where

$$\begin{aligned} \tilde{\alpha} &= \alpha_{xx}\bar{C}_{11} + \alpha_{yy}\bar{C}_{12} + \alpha_{zz}\bar{C}_{13} + \alpha_{xy}\bar{C}_{16} \\ \tilde{\alpha}_2 &= \alpha_{xx}\bar{C}_{12} + \alpha_{yy}\bar{C}_{22} + \alpha_{zz}\bar{C}_{23} + \alpha_{xy}\bar{C}_{26} \\ \tilde{\alpha}_3 &= \alpha_{xx}\bar{C}_{13} + \alpha_{yy}\bar{C}_{23} + \alpha_{zz}\bar{C}_{33} + \alpha_{xy}\bar{C}_{36} \\ \tilde{\alpha}_6 &= \alpha_{xx}\bar{C}_{16} + \alpha_{yy}\bar{C}_{26} + \alpha_{zz}\bar{C}_{36} + \alpha_{xy}\bar{C}_{66} \end{aligned} \tag{2.74}$$

The equilibrium equations can be written in terms of the displacements:

$$\begin{aligned} &\bar{C}_{66}\frac{\partial^2 u}{\partial y^2} + \bar{C}_{55}\frac{\partial^2 u}{\partial z^2} + \bar{C}_{26}\frac{\partial^2 v}{\partial y^2} + \bar{C}_{36}\frac{\partial^2 w}{\partial y\,\partial z} = \tilde{\alpha}_6\frac{\partial T}{\partial y} \\ &\bar{C}_{26}\frac{\partial^2 u}{\partial y^2} + \bar{C}_{22}\frac{\partial^2 v}{\partial y^2} + \bar{C}_{44}\frac{\partial^2 v}{\partial z^2} + (\bar{C}_{23} + \bar{C}_{44})\frac{\partial^2 w}{\partial y\,\partial z} = \tilde{\alpha}_2\frac{\partial T}{\partial y} \\ &\bar{C}_{36}\frac{\partial^2 u}{\partial y\,\partial z} + (\bar{C}_{44} + \bar{C}_{23})\frac{\partial^2 v}{\partial y\,\partial z} + \bar{C}_{44}\frac{\partial^2 w}{\partial y^2} + \bar{C}_{33}\frac{\partial^2 w}{\partial z^2} = 0 \end{aligned} \tag{2.75}$$

The equilibrium equations can be solved by a singular perturbation technique (Van Dyke 1975). It is assumed by Y. Wang and Chou (1988, 1989) that, for h/b sufficiently small, i.e. $<10\%$ (see Fig. 2.12), the linear and higher order terms of h/b can be neglected and a zeroth order perturbation approach (Hsu and Herakovich 1976, 1977) is applied. The solution of Eqs. (2.75) for the kth layer in the interior region ($Y = y/b < 1$) of a laminate is

$$\begin{aligned} U^{(k)} &= \frac{2u}{h} = B(Y, t) \\ V^{(k)} &= \frac{2v}{h} = D(Y, t) \\ W^{(k)} &= \frac{2w}{h} = E(Y, t)Z \end{aligned} \tag{2.76}$$

where

$$\begin{aligned} Z &= \frac{2z}{h} \\ \left(\frac{h}{b}\right)B(Y, t) &= \frac{q_2 Q_1(Y, t) - q_1 Q_2(Y, t)}{q_1 q_3 - q_2^2} \\ \left(\frac{h}{b}\right)D(Y, t) &= \frac{q_3 Q_1(Y, t) - q_2 Q_2(Y, t)}{q_2^2 - q_1 q_3} \\ E(Y, t) &= \frac{\tilde{\alpha}_3}{\bar{C}_{33}} T(Y, t) \\ Q_1(Y, t) &= \sum_{k=1}^{n} \left(\frac{\bar{C}_{23}}{\bar{C}_{33}} \tilde{\alpha}_3 - \tilde{\alpha}_2\right)_k \bar{T}_k(Y, t) h^{(k)} \\ Q_2(Y, t) &= \sum_{k=1}^{n} \left(\frac{\bar{C}_{36}}{\bar{C}_{33}} \tilde{\alpha}_3 - \tilde{\alpha}_6\right)_k \bar{T}_k(Y, t) h^{(k)} \\ q_1 &= \sum_{k=1}^{n} (\bar{C}_{22})_k h^{(k)} \qquad q_2 = \sum_{k=1}^{n} (\bar{C}_{26})_k h^{(k)} \\ q_3 &= \sum_{k=1}^{n} (\bar{C}_{66})_k h^{(k)} \qquad \bar{T}_k(Y, t) = \int_0^Y T_k(Y, t)\, \mathrm{d}Y \\ h^{(k)} &= k\text{th layer thickness} \end{aligned} \tag{2.77}$$

