CHAPTER SEVEN

Beams with Shear Deformation

Frequently, beams undergoing small deformations may be analyzed by the Bernoulli–Navier hypothesis, namely, by the assumptions that planes of the cross section remain plane and perpendicular to the axis (Fig. 7.1, left).

In this chapter we treat beams for which the Bernoulli–Navier hypothesis is invalid. In the analysis we employ an x, y, z, coordinate system with the origin in the centroid. Furthermore, to simplify the notation, we use single subscripts y and z describing shear in the x–y and x–z planes.

In the x–z plane the deflection of the beam's axis w is related to the rotation of the cross section χ_z by

$$\frac{dw}{dx} = \chi_z \qquad \begin{matrix}\text{no shear deformation}\\ x\text{–}z \text{ plane.}\end{matrix} \tag{7.1}$$

For thick solid cross-section beams, sandwich beams, and thin-walled beams the first assumption (that planes of the cross section remain plane) is reasonable. The second assumption may no longer be valid because cross sections do not necessarily remain perpendicular to the axis (Fig. 7.1, right). In this case the deflection of the beam is

$$\frac{dw}{dx} = \chi_z + \gamma_z \qquad \begin{matrix}\text{with shear deformation}\\ x\text{–}z \text{ plane,}\end{matrix} \tag{7.2}$$

where γ_z is the transverse shear strain in the x–z plane. The theory, based on the assumption that cross sections remain plane but not perpendicular to the axis is frequently called first-order shear theory. A beam, in which shear deformation is taken into account is called a Timoshenko beam.

Similarly, in the x–y plane, we have

$$\frac{dv}{dx} = \chi_y + \gamma_y \qquad \begin{matrix}\text{with shear deformation}\\ x\text{–}y \text{ plane,}\end{matrix} \tag{7.3}$$

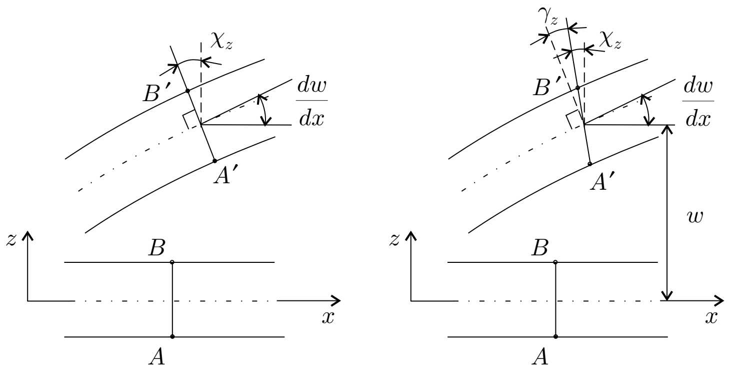


Figure 7.1: Deformation of a beam in the x–z plane without shear deformation (left) and with shear deformation (right).

The rotations of the cross sections χ_z, χ_y are caused by the bending moments $\widehat{M}_y$, $\widehat{M}_z$, and the transverse shear strains γ_z, γ_y are caused by the transverse shear forces $\widehat{V}_z$, $\widehat{V}_y$. The relationships between χ and $\widehat{M}$ and between γ and $\widehat{V}$ are presented in Section 7.1.2 for orthotropic beams.

In torsion, when shear deformation of the wall is neglected, the rate of twist ϑ^{B} is related to the twist of the cross section about the beam's axis ψ by (Fig. 7.2, left)

$$\frac{d\psi}{dx} = \vartheta^{\mathrm{B}} \qquad \text{no shear deformation.} \tag{7.4}$$

When shear deformation is not negligible, there is an additional rate of twist of the cross section ϑ^{S}, as shown in Figure 7.2, right. Thus, in the presence of shear deformation, the rate of twist of the beam is[1]

$$\frac{d\psi}{dx} = \vartheta^{\mathrm{B}} + \vartheta^{\mathrm{S}} \qquad \text{with shear deformation.} \tag{7.5}$$

For open-section beams, ϑ^{B} is the rate of twist due to warping when the shear strain γ is zero, and ϑ^{S} is the rate of twist due to the shear deformation when warping is zero (Fig. 7.2). For closed-section beams the interpretation of ϑ^{B} and ϑ^{S} is more complicated and is not given here.

7.1 Governing Equations

The response of a beam to the applied forces is described by the strain–displacement, force–strain, and equilibrium equations. These equations are given in this section for orthotropic beams, including the effect of restrained warping.

[1] X. Wu and C. T. Sun, Simplified Theory for Composite Thin-Walled Beams. *AIAA Journal*, Vol. 30, 2945–2951, 1992.

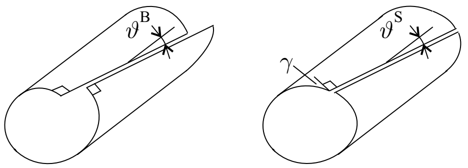


Figure 7.2: The rate of twist ϑ^{B} due to warping when the shear strain is zero (left) and the rate of twist ϑ^{S} due to shear deformation when warping is zero (right).

7.1.1 Strain–Displacement Relationships

There are seven independent displacements, of which the following four are identical to the displacements of beams without shear deformation (Fig. 6.3): the axial displacement u, the transverse displacements v and w in the y and z directions, respectively, and the twist of the cross section ψ. When the shear deformation is taken into account, there are three additional displacements, namely, the rotations of the cross section χ_z, χ_y in the x–z and x–y planes, respectively, and the rate of twist due to warping, ϑ^{B}, when the shear strain γ is zero. We define the following generalized strain components (hereafter referred to as strain):

$$\epsilon_x^{o} = \frac{\partial u}{\partial x} \qquad \frac{1}{\rho_z} = -\frac{\partial \chi_y}{\partial x} \qquad \frac{1}{\rho_y} = -\frac{\partial \chi_z}{\partial x} \qquad \Gamma = -\frac{\partial \vartheta^{B}}{\partial x} \qquad \vartheta = \frac{d\psi}{dx} \tag{7.6}$$

We note that ρ_z and ρ_y are not the radii of curvatures of the beam's axis; they are the radii of curvatures only when shear deformation is neglected. Equations (7.2), (7.3), and (7.5) yield

$$\gamma_z = \frac{dw}{dx} - \chi_z \qquad \gamma_y = \frac{dv}{dx} - \chi_y \qquad \vartheta^{S} = \frac{d\psi}{dx} - \vartheta^{B}. \tag{7.7}$$

7.1.2 Force–Strain Relationships

The force–strain relationships are first presented in detail for an orthotropic I-beam with doubly symmetrical cross section subjected to bending moment $\widehat{M}_y$, shear force acting through the plane of symmetry $\widehat{V}_z$, and torque $\widehat{T}$ (Fig. 7.3). The relationships thus obtained are then generalized to beams with arbitrary cross sections.

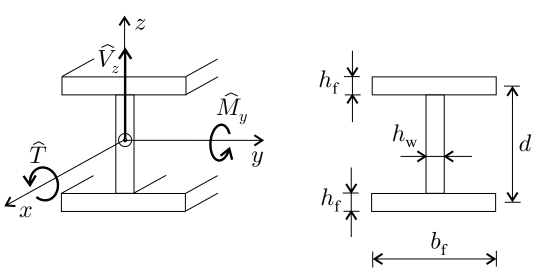


Figure 7.3: Thin-walled I-beam.

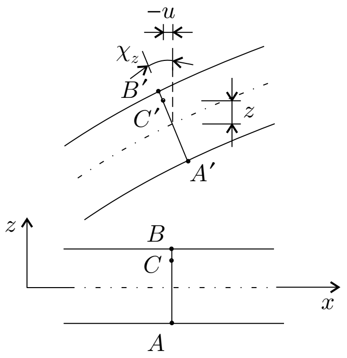


Figure 7.4: Displacement u of point C in the axial direction.

I-*beam.* In the absence of twist, the displacement u in the x direction at a point located at distance z from the centroid is (Fig. 7.4, $\chi_z \approx \tan \chi_z = -\frac{u}{z}$)

$$u = -z\chi_z, \tag{7.8}$$

where χ_z is the rotation of the cross section in the x–z plane.

The strain–displacement relationship (Eq. 2.2), together with Eq. (7.8), gives the axial strain

$$\epsilon_x = \frac{du}{dx} = -z\frac{d\chi_z}{dx}. \tag{7.9}$$

The bending moment about the y-axis is defined as

$$\widehat{M}_y = \int_A z\sigma_x dA. \tag{7.10}$$

For an isotropic material $\sigma_x = E\epsilon_x$, and we have

$$\widehat{M}_y = \int_A zE\left(-z\frac{d\chi_z}{dx}\right) dA = -E\underbrace{\int_A z^2 dA}_{I_{yy}}\frac{d\chi_z}{dx}. \tag{7.11}$$

Recalling the analyses in Sections 6.3 and 6.4, we replace EI by $\widehat{EI}$ for composite beams and write

$$\widehat{M}_y = \widehat{EI}_{yy}\underbrace{\left(-\frac{d\chi_z}{dx}\right)}_{1/\rho_y} \qquad \text{orthotropic beam.} \tag{7.12}$$

where $\widehat{EI}_{yy}$ is the replacement bending stiffness. Replacement bending stiffnesses for beams are given in Tables 6.3–6.8 (page 231) and A.1–A.4.

The shear strain varies linearly with the shear force. Thus, formalistically, we write

$$\widehat{V}_z = \widehat{S}_{zz}\gamma_z \qquad \text{orthotropic beam,} \tag{7.13}$$

where $\widehat{S}_{zz}$ is the shear stiffness.

Next, we consider an orthotropic beam subjected to a torque $\widehat{T}$ (Eq. 6.240):

$$\widehat{T} = \widehat{T}_{\text{sv}} + \widehat{T}_{\omega}. \tag{7.14}$$

The torque $\widehat{T}_{\text{sv}}$ (Saint-Venant torque, Fig. 6.56, top) is (Eq. 6.240)

$$\widehat{T}_{\text{sv}} = \widehat{GI}_{\text{t}}\vartheta \qquad \text{Saint-Venant torque.} \tag{7.15}$$

The torque $\widehat{T}_{\omega}$ (restrained–warping-induced torque, Fig 6.56, bottom) is derived below following the reasoning used for an I-beam without shear deformation (Section 6.5.5).

The displacement of the flange v_{f} is (Fig. 6.57)

$$v_{\text{f}} = \psi\frac{d}{2}, \tag{7.16}$$

where ψ is the twist of the cross section about the beam's axis and d is the distance between the midplanes of the flanges. The rate of twist is $\vartheta = d\psi/dx$ (Eq. 6.1), and we write

$$\frac{dv_{\text{f}}}{dx} = \frac{d}{2}\vartheta. \tag{7.17}$$

On the basis of Eq. (7.3), the first derivative of the displacement is written as

$$\frac{dv_{\text{f}}}{dx} = (\chi)_{\text{f}} + (\gamma)_{\text{f}}, \tag{7.18}$$

where $(\chi)_{\text{f}}$ is the rotation of the cross section of the flange about the z-axis (Fig. 7.3), and $(\gamma)_{\text{f}}$ is the shear strain in the flange.

We express the rate of twist in the form[2] (Eq. 7.5)

$$\vartheta = \vartheta^{\text{B}} + \vartheta^{\text{S}}. \tag{7.19}$$

The first term represents the rate of twist in the absence of shear deformation, and the second term is the rate of twist due to shear deformation. Equations (7.17)–(7.19) give

$$(\chi)_{\text{f}} = \frac{d}{2}\vartheta^{\text{B}} \qquad (\gamma)_{\text{f}} = \frac{d}{2}\vartheta^{\text{S}}. \tag{7.20}$$

Recalling Eq. (7.12), we write the bending moment $\widehat{M}_{\text{f}}$ for an orthotropic flange in the presence of shear deformation as

$$\widehat{M}_{\text{f}} = \widehat{EI}_{\text{f}}\left(-\frac{d(\chi)_{\text{f}}}{dx}\right) = -\widehat{EI}_{\text{f}}\frac{d}{2}\frac{d\vartheta^{\text{B}}}{dx}, \tag{7.21}$$

where the second equality is written by virtue of Eq. (7.20), and $\widehat{EI}_{\text{f}}$ is the bending stiffness of the flange about the z-axis.

For an I-beam the bimoment is (Eq. 6.232)

$$\widehat{M}_{\omega} = \widehat{M}_{\text{f}}d. \tag{7.22}$$

[2] X. Wu and C. T. Sun, Simplified Theory for Composite Thin-Walled Beams. *AIAA Journal*, Vol. 30, 2945–2951, 1992.

The last two equations give

$$\widehat{M}_\omega = \underbrace{\left(\frac{d^2}{2}\widehat{EI}_\mathrm{f}\right)}_{\widehat{EI}_\omega}\underbrace{\left(-\frac{d\vartheta^\mathrm{B}}{dx}\right)}_{\Gamma}, \tag{7.23}$$

where the term in the first parentheses $\widehat{EI}_\omega(=\frac{d^2}{2}\widehat{EI}_\mathrm{f})$ is the warping stiffness, which for an isotropic beam is given by Eq. (6.238) and for an orthotropic I-beam by Eq. (6.244).

The shear force in the flange is related to $\widehat{M}_\mathrm{f}$ by

$$\widehat{V}_\mathrm{f} = \frac{d\widehat{M}_\mathrm{f}}{dx}. \tag{7.24}$$

By referring to Eq. (7.13), we write $\widehat{V}_\mathrm{f}$ as

$$\widehat{V}_\mathrm{f} = (\widehat{S}_{yy})_\mathrm{f}\,(\gamma)_\mathrm{f}\,, \tag{7.25}$$

where $(\widehat{S}_{yy})_\mathrm{f}$ is the shear stiffness of the flange in the x–y plane. The warping-induced torque is (Eq. 6.235)

$$\widehat{T}_\omega = \widehat{V}_\mathrm{f} d. \tag{7.26}$$

Equations (7.22), (7.24), and (7.26) result in

$$\widehat{T}_\omega = \frac{d\widehat{M}_\omega}{dx}. \tag{7.27}$$

From Eqs. (7.26), (7.25), and (7.20) we obtain

$$\widehat{T}_\omega = \underbrace{\left[(\widehat{S}_{yy})_\mathrm{f}\frac{d^2}{2}\right]}_{\widehat{S}_{\omega\omega}}\vartheta^\mathrm{S} \qquad \text{restrained–warping-induced torque}, \tag{7.28}$$

where $\widehat{S}_{\omega\omega}$ is the rotational shear stiffness defined as

$$\widehat{S}_{\omega\omega} = (\widehat{S}_{yy})_\mathrm{f}\frac{d^2}{2}. \tag{7.29}$$

Equations (7.12), (7.13), (7.15), (7.23), and (7.28) are the force–strain relationships for an orthotropic I-beam with doubly symmetrical cross section subjected to a bending moment $\widehat{M}_y$, a shear force acting in the plane of symmetry $\widehat{V}_z$, and a torque $\widehat{T}(=\widehat{T}_\mathrm{sv}+\widehat{T}_\omega)$.

Arbitrary cross-section beams. We now consider orthotropic beams of arbitrary cross sections with internal forces $\widehat{N}$, $\widehat{M}_y$, $\widehat{M}_z$, $\widehat{V}_y$, $\widehat{V}_z$, and $\widehat{M}_\omega$, $\widehat{T}$.

The relationship between the axial force (acting at the centroid) and the axial strain is (Eqs. 6.7 and 6.8)

$$\widehat{N} = \widehat{EA}\epsilon_x^\mathrm{o}. \tag{7.30}$$

The bending moments and bimoment for a (symmetrical) I-beam are (see Eqs. 7.12 and 7.23)

$$\begin{Bmatrix} \widehat{M}_z \\ \widehat{M}_y \\ \widehat{M}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{EI}_{zz} & 0 & 0 \\ 0 & \widehat{EI}_{yy} & 0 \\ 0 & 0 & \widehat{EI}_\omega \end{bmatrix} \begin{Bmatrix} -\frac{d\chi_y}{dx} \\ -\frac{d\chi_z}{dx} \\ -\frac{d\vartheta^{\mathrm{B}}}{dx} \end{Bmatrix}, \tag{7.31}$$

where the equation for $\widehat{M}_z$ is analogous to the equation for $\widehat{M}_y$. For an arbitrary cross-section orthotropic beam, the 12 and 21 elements are not zero. Hence, we write

$$\begin{Bmatrix} \widehat{M}_z \\ \widehat{M}_y \\ \widehat{M}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{EI}_{zz} & \widehat{EI}_{yz} & 0 \\ \widehat{EI}_{yz} & \widehat{EI}_{yy} & 0 \\ 0 & 0 & \widehat{EI}_\omega \end{bmatrix} \begin{Bmatrix} -\frac{d\chi_y}{dx} \\ -\frac{d\chi_z}{dx} \\ -\frac{d\vartheta^{\mathrm{B}}}{dx} \end{Bmatrix}. \tag{7.32}$$

By utilizing the definitions given in Eq. (7.6), these equations may be written as

$$\begin{Bmatrix} \widehat{M}_z \\ \widehat{M}_y \\ \widehat{M}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{EI}_{zz} & \widehat{EI}_{yz} & 0 \\ \widehat{EI}_{yz} & \widehat{EI}_{yy} & 0 \\ 0 & 0 & \widehat{EI}_\omega \end{bmatrix} \begin{Bmatrix} \frac{1}{\rho_z} \\ \frac{1}{\rho_y} \\ \Gamma \end{Bmatrix}. \tag{7.33}$$

The relationship between $\widehat{T}_{\mathrm{sv}}$ and the rate of twist ϑ is (Eq. 7.15)

$$\widehat{T}_{\mathrm{sv}} = \widehat{GI}_{\mathrm{t}}\vartheta. \tag{7.34}$$

The shear forces and the restrained–warping-induced torque for a (doubly symmetrical) I-beam is (see Eqs. 7.13 and 7.28)

$$\begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & 0 & 0 \\ 0 & \widehat{S}_{zz} & 0 \\ 0 & 0 & \widehat{S}_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{Bmatrix}, \tag{7.35}$$

where the first equality is analogous to the second one. Shear is introduced by $\widehat{T}_\omega$ and by $\widehat{V}_y$ and $\widehat{V}_z$. Therefore, the force–strain relationships are coupled. We now extend the preceding equations to include these couplings and (for arbitrary cross-section beams) write the force–strain relationships as

$$\begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix} \begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{Bmatrix} = \left[\, \widehat{S}_{ij} \right] \begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{Bmatrix}, \tag{7.36}$$

where $[\widehat{S}_{ij}]$ is the shear stiffness matrix of the beam. The inverse of Eq. (7.36) is

$$\begin{Bmatrix} \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{Bmatrix} = [\widehat{s}_{ij}] \begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix}, \tag{7.37}$$

where $[\widehat{s}_{ij}]$ is the shear compliance matrix

$$\begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \widehat{s}_{yy} & \widehat{s}_{yz} & \widehat{s}_{y\omega} \\ \widehat{s}_{yz} & \widehat{s}_{zz} & \widehat{s}_{z\omega} \\ \widehat{s}_{y\omega} & \widehat{s}_{z\omega} & \widehat{s}_{\omega\omega} \end{bmatrix}^{-1}. \tag{7.38}$$

Equations (7.30), (7.33), (7.34), and (7.36) are the force–strain relationships for a beam with shear deformation. The way in which the elements of the stiffness and compliance matrices are determined is discussed in Section 7.2.

7.1.3 Equilibrium Equations

In the presence of shear deformation the equilibrium equations for a straight beam subjected to the loads shown in Figure 6.1 are identical to those of a beam without shear deformation (see Eq. 6.3):

$$\begin{aligned} \frac{\partial \widehat{N}}{\partial x} &= -p_x & \frac{\partial(\widehat{T}_{\mathrm{sv}} + \widehat{T}_\omega)}{\partial x} &= -t \\ \frac{\partial \widehat{V}_y}{\partial x} &= -p_y & \frac{\partial \widehat{V}_z}{\partial x} &= -p_z \\ \frac{\partial \widehat{M}_y}{\partial x} &= \widehat{V}_z & \frac{\partial \widehat{M}_z}{\partial x} &= \widehat{V}_y. \end{aligned} \tag{7.39}$$

When the beam is axially constrained, we have the additional equation (Eq. 7.27) as follows:

$$\frac{d\widehat{M}_\omega}{dx} = \widehat{T}_\omega. \tag{7.40}$$

7.1.4 Summary of Equations

In summary, we have 23 unknowns: 7 generalized displacements (u, v, w, ψ, χ_z, χ_y, ϑ^{B}), 8 generalized strain components (ϵ_x^{o}, $1/\rho_z$, $1/\rho_y$, Γ, γ_y, γ_z, ϑ^{S}, ϑ), and 8 generalized internal forces ($\widehat{N}$, $\widehat{M}_z$, $\widehat{M}_y$, $\widehat{M}_\omega$, $\widehat{V}_y$, $\widehat{V}_z$, $\widehat{T}_\omega$, $\widehat{T}_{\mathrm{sv}}$).

The equations that yield the solution are as follows:

	Number of equations	Equation number
strain–displacement	8	7.6, 7.7
force–strain	8	7.30, 7.33, 7.34, 7.36
equilibrium	7	7.39, 7.40

Table 7.1. Boundary conditions for beams with shear stiffnesses

	$x-z$ plane		$x-y$ plane	
Built-in	$w=0$	$\chi_z=0$	$v=0$	$\chi_y=0$
Simply supported	$w=0$	$\widehat{M}_y=0$	$v=0$	$\widehat{M}_z=0$
Free	$\widehat{V}_z=0$	$\widehat{M}_y=0$	$\widehat{V}_y=0$	$\widehat{M}_z=0$
Axially restrained	$u=0$			
unrestrained	$\widehat{N}=0$			
Warping restrained	$\vartheta^{\mathrm{B}}=0$			
unrestrained	$\widehat{M}_\omega=0$			
Rotationally restrained	$\psi=0$			
unrestrained	$\widehat{T}=0$			

In general the strain–displacement, force–strain, and equilibrium equations must be solved simultaneously to determine the forces in the beam. The analysis becomes simpler when the beam is orthotropic. When the beam is orthotropic the internal forces can be determined by substituting the replacement stiffnesses into the corresponding isotropic beam expressions. The replacement stiffnesses are discussed in Section 7.2.

7.1.5 Boundary Conditions

At a built-in end the displacement and the rotations of the cross section are zero. At a simply supported end the displacement and the moment are zero. At a free end the moments and the shear force are zero.

When the end of the beam is restrained so that it cannot move axially, the axial displacement is zero. When the end is not restrained axially, the axial force is zero. When warping of the cross section is restrained, the distortion is zero. When warping is not restrained, the bimoment is zero.

When the end may rotate, the torque is zero. When the end is rotationally restrained, the rate of twist is zero.

The preceding boundary conditions are summarized in Table 7.1.

7.2 Stiffnesses and Compliances of Beams

The stiffnesses that characterize orthotropic beams with shear deformation are shown in Table 7.2. For thin-walled beams these stiffnesses are given in Sections 6.2–6.6 except for the shear stiffnesses derived below in Sections 7.2.1 and 7.2.2. The stiffnesses of sandwich beams are presented in Section 7.2.3.

The shear stiffness of thick solid-section beams are not discussed here; they are given by Whitney.[3]

[3] J. M. Whitney, *Structural Analysis of Laminated Anisotropic Plates.* Technomic, Lancaster, Pennsylvania, 1987, p. 270.

Table 7.2. Stiffnesses of orthotropic beams with shear deformation

Tensile stiffness	$\widehat{EA}$	Sections 6.2–6.4
Bending stiffnesses	$\widehat{EI}_{zz}$, $\widehat{EI}_{yy}$, $\widehat{EI}_{yz}$	Sections 6.2–6.4
Torsional stiffness	$\widehat{GI}_{\mathrm{t}}$	Sections 6.5.1–6.6.2
Warping stiffness	$\widehat{EI}_{\omega}$	Section 6.6.4
Shear stiffnesses	$\widehat{S}_{yy}$, $\widehat{S}_{zz}$, $\widehat{S}_{\omega\omega}$, $\widehat{S}_{yz}$, $\widehat{S}_{y\omega}$, $\widehat{S}_{z\omega}$	Sections 7.2.1–7.2.3

7.2.1 Shear Stiffnesses and Compliances of Thin-Walled Open-Section Beams

To determine the shear stiffness and shear compliance matrices of thin-walled open-section beams we consider an element of length ΔL of a thin-walled open-section beam (Fig. 7.5). On the two faces of the element there are equal and opposite shear forces $\widehat{V}_y$, equal and opposite shear forces $\widehat{V}_z$, equal and opposite torques $\widehat{T}(= \widehat{T}_{\mathrm{sv}} + \widehat{T}_{\omega})$, and axial stresses σ_x, which are different at the two faces. We neglect the Saint-Venant torque; hence, $\widehat{T} = \widehat{T}_{\omega}$.

The shear stresses are represented by the shear flow q,

$$q = \widehat{V}_y q_y^* + \widehat{V}_z q_z^* + \widehat{T}_{\omega} q_{\omega}^*, \tag{7.41}$$

where q_y^* and q_z^* are shear flows (per unit force) and are calculated by setting either $\widehat{V}_y = 1$ and $\widehat{V}_z = 0$ or $\widehat{V}_y = 0$ and $\widehat{V}_z = 1$ in Eq. (6.281) and q_{ω}^* is the shear flow (per unit torque) introduced by a unit torque $\widehat{T}_{\omega}$; q_{ω}^* is evaluated by replacing Eh by $1/\alpha_{11}^{\varrho}$ (see page 264) in the expression of q_{ω}^* given[4] for the corresponding isotropic beams.

In the following we take into account only shear deformation caused by shear. Thus, the strain energy is (Eq. 2.200)

$$U = \frac{1}{2} \int \int \int (\tau_{\xi\eta} \gamma_{\xi\eta})\, d\zeta\, d\eta\, d\xi, \tag{7.42}$$

where ζ is the coordinate perpendicular to the wall, η is the coordinate along the circumference of the cross section, and $\xi = x$. We assume that the shear flow $q(= \int \tau_{\xi\eta} d\zeta)$ does not vary along the ΔL long element and the shear strain $\gamma_{\xi\eta}$ is uniform across the thickness of the wall. Thus, the strain energy is

$$U = \frac{1}{2} \int \left(\gamma_{\xi\eta}^{\mathrm{o}} q\right) d\eta\ \Delta L, \tag{7.43}$$

where $\gamma_{\xi\eta}^{\mathrm{o}}$ is the shear strain in the wall's reference surface and is related to the shear flow in the composite wall by $\gamma_{\xi\eta}^{\mathrm{o}} = \alpha_{66}^{\nu} q$ (see Eq. 6.195), where α_{66}^{ν} is given by $\alpha_{66}^{\nu} = \alpha_{66}^{\varrho} - \frac{(\beta_{66}^{\varrho})^2}{\delta_{66}^{\varrho}}$ (see Eq. 6.196). The superscript ϱ indicates that the compliances

[4] T. H. G. Megson, *Aircraft Structures for Engineering Students*. 3rd edition. Halsted Press, John Wiley & Sons, New York, 1999, pp. 465–472.

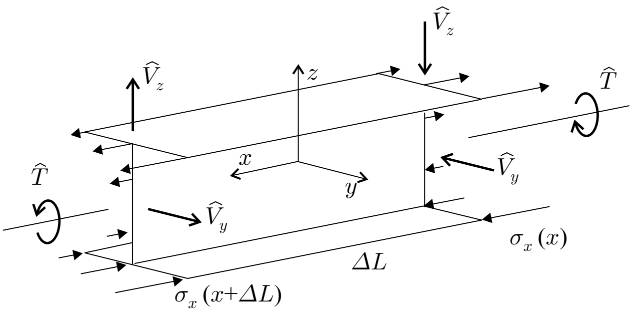


Figure 7.5: The loads applied to a thin-walled, open-section beam.

are evaluated in the ϱ reference plane defined in Figure 6.26. When this expression is used, Eq. (7.43) becomes

$$U = \frac{1}{2}\int \left(\alpha_{66}^{\nu} q^2\right) d\eta \Delta L. \tag{7.44}$$

The work done by the external forces due to shear deformation is

$$W = \underbrace{\frac{1}{2}\widehat{V}_y\gamma_y\Delta L}_{\text{due to displacement in the } y \text{ direction}} + \underbrace{\frac{1}{2}\widehat{V}_z\gamma_z\Delta L}_{\text{due to displacement in the } z \text{ direction}} + \underbrace{\frac{1}{2}\widehat{T}_\omega\vartheta^{\mathrm{S}}\Delta L}_{\text{due to the twist-induced displacement}}$$
$$+ \underbrace{\frac{1}{2}\left(\int\limits_{\text{at } x} \sigma_x u_x dA - \int\limits_{\text{at } x+\Delta L} \sigma_x u_x dA\right)}_{\text{due to warping (neglected)}}. \tag{7.45}$$

We neglect the work done by the axial stresses on the shear–force-induced warping.

The law of conservation of energy gives

$$U = W. \tag{7.46}$$

By introducing Eqs. (7.44), (7.45), (7.37), and (7.41) into Eq. (7.46), and by performing the algebra, we obtain

$$\begin{aligned}
&\widehat{V}_y^2\tfrac{1}{2}\widehat{s}_{yy} + \widehat{V}_z^2\tfrac{1}{2}\widehat{s}_{zz} + \widehat{T}_\omega^2\tfrac{1}{2}\widehat{s}_{\omega\omega} + \widehat{V}_y\widehat{V}_z\widehat{s}_{yz} + \widehat{V}_y\widehat{T}_\omega\widehat{s}_{y\omega} + \widehat{V}_z\widehat{T}_\omega\widehat{s}_{z\omega} = \\
&\quad \widehat{V}_y^2\tfrac{1}{2}\textstyle\int_{(S)}\alpha_{66}^{\nu}(q_y^*)^2 d\eta + \widehat{V}_z^2\tfrac{1}{2}\int_{(S)}\alpha_{66}^{\nu}(q_z^*)^2 d\eta + \widehat{T}_\omega^2\tfrac{1}{2}\int_{(S)}\alpha_{66}^{\nu}(q_\omega^*)^2 d\eta \\
&\qquad + \widehat{V}_y\widehat{V}_z\textstyle\int_{(S)}\alpha_{66}^{\nu}q_y^*q_z^* d\eta + \widehat{V}_y\widehat{T}_\omega\int_{(S)}\alpha_{66}^{\nu}q_y^*q_\omega^* d\eta + \widehat{V}_z\widehat{T}_\omega\int_{(S)}\alpha_{66}^{\nu}q_z^*q_\omega^* d\eta,
\end{aligned} \tag{7.47}$$

where S indicates integration around the entire circumference. This equation is

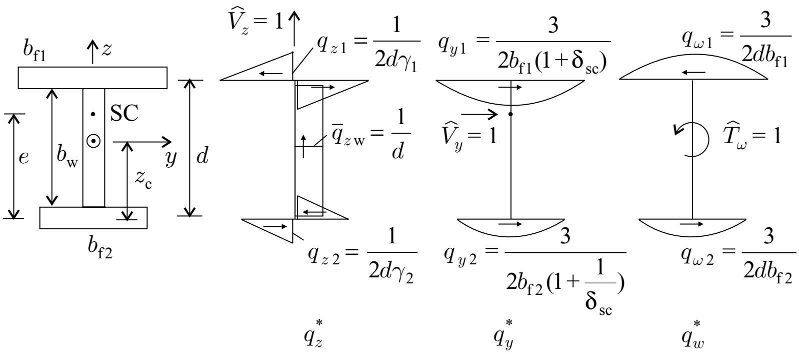


Figure 7.6: I-beam and the shear flows from unit shear forces in the z and y directions and from a unit torque.

valid for arbitrary values of $\widehat{V}_y$, $\widehat{V}_z$, and $\widehat{T}_\omega$. Hence, the elements of the shear compliance matrix are

$$\widehat{s}_{yy} = \int_{(S)} \alpha_{66}^{\nu}(q_y^*)^2 d\eta \qquad \widehat{s}_{zz} = \int_{(S)} \alpha_{66}^{\nu}(q_z^*)^2 d\eta \qquad \widehat{s}_{\omega\omega} = \int_{(S)} \alpha_{66}^{\nu}(q_\omega^*)^2 d\eta$$
$$\widehat{s}_{z\omega} = \int_{(S)} \alpha_{66}^{\nu} q_z^* q_\omega^* d\eta \qquad \widehat{s}_{y\omega} = \int_{(S)} \alpha_{66}^{\nu} q_y^* q_\omega^* d\eta \qquad \widehat{s}_{yz} = \int_{(S)} \alpha_{66}^{\nu} q_z^* q_y^* d\eta. \tag{7.48}$$

The stiffness is given by Eq. (7.38).

I-beam. We consider an I-beam symmetrical about the z-axis (Fig. 7.6). The layup of the web is symmetrical, although the layups of the top and bottom flanges may be unsymmetrical. Unit forces $\widehat{V}_z$ and $\widehat{V}_y$ and a unit torque $\widehat{T}_\omega$ are applied at the shear center.

The shear flows, resulting from unit shear forces $\widehat{V}_z = 1$ and $\widehat{V}_y = 1$ acting at the shear center and from a unit torque $\widehat{T} = \widehat{T}_\omega = 1$, are shown in Figure 7.6. Details of the calculations are not presented here. We merely note that the shear flow along the length of the web due to $\widehat{V}_z$ is taken to be constant. We now define γ_1, γ_2, δ_c, and δ_{sc} as follows:

$$\gamma_1 = 1 + \frac{1}{3}\frac{(\alpha_{11})_{f1}}{(\alpha_{11})_w}\frac{d}{b_{f1}\left(1+\frac{1}{\delta_c}\right)} \qquad \delta_c = \frac{d - z_c}{z_c}$$
$$\gamma_2 = 1 + \frac{1}{3}\frac{(\alpha_{11})_{f2}}{(\alpha_{11})_w}\frac{d}{b_{f2}(1+\delta_c)} \qquad \delta_{sc} = \frac{d-e}{e}, \tag{7.49}$$

where e is the location of the shear center, z_c is the location of the centroid, d is the distance between the midplanes of the flanges, and b_{f1} and b_{f2} are the widths of the flanges (Fig. 7.6.)