The subscript k indicates the kth lamina of the n-layer laminate.

Following Hsu and Herakovich (1976, 1977), a stretching transformation parameter is introduced to obtain the solution for the boundary layer region ($Y \approx 1$):

$$\eta = (1 - Y)/(h/b) \tag{2.78}$$

Then the equilibrium equations (2.75) become

$$\tilde{C}_{66}\frac{\partial^2 U}{\partial \eta^2}+\tilde{C}_{55}\frac{\partial^2 U}{\partial Z^2}+\tilde{C}_{26}\frac{\partial^2 V}{\partial \eta^2}-\tilde{C}_{36}\frac{\partial^2 W}{\partial \eta\,\partial Z}=\left(\frac{h}{b}\right)\left(\frac{\tilde{\alpha}_6}{\bar{C}_{\max}}\right)\frac{\partial T}{\partial Y}$$

$$\begin{aligned}&\tilde{C}_{26}\frac{\partial^2 U}{\partial \eta^2}+\tilde{C}_{22}\frac{\partial^2 V}{\partial \eta^2}+\tilde{C}_{44}\frac{\partial^2 V}{\partial Z^2}-(\tilde{C}_{23}+\tilde{C}_{44})\frac{\partial^2 W}{\partial \eta\,\partial Z}\\&\quad=\left(\frac{h}{b}\right)\left(\frac{\tilde{\alpha}_2}{\bar{C}_{\max}}\right)\frac{\partial T}{\partial Y}-\tilde{C}_{36}\frac{\partial^2 U}{\partial \eta\,\partial Z}-(\tilde{C}_{23}+\tilde{C}_{44})\frac{\partial^2 V}{\partial \eta\,\partial Z}\\&\quad+\tilde{C}_{44}\frac{\partial^2 W}{\partial \eta^2}+\tilde{C}_{33}\frac{\partial^2 W}{\partial Z^2}=0\end{aligned}\tag{2.79}$$

where $\tilde{C}_{ij}=\bar{C}_{ij}/\bar{C}_{\max}$, and $\bar{C}_{\max}$ is the largest among all the $\bar{C}_{ij}$ values. The following expressions of the displacement field are assumed for matching the solutions in both interior and boundary layer regions, based upon Prandtl's matching principle:

$$\begin{aligned}U^{(k)}&=B(Y,t)+P\mathrm{e}^{\lambda\eta}\cos(\delta Z)\\V^{(k)}&=D(Y,t)+R\mathrm{e}^{\lambda\eta}\cos(\delta Z)\\W^{(k)}&=E(Y,t)Z+S\mathrm{e}^{\lambda\eta}\sin(\delta Z)\end{aligned}\tag{2.80}$$

Here, $B(Y, t)$, $D(Y, t)$ and $E(Y, t)$ are the interior region solutions (Eqs. 2.77); P, R and S are coefficients to be determined for the correction terms; δ is an undetermined positive constant; λ is the negative characteristic of Eqs. (2.79). It is seen from Eqs. (2.80) that away from the boundary layer region ($\eta \gg 1$), the correction terms have no influence on the displacement field; their effects become significant in and near the boundary layer region.