To calculate $\widehat{s}_{yy}$ we apply $\widehat{V}_y$ and perform the integration indicated in Eq. (7.48) for the web and for the top and bottom flanges. When only $\widehat{V}_y$ acts, there is no shear flow in the web (Fig. 7.6), and hence the integration needs to be performed only for the two flanges as follows:

$$\widehat{s}_{yy} = \int_{(S)} \alpha_{66}^{\nu} (q_y^*)^2 d\eta = \int_0^{b_{f1}} \left\{ (\alpha_{66}^{\nu})_{f1} \left[\frac{3}{2b_{f1}(1+\delta_{sc})} 4\frac{\eta}{b_{f1}} \left(1 - \frac{\eta}{b_{f1}}\right) \right]^2 \right\} d\eta$$

$$+ \int_0^{b_{f2}} \left\{ (\alpha_{66}^{\nu})_{f2} \left[\frac{3}{2b_{f2}\left(1+\frac{1}{\delta_{sc}}\right)} 4\frac{\eta}{b_{f2}} \left(1 - \frac{\eta}{b_{f2}}\right) \right]^2 \right\} d\eta. \tag{7.50}$$

By performing the integrations, we obtain

$$\widehat{s}_{yy} = 1.2 \left[\frac{(\alpha_{66}^{\nu})_{f1}}{b_{f1}(1+\delta_{sc})^2} + \frac{(\alpha_{66}^{\nu})_{f2}}{b_{f2}\left(1+\frac{1}{\delta_{sc}}\right)^2} \right], \tag{7.51}$$

where the subscripts f1 and f2 refer to the top and bottom flanges. The other elements of the shear compliance matrix are calculated similarly from Eq. (7.48) with the shear flows shown in Figure 7.6. The results are

$$\widehat{s}_{zz} = \int_{(S)} \alpha_{66}^{\nu} (q_z^*)^2 d\eta = \frac{(\alpha_{66}^{\nu})_{w}}{d} + \frac{1}{12} \frac{(\alpha_{66}^{\nu})_{f1} b_{f1}}{d^2 \gamma_1^2} + \frac{1}{12} \frac{(\alpha_{66}^{\nu})_{f2} b_{f2}}{d^2 \gamma_2^2} \tag{7.52}$$

$$\widehat{s}_{\omega\omega} = \int_{(S)} \alpha_{66}^{\nu} (q_\omega^*)^2 d\eta = \frac{1.2}{d^2} \left(\frac{(\alpha_{66}^{\nu})_{f1}}{b_{f1}} + \frac{(\alpha_{66}^{\nu})_{f2}}{b_{f2}} \right) \tag{7.53}$$

$$\widehat{s}_{y\omega} = \int_{(S)} \alpha_{66}^{\nu} q_y^* q_\omega^* d\eta = \frac{1.2}{d} \left(-\frac{(\alpha_{66}^{\nu})_{f1}}{b_{f1}(1+\delta_{sc})} + \frac{(\alpha_{66}^{\nu})_{f2}}{b_{f2}\left(1+\frac{1}{\delta_{sc}}\right)} \right) \tag{7.54}$$

$$\widehat{s}_{yz} = \int_{(S)} \alpha_{66}^{\nu} q_y^* q_z^* d\eta = 0 \qquad \widehat{s}_{z\omega} = \int \alpha_{66}^{\nu} q_z^* q_\omega^* d\eta = 0. \tag{7.55}$$

The stiffness is given by Eq. (7.38).

Arbitrary cross-section beam. The shear compliances of selected thin-walled composite beams[5] are presented in Tables A.8 and A.9. The shear stiffness matrix is determined by Eq. (7.38).

7.2.2 Shear Stiffnesses and Compliances of Thin-Walled Closed-Section Beams

When there is no restrained warping ($\widehat{T}_\omega = 0$, and consequently $\vartheta^{\mathrm{S}} = 0$, see Eq. 7.28), the shear-force–strain relationships (Eq. 7.36) become

$$\begin{Bmatrix} \widehat{V}_y \\ \widehat{V}_z \end{Bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} \\ \widehat{S}_{yz} & \widehat{S}_{zz} \end{bmatrix} \begin{Bmatrix} \gamma_y \\ \gamma_z \end{Bmatrix}. \tag{7.56}$$

[5] L. P. Kollár, Flexural-Torsional Vibration of Open Section Composite Beams with Shear Deformation. *International Journal of Solids and Structures*, Vol. 38, 7543–7558, 2001.

The shear compliance matrix is (see Eq. 7.38)

$$\begin{bmatrix} \widehat{s}_{yy} & \widehat{s}_{yz} \\ \widehat{s}_{yz} & \widehat{s}_{zz} \end{bmatrix} = \begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} \\ \widehat{S}_{yz} & \widehat{S}_{zz} \end{bmatrix}^{-1}. \tag{7.57}$$

The elements of the shear stiffness matrix of orthotropic thin-walled closed-section beams are obtained by reasoning similar to that used for open-section beams. Thus, from Eq. (7.48) we have

$$\widehat{s}_{yy} = \oint \alpha_{66}^{\nu}(q_y^*)^2 d\eta \qquad \widehat{s}_{zz} = \oint \alpha_{66}^{\nu}\,(q_z^*)^2\, d\eta \qquad \widehat{s}_{yz} = \oint \alpha_{66}^{\nu} q_z^* q_y^* d\eta, \tag{7.58}$$

where q_y^* and q_z^* are shear flows (per unit force) calculated by setting either $\widehat{V}_y = 1$ and $\widehat{V}_z = 0$ or $\widehat{V}_y = 0$ and $\widehat{V}_z = 1$. These shear flows are calculated according to the analysis of thin-walled closed-section beams (Section 6.7). Shear compliances of rectangular and circular cross-section thin-walled beams are given in Table A.7.

The shear stiffness are given by Eq. (7.38).

7.2.3 Stiffnesses of Sandwich Beams

Like sandwich plates (page 169), sandwich beams consist of a foam or honeycomb core covered by facesheets (Fig. 7.7). The core acts much like the web in an I-beam; it separates the two facesheets, resulting in high bending stiffness.

In this section we consider orthotropic sandwich beams consisting of an orthotropic core and orthotropic facesheets (Fig. 7.7).

To determine the shear stiffness in the x–z plane, we apply a shear force $\widehat{V}_z$. For an orthotropic sandwich plate the relationships between the shear forces (per unit length) and the shear strains are (Eqs. 5.15, 5.34)

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad \text{sandwich plate.} \tag{7.59}$$

The total shear force $\widehat{V}_z$ corresponds to bV_x in the plate (where b is the width), and $V_y = 0$ (only $\widehat{V}_z$ acts). Thus, we can apply the laminate plate theory expressions

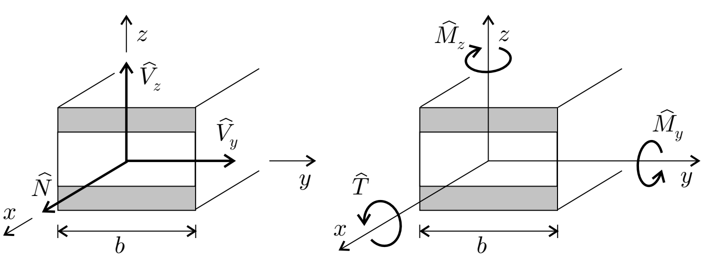


Figure 7.7: Sandwich beam.

to sandwich beams by making the following substitutions:

$$V_x = \frac{\widehat{V}_z}{b} \qquad V_y = 0, \tag{7.60}$$

where b is the width of the beam. The shear strain in the plate γ_{xz} corresponds to the transverse shear strain γ_z in the beam (Fig. 7.1). Thus, Eqs. (7.59) and (7.60) yield

$$\gamma_z = \left(\frac{1}{\widetilde{S}_{11}}\frac{1}{b}\right)\widehat{V}_z \qquad \begin{array}{c}\text{sandwich}\\ \text{beam.}\end{array} \tag{7.61}$$

The term in parentheses is the shear compliance of a sandwich beam in the x–z plane (see Eq. 7.37)

$$\widehat{s}_{zz} = \frac{1}{\widetilde{S}_{11} b}. \tag{7.62}$$

To determine the bending stiffness in the x–z plane, we again refer to a sandwich plate for which Eqs. (5.11) and (5.12) yield

$$\begin{Bmatrix} \epsilon_x^{\mathrm{o}} \\ \epsilon_y^{\mathrm{o}} \\ \gamma_{xy}^{\mathrm{o}} \end{Bmatrix} = [\alpha] \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + [\beta] \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{7.63}$$

$$\begin{Bmatrix} -\frac{\partial \chi_{xz}}{\partial x} \\ -\frac{\partial \chi_{yz}}{\partial y} \\ -\frac{\partial \chi_{xz}}{\partial y} - \frac{\partial \chi_{yz}}{\partial x} \end{Bmatrix} = [\beta]^T \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + [\delta] \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}, \tag{7.64}$$

where M_x, M_y, M_{xy}, N_x, N_y, N_{xy} are bending moments and in-plane forces per unit length and $[\delta]$, $[\beta]$, and $[\delta]$ are evaluated in the x, y, z coordinate system. We now apply only a bending moment $\widehat{M}_y$ to the sandwich beam (Fig. 7.7). The relationship between the bending moment in the plate M_x and the beam $\widehat{M}_y$ is (Eq. 6.20)

$$M_x = \frac{\widehat{M}_y}{b}. \tag{7.65}$$

Since only $\widehat{M}_y$ acts, we have

$$M_y = M_{xy} = N_x = N_y = N_{xy} = 0. \tag{7.66}$$

Equations (7.64), (7.65), and (7.66) yield ($\chi_{xz} = \chi_x$)

$$-\frac{\partial \chi_x}{\partial x} = \left(\frac{\delta_{11}}{b}\right)\widehat{M}_y. \tag{7.67}$$

We compare Eq. (7.67) with Eq. (7.32) and note that an orthotropic sandwich beam is symmetrical with respect to the z-axis and hence $\widehat{EI}_{yz}$ is zero. Thus, the bending stiffness in the x–z plane is

$$\widehat{EI}_{yy} = \frac{b}{\delta_{11}}. \tag{7.68}$$

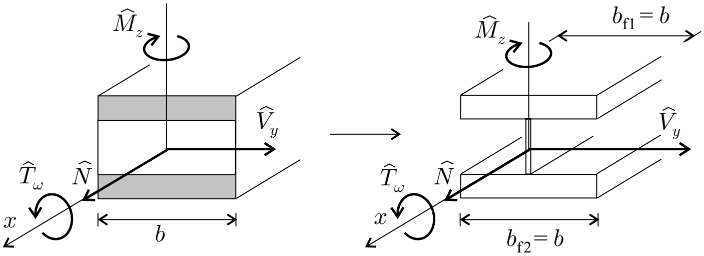


Figure 7.8: Sandwich beam and the equivalent I-beam.

The presence of the core has little effect (i) on the bending stiffnesses in the x–y plane, (ii) on the tensile stiffness, (iii) on the warping stiffness, and (iv) on every element but the $\widehat{s}_{zz}$ element of the shear compliance matrix. We utilize this observation and determine these properties by neglecting the core.

When the core is neglected, the sandwich beam behaves like an I-beam with a thin web (Fig. 7.8). We now use the expressions for an I-beam in Tables A.2, A.5, and A.8 and substitute b for both b_{f1} and b_{f2}, the superscripts t and b for subscripts f1 and f2, and set the terms referring to the web to zero. The resulting bending, tensile, and warping stiffnesses are

$$\widehat{EI}_{zz} = \frac{1}{\alpha_{11}^{t}}\frac{b^3}{12} + \frac{1}{\alpha_{11}^{b}}\frac{b^3}{12} \qquad \widehat{EA} = \frac{1}{\alpha_{11}^{t}}b + \frac{1}{\alpha_{11}^{b}}b \qquad \widehat{EI}_{\omega} = \frac{b^3\frac{1}{\alpha_{11}^{t}}}{12}ed, \tag{7.69}$$

where α_{11}^{t} and α_{11}^{b} are evaluated in the coordinate systems whose origins are at the "neutral" reference planes (which is at $\widetilde{\varrho}$, Eq. 6.105) in the top and bottom flanges, respectively. When the core thickness is so low as to prevent distortion of the cross section, the warping stiffness is lower than the value given in Eq. (7.69). A conservative estimate of the warping stiffness is its lower limit, which is $\widehat{EI}_{\omega} = 0$. The locations of the centroid z_c and the shear center e are (Fig. 7.9)

$$z_c = \frac{1}{\widehat{EA}}\left(\frac{1}{\alpha_{11}^{t}}bd\right) \qquad e = d\frac{\frac{1}{\alpha_{11}^{t}}}{\frac{1}{\alpha_{11}^{t}} + \frac{1}{\alpha_{11}^{b}}} = z_c. \tag{7.70}$$

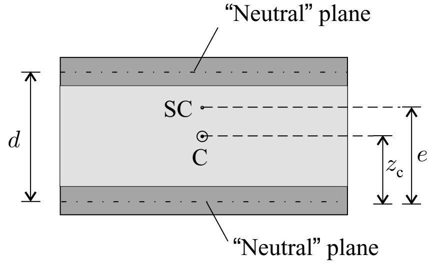


Figure 7.9: Location of the shear center and the centroid of a sandwich beam.

The shear compliances are

$$\widehat{s}_{yy} = \frac{1.2}{b}\left(\frac{\left(\alpha_{66}^{\nu}\right)^{\mathrm{t}}}{\left(1+\delta_{\mathrm{sc}}\right)^{2}} + \frac{\left(\alpha_{66}^{\nu}\right)^{\mathrm{b}}}{\left(1+\frac{1}{\delta_{\mathrm{sc}}}\right)^{2}}\right) \tag{7.71}$$

$$\widehat{s}_{\omega\omega} = \frac{1.2}{d^{2}b}\left(\left(\alpha_{66}^{\nu}\right)^{\mathrm{t}} + \left(\alpha_{66}^{\nu}\right)^{\mathrm{b}}\right) \tag{7.72}$$

$$\widehat{s}_{y\omega} = \frac{1.2}{db}\left(-\frac{\left(\alpha_{66}^{\nu}\right)^{\mathrm{t}}}{1+\delta_{\mathrm{sc}}} + \frac{\left(\alpha_{66}^{\nu}\right)^{\mathrm{b}}}{1+\frac{1}{\delta_{\mathrm{sc}}}}\right) \tag{7.73}$$

$$\widehat{s}_{yz} = 0 \qquad \widehat{s}_{z\omega} = 0, \tag{7.74}$$

where α_{66}^{ν} is given by Eq. (6.196) and $\delta_{\mathrm{sc}} = (d-e)/d$. The elements of the shear stiffness matrix are determined by substituting the elements of the shear compliance matrix into Eq. (7.38).

Next we present upper and lower bounds of the torsional stiffness of a sandwich beam. The upper bound is the torsional stiffness of a solid (not sandwich) beam with the same cross section as the sandwich beam (Eq. 6.169)

$$\widehat{GI}_{\mathrm{t}} = \frac{4b}{\delta_{66}} \qquad \text{upper bound.} \tag{7.75}$$

The lower bound of the torsional stiffness corresponds to the torsional stiffness of a sandwich beam in which the shear stiffness of the core in the y–z plane is neglected. In this case we need to consider only the two facesheets. The torsional stiffnesses of the facesheets are (Eq. 6.169) $4b/\delta_{66}^{\mathrm{t}}$ and $4b/\delta_{66}^{\mathrm{b}}$, where superscripts t and b refer to the top and bottom facesheets. The torsional stiffness of the sandwich beam with two facesheets is

$$\widehat{GI}_{\mathrm{t}} = \frac{4b}{\delta_{66}^{\mathrm{t}}} + \frac{4b}{\delta_{66}^{\mathrm{b}}} \qquad \text{lower bound.} \tag{7.76}$$

7.3 Transversely Loaded Beams

In this section we present the deflections of orthotropic beams, with cross sections symmetrical with respect to the x–z plane. A transverse load p_z (per unit length) acts in the plane of symmetry (Fig. 6.1). The relevant equilibrium equations are (Eq. 7.39)

$$\frac{d\widehat{V}_z}{dx} + p_z = 0 \tag{7.77}$$

$$\frac{d\widehat{M}_y}{dx} - \widehat{V}_z = 0, \tag{7.78}$$

where $\widehat{V}_z$ is the shear force, and $\widehat{M}_y$ is the bending moment in the x–z plane. For beams with symmetrical cross sections, $\widehat{EI}_{yz} = 0$, $\widehat{S}_{yz} = 0$, and $\widehat{S}_{z\omega} = 0$, and

Eqs. (7.32) and (7.36) give

$$\widehat{M}_y = -\widehat{EI}_{yy}\frac{d\chi_z}{dx} \qquad \widehat{V}_z = \widehat{S}_{zz}\gamma_z. \tag{7.79}$$

For simplicity, we introduce the following notations:

$$\begin{aligned} \widehat{M}_y &\equiv \widehat{M} & \widehat{V}_z &\equiv \widehat{V} \\ \widehat{EI}_{yy} &\equiv \widehat{EI} & \widehat{S}_{zz} &\equiv \widehat{S} \\ \chi_z &\equiv \chi & \gamma_z &\equiv \gamma \\ p_z &= p. \end{aligned} \tag{7.80}$$

With these notations we have

$$\widehat{M} = -\widehat{EI}\frac{d\chi}{dx} \qquad \widehat{V} = \widehat{S}\gamma. \tag{7.81}$$

Equations (7.78) and (7.81) yield

$$\gamma = -\frac{\widehat{EI}}{\widehat{S}}\frac{d^2\chi}{dx^2}. \tag{7.82}$$

By combining Eqs. (7.77), (7.78), (7.79) and by using Eq. (7.2), we obtain

$$-\widehat{EI}\frac{d^3\chi}{dx^3} + p = 0 \tag{7.83}$$

$$\widehat{EI}\frac{d^2\chi}{dx^2} + \widehat{S}\left(\frac{dw}{dx} - \chi\right) = 0. \tag{7.84}$$

Equations (7.83) and (7.84), together with the boundary conditions listed in Table 7.1 (page 321), provide the deflection w and the rotation of the cross section χ.

Known bending moment distribution. When the bending moment distribution along the beam is known, the deflection may be obtained by the following procedure. The deflection is due to both bending and shear deformation

$$w = w^{\mathrm{B}} + w^{\mathrm{S}}. \tag{7.85}$$

We recall that the deflection is related to the rotation of the cross section and to the shear strain by (Eq. 7.2)

$$\frac{dw}{dx} = \chi + \gamma. \tag{7.86}$$

We combine the two preceding equations and write

$$\frac{dw}{dx} = \frac{dw^{\mathrm{B}}}{dx} + \frac{dw^{\mathrm{S}}}{dx} = \chi + \gamma. \tag{7.87}$$

Thus, we have

$$\frac{dw^{\mathrm{B}}}{dx} = \chi \qquad \frac{dw^{\mathrm{S}}}{dx} = \gamma. \tag{7.88}$$

Substitution of the first of these expressions into Eq. (7.79) gives

$$\frac{d^2w^{\mathrm{B}}}{dx^2} = -\frac{\widehat{M}}{\widehat{EI}}. \tag{7.89}$$

From the second expression in Eq. (7.88) we obtain

$$w^{S} = \int \gamma\, dx. \tag{7.90}$$

By replacing γ with $\widehat{V}/\widehat{S}$ (Eq. 7.79) and by performing the integration, we have

$$w^{S} = \int \frac{\widehat{V}}{\widehat{S}} dx = C_o + \frac{\widehat{M}}{\widehat{S}}, \tag{7.91}$$

where C_o is an integration constant. The deflections w^B and w^S are now determined by the following steps:

1. For the known bending moment $\widehat{M}$, the deflection w^B is determined for a beam without shear deformation by using the expression for the corresponding isotropic beam and by replacing the isotropic bending stiffness EI by $\widehat{EI}$.
2. The deflection w^S is calculated by the expression

$$w^{S} = C_o + \frac{\widehat{M}}{\widehat{S}}. \tag{7.92}$$

The constant C_o is determined from the boundary condition at the support where the deflection w is zero ($w^S = w^B = 0$). Thus, C_o is

$$C_o = - \left.\frac{\widehat{M}}{\widehat{S}}\right|_{\text{where } w=0}. \tag{7.93}$$

3. The total deflection is

$$w = w^B + w^S. \tag{7.94}$$

Maximum deflections of selected beams are given in Table 7.3.

Sandwich beams with arbitrary layup. The preceding analysis for orthotropic beams can be extended to transversely loaded nonorthotropic sandwich beams provided that the beam is free to rotate about its axis. An important practical example is a beam built-in at one end and free at the other end (cantilever beam).

The internal forces arising from the applied loads are $\widehat{M}_y$ and $\widehat{V}_z$ (Fig. 7.10). For sandwich plates the relationships between the shear forces per unit length and the shear strains are (Eq. 5.15)

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{12} & \widetilde{S}_{22} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \qquad \begin{matrix}\text{sandwich} \\ \text{plate.}\end{matrix} \tag{7.95}$$

Equation (7.95) with $V_y = 0$ yields

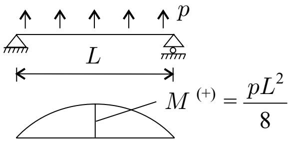
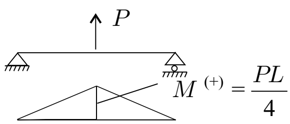
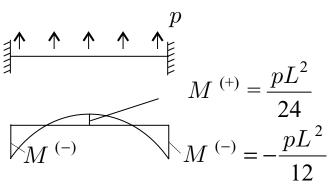
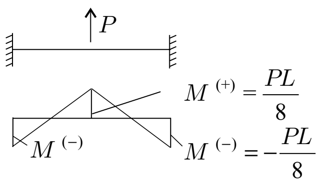
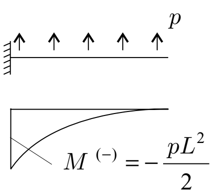
$$V_x = \left(\frac{\widetilde{S}_{11}\widetilde{S}_{22} - \widetilde{S}_{12}^2}{\widetilde{S}_{22}} \right) \gamma_{xz}. \tag{7.96}$$

The shear forces in a sandwich beam are (Eq. 7.60)

$$V_x = \frac{\widehat{V}_z}{b} \qquad V_y = 0, \tag{7.97}$$

where b is the width of the beam.

Table 7.3. Bending moments and maximum deflections of beams with bending stiffness $\widehat{EI}$ and shear stiffness $\widehat{S}$

Geometry, loading, bending moment	Maximum deflection $\widetilde{w} = \widetilde{w}^{\mathrm{B}} + \widetilde{w}^{\mathrm{S}}$	$\frac{\widetilde{w}^{\mathrm{B}}}{\widetilde{w}^{\mathrm{S}}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{5}{384}\frac{pL^4}{\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{pL^2}{8\widehat{S}}$	$\frac{5}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{PL^3}{48\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{PL}{4\widehat{S}}$	$\frac{4}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{1}{384}\frac{pL^4}{\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{pL^2}{8\widehat{S}}$	$\frac{1}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{PL^3}{192\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{PL}{4\widehat{S}}$	$\frac{1}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{1}{8}\frac{pL^4}{\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{pL^2}{2\widehat{S}}$	$\frac{12}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$
	$\widetilde{w}^{\mathrm{B}} = \frac{PL^3}{3\widehat{EI}}$ $\widetilde{w}^{\mathrm{S}} = \frac{PL}{\widehat{S}}$	$\frac{16}{48}\frac{L^2\widehat{S}}{\widehat{EI}}$

The shear strain in the plate γ_{xz} corresponds to the transverse shear strain γ_z in the beam, and $\widehat{V}_z = V_x b$. Thus, we have for a beam

$$\widehat{V}_z = \left(\frac{\widetilde{S}_{11}\widetilde{S}_{22} - \widetilde{S}_{12}^2}{\widetilde{S}_{22}} b\right)\gamma_z \qquad \text{sandwich beam.} \tag{7.98}$$

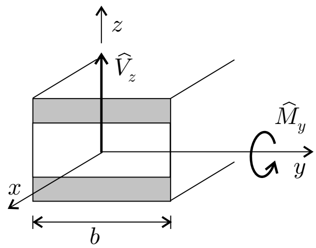


Figure 7.10: Sandwich beam subjected to bending and shear.

The term in parentheses is the shear stiffness (Eq. 7.81):

$$\widehat{S} = \frac{\widetilde{S}_{11}\widetilde{S}_{22} - \widetilde{S}_{12}^2}{\widetilde{S}_{22}} b. \tag{7.99}$$

where $\widetilde{S}_{11}$, $\widetilde{S}_{22}$ and $\widetilde{S}_{12}$ are given by Eq. (5.32).

To determine the bending stiffness in the x–z plane we again start the analysis with sandwich plates. From Eqs. (5.11) and (5.12), for a sandwich plate we have

$$\left\{\begin{array}{c} -\frac{\partial \chi_{xz}}{\partial x} \\ -\frac{\partial \chi_{yz}}{\partial y} \\ -\frac{\partial \chi_{xz}}{\partial y} - \frac{\partial \chi_{yz}}{\partial x} \end{array}\right\} = [\delta] \left\{\begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array}\right\} + [\beta] \left\{\begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array}\right\} \tag{7.100}$$

$$\left\{\begin{array}{c} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{array}\right\} = [\beta] \left\{\begin{array}{c} M_x \\ M_y \\ M_{xy} \end{array}\right\} + [\alpha] \left\{\begin{array}{c} N_x \\ N_y \\ N_{xy} \end{array}\right\}, \tag{7.101}$$

where M_x, M_y, M_{xy}, N_x, N_y, N_{xy} are moments and forces per unit length, and $[\delta]$, $[\beta]$ and $[\delta]$ are evaluated in the x, y, z coordinate system. We now apply a bending moment $\widehat{M}_y$ to the sandwich beam. The relationship between the bending moment in the sandwich plate M_x and the sandwich beam $\widehat{M}_y$ is (Eq. 6.20)

$$M_x = \frac{\widehat{M}_y}{b}. \tag{7.102}$$

Since only $\widehat{M}_y$ acts, we have

$$M_y = M_{xy} = N_x = N_y = N_{xy} = 0. \tag{7.103}$$

From Eqs. (7.100), (7.102), and (7.103), $\partial \chi_x / \partial x = (\partial \chi_{xz}/\partial x)$ is

$$-\frac{\partial \chi_x}{\partial x} = \left(\frac{\delta_{11}}{b}\right) \widehat{M}_y. \tag{7.104}$$

By comparing Eq. (7.104) with Eq. (7.81), we see that the term in parentheses is the inverse of the replacement bending stiffness:

$$\widehat{EI} = \frac{b}{\delta_{11}}. \tag{7.105}$$

The deflections in the x–z plane are calculated by the three steps described on page 331 with $\widehat{S}$ given by Eq. (7.99) and $\widehat{EI}$ by Eq. (7.105).

We note that a nonorthotropic sandwich beam under the action of a bending moment in the x–z plane may also bend in the x–y plane and may also twist.

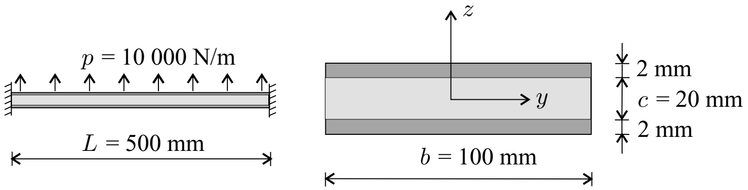


Figure 7.11: The sandwich beam in Example 7.1

7.1 Example. *An $L = 0.5$-m-long and $b = 0.01$-m-wide rectangular sandwich beam is made of a 0.02-m-thick core covered on both sides by graphite epoxy facesheets (Fig. 7.11). The material properties are given in Table 3.6 (page 81). The layup of each facesheet is $[\pm 45^{\mathrm{f}}_{2}/0_{12}/\pm 45^{\mathrm{f}}_{2}]$. The thickness of each facesheet is 0.002 m. The beam, built-in at both ends, is subjected to a uniformly distributed load $p =$ 10 000 N/m. Calculate the maximum deflection of the beam. The core is isotropic ($E_{\mathrm{c}} = 2 \times 10^6$ kN/m^2, $\nu_{\mathrm{c}} = 0.3$).*

Solution. The bending compliance matrix is (see Eq. 5.53)

$$[d] = \begin{bmatrix} 52.16 & 7.96 & 0 \\ 7.96 & 11.71 & 0 \\ 0 & 0 & 8.76 \end{bmatrix}^{-1} 10^{-3} = \begin{bmatrix} 21.39 & -14.55 & 0 \\ -14.55 & 95.31 & 0 \\ 0 & 0 & 114.21 \end{bmatrix} 10^{-6} \frac{1}{\mathrm{N \cdot m}}. \tag{7.106}$$

For symmetrical layup $[\delta] = [d]$, and the bending stiffness is (Eq. 7.68)

$$\widehat{EI}_{yy} = \frac{b}{\delta_{11}} = \frac{0.1}{21.39 \times 10^{-6}} = 4\,675\ \mathrm{N \cdot m^2}. \tag{7.107}$$

From Eq. (7.62), (with $\widetilde{S}_{11} = 18.615 \times 10^6 \frac{\mathrm{N}}{\mathrm{m}}$, Eq. 5.54) we have

$$\widehat{s}_{zz} = \frac{1}{\widetilde{S}_{11} b} = 537.17 \times 10^{-9} \frac{1}{\mathrm{N}}. \tag{7.108}$$

The beam is orthotropic, and $\widehat{s}_{yz} = \widehat{s}_{z\omega} = 0$ (Eq. 7.74). Thus, from Eqs. (7.38) and (7.108) we have

$$\widehat{S}_{zz} = \frac{1}{\widehat{s}_{zz}} = 1\,862 \times 10^3\ \mathrm{N}. \tag{7.109}$$

From Table 7.3, third row (page 332), with $\widehat{EI} = \widehat{EI}_{yy}$ and $\widehat{S} = \widehat{S}_{zz}$, the maximum deflection is

$$\widetilde{w} = \frac{1}{384} \frac{pL^4}{\widehat{EI}} + \frac{pL^2}{8\widehat{S}} = \underbrace{0.000\,348}_{\substack{\text{without shear} \\ \text{deformation}}} + 0.000\,168 = 0.000\,516\ \mathrm{m} = 0.52\ \mathrm{mm}.$$

7.4 Buckling of Beams

In this section we present the buckling loads of orthotropic beams subjected to either axial or transverse loads.

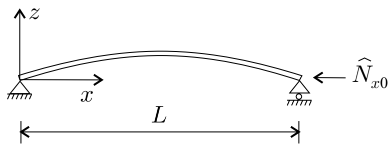


Figure 7.12: Buckling in the x–z plane.

7.4.1 Axially Loaded Beams with Doubly Symmetrical Cross Sections (Flexural and Torsional Buckling)

The beam is orthotropic and its cross section is symmetrical with respect to both the y- and the z-axes ($\widehat{EI}_{yz} = 0$). A compressive axial load $\widehat{N}_{x0}$ is applied at the centroid of the beam (Fig. 7.12). We wish to determine the axial load at which the beam buckles.

When the cross section of the beam has two axes of symmetry (doubly symmetrical cross section), it may buckle (a) in one of the planes of symmetry, (b) in the other plane of symmetry, (c) torsionally (Fig. 6.84). Accordingly, there are three sets of buckling loads

$$\widehat{N}_{\text{cr}1} = \widehat{N}_{\text{cr}y} \qquad \widehat{N}_{\text{cr}2} = \widehat{N}_{\text{cr}z} \qquad \widehat{N}_{\text{cr}3} = \widehat{N}_{\text{cr}\psi}, \tag{7.110}$$

where $\widehat{N}_{\text{cr}y}$ and $\widehat{N}_{\text{cr}z}$ correspond to buckling in the x–z and x–y planes and $\widehat{N}_{\text{cr}\psi}$ to torsional buckling (in which case the axis of the beam rotates but does not bend).

Buckling in the x–z plane. We consider a beam of length L that buckles in the x–z plane (Fig. 7.12). The equilibrium equations are identical to those of beams without shear deformation and are[6]

$$\frac{d\widehat{V}_z}{dx} - \widehat{N}_{x0}\frac{d^2 w}{dx^2} = 0 \tag{7.111}$$

$$\frac{d\widehat{M}_y}{dx} - \widehat{V}_z = 0, \tag{7.112}$$

where w is the deflection of the beam's axis in the x–z symmetry plane, $\widehat{V}_z$ is the shear force, and $\widehat{M}_y$ is the bending moment. By substituting Eq. (7.79) into (7.111) and (7.112), and by using Eq. (7.2), we obtain the following equilibrium equations for a composite beam:

$$-\widehat{EI}_{yy}\frac{d^3\chi_z}{dx^3} - \widehat{N}_{x0}\frac{d^2 w}{dx^2} = 0 \tag{7.113}$$

$$-\widehat{EI}_{yy}\frac{d^2\chi_z}{dx^2} - \widehat{S}_{zz}\left(\frac{dw}{dx} - \chi_z\right) = 0. \tag{7.114}$$

[6] S. P. Timoshenko and J. Gere, *Theory of Elastic Stability.* 2nd edition. McGraw-Hill, New York, 1961, p. 3.

With the notation $\widehat{EI}_{yy} \equiv \widehat{EI}$, $\widehat{S}_{zz} \equiv \widehat{S}$, $\chi_z \equiv \chi$, these equations may be rearranged to yield

$$-\widehat{EI}\frac{d^3\chi}{dx^3} - \widehat{N}_{x0}\left(\frac{d\chi}{dx} - \frac{\widehat{EI}}{\widehat{S}}\frac{d^3\chi}{dx^3}\right) = 0 \tag{7.115}$$

$$\frac{dw}{dx} = \chi - \frac{\widehat{EI}}{\widehat{S}}\frac{d^2\chi}{dx^2}. \tag{7.116}$$

Solution of the first of these equations is[7]

$$\chi = C_1 \sin\frac{\mu}{L}x + C_2 \cos\frac{\mu}{L}x + C_3. \tag{7.117}$$

We substitute this expression into Eq. (7.116) and integrate the equation. The result is

$$w = \left(-C_1 \cos\frac{\mu}{L}x + C_2 \sin\frac{\mu}{L}x\right)\frac{L}{\mu}\left(1 + \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\right) + C_3 x + C_4, \tag{7.118}$$

where C_1–C_4 are constants determined from the boundary conditions and μ is a parameter defined as

$$\mu = L\sqrt{\frac{1}{\widehat{EI}}\frac{1}{\frac{1}{\widehat{N}_{x0}} - \frac{1}{\widehat{S}}}}. \tag{7.119}$$

Equations (7.117)–(7.119) together with the boundary conditions are used to determine the buckling load. In the following we derive the buckling loads of beams built-in at both ends and of beams built-in at one end and simply supported at the other end.