Substituting the $U^{(k)}$, $V^{(k)}$ and $W^{(k)}$ expressions into the equilibrium equations (2.79), the six roots of λ for non-trivial solutions of P, R and S are obtained:

$$\begin{aligned}\lambda_{1,2}&=\pm a_k\delta\\\lambda_{3,4}&=\pm b_k\delta\\\lambda_{5,6}&=\pm c_k\delta\end{aligned}\tag{2.81}$$

where a_k, b_k and c_k are three positive constants. The positive roots of λ are dropped to avoid divergence in the displacement field. Thus, the displacements for both the interior and boundary layer regions can be written as follows:

$$
\begin{aligned}
U^{(k)} &= B(Y, t) + (P_1 e^{-a_k \delta \eta} + P_2 e^{-b_k \delta \eta} + P_3 e^{-c_k \delta \eta}) \cos(\delta Z) \\
V^{(k)} &= D(Y, t) + (R_1 e^{-a_k \delta \eta} + R_2 e^{-b_k \delta \eta} + R_3 e^{-c_k \delta \eta}) \cos(\delta Z) \\
W^{(k)} &= E(Y, t) Z + (S_1 e^{-a_k \delta \eta} + S_2 e^{-b_k \delta \eta} + S_3 e^{-c_k \delta \eta}) \sin(\delta Z)
\end{aligned}
\tag{2.82}
$$

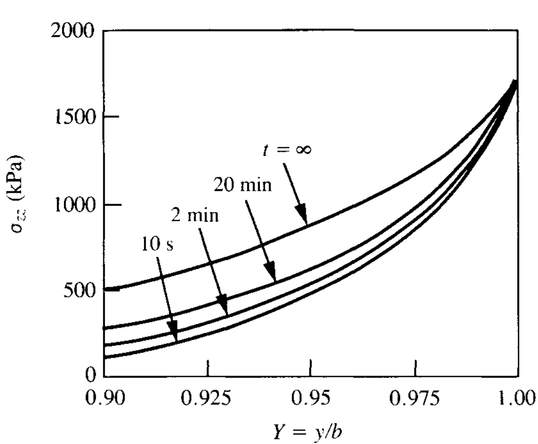
There are ten unknowns for the displacement solution of the kth layer (P_1, P_2, P_3, R_1, R_2, R_3, S_1, S_2, S_3 and δ).

The available equations for the solution of these constants are: (i) three stress boundary conditions, $\sigma_{yy}(b, z) = \sigma_{xy}(b, z) = \sigma_{yz}(b, z) = 0$; (ii) six equilibrium equations (2.79); and (iii) the integrated equilibrium condition

$$
\int_0^{1/2} \sigma_{xy}(0, Z) \frac{h}{2} \, dZ = \int_0^1 \sigma_{xz}\left(Y, \frac{1}{2}\right) b \, dY \tag{2.83}
$$

A four-layer angle-ply composite is taken as a numerical example. Each layer is 5 mm in thickness $h^{(k)}$, 200 mm in width (b). The SiC/borosilicate glass laminate is used as a baseline composite system for demonstration of the results. The transient interlaminar normal stress distribution of a $[-45°/45°]_s$ SiC/borosilicate glass laminate, which is subjected to a sudden edge heating of the magnitude $T_o = 1°C$ at $t = 0^+$, is demonstrated in Fig. 2.13. No stress singularity is found as a consequence of the assumed displacement

Fig. 2.13. Transient interlaminar thermal stress of a SiC/borosilicate glass $[-45°/45°]_s$ laminate for $V_f = 30\%$ and $T_o = 1°C$. (After Y. Wang and Chou 1989.)



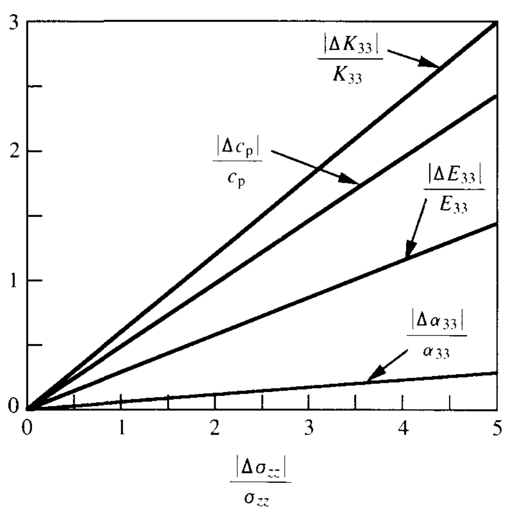
field, but it is apparent that the interlaminar normal stress concentration increases very significantly as approaching to the free edge of the plate ($Y = 1$). The stress at $Y = 1.0$ is about three to twenty times higher than that at $Y = 0.90$ for $t = 10$ s to ∞. As the heating proceeds, the overall interlaminar normal stress increases smoothly, while the stress which is very close to the boundary remains almost constant. Also, the interlaminar normal stress tends to zero away from the free edge of the laminate due to the adoption of the classical lamination theory in the interior region.

Figure 2.14 shows the results of a parametric study of the stress solution sensitivity to the composite elastic and thermal properties. Here the $[-45°/45°]_s$ SiC/borosilicate glass laminate is taken as the baseline system, and Δ indicates an increment. The Young's modulus (E_{33}) and thermal expansion coefficient (α_{33}) along the plate thickness direction have a more significant effect on the stress σ_{zz} than the thermal conductivity (K_{33}) and specific heat (C_p). The transient thermal stress analysis can be applied for the characterization of thermal shock resistance capability of composite materials. (See, for example, Cheng 1951; Kingery 1955; Y.Wang and Chou 1991.)

2.5.4 *Transient in-plane thermal stress*

Having discussed the thermoelastic field due to one-dimensional heat and moisture diffusion in Sections 2.5.2 and 2.5.3,

Fig. 2.14. Parametric studies of stress solution sensitivity to composite elastic and thermal properties. The base material is a SiC/borosilicate glass $[-45°/45°]_s$ laminate. Calculations of $|\Delta\sigma_{zz}|/\sigma_{zz}$ are based upon $t = 2$ min and $Y = 0.99$. (After Wang and Chou 1989.)



a two-dimensional transient heat conduction problem is examined in the following. The model material considered is a unidirectional lamina. The interaction of thermal stresses among the layers of a laminate is thus not included in order to clearly demonstrate the effect of transient heat conduction.

2.5.4.1 *Transient temperature field*

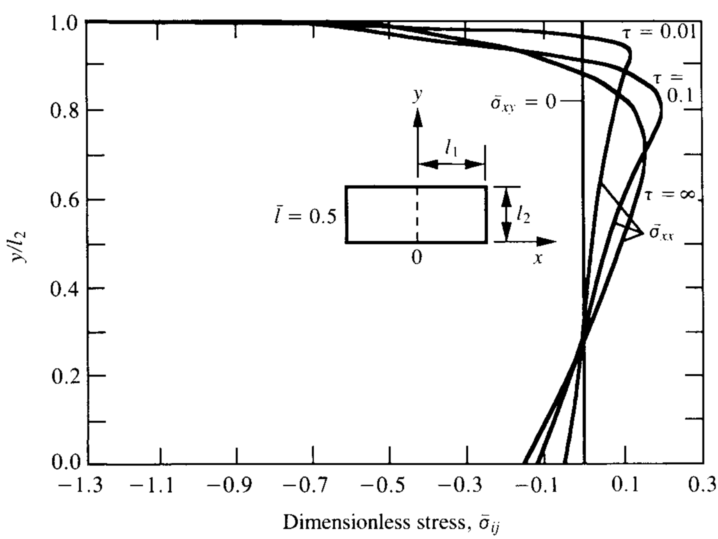
Consider the two-dimensional problem of an orthotropic slab with a rectangular region ($0 \le x \le l_1$, $0 \le y \le l_2$) as shown in Fig. 2.15. The slab is initially held at a uniform temperature and then the edge $y = l_2$ is suddenly subjected to an arbitrary temperature distribution or heat flux $f(x)$. The two-dimensional temperature distribution, $T(x, y; t)$ in the rectangular region is assumed to satisfy the heat conduction equation (2.63).