Beams with both ends built in. When both ends of the beam are built-in, the displacement and the rotation of the cross section at each end are zero (Table 7.1, page 321) as follows:

$$w(L) = 0 \qquad \chi(L) = 0 \qquad w(0) = 0 \qquad \chi(0) = 0. \tag{7.120}$$

By substituting w and χ (given by Eqs. 7.117 and 7.118) into these boundary conditions, we obtain

$$\begin{bmatrix} -\Phi\cos\mu & \Phi\sin\mu & L & 1 \\ \sin\mu & \cos\mu & 1 & 0 \\ -\Phi & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{7.121}$$

where Φ is defined as

$$\Phi = \frac{L}{\mu}\left(1 + \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\right). \tag{7.122}$$

When the beam buckles, the deflection is nonzero and, hence, at least one of the constants C_1–C_4 must be nonzero. Thus, at buckling, Eq. (7.121) is satisfied

[7] E. Kreyszig, *Advanced Engineering Mathematics.* 7th edition. John Wiley & Sons, New York, 1993, pp. 136–138.

only when the determinant of the matrix is zero, which is a requirement that is met when

$$2\Phi(1-\cos\mu) - L\sin\mu = 0. \tag{7.123}$$

This equation is satisfied when μ is

$$\mu = \mu_i = 2i\pi, \tag{7.124}$$

where i is an integer ($i = 1, 2, 3, \dots$). With these values of μ_i, Eq. (7.119) gives the buckling loads ($\widehat{N}_{x0} = \widehat{N}_{\text{cr}yi}$)

$$\frac{1}{\widehat{N}_{\text{cr}yi}} = \frac{\left(\frac{L}{2i}\right)^2}{\pi^2 \widehat{EI}} + \frac{1}{\widehat{S}}. \tag{7.125}$$

We are interested in the lowest value of the buckling load. For the built-in beam under consideration the lowest value of $\widehat{N}_{\text{cr}yi}$ is obtained when $i = 1$. This value is denoted by $\widehat{N}_{\text{cr}y}$

$$\frac{1}{\widehat{N}_{\text{cr}y}} = \frac{1}{\frac{\pi^2 \widehat{EI}}{\left(\frac{L}{2}\right)^2}} + \frac{1}{\widehat{S}}. \tag{7.126}$$

The first term in this expression is the inverse of the buckling load of a beam without shear deformation (Eq. 6.337, Table 6.11, page 293):

$$\widehat{N}^{\text{B}}_{\text{cr}y} = \frac{\pi^2 \widehat{EI}}{\left(\frac{L}{2}\right)^2}. \tag{7.127}$$

We now can write Eq. (7.126) as

$$\widehat{N}_{\text{cr}y} = \frac{1}{\frac{1}{\widehat{N}^{\text{B}}_{\text{cr}y}} + \frac{1}{\widehat{S}}}. \tag{7.128}$$

This equation shows that the buckling load decreases with decreasing shear stiffness and, correspondingly, with increasing shear deformation.

Beams with one end built-in the other end simply supported. At the built-in end ($x = L$) the deflection w and the rotation χ are zero (Table 7.1, page 321); at the simply supported end ($x = 0$), the deflection w and the first derivative of χ are zero

$$w(L) = 0 \qquad \chi(L) = 0 \qquad w(0) = 0 \qquad \frac{d\chi(0)}{dx} = 0. \tag{7.129}$$

By substituting w and χ (given by Eqs. 7.117 and 7.118) into these boundary conditions, we obtain

$$\begin{bmatrix} -\Phi\cos\mu & \Phi\sin\mu & L & 1 \\ \sin\mu & \cos\mu & 1 & 0 \\ -\Phi & 0 & 0 & 1 \\ \mu & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{7.130}$$

where Φ is given by Eq. (7.122). When the beam buckles, the deflection is nonzero, and, hence, at least one of the constants C_1–C_4 must be nonzero. Thus, at buckling, Eq. (7.130) is satisfied only when the determinant of the matrix is zero. This requirement is met when

$$\left(1 + \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\right)\tan\mu = \mu. \tag{7.131}$$

The values of μ that satisfy this equation are denoted by μ_i $(i = 1, 2, \ldots)$. The values of μ_i must be obtained numerically. The buckling loads $\widehat{N}_{\mathrm{cr}yi} (= \widehat{N}_{x0})$ are obtained from Eq. (7.119) and are expressed as

$$\frac{1}{\widehat{N}_{\mathrm{cr}yi}} = \frac{1}{\frac{\mu_i^2 \widehat{EI}}{L^2}} + \frac{1}{\widehat{S}}. \tag{7.132}$$

The lowest value of μ_i results in the buckling load. The lowest value of μ_i may be approximated by setting $(1 + \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2)$ equal to unity in Eq. (7.131). With this approximation the lowest value of μ_i is $\mu_1 = \pi/0.70$, and the buckling load becomes

$$\frac{1}{\widehat{N}_{\mathrm{cr}y}} \simeq \frac{1}{\frac{\pi^2 \widehat{EI}}{(0.7L)^2}} + \frac{1}{\widehat{S}}. \tag{7.133}$$

This expression overestimates the buckling load by less than 6 percent.

The first term in the expression above is the inverse of the buckling load of a beam without shear deformation (see Eq. 6.337 and Table 6.11):

$$\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y} = \frac{\pi^2 \widehat{EI}}{(0.7L)^2}. \tag{7.134}$$

We now write

$$\widehat{N}_{\mathrm{cr}y} \simeq \frac{1}{\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y}} + \frac{1}{\widehat{S}}}. \tag{7.135}$$

This equation shows that the buckling load decreases with decreasing shear stiffness and, correspondingly, with increasing shear deformation.

Different types of end supports. Equations (7.128) and (7.135) show that for beams with one end built-in and the other simply supported or with both ends built-in, the buckling load in the x–z plane may be written in the form $(\widehat{S} = \widehat{S}_{zz})$ as

$$\widehat{N}_{\mathrm{cr}y} = \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y}} + \frac{1}{\widehat{S}_{zz}}\right)^{-1}, \tag{7.136}$$

where $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y}$ is the buckling load of the beam without shear deformation.

By analyses similar to those presented above, it can be shown that this expression is also applicable to beams with both ends simply supported or with one end built-in and the other end free. In applying Eq. (7.136), $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y}$ is calculated by

Eq. (6.337) with the k values listed in Table 6.11 (page 293). Equation (7.136) is accurate for cases (a), (c), and (d) in Table 6.11 and may overestimate the buckling load by up to 6 percent for case (b).

Buckling in the x–y plane. We express the buckling load in the x–y plane similarly to the buckling load in the x–z plane as

$$\widehat{N}_{\mathrm{cr}z} = \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}z}} + \frac{1}{\widehat{S}_{yy}} \right)^{-1}, \tag{7.137}$$

where $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}z}$ is the buckling load for the beam without shear deformation (see Eq. 6.337).

Torsional buckling. We consider a doubly symmetrical cross-section beam subjected to an axial load $\widehat{N}_{x0}$. Under this load the beam may rotate about the axis. For such a beam the equilibrium equation is[8]

$$\frac{d\widehat{T}}{dx} - \widehat{N}_{x0} i_\omega^2 \frac{d^2\psi}{dx^2} = 0, \tag{7.138}$$

where ψ is the twist of the cross section about the axis, $\widehat{T}$ is the torque, and i_ω is the polar radius of gyration defined by Eq. (6.340),

$$i_\omega^2 = z_{\mathrm{sc}}^2 + y_{\mathrm{sc}}^2 + \frac{\widehat{EI}_{zz} + \widehat{EI}_{yy}}{\widehat{EA}}, \tag{7.139}$$

and $\widehat{T}$ is given by Eq. (7.14) as

$$\widehat{T} = \widehat{T}_{\mathrm{sv}} + \widehat{T}_\omega. \tag{7.140}$$

From Eq. (7.40) we have

$$\frac{d\widehat{M}_\omega}{dx} - \widehat{T}_\omega = 0. \tag{7.141}$$

The force–strain relationships for a beam with doubly symmetrical cross section are (Eqs. 7.32, 7.34, 7.36)

$$\widehat{M}_\omega = -\widehat{EI}_\omega \frac{d\vartheta^{\mathrm{B}}}{dx} \qquad \widehat{T}_{\mathrm{sv}} = \widehat{GI}_{\mathrm{t}} \vartheta \qquad \widehat{T}_\omega = \widehat{S}_{\omega\omega} \vartheta^{\mathrm{S}}, \tag{7.142}$$

where ϑ, and ϑ^{S} are (Eqs. 7.6 and 7.7)

$$\vartheta = \frac{d\psi}{dx} \qquad \vartheta^{\mathrm{S}} = \frac{d\psi}{dx} - \vartheta^{\mathrm{B}}. \tag{7.143}$$

Equations (7.138)–(7.143) may be combined to yield

$$\widehat{S}_{\omega\omega} \left(\frac{d^2\psi}{dx^2} - \frac{d\vartheta^{\mathrm{B}}}{dx} \right) + \widehat{GI}_{\mathrm{t}} \frac{d^2\psi}{dx^2} - \widehat{N}_{x0} i_\omega^2 \frac{d^2\psi}{dx^2} = 0 \tag{7.144}$$

$$-\widehat{EI}_\omega \frac{d^2\vartheta^{\mathrm{B}}}{dx^2} - \widehat{S}_{\omega\omega} \left(\frac{d\psi}{dx} - \vartheta^{\mathrm{B}} \right) = 0. \tag{7.145}$$

[8] S. P. Timoshenko and J. Gere, *Theory of Elastic Stability.* 2nd edition. New York, McGraw-Hill, 1961, p. 227, Eqs. (b) and (d).

After algebraic manipulations these equations become
torsional buckling:

$$-\widehat{EI}_{\omega}\frac{d^3\vartheta^{\mathrm{B}}}{dx^3}-\left(\widehat{N}_{x0}i_{\omega}^2-\widehat{GI}_{\mathrm{t}}\right)\left(\frac{d\vartheta^{\mathrm{B}}}{dx}-\frac{\widehat{EI}}{\widehat{S}_{\omega\omega}}\frac{d^3\vartheta^{\mathrm{B}}}{dx^3}\right)=0 \tag{7.146}$$

$$\frac{d\psi}{dx}=\vartheta^{\mathrm{B}}-\frac{\widehat{EI}}{\widehat{S}_{\omega\omega}}\frac{d^2\vartheta^{\mathrm{B}}}{dx^2}. \tag{7.147}$$

The corresponding equations for buckling in the x–z plane were derived previously (Eqs. 7.115 and 7.116) and, for convenience, are repeated below
in-plane buckling:

$$-\widehat{EI}\frac{d^3\chi}{dx^3}-\widehat{N}_{x0}\left(\frac{d\chi}{dx}-\frac{\widehat{EI}}{\widehat{S}}\frac{d^3\chi}{dx^3}\right)=0 \tag{7.148}$$

$$\frac{dw}{dx}=\chi-\frac{\widehat{EI}}{\widehat{S}}\frac{d^2\chi}{dx^2}. \tag{7.149}$$

It is readily seen that these two equations are identical to Eqs. (7.146) and (7.147) when the following substitutions are made:

$$\begin{array}{rl} \text{In-plane buckling} & \text{Torsional buckling} \\ \widehat{N}_{x0} \Longrightarrow & \widehat{N}_{x0}i_{\omega}^2-\widehat{GI}_{\mathrm{t}} \\ \widehat{EI} \Longrightarrow & \widehat{EI}_{\omega} \\ \widehat{S} \Longrightarrow & \widehat{S}_{\omega\omega} \\ w \Longrightarrow & \psi \\ \chi \Longrightarrow & \vartheta^{\mathrm{B}}. \end{array} \tag{7.150}$$

For in-plane buckling the load $\widehat{N}_{x0}$ at which the beam buckles is denoted by $\widehat{N}_{\mathrm{cr}y}$. For torsional buckling, the load at which the beam buckles is denoted by $\widehat{N}_{\mathrm{cr}\psi}$. Thus, the solutions for buckling in the x–z plane (Eq. 7.136) are applicable to torsional buckling with the following substitutions:

$$\begin{array}{rl} \text{In-plane buckling} & \text{Torsional buckling} \\ \widehat{N}_{\mathrm{cr}y} \Longrightarrow & \widehat{N}_{\mathrm{cr}\psi}i_{\omega}^2-\widehat{GI}_{\mathrm{t}} \\ \widehat{EI} \Longrightarrow & \widehat{EI}_{\omega} \\ \widehat{S} \Longrightarrow & \widehat{S}_{\omega\omega}. \end{array} \tag{7.151}$$

The parameters w and χ are not included here because they do not appear in the expression for the buckling load.

From Eqs. (7.136) and (6.337) we obtain

$$\frac{1}{\widehat{N}_{\mathrm{cr}y}}=\frac{1}{\frac{\pi^2\widehat{EI}_{yy}}{(kL)^2}}+\frac{1}{\widehat{S}_{zz}}. \tag{7.152}$$

By replacing $\widehat{N}_{\mathrm{cr}y}$, $\widehat{EI}$, and $\widehat{S}_{zz}$ $(=\widehat{S})$ with the expressions in Eq. (7.151), we have

$$\frac{1}{\widehat{N}_{\mathrm{cr}\psi}i_{\omega}^2-\widehat{GI}_{\mathrm{t}}}=\frac{1}{\frac{\pi^2\widehat{EI}_{\omega}}{(kL)^2}}+\frac{1}{\widehat{S}_{\omega\omega}}, \tag{7.153}$$

where $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega}$ is defined as in Eq.(6.338)

$$\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega} = \frac{1}{i_\omega^2}\frac{\pi^2 \widehat{EI}_\omega}{(kL)^2}. \tag{7.154}$$

Equations (7.153) and (7.154) may be rearranged to yield

$$\widehat{N}_{\mathrm{cr}\psi} = \widehat{N}_{\mathrm{cr}\omega} + \frac{1}{i_\omega^2}\widehat{GI}_{\mathrm{t}}, \tag{7.155}$$

where $\widehat{N}_{\mathrm{cr}\omega}$ is defined as

$$\widehat{N}_{\mathrm{cr}\omega} = \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega}} + \frac{1}{\frac{1}{i_\omega^2}\widehat{S}_{\omega\omega}}\right)^{-1}, \tag{7.156}$$

and where $\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega}$ is the buckling load without shear deformation given by Eq. (6.338) with the k values presented in Table 6.11 (page 293). Equation (7.155) is accurate for cases (a), (c), and (d) in Table 6.11 and overestimates the buckling load by less than 6 percent for case (b).

The buckling loads of axially loaded beams with shear deformation are summarized in Appendix B.

7.4.2 Axially Loaded Beams with Symmetrical or Unsymmetrical Cross Sections (Flexural–Torsional Buckling)

We consider orthotropic beams subjected to an axial load. The cross section may be symmetrical or unsymmetrical. The coordinate directions y and z must be in the principal directions (page 208), and, consequently, $\widehat{EI}_{yz} = 0$.

When the cross section of the beam has one axis of symmetry (about the z-axis), it may undergo (a) buckling in the plane of symmetry (x–z plane), or (b) combined flexural–torsional buckling. When the cross section is unsymmetrical, an orthotropic beam undergoes combined flexural–torsional buckling.

In general, for symmetrical and unsymmetrical cross sections, every element of the stiffness matrix must be considered in the calculation of the buckling load. When the off-diagonal terms of the shear stiffness matrix (Eq. 7.36) either are zero (cross section is doubly symmetrical) or are negligible, the buckling load may be approximated by the solution of the following equations[9]

unsymmetrical cross section:

$$\left| \begin{bmatrix} \widehat{N}_{\mathrm{cr}z} & 0 & 0 \\ 0 & \widehat{N}_{\mathrm{cr}y} & 0 \\ 0 & 0 & \widehat{N}_{\mathrm{cr}\psi} i_\omega^2 \end{bmatrix} - \widehat{N}_{\mathrm{cr}} \begin{bmatrix} 1 & 0 & z_{\mathrm{sc}} \\ 0 & 1 & -y_{\mathrm{sc}} \\ z_{\mathrm{sc}} & -y_{\mathrm{sc}} & i_\omega^2 \end{bmatrix} \right| = 0, \tag{7.157}$$

[9] L. P. Kollár, Flexural-Torsional Buckling of Open Section Composite Columns with Shear Deformation. *International Journal of Solids and Structures,* Vol. 38, 7525–7541, 2001.

where $||$ denotes determinant. The buckling load $\widehat{N}_{\mathrm{cr}}$ corresponds to flexural–torsional buckling. The solution of Eq. (7.157) results in three values of $\widehat{N}_{\mathrm{cr}}$, of which the lowest value is of interest.

symmetrical cross section:

$$\widehat{N}_{\mathrm{cr1}} = \widehat{N}_{\mathrm{cr}y} \tag{7.158}$$

$$\left| \begin{bmatrix} \widehat{N}_{\mathrm{cr}z} & \\ & \widehat{N}_{\mathrm{cr}\psi} i_\omega^2 \end{bmatrix} - \widehat{N}_{\mathrm{cr}} \begin{bmatrix} 1 & z_{\mathrm{sc}} \\ z_{\mathrm{sc}} & i_\omega^2 \end{bmatrix} \right| = 0, \tag{7.159}$$

where $\widehat{N}_{\mathrm{cr1}}$ is the buckling load in the x–z plane and $\widehat{N}_{\mathrm{cr}}$ corresponds to flexural–torsional buckling. Equation (7.158) yields the value of $\widehat{N}_{\mathrm{cr1}}$ and Eq. (7.159) yields two values of $\widehat{N}_{\mathrm{cr}}$ denoted by $\widehat{N}_{\mathrm{cr2}}$ and $\widehat{N}_{\mathrm{cr3}}$. The buckling load of interest is the lowest of these three values.

We observe that the preceding equations (Eqs. 7.157–7.159) are identical to those given for beams without shear deformation (Eqs. 6.343–6.346). The difference is in the expressions used to calculate $\widehat{N}_{\mathrm{cr}z}$, $\widehat{N}_{\mathrm{cr}y}$, and $\widehat{N}_{\mathrm{cr}\psi}$. Here we use the expressions given by Eqs. (7.136), (7.137), and (7.155).

Equations (7.157)–(7.159) give the exact buckling load when $\widehat{S}_{ij} = 0$ $(i \neq j)$ and either (a) both ends are simply supported, (b) both ends are built-in, or (c) when one end is built-in and the other end is free. When one end is built-in and one is simply supported (case (b) in Table 6.11, page 293) the equations above yield only an approximate value of the buckling load.

The buckling loads of axially loaded beams with shear deformation are summarized in Appendix B.

7.2 Example. *An* $L = 0.5$*-m-long and* $b = 0.1$*-m-wide rectangular sandwich beam (Fig. 7.13) is made of a* 0.02*-m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each face is* $[\pm 45_2^{\mathrm{f}}/0_{12}/\pm 45_2^{\mathrm{f}}]$, *and the thickness of each facesheet is 0.002 m. The beam, built-in at both ends, is subjected to an axial compressive load. Calculate the buckling load. The core is isotropic (*$E_{\mathrm{c}} = 2 \times 10^6$ *kN/m*^{2}, $\nu_{\mathrm{c}} = 0.3$*).*

Solution. The beam buckles in the x–z plane. With the values of $\widehat{EI}_{yy} = 4\,675\ \mathrm{N \cdot m^2}$, $\widehat{S}_{zz} = \frac{1}{\widehat{s}_{zz}} = 1\,862$ kN (Eqs. 7.107 and 7.109), and $L = 0.5$ m, the

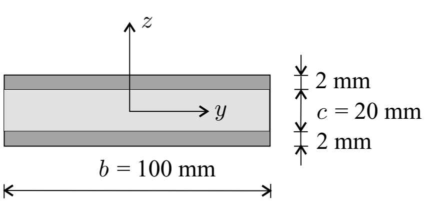


Figure 7.13: The cross section of the sandwich beam in Example 7.2.

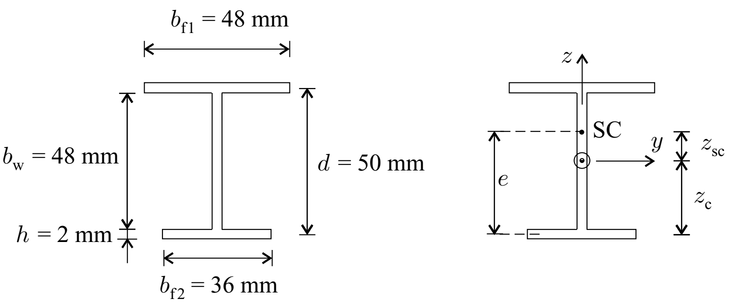


Figure 7.14: The cross-section of the beam in Example 7.3.

buckling load is (see Eqs. 7.136 and 6.337)

$$\widehat{N}_{cry} = \left(\frac{1}{\widehat{N}^{B}_{cry}} + \frac{1}{\widehat{S}_{zz}}\right)^{-1} = \left(\frac{1}{\frac{4\pi^2 \widehat{EI}_{yy}}{L^2}} + \frac{1}{\widehat{S}_{zz}}\right)^{-1}$$
$$= \left(\frac{1}{738} + \frac{1}{1\,862}\right)^{-1} = 529 \text{ kN}. \tag{7.160}$$

7.3 Example. *An* $L = 0.5$*-m-long I-section beam with the cross section shown in Figure 7.14 is made of graphite epoxy unidirectional plies. The material properties are given in Table 3.6 (page 81). The layup is* $[0_{20}]$*. The beam is simply supported at each end. Determine the buckling load when the beam is subjected to a compressive axial load.*

Solution. The shear compliances of symmetrical I-beams are given in Table A.8. The layup of the web and the flanges is identical. For a symmetrical layup, $(\alpha^{\nu}_{66})_f$ and $(\alpha^{\nu}_{66})_w$ are replaced by a_{66}, and we write

$$\begin{aligned}
\widehat{s}_{yy} &= 1.2\left(\frac{a_{66}}{b_{f1}(1+\delta_{sc})^2} + \frac{a_{66}}{b_{f2}\left(1+\frac{1}{\delta_{sc}}\right)^2}\right) = 1.681 \times 10^{-6}\,\frac{1}{\text{N}} \\
\widehat{s}_{zz} &= \frac{a_{66}}{d} + \frac{1}{12}\frac{a_{66}b_{f1}}{d^2\gamma_1^2} + \frac{1}{12}\frac{a_{66}b_{f2}}{d^2\gamma_2^2} = 2.413 \times 10^{-6}\,\frac{1}{\text{N}} \\
\widehat{s}_{\omega\omega} &= \frac{1.2}{d^2}\left(\frac{a_{66}}{b_{f1}} + \frac{a_{66}}{b_{f2}}\right) = 2\,564 \times 10^{-6}\,\frac{1}{\text{N}\cdot\text{m}^2} \\
\widehat{s}_{y\omega} &= \frac{1.2}{d}\left(-\frac{a_{66}}{b_{f1}(1+\delta_{sc})} + \frac{a_{66}}{b_{f2}\left(1+\frac{1}{\delta_{sc}}\right)}\right) = -16.906 \times 10^{-6}\,\frac{1}{\text{N}\cdot\text{m}} \\
\widehat{s}_{yz} &= \widehat{s}_{z\omega} = 0
\end{aligned} \tag{7.161}$$

$$\begin{aligned}
\delta_c &= \frac{d - z_c}{z_c} = 0.833 \qquad & \gamma_1 &= 1 + \frac{1}{3}\frac{(\alpha_{11})_{f1}}{(\alpha_{11})_w}\frac{d}{b_{f1}\left(1+\frac{1}{\delta_c}\right)} = 1.16 \\
\delta_{sc} &= \frac{d - e}{e} = 0.4219 \qquad & \gamma_2 &= 1 + \frac{1}{3}\frac{(\alpha_{11})_{f2}}{(\alpha_{11})_w}\frac{d}{b_{f2}(1+\delta_c)} = 1.25.
\end{aligned} \tag{7.162}$$

The parameter a_{66} for both the web and the flanges is $a_{66} = 109.89 \times 10^{-9} \frac{\text{m}}{\text{N}}$ (Table 3.8, page 85), $b_{\text{f1}} = 0.048$ m, $b_{\text{f2}} = 0.036$ m, $b_{\text{w}} = 0.048$ m, $d = 0.050$ m, and $e = 0.035\ 17$ m (Eq. 6.348). The shear stiffnesses are (Eqs. 7.38 and 7.161)

$$\begin{bmatrix} \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \widehat{s}_{yy} & 0 & \widehat{s}_{y\omega} \\ 0 & \widehat{s}_{zz} & 0 \\ \widehat{s}_{y\omega} & 0 & \widehat{s}_{\omega\omega} \end{bmatrix}^{-1} = \begin{bmatrix} 637.0 & 0 & 4.200 \\ 0 & 414.4 & 0 \\ 4.200 & 0 & 0.4177 \end{bmatrix} \times 10^3. \tag{7.163}$$

From Example 6.8 (page 294) we have $\widehat{N}^{\text{B}}_{\text{cr}y} = 713.55$ kN, $\widehat{N}^{\text{B}}_{\text{cr}z} = 153.32$ kN, $\widehat{N}^{\text{B}}_{\text{cr}\omega} = 127.96$ kN (Eqs. 6.351 and 6.352), $\widehat{GI}_{\text{t}} = 1.602$ N $\cdot$ m^2 (Eq. 6.348), $i_\omega = 0.024\ 99$ m (Eq.6.350). The buckling loads are (Eqs. 7.128, 7.137, 7.155, and 7.156)

$$\begin{aligned} \widehat{N}_{\text{cr}z} &= \left(\frac{1}{\widehat{N}^{\text{B}}_{\text{cr}z}} + \frac{1}{\widehat{S}_{yy}} \right)^{-1} = 123.57\,\text{kN} \\ \widehat{N}_{\text{cr}y} &= \left(\frac{1}{\widehat{N}^{\text{B}}_{\text{cr}y}} + \frac{1}{\widehat{S}_{zz}} \right)^{-1} = 262.16\,\text{kN} \\ \widehat{N}_{\text{cr}\psi} &= \widehat{N}_{\text{cr}\omega} + \frac{1}{i_\omega^2} \widehat{GI}_{\text{t}} = 109.98\,\text{kN}, \end{aligned} \tag{7.164}$$

where

$$\widehat{N}_{\text{cr}\omega} = \left(\frac{1}{\widehat{N}^{\text{B}}_{\text{cr}\omega}} + \frac{1}{\frac{1}{i_\omega^2} \widehat{S}_{\omega\omega}} \right)^{-1} = 107.42\ \text{kN}. \tag{7.165}$$

The buckling load $\widehat{N}_{\text{cr1}}$ is (Eq. 7.158)

$$\widehat{N}_{\text{cr1}} = \widehat{N}_{\text{cr}y} = 262.16\ \text{kN} \qquad \text{buckling in the } x\text{–}z \text{ plane.} \tag{7.166}$$

$\widehat{N}_{\text{cr2}}$ and $\widehat{N}_{\text{cr3}}$ are the roots of Eq. (7.159),

$$\left\| \begin{bmatrix} \widehat{N}_{\text{cr}z} & \\ & \widehat{N}_{\text{cr}\psi} i_\omega^2 \end{bmatrix} - \widehat{N}_{\text{cr}} \begin{bmatrix} 1 & -z_{\text{sc}} \\ -z_{\text{sc}} & i_\omega^2 \end{bmatrix} \right\| = 0, \tag{7.167}$$

where $z_{\text{sc}} = 0.000\ 789$ m (Eq. 6.349). Solution of this equation yields

$$\widehat{N}_{\text{cr2}} = 169.17\ \text{kN} \qquad \widehat{N}_{\text{cr3}} = 87.06\ \text{kN}. \tag{7.168}$$

Calculations that neglect shear deformation give the buckling loads as (Eqs. 6.353 and 6.355)

$$\widehat{N}_{\text{cr1}} = 713.55\ \text{kN} \qquad \widehat{N}_{\text{cr2}} = 208.88\ \text{kN} \qquad \widehat{N}_{\text{cr3}} = 106.43\ \text{kN}. \tag{7.169}$$

We now compare these values with the buckling loads calculated above, taking shear deformation into account. The buckling loads calculated with and without shear deformation differ significantly.

7.4.3 Lateral–Torsional Buckling of Beams with Symmetrical Cross Section

We consider orthotropic beams. The cross section of the beam is symmetrical with respect to the z-axis ($\widehat{EI}_{yz} = 0$). The beam is simply supported at each end. At the simple support the beam is restrained from rotating about the x-axis, but the cross section is free to rotate about the y- and z-axes and free to warp.

The beam is subjected to two equal and opposite bending moments at the two ends (Fig. 6.86, top), a uniformly distributed load p in the plane of symmetry (Fig. 6.86, middle), or a concentrated force P in the plane of symmetry at the midspan of the beam (Fig. 6.86, bottom). The distance between the shear center and the point where the load acts is denoted by Δ.

At a certain applied load the beam buckles laterally while the cross sections of the beam rotate (Fig. 6.87). This phenomenon is called lateral buckling or lateral–torsional buckling.

In general, for symmetrical cross sections, every element of the shear stiffness matrix must be considered in the calculation of the buckling load. When the off-diagonal terms of the shear stiffness matrix (Eq. 7.36) either are zero (cross section is doubly symmetrical) or are negligible, the buckling load may be approximated by the following equation[10]:

$$\widehat{Q}_{\text{cr}} = F_1 \widehat{N}_{\text{cr}z} \left(F_2\Delta + F_3\beta_1 \pm \sqrt{(F_2\Delta + F_3\beta_1)^2 + \frac{\widehat{N}_{\text{cr}\psi} i_\omega^2}{\widehat{N}_{\text{cr}z}}} \right), \tag{7.170}$$

where $\widehat{Q}_{\text{cr}}$ is the critical value of the bending moment and is related to the applied loads, as shown in Table 6.12 (page 298). The positive sign before the square root results in a positive load (which acts upward), whereas the negative sign results in a negative load (which acts downward). The values of the constants F_1, F_2, F_3 are also listed in Table 6.12. The parameter β_1 depends on the shape of the cross section (Eq. 6.360).

We observe that Eq. (7.170) is identical to that given for beams without shear deformation (Eq. 6.359). The difference is in the expressions used to calculate $\widehat{N}_{\text{cr}z}$ and $\widehat{N}_{\text{cr}\psi}$. Here we use the expressions given by Eqs. (7.137) and (7.155). These equations contain $\widehat{N}^{\text{B}}_{\text{cr}z}$ and $\widehat{N}^{\text{B}}_{\text{cr}\psi}$, which are to be calculated by Eq. (6.337) with $k = 1$.

7.4 Example. *An $L = 0.5$-m-long I-section beam with the cross section shown in Figure 7.14 is made of graphite epoxy unidirectional plies. The material properties*

[10] A. Sapkás and L. P. Kollár, Lateral-Torsional Buckling of Composite Beams. *International Journal of Solids and Structures,* Vol. 39, 2939–2963, 2002.

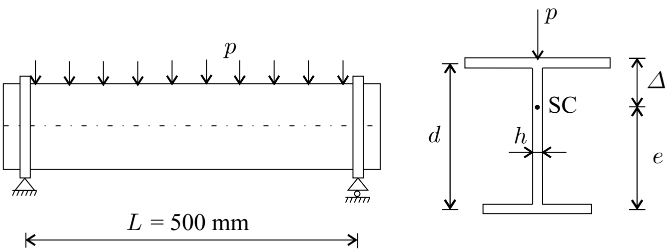


Figure 7.15: The beam in Example 7.4.

are given in Table 3.6 (page 81). The layup is $[0_{20}]$. *The beam, simply supported at each end, is subjected to a distributed load along the top flange (Fig. 7.15). Determine the buckling load.*

Solution. For a transversely loaded beam the buckling load is (Table 6.12, page 298)

$$p_{\mathrm{cr}} = \frac{8\widehat{Q}_{\mathrm{cr}}}{L^2}, \tag{7.171}$$

where $\widehat{Q}_{\mathrm{cr}}$ is given by Eq. (7.170) as follows:

$$\widehat{Q}_{\mathrm{cr}} = F_1 \widehat{N}_{\mathrm{cr}z} \left(F_2\Delta + F_3\beta_1 - \sqrt{(F_2\Delta + F_3\beta_1)^2 + \frac{\widehat{N}_{\mathrm{cr}\psi} i_\omega^2}{\widehat{N}_{\mathrm{cr}z}}} \right). \tag{7.172}$$

The sign before the square root is negative because the load is downward. The parameters are $F_1 = 1.13$, $F_2 = 0.45$, $F_3 = 0.267$ (Table 6.12, page 298), $\Delta = 0.015\,84\,\mathrm{m}$, (Example 6.9, page 298), $\widehat{N}_{\mathrm{cr}z} = 123.57\,\mathrm{kN}$, $\widehat{N}_{\mathrm{cr}\psi} = 109.98\,\mathrm{kN}$ (Eq. 7.164), $i_\omega = 0.024\,99$ m (Eq. 6.350), and $\beta_1 = -0.017\,86$ m (Eq. 6.369).

With these values $\widehat{Q}_{\mathrm{cr}}$ is

$$\widehat{Q}_{\mathrm{cr}} = -2\,979\ \mathrm{N \cdot m}. \tag{7.173}$$

Equations (7.171) and (7.173) give

$$p_{\mathrm{cr}} = \frac{8\widehat{Q}_{\mathrm{cr}}}{L^2} = -95\,320\ \frac{\mathrm{N}}{\mathrm{m}}. \tag{7.174}$$

Calculations that neglect shear deformation give the buckling load as $p_{\mathrm{cr}} = -115\,407\,\mathrm{N/m}$ (Eq. 6.372). The buckling load calculated with and without shear deformation differs significantly.

7.4.4 Summary

In summary, when the cross section is such that the off-diagonal elements of the stiffness matrix are negligible ($\widehat{S}_{ij} = 0, i \neq j$), the buckling loads may be approximated by the expressions derived for beams without shear deformation by the following substitutions:

$$\begin{array}{ll} \text{No shear deformation} & \text{With shear deformation} \\ \widehat{N}^{\mathrm{B}}_{\mathrm{cr}z} & \Longrightarrow \left(\dfrac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}z}} + \dfrac{1}{\widehat{S}_{yy}} \right)^{-1} \\ \widehat{N}^{\mathrm{B}}_{\mathrm{cr}y} & \Longrightarrow \left(\dfrac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}y}} + \dfrac{1}{\widehat{S}_{zz}} \right)^{-1} \\ \widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega} & \Longrightarrow \left(\dfrac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega}} + \dfrac{1}{\frac{1}{i_\omega^2}\widehat{S}_{\omega\omega}} \right)^{-1}. \end{array} \tag{7.175}$$

Numerical examples[11,12] show that these substitutions are reasonable even when $\widehat{S}_{ij} \neq 0$.

7.5 Free Vibration of Beams

In this section we present the circular frequencies ω of orthotropic beams. The parameter ω is related to the period of vibration T and to the natural frequency f by

$$T = \frac{2\pi}{\omega} \qquad f = \frac{\omega}{2\pi}. \tag{7.176}$$

7.5.1 Beams with Doubly Symmetrical Cross Sections

The cross section of the beam has two axes of symmetry y and z. The mass is symmetrical with respect to these axes, and, accordingly, the center of mass coincides with the origin of the y–z coordinate system.