The initial condition is

$$T(x, y; 0) = 0 \qquad \text{for } t = 0 \tag{2.84}$$

The boundary conditions of the rectangle assume the following

Fig. 2.15. Thermal stress variations with time for $K = 0.1$ at the cross-section $x = 0$. (After H. Wang and Chou 1985.)



general forms:

$$-a_1 \frac{\partial T}{\partial x} + b_1 T = 0 \qquad \text{for } x = 0 \tag{2.85a}$$

$$a_2 \frac{\partial T}{\partial x} + b_2 T = 0 \qquad \text{for } x = l_1 \tag{2.85b}$$

$$-a_3 \frac{\partial T}{\partial y} + b_3 T = 0 \qquad \text{for } y = 0 \tag{2.85c}$$

$$a_4 \frac{\partial T}{\partial y} + b_4 T = f(x) \qquad \text{for } y = l_2 \tag{2.85d}$$

Here, a_i $(i = 1, 2, 3, 4)$ are conductivities for the respective directions, and b_i are the coefficients of surface heat transfer. The various types of boundary conditions can be obtained through the proper selections of the constant ratio b_i/a_i (see, for example, Carslaw and Jaeger 1959). Equations (2.85a)–(2.85c) correspond to zero surface temperature or heat flow, whereas the nonhomogeneous boundary condition of Eq. (2.85d) is for an arbitrary variation of surface thermal condition.

Equation (2.85d) suggests the use of the principle of superposition. The problem has a steady-state solution as $t \to \infty$. It is assumed that

$$T(x, y; t) = \phi(x, y) + \psi(x, y; t) \tag{2.86}$$

such that $\phi(x, y)$ and $\psi(x, y; t)$ satisfy

$$K_{xx}^{\mathrm{T}} \frac{\partial^2 \phi}{\partial x^2} + K_{yy}^{\mathrm{T}} \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.87}$$

$$-a_1 \frac{\partial \phi}{\partial x} + b_1 \phi = 0 \qquad \text{for } x = 0 \tag{2.88a}$$

$$a_2 \frac{\partial \phi}{\partial x} + b_2 \phi = 0 \qquad \text{for } x = l_1 \tag{2.88b}$$

$$-a_3 \frac{\partial \phi}{\partial y} + b_3 \phi = 0 \qquad \text{for } y = 0 \tag{2.88c}$$

$$a_4 \frac{\partial \phi}{\partial y} + b_4 \phi = f(x) \qquad \text{for } y = l_2 \tag{2.88d}$$

and

$$K_{xx}^{\mathrm{T}} \frac{\partial^2 \psi}{\partial x^2} + K_{yy}^{\mathrm{T}} \frac{\partial^2 \psi}{\partial y^2} = \rho C_{\mathrm{p}} \frac{\partial \psi}{\partial t} \tag{2.89}$$

$$\psi(x, y; 0) = -\phi(x, y) \qquad \text{for } t = 0 \tag{2.90}$$

$$-a_1 \frac{\partial \psi}{\partial x} + b_1 \psi = 0 \qquad \text{for } x = 0 \tag{2.91a}$$

$$a_2 \frac{\partial \psi}{\partial x} + b_2 \psi = 0 \qquad \text{for } x = l_1 \tag{2.91b}$$

$$-a_3 \frac{\partial \psi}{\partial y} + b_3 \psi = 0 \qquad \text{for } y = 0 \tag{2.91c}$$

$$a_4 \frac{\partial \psi}{\partial y} + b_4 \psi = 0 \qquad \text{for } y = l_2 \tag{2.91d}$$

H. Wang and Chou (1986) have obtained the general solution of ϕ and ψ with the unknown constants in the infinite series expressions to be determined by the boundary conditions of Eqs. (2.88) and (2.91) and initial condition of Eq. (2.90).

An example of this solution technique is given by H. Wang and Chou (1985) for a slab initially held at a constant temperature and suddenly subjected to an arbitrary temperature variation along one of its edges. The constants in Eqs (2.85) are $a_1 = -1$, $a_2 = a_3 = b_1 = 0$, and $b_2 = b_3 = 1$. The temperature field solution is

$$\begin{aligned} T(x, y; t) = \sum_{n=1}^{\infty} \Bigg\{ & I_n \cos \delta_n x \sinh \frac{\delta_n}{K} y \\ & + \sum_{m=1}^{\infty} I_{nm} \cos \delta_n x \sin \frac{\mu_{\mathrm{m}}}{K} y \exp[-d(\delta_n^2 + \mu_m^2)t] \Bigg\} \end{aligned} \tag{2.92}$$

where

$$d = K_{xx}/\rho C_{\mathrm{p}}, K^2 = K_{yy}/K_{xx}$$

$$I_{nm}(\delta_{\mathrm{n}}, \mu_{\mathrm{m}}) = \frac{2}{l_2^2} I_{\mathrm{n}}(\delta_{\mathrm{n}}) \frac{(-1)^m \dfrac{\mu_{\mathrm{m}}}{K} l_2}{\left(\dfrac{\delta_n}{K}\right)^2 + \left(\dfrac{\mu_m}{K}\right)^2} \sinh \frac{\delta_n}{K} l_2$$

$$I_n(\delta_n) = \frac{2}{a_4 \dfrac{\delta_n}{K} \cosh \dfrac{\delta_n l_2}{K} + b_4 \sinh \dfrac{\delta_n l_2}{K}} \frac{1}{l_1} \int_0^{l_1} f(x) \cos \delta_n x \, \mathrm{d}x \tag{2.93}$$

$$\delta_n = \frac{2n-1}{2l_1} \pi \qquad n = 1, 2, 3, \ldots, \infty$$

Also, μ_m/K are the positive roots of

$$\left(\frac{\mu}{K} l_2\right) \cot\left(\frac{\mu}{K} l_2\right) + \frac{b_4}{a_4} l_2 = 0$$

If $a_4 = 0$ and $b_4 = 1$,

$$\frac{\mu_m}{K} = \frac{m}{l_2} \pi = \delta_m \qquad m = 1, 2, 3, \ldots, \infty \tag{2.94}$$

H. Wang and Chou (1986) have tabulated the solution of temperature field from the various combinations of a_1, a_2, a_3, b_1, b_2 and b_3 values of Eqs. (2.85).

2.5.4.2 *Thermal stress field*

Consider a unidirectional fiber composite; let the principal material directions x_1 and x_2 coincide with the reference axes x and y, respectively. The stress–strain relations follow Eq. (2.49) with the $\bar{C}_{ij}$ replaced by C_{ij}. Depending upon the thickness of the elastic medium in the z direction, the thermoelastic problem is in the state of either plane strain or plane stress. In the case of plane strain, the stress components in the x–y plane are related to the in-plane displacements, $u(x, y; t)$ and $v(x, y; t)$, and the temperature, $T(x, y; t)$, by substituting the strain–displacement relations into the stress–strain relations of Eqs. (2.73). The results are

$$\begin{aligned}
\sigma_{xx} &= C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} - \tilde{\alpha}_1 T(x, y; t) \\
\sigma_{yy} &= C_{12}\frac{\partial u}{\partial x} + C_{22}\frac{\partial v}{\partial y} - \tilde{\alpha}_2 T(x, y; t) \\
\sigma_{zz} &= C_{13}\frac{\partial u}{\partial x} + C_{23}\frac{\partial v}{\partial y} - \tilde{\alpha}_3 T(x, y; t) \\
\sigma_{xy} &= C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned} \tag{2.95}$$

where

$$\begin{aligned}
\tilde{\alpha}_1 &= C_{11}\alpha_{11} + C_{12}\alpha_{22} + C_{13}\alpha_{33} \\
\tilde{\alpha}_2 &= C_{12}\alpha_{11} + C_{22}\alpha_{22} + C_{23}\alpha_{33} \\
\tilde{\alpha}_3 &= C_{13}\alpha_{11} + C_{23}\alpha_{22} + C_{33}\alpha_{33}
\end{aligned} \tag{2.96}$$

The relations corresponding to Eqs. (2.95) for plane stress condition are obtained by replacing C_{ij} and $\tilde{\alpha}_i$ by $C_{ij} - C_{3j}/C_{33}$ and $\tilde{\alpha}_i - \tilde{\alpha}_3 C_{3i}/C_{33}$, respectively.