A beam with two cross-sectional planes of symmetry may undergo flexural vibration in either of the two planes of symmetry and torsional vibration (Fig. 6.94). There are three sets of vibrations modes[13],

$$\omega_1 = \omega_y \qquad \omega_2 = \omega_z \qquad \omega_3 = \omega_\psi, \tag{7.177}$$

where ω_y and ω_z correspond to vibration in the x–z and x–y planes, respectively, and ω_ψ corresponds to the torsional mode.

Vibration in the x–z plane. We recall the equilibrium equations of beams subjected to transverse loads (Eqs. 7.83 and 7.84):

$$-\widehat{EI}\frac{d^3\chi}{dx^3} + p = 0 \tag{7.178}$$

$$\widehat{EI}\frac{d^2\chi}{dx^2} + \widehat{S}\left(\frac{dw}{dx} - \chi\right) = 0. \tag{7.179}$$

[11] Ibid.

[12] L. P. Kollár, Flexural-Torsional Vibration of Open Section Composite Beams with Shear Deformation. *International Journal of Solids and Structures,* Vol. 38, 7543–7558, 2001.

[13] W. Weaver, S. P. Timoshenko, and D. H. Young, *Vibration Problems in Engineering.* 5th edition. John Wiley & Sons, New York, 1990, pp. 422–432.

In writing these equations we used the symbols specified in Eq. (7.80). In the present context p is the inertia force given by

$$p = \rho\omega^2 w, \tag{7.180}$$

where ρ is the mass per unit length. Equations (7.178)–(7.180) may be rearranged to yield

$$\frac{1}{\rho\omega^2}\frac{d^4\chi}{dx^4} + \frac{1}{\widehat{S}}\frac{d^2\chi}{dx^2} - \frac{1}{\widehat{EI}}\chi = 0 \tag{7.181}$$

$$w = \frac{\widehat{EI}}{\rho\omega^2}\frac{d^3\chi}{dx^3}. \tag{7.182}$$

The solution of the first of these equations is[14]

$$\chi = C_1 \cosh\frac{\nu}{L}x + C_2 \sinh\frac{\nu}{L}x + C_3 \cos\frac{\mu}{L}x + C_4 \sin\frac{\mu}{L}x. \tag{7.183}$$

We substitute this expression into Eq. (7.182). The result is

$$w = \frac{\widehat{EI}_{yy}}{\rho\omega^2}\left(C_1\frac{\nu^3}{L^3}\sinh\frac{\nu}{L}x + C_2\frac{\nu^3}{L^3}\cosh\frac{\nu}{L}x + C_3\frac{\mu^3}{L^3}\sin\frac{\mu}{L}x - C_4\frac{\mu^3}{L^3}\cos\frac{\mu}{L}x\right), \tag{7.184}$$

where L is the length of the beam, C_1–C_4 are constants determined from the boundary conditions, and μ and ν are parameters defined as

$$\mu = L\sqrt{\left(\frac{\rho\omega^2}{2\widehat{S}}\right) + \sqrt{\left(\frac{\rho\omega^2}{2\widehat{S}}\right)^2 + \frac{\rho\omega^2}{\widehat{EI}}}} \tag{7.185}$$

$$\nu = L\sqrt{-\left(\frac{\rho\omega^2}{2\widehat{S}}\right) + \sqrt{\left(\frac{\rho\omega^2}{2\widehat{S}}\right)^2 + \frac{\rho\omega^2}{\widehat{EI}}}}. \tag{7.186}$$

From the first of these equations, we have

$$\frac{1}{\omega^2} = \rho\left[\left(\frac{L}{\mu}\right)^4\frac{1}{\widehat{EI}} + \left(\frac{L}{\mu}\right)^2\frac{1}{\widehat{S}}\right]. \tag{7.187}$$

By combining Eqs. (7.185) and (7.186), we obtain

$$\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0. \tag{7.188}$$

Equations (7.183), (7.184), (7.187), and (7.188) together with the boundary conditions provide the circular frequencies. The process by which ω is determined is illustrated by two examples: a beam simply supported at both ends and a beam built-in at both ends.

[14] E. Kreyszig, *Advanced Engineering Mathematics.* 7th edition. John Wiley & Sons, New York, 1993, pp. 136–138.

Both ends simply supported. When both ends of the beam are simply supported, at the ends the deflection w and the first derivative of χ are zero (Table 7.1, page 321) as follows:

$$w(L) = 0 \qquad \frac{d\chi(L)}{dx} = 0 \qquad w(0) = 0 \qquad \frac{d\chi(0)}{dx} = 0. \tag{7.189}$$

By substituting w and χ (given by Eqs. 7.183 and 7.184) into these boundary conditions, we obtain

$$\begin{bmatrix} \nu^3 \sinh\nu & \nu^3\cosh\nu & \mu^3 \sin\mu & -\mu^3\cos\mu \\ \nu\sinh\nu & \nu\cosh\nu & -\mu\sin\mu & \mu\cos\mu \\ 0 & \nu^3 & 0 & -\mu^3 \\ 0 & \nu & 0 & \mu \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{7.190}$$

For the deflection to be nonzero, at least one of the constants C_1–C_4 must be nonzero. Therefore, Eq. (7.190) is satisfied only when the determinant of the matrix is zero. This requirement is met when

$$\mu^2\nu^2(\nu^2 + \mu^2)^2 \sinh\nu \sin\mu = 0. \tag{7.191}$$

This equation requires that the sine term be zero. This condition is satisfied when

$$\mu = \mu_i = i\pi. \tag{7.192}$$

Equations (7.187) and (7.192) give the following expression for the circular frequencies of the beam vibrating in the plane of symmetry:

$$\frac{1}{\omega_i^2} = \rho\left[\frac{1}{\widehat{EI}}\left(\frac{L}{i\pi}\right)^4 + \frac{1}{\widehat{S}}\left(\frac{L}{i\pi}\right)^2\right]. \tag{7.193}$$

Both ends built-in. When both ends of the beam are built-in, the displacement and the rotation of the cross section at each end are zero (Table 7.1, page 321) as follows:

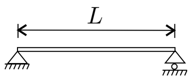
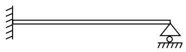
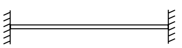
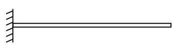
$$w(L) = 0 \qquad \chi(L) = 0 \qquad w(0) = 0 \qquad \chi(0) = 0. \tag{7.194}$$

By substituting w and χ (given by Eqs. 7.183 and 7.184) into these boundary conditions, we obtain

$$\begin{bmatrix} \nu^3 \sinh\nu & \nu^3\cosh\nu & \mu^3 \sin\mu & -\mu^3\cos\mu \\ \cosh\nu & \sinh\nu & \cos\mu & \sin\mu \\ 0 & \nu^3 & 0 & -\mu^3 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{7.195}$$

Table 7.4. The equations to determine the circular frequencies ω of beams with shear deformation. For vibration in the x–z plane $\widehat{EI} = \widehat{EI}_{yy}$, and $\widehat{S} = \widehat{S}_{zz}$. For vibration in the x–y plane $\widehat{EI} = \widehat{EI}_{zz}$, and $\widehat{S} = \widehat{S}_{yy}$.

$$\frac{1}{\omega_i^2} = \left(\frac{L}{\mu_i}\right)^4 \frac{\rho}{\widehat{EI}} + \left(\frac{L}{\mu_i}\right)^2 \frac{\rho}{\widehat{S}}$$

Geometry	μ_i
(a) 	$\sin\mu = 0$ (solution : $\mu_i = i\pi$)
(b) 	$\nu^3 \tanh\nu - \mu^3 \tan\mu = 0$ $\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0$
(c) 	$\cos\mu\cosh\nu + \frac{1}{2}\left[\left(\frac{\nu}{\mu}\right)^3 - \left(\frac{\mu}{\nu}\right)^3\right]\sin\mu\sinh\nu - 1 = 0$ $\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0$
(d) 	$2 + \left(\frac{\nu}{\mu} - \frac{\mu}{\nu}\right)\sin\mu\sinh\nu$ $+\left[\left(\frac{\nu}{\mu}\right)^2 + \left(\frac{\mu}{\nu}\right)^2\right]\cos\mu\cosh\nu = 0$ $\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0$
(e)	$-\cos\mu\cosh\nu + \frac{1}{2}\left(\frac{\nu}{\mu} - \frac{\mu}{\nu}\right)\sin\mu\sinh\nu + 1 = 0$ $\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0$

For the deflection to be nonzero, at least one of the constants C_1–C_4 must be nonzero. Thus, Eq. (7.195) is satisfied only when the determinant of the matrix is zero. This requirement is met when

$$\cos\mu\cosh\nu + \frac{1}{2}\left[\left(\frac{\nu}{\mu}\right)^3 - \left(\frac{\mu}{\nu}\right)^3\right]\sin\mu\sinh\nu = 1. \tag{7.196}$$

We now recall Eq. (7.188) as follows:

$$\mu^2 - \frac{\widehat{EI}}{\widehat{S}L^2}\mu^2\nu^2 - \nu^2 = 0. \tag{7.197}$$

Equations (7.196) and (7.197) can be solved numerically for μ and ν; the resulting values are denoted by μ_i and ν_i. The circular frequencies ω_i are then calculated from Eq. (7.187).

Different types of end supports. The equations needed to calculate μ (and the circular frequencies, Eq. 7.187) of beams with different end supports can be obtained in a similar manner. The results are summarized in Table 7.4. The calculation of the circular frequencies requires a numerical procedure except when both ends are simply supported.

Approximate solution. The analysis just described requires a numerical procedure to calculate the circular frequencies. In the following an approximate solution is presented, which results in a closed-form expression for the circular frequencies.

We apply Föppl's theorem developed for estimating the buckling load of elastic structures,[15] and write the circular frequency ω as

$$\frac{1}{\omega^2} \simeq \left(\frac{1}{\omega^{\mathrm{B}}}\right)^2 + \left(\frac{1}{\omega^{\mathrm{S}}}\right)^2, \tag{7.198}$$

where ω^{B} is the circular frequency of a beam undergoing bending deformation only, and ω^{S} is the circular frequency of a beam undergoing shear deformation only.

We illustrate the use of the preceding equation through the example of a beam with both ends built-in. We then generalize the results to beams with different end supports.

To calculate ω^{B} we recall that, when there is no shear deformation the shear stiffness is infinity ($\widehat{S} \to \infty$). In this limit, Eqs. (7.185) and (7.186) yield

$$\mu = \nu. \tag{7.199}$$

By denoting this value of μ by μ_{B} and by substituting $\mu_{\mathrm{B}} = \mu = \nu$ into Eq. (7.196), we obtain

$$\cos \mu_{\mathrm{B}} \cosh \mu_{\mathrm{B}} = 1. \tag{7.200}$$

Numerical solution of this equation yields

$$\begin{aligned} \mu_{B1} &= 4.730 \\ \mu_{B2} &= 7.853 \\ &\vdots \\ \mu_{Bi} &\simeq (i + 0.5)\pi. \end{aligned} \tag{7.201}$$

With these approximate values of $\mu_{\mathrm{B}i}$ the circular frequencies are (Eq. 7.187 with $\widehat{S} \longrightarrow \infty$)

$$\left(\frac{1}{\omega^{\mathrm{B}}}\right)^2 = \rho \frac{1}{\widehat{EI}} \left(\frac{L}{\mu_{\mathrm{B}i}}\right)^4. \tag{7.202}$$

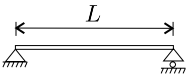
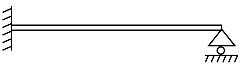
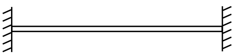
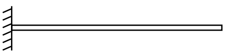
To determine ω^{S} we note that when there is no bending deformation the bending stiffness is infinity ($\widehat{EI} \longrightarrow \infty$). In this limit Eq. (7.186) yields $\nu = 0$. With this value of ν and with the substitution $\mu = \mu_{\mathrm{S}}$ Eq. (7.196) gives

$$\sin \mu_{\mathrm{S}} = 0. \tag{7.203}$$

[15] T. Tarnai, Summation Theorems Concerning Critical Loads of Bifurcation. In: *Structural Stability in Engineering Practice.* L. Kollár, ed., E & FN Spon, London, 1999, pp. 23–58.

Table 7.5. The circular frequencies of beams with shear deformation only. The equations for calculating μ_{Si} (left column) and the values of μ_{Si} (right column). For vibration in the x–z and x–y planes $\widehat{S}$ is $\widehat{S}_{zz}$ and $\widehat{S}_{yy}$ respectively.

$$(\omega_i^{S})^2 = \frac{\widehat{S}}{\rho}\left(\frac{\mu_{Si}}{L}\right)^2$$

Geometry	μ_{Si}	
	$\sin\mu_S = 0$	$\mu_{Si} = i\pi$
	$\tan\mu_S = 0$	$\mu_{Si} = i\pi$
	$\sin\mu_S = 0$	$\mu_{Si} = i\pi$
	$\cos\mu_S = 0$	$\mu_{Si} = (i - 0.5)\pi$
	$-\cos\mu_S + 1 - \frac{\mu_S}{2}\sin\mu_S = 0$	$\mu_{S1} = 6.283$ $\mu_{S2} = 8.987$ $\mu_{S3} = 12.566$ Approximate: $\mu_{Si} \approx (i+1)\pi$

Solutions to this equation are

$$\mu_S = \mu_{Si} = i\pi \qquad i = 1, 2, \ldots. \tag{7.204}$$

With these values of μ_S Eq. (7.187) becomes ($\widehat{EI} \longrightarrow \infty$)

$$\left(\frac{1}{\omega^S}\right)^2 = \rho\left(\frac{L}{\mu_{Si}}\right)^2 \frac{1}{\widehat{S}}. \tag{7.205}$$

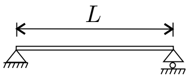
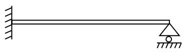
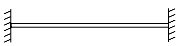
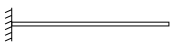
Equations (7.198), (7.202), and (7.205) give the circular frequencies in the x–z plane of symmetry of orthotropic beams with built-in ends:

$$\frac{1}{\omega_y^2} \simeq \frac{1}{\left(\omega_y^B\right)^2} + \frac{1}{\left(\omega_y^S\right)^2} = \frac{\rho}{\widehat{EI}_{yy}}\left(\frac{L}{\mu_{Bi}}\right)^4 + \frac{\rho}{\widehat{S}_{zz}}\left(\frac{L}{\mu_{Si}}\right)^2. \tag{7.206}$$

The subscript y is introduced to indicate that the beam vibrates in the x–z plane.

It can be shown that Eq. (7.206) is also applicable to beams with end supports other than ones built-in at both ends provided that the appropriate values of μ_{Bi} and μ_{Si} are used. Values for μ_{Bi} for different end supports are given in Table 6.13 (page 308). Values for μ_{Si} are determined from the expressions given in Table 7.4 (page 350) by setting $\nu = 0$. (Recall that when there is no bending deformation $\widehat{EI} \longrightarrow \infty$ and $\nu = 0$.) The resulting equations are listed in Table 7.5. Solutions of these equations are also given in this table. By comparing the results in Tables 7.5 and 6.13 we see that $\mu_{Si} = \mu_{Gi}$.

Table 7.6. The errors in the circular frequencies (Table 7.4) calculated by the approximate expressions (Eqs. 7.206 and 7.207).

Geometry		$\text{Er} = \frac{\omega^{\text{accurate}} - \omega^{\text{approximate}}}{\omega^{\text{accurate}}} 100$
(a) 		$\text{Er} = 0$
(b) 	$i = 1$	$0 \leq \text{Er} < 4.2$
	$i = 2$	$0 \leq \text{Er} < 3.1$
	$i > 2$	$0 \leq \text{Er} < 2.4$
(c) 	$i = 1$	$-0.32 < \text{Er} \leq 0$
	$i = 2$	$0 \leq \text{Er} < 4.2$
	$i > 2$	$0 \leq \text{Er} < 3.5$
(d) 	$i = 1$	$-1.0 < \text{Er} \leq 0$
	$i = 2$	$0 \leq \text{Er} < 10.3$
	$i > 2$	$0 \leq \text{Er} < 5.1$
(e)	$i = 1$	$-0.43 < \text{Er} \leq 0$
	$i = 2$	$-0.32 < \text{Er} \leq 0$
	$i > 2$	$0 \leq \text{Er} < 2.2$

Circular frequencies of different types of beams were calculated numerically by the accurate (Table 7.4) and by the approximate solutions (Eq. 7.206 or Table B.2). The errors in the approximate solutions are summarized in Table 7.6.

Vibration in the x–y ***plane.*** Like vibration in the x–z plane we may express the circular frequency in the x–y plane as

$$\frac{1}{\omega_z^2} \simeq \frac{1}{(\omega_z^{\text{B}})^2} + \frac{1}{(\omega_z^{\text{S}})^2} = \frac{\rho}{\widehat{EI}_{zz}} \left(\frac{L}{\mu_{\text{B}i}}\right)^4 + \frac{\rho}{\widehat{\text{S}}_{yy}} \left(\frac{L}{\mu_{Si}}\right)^2, \tag{7.207}$$

where ω^{B} is the circular frequency of a beam undergoing bending deformation only, and ω^{S} is the circular frequency of a beam undergoing shear deformation only.