The displacement equations of equilibrium governing the plane strain conditions are

$$C_{11}\frac{\partial^2 u}{\partial x^2}+C_{66}\frac{\partial^2 u}{\partial y^2}+(C_{12}+C_{66})\frac{\partial^2 v}{\partial x\,\partial y}=\tilde{\alpha}_1\frac{\partial T}{\partial x}$$
$$C_{66}\frac{\partial^2 v}{\partial x^2}+C_{22}\frac{\partial^2 v}{\partial y^2}+(C_{12}+C_{66})\frac{\partial^2 u}{\partial x\,\partial y}=\tilde{\alpha}_2\frac{\partial T}{\partial y} \tag{2.97}$$

The equilibrium equations are solved by introducing the displacement potentials ψ_1, ψ_2 and ϕ defined by

$$u(x, y; t)=\frac{\partial \psi_1}{\partial x}+\frac{\partial \psi_2}{\partial x}+\frac{\partial \phi}{\partial x}$$
$$v(x, y; t)=\nu_1\frac{\partial \psi_1}{\partial y}+\nu_2\frac{\partial \psi_2}{\partial y}+\lambda\frac{\partial \phi}{\partial y} \tag{2.98}$$

where ν_1, ν_2 and λ are unknown constants. Also, ϕ is the homogeneous solution and ψ_1 and ψ_2 are particular solutions of Eqs. (2.97).

An example of the transient thermal stress solution is given by H. Wang and Chou (1985) for a rectangular slab ($-l_1 \le x \le l_1$ and $0 \le y \le l_2$) with fibers oriented in the x direction. The initial temperature of the slab is $T=0$. Then the following form of temperature rise at the upper edge ($y=l_2$) is adopted:

$$T=f(x)=T_0\cos\frac{\pi}{2l_1}x \qquad \text{for } t>0 \tag{2.99}$$

while the temperature over the remainder of the boundary is maintained at the initial value. All edges of the rectangle are assumed to be traction free:

$$\sigma_{xx}=\sigma_{xy}=0 \qquad \text{for } x=\pm l_1$$
$$\sigma_{yy}=\sigma_{xy}=0 \qquad \text{for } y=0,\ l_2 \tag{2.100}$$

The thermal and elastic properties as given by Aköz and Tauchert (1978) simulating a boron/epoxy composite are adopted for the numerical calculations. Owing to the symmetry of the assumed temperature rise, only one half of the rectangle ($0 \le x \le l_1$) needs to be considered. Thus, the boundary condition Eq. (2.85d) is reduced to $a_4=0$ and $b_4=1$. For the convenience of presenting the numerical results, the following dimensionless quantities are introduced for temperature stress, time, and lamina dimension, respectively: $\bar{T} = T(x, y; t)/T_o$; $\bar{\sigma}_{ij} = \sigma_{ij}(x, y; t)/\tilde{\alpha}_2 T_o$, $\tau = dt/l_1^2$, and $\bar{l} = l_2/l_1$.

Figure 2.15 shows the y direction variation of thermal stresses at the cross-section $x = 0$ for the various dimensionless time intervals. It is clear that large longitudinal stresses σ_{xx} occur in the vicinity of the heated boundary, where the relatively large temperature gradient, $\partial T/\partial y$, exists. On the other hand, the transverse stresses, σ_{yy}, and the shear stresses σ_{xy} are fairly small. Also, for σ_{xx}, the maximum transient tensile stress is 25% higher than that in the steady state; the maximum transient compressive stress near the upper edge ($y = l_2$) is 78% higher than the corresponding steady-state stress. An examination of the plots of σ_{xx} and σ_{xy} at a given time interval indicates that each stress is in self-equilibrium when the slab is free to deform, i.e. no boundary constraints. This is consistent with the nature of thermal residual stresses.