Torsional vibration. For a doubly symmetrical beam vibrating about its axis the equilibrium equations are (see Eqs. 7.39 and 7.40)

$$\frac{d\widehat{T}}{dx} + t = 0 \tag{7.208}$$

$$\frac{d\widehat{M}_\omega}{dx} - \widehat{T}_\omega = 0, \tag{7.209}$$

where t is the inertia force defined as[16]

$$t = \Theta\omega^2\psi, \tag{7.210}$$

[16] W. Weaver, S. P. Timoshenko, and D. H. Young, *Vibration Problems in Engineering.* 5th edition. John Wiley & Sons, New York, 1990, pp. 401–402.

where Θ is the polar moment of mass per unit length about the shear center (Eq. 6.402), ψ is the twist of the cross section about the beam's axis, and $\widehat{M}_\omega$ is the bimoment; $\widehat{T}$ is given by Eq. (7.14) as follows:

$$\widehat{T} = \widehat{T}_{\mathrm{sv}} + \widehat{T}_\omega. \tag{7.211}$$

The force–strain relationships are (Eqs. 7.32, 7.34, 7.36)

$$\widehat{M}_\omega = -\widehat{EI}_\omega \frac{d\vartheta^{\mathrm{B}}}{dx} \qquad \widehat{T}_{\mathrm{sv}} = \widehat{GI}_{\mathrm{t}}\vartheta \qquad \widehat{T}_\omega = \widehat{\mathrm{S}}_{\omega\omega}\vartheta^{\mathrm{S}}, \tag{7.212}$$

where ϑ and ϑ^{S} are (Eqs. 7.6 and 7.7)

$$\vartheta = \frac{d\psi}{dx} \qquad \vartheta^{\mathrm{S}} = \frac{d\psi}{dx} - \vartheta^{\mathrm{B}}. \tag{7.213}$$

Equations (7.208)–(7.213) may be combined to yield

$$\widehat{\mathrm{S}}_{\omega\omega}\left(\frac{d^2\psi}{dx^2} - \frac{d\vartheta^{\mathrm{B}}}{dx}\right) + \widehat{GI}_{\mathrm{t}}\frac{d^2\psi}{dx^2} + \Theta\omega^2\psi = 0 \tag{7.214}$$

$$-\widehat{EI}_\omega \frac{d^2\vartheta^{\mathrm{B}}}{dx^2} - \widehat{\mathrm{S}}_{\omega\omega}\left(\frac{d\psi}{dx} - \vartheta^{\mathrm{B}}\right) = 0. \tag{7.215}$$

After algebraic manipulations these equations become

$$\frac{1}{\Theta\omega^2}\frac{d^4\vartheta^{\mathrm{B}}}{dx^4} + \frac{1}{\widehat{\mathrm{S}}_{\omega\omega}}\frac{d^2\vartheta^{\mathrm{B}}}{dx^2} - \frac{1}{\widehat{EI}_\omega}\vartheta^{\mathrm{B}} - \frac{\widehat{GI}_{\mathrm{t}}}{\Theta\omega^2}\left(\frac{1}{\widehat{EI}_\omega}\frac{d^2\vartheta^{\mathrm{B}}}{dx^2} - \frac{1}{\widehat{\mathrm{S}}_{\omega\omega}}\frac{d^4\vartheta^{\mathrm{B}}}{dx^4}\right) = 0 \tag{7.216}$$

$$\psi = \frac{1}{\Theta\omega^2}\left[\widehat{EI}_\omega\frac{d^3\vartheta^{\mathrm{B}}}{dx^3} - \widehat{GI}_{\mathrm{t}}\left(\frac{d\vartheta^{\mathrm{B}}}{dx} - \frac{\widehat{EI}_\omega}{\widehat{\mathrm{S}}_{\omega\omega}}\frac{d^3\vartheta^{\mathrm{B}}}{dx^3}\right)\right]. \tag{7.217}$$

Short beam. For short beams ($\widehat{GI}_{\mathrm{t}}L^2 \ll \widehat{EI}_\omega$) Eqs. (7.216) and (7.217) simplify to *torsional vibration:*

$$\frac{1}{\Theta\omega^2}\frac{d^4\vartheta^{\mathrm{B}}}{dx^4} + \frac{1}{\widehat{\mathrm{S}}_{\omega\omega}}\frac{d^2\vartheta^{\mathrm{B}}}{dx^2} - \frac{1}{\widehat{EI}_\omega}\vartheta^{\mathrm{B}} = 0 \tag{7.218}$$

$$\psi = \frac{\widehat{EI}_\omega}{\Theta\omega^2}\frac{d^3\vartheta^{\mathrm{B}}}{dx^3}. \tag{7.219}$$

The equations for vibration in the x–z plane are given by Eqs. (7.181) and (7.182) and, for convenience, are repeated here *in-plane vibration:*

$$\frac{1}{\rho\omega^2}\frac{d^4\chi}{dx^4}+\frac{1}{\widehat{S}}\frac{d^2\chi}{dx^2}-\frac{1}{\widehat{EI}}\chi=0 \tag{7.220}$$

$$w=\frac{\widehat{EI}}{\rho\omega^2}\frac{d^3\chi}{dx^3}. \tag{7.221}$$

It can be seen that these equations are identical to Eqs. (7.218) and (7.219) when the following substitutions are made:

$$\begin{array}{rl} \text{Vibration in the } x\text{–}z \text{ plane} & \text{Torsional vibration} \\ \rho \Longrightarrow & \Theta \\ \widehat{EI} \Longrightarrow & \widehat{EI}_\omega \\ \widehat{S} \Longrightarrow & \widehat{S}_{\omega\omega} \\ w \Longrightarrow & \psi \\ \chi \Longrightarrow & \vartheta^{\mathrm{B}} \end{array} \tag{7.222}$$

For in-plane vibration of short beams the circular frequency is denoted by ω_y, and for torsional vibration by ω_ψ. Thus, the solution for vibration in the x–z plane (Eq. 7.206) is also applicable to torsional vibration of short beams with the following substitutions:

$$\begin{array}{rl} \text{Vibration in the } x\text{–}z \text{ plane} & \text{Torsional vibration} \\ \omega_y \Longrightarrow & \omega_\psi \\ \rho \Longrightarrow & \Theta \\ \widehat{EI} \Longrightarrow & \widehat{EI}_\omega \\ \widehat{S} \Longrightarrow & \widehat{S}_{\omega\omega} \end{array} \tag{7.223}$$

The parameters w and χ are not included in this equation because they do not appear in the expression for the circular frequency.

For beams with different end supports the circular frequency is given by Eq. (7.206). By replacing ρ, $\widehat{EI}\,(=\widehat{EI}_{yy})$, and $\widehat{S}\,(=\widehat{S}_{zz})$ with the expressions in Eq. (7.223), we have

$$\frac{1}{\omega_{\psi i}^2}=\frac{\Theta}{\widehat{EI}_\omega}\left(\frac{L}{\mu_{\mathrm{B}i}}\right)^4+\frac{\Theta}{\widehat{S}_{\omega\omega}}\left(\frac{L}{\mu_{\mathrm{S}i}}\right)^2 \qquad \begin{array}{c}\text{torsional vibration}\\ \text{short beam,}\end{array} \tag{7.224}$$

where $\mu_{\mathrm{B}i}$ and $\mu_{\mathrm{S}i}\,(=\mu_{\mathrm{G}i})$ are given in Tables 6.13 and B.2 (pages 308 and 463).

Long beam. For long beams ($\widehat{GI}_{\mathrm{t}}L^2 \gg \widehat{EI}_\omega$ and $\widehat{S}_{\omega\omega}L^2 \gg \widehat{EI}_\omega$) Eqs. (7.216) and (7.217) simplify to

$$\frac{\widehat{GI}_{\mathrm{t}}}{\Theta\omega^2}\frac{d^2\vartheta^{\mathrm{B}}}{dx^2}+\vartheta^{\mathrm{B}}=0 \tag{7.225}$$

$$\frac{d\psi}{dx}=\vartheta^{\mathrm{B}}. \tag{7.226}$$

These equations are independent of the torsional shear stiffness $\widehat{S}_{\omega\omega}$ and are identical to the differential equations of long beams without shear deformation.

Hence, the circular frequency is identical to Eq. (6.400):

$$(\omega_{\psi i})^2 = \frac{\widehat{GI}_{\mathrm{t}}}{\Theta}\frac{\mu_{Gi}^2}{L^2} \qquad \begin{matrix}\text{torsional vibration}\\ \text{long beam,}\end{matrix} \tag{7.227}$$

Arbitrary length. We now approximate the torsional circular frequency of a beam of arbitrary length by

$$(\omega_{\psi i})^2 = (\omega_{\psi i})^2\big|_{\mathrm{short}} + (\omega_{\psi i})^2\big|_{\mathrm{long}} \qquad \text{torsional vibration,} \tag{7.228}$$

By using Eqs. (7.224) and (7.227) we have

$$(\omega_{\psi i})^2 = \left[\frac{\Theta}{\widehat{EI}_\omega}\left(\frac{L}{\mu_{\mathrm{B}i}}\right)^4 + \frac{\Theta}{\widehat{S}_{\omega\omega}}\left(\frac{L}{\mu_{\mathrm{S}i}}\right)^2\right]^{-1} + \frac{\widehat{GI}_{\mathrm{t}}}{\Theta}\frac{\mu_{Gi}^2}{L^2}. \tag{7.229}$$

We introduce the notation

$$\frac{1}{\omega_{\omega i}^2} = \frac{1}{\left(\omega_{\omega i}^{\mathrm{B}}\right)^2} + \frac{\Theta}{\widehat{S}_{\omega\omega}}\left(\frac{L}{\mu_{\mathrm{S}i}}\right)^2, \tag{7.230}$$

where $\omega_{\omega i}^{\mathrm{B}}$ is given by (Eq. 6.405)

$$\left(\omega_{\omega i}^{\mathrm{B}}\right)^2 = \frac{\widehat{EI}_\omega}{\Theta}\frac{\mu_{\mathrm{B}i}^4}{L^4}. \tag{7.231}$$

When the notations above are used, Eq. (7.229) becomes

$$(\omega_{\psi i})^2 = \omega_{\omega i}^2 + \frac{\widehat{GI}_{\mathrm{t}}}{\Theta}\frac{\mu_{Gi}^2}{L^2} \qquad \text{torsional vibration,} \tag{7.232}$$

where $\mu_{\mathrm{B}i}$ and $\mu_{\mathrm{S}i}$ ($= \mu_{\mathrm{G}i}$) are given in Table 6.13 (page 308).

Although Eq. (7.232) is an approximation, it is accurate when both ends are simply supported.

The circular frequencies are summarized in Table B.2.

7.5.2 Beams with Symmetrical or Unsymmetrical Cross Sections

We consider freely vibrating orthotropic beams. The cross section may be symmetrical or unsymmetrical. The coordinate directions y and z are in the principal directions (page 208), and, consequently, $\widehat{EI}_{yz} = 0$.

When the cross section of the beam has one axis of symmetry (about the z-axis), it may undergo vibration in the plane of symmetry (x–z plane) or combined flexural–torsional vibration. When the cross section is unsymmetrical, the beam undergoes combined flexural–torsional vibration.

In general, for symmetrical and unsymmetrical cross sections, every element of the stiffness matrix must be considered in the calculation of the circular frequency. When the off-diagonal terms of the shear stiffness matrix (Eq. 7.36) either are zero (cross section is doubly symmetrical) or are negligible, the circular frequency may

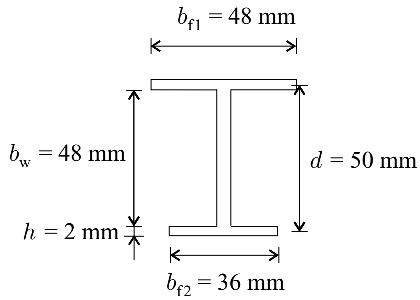


Figure 7.16: The cross section of the beam in Example 7.5.

be approximated by the solution of the following sets of equations[17]:
unsymmetrical cross section:

$$\left\| \begin{bmatrix} \omega_z^2 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & \omega_\psi^2 \frac{\Theta}{\rho} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & -(z_G - z_{sc}) \\ 0 & 1 & (y_G - y_{sc}) \\ -(z_G - z_{sc}) & (y_G - y_{sc}) & \frac{\Theta}{\rho} \end{bmatrix} \right\| = 0, \tag{7.233}$$

where || denotes determinant and ω is the circular frequency corresponding to flexural–torsional vibration. Equation (7.233) yields three sets of ω.
symmetrical cross-section:

$$\omega_1 = \omega_y \tag{7.234}$$

$$\left\| \begin{bmatrix} \omega_z^2 & 0 \\ 0 & \omega_\psi^2 \frac{\Theta}{\rho} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & -(z_G - z_{sc}) \\ -(z_G - z_{sc}) & \frac{\Theta}{\rho} \end{bmatrix} \right\| = 0. \tag{7.235}$$

Equations (7.234) and (7.235) yield three sets of ω. The first set is ω_1, which corresponds to the vibration in the x–z plane. The second and third sets of circular frequencies, corresponding to the flexural–torsional vibration, are the two roots of Eq. (7.235), and are denoted by ω_2 and ω_3.

We observe that the preceding equations (Eqs. 7.233–7.235) are identical to those given for beams without shear deformation (Eqs. 6.407–6.410). The difference is in the expressions used to calculate ω_y, ω_z, and ω_ψ. Here these circular frequencies are to be calculated by the expressions that include the shear deformation (Eqs. 7.206, 7.207, and 7.232).

Equations (7.233) and (7.235) give the exact circular frequency when $\widehat{S}_{ij} = 0$ $(i \neq j)$ and the two ends are simply supported (case (a) in Table 6.13, page 308). When both ends are built-in, when one end is built-in and one is simply supported, or when one end is built-in and the other end is free, the equations above yield only approximate values of the circular frequencies.

7.5 Example. *An* $L = 0.5$*-m-long I-section beam with the cross section shown in Figure 7.16 is made of graphite epoxy unidirectional plies. The material properties are given in Table 3.6 (page 81). The density of the composite is 1.6 g/cm*^{3}*. The*

[17] L. P. Kollár, Flexural-Torsional Vibration of Open Section Composite Beams with Shear Deformation. *International Journal of Solids and Structures,* Vol. 38, 7543–7558, 2001.

layup is $[0_{20}]$. *The beam is simply supported at each end. Calculate the natural frequencies of the beam.*

Solution. The circular frequencies are (Eqs. 7.206, 7.207, 7.230, and 7.232)

$$\begin{aligned}
\omega_{y1} &= \sqrt{\left(\frac{1}{\left(\omega_{y1}^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\widehat{S}_{zz}}{\rho}\frac{\mu_{S1}^2}{L^2}}\right)^{-1}} \\
\omega_{z1} &= \sqrt{\left(\frac{1}{\left(\omega_{z1}^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\widehat{S}_{yy}}{\rho}\frac{\mu_{S1}^2}{L^2}}\right)^{-1}} \\
\omega_{\omega 1} &= \sqrt{\left(\frac{1}{\left(\omega_{\omega 1}^{\mathrm{B}}\right)^2} + \frac{1}{\frac{\widehat{S}_{\omega\omega}}{\Theta}\frac{\mu_{S1}^2}{L^2}}\right)^{-1}} \\
\omega_{\psi 1} &= \sqrt{\omega_{\omega 1}^2 + \frac{\widehat{GI}_{\mathrm{t}}}{\Theta}\frac{\mu_{G1}^2}{L^2}}.
\end{aligned} \tag{7.236}$$

The parameters are $\omega_{y1}^{\mathrm{B}} = 8\,166\,\frac{1}{\mathrm{s}}$, $\omega_{z1}^{\mathrm{B}} = 3\,785\,\frac{1}{\mathrm{s}}$, $\omega_{\omega 1}^{\mathrm{B}} = 3\,458\,\frac{1}{\mathrm{s}}$ (Eq. 6.415), $\widehat{GI}_{\mathrm{t}} = 1.602\ \mathrm{N \cdot m^2}$ (Eq. 6.348), $i_\omega = 0.024\,99$ m (Eq. 6.350), $\mu_{\mathrm{B}1} = \mu_{G1} = \pi$ (Table 6.13, page 308), $\rho = 0.4224\ \frac{\mathrm{kg}}{\mathrm{m}}$ (Eq. 6.412), $\Theta = 0.000\,263\,7\ \mathrm{kg \cdot m}$ (Eq. 6.414), $z_{\mathrm{G}} = 0$ (Eq. 6.413), $z_{\mathrm{sc}} = 0.000\,789$ m (Eq. 6.349). $\hat{S}_{zz} = 414.4 \times 10^3$ N, $\hat{S}_{yy} = 637.0 \times 10^3$ N, $\hat{S}_{\omega\omega} = 0.4177\ \mathrm{Nm^2}$ (Eq. 7.163). With these values we have

$$\omega_{y1} = 4\,950\frac{1}{\mathrm{s}} \qquad \omega_{z1} = 3\,398\frac{1}{\mathrm{s}} \qquad \omega_{\omega 1} = 3\,169\frac{1}{\mathrm{s}} \qquad \omega_{\psi 1} = 3\,206\frac{1}{\mathrm{s}}\cdot \tag{7.237}$$

The circular frequency ω_1 is (Eq. 7.234)

$$\omega_1 = \omega_{y1} = 4\,950\frac{1}{\mathrm{s}}, \tag{7.238}$$

where ω_2 and ω_3 are the roots of Eq. (7.235):

$$\left\| \begin{bmatrix} \omega_{z1}^2 & 0 \\ 0 & \omega_{\psi 1}^2 \frac{\Theta}{\rho} \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & (z_{\mathrm{G}} - z_{\mathrm{sc}}) \\ (z_{\mathrm{G}} - z_{\mathrm{sc}}) & \frac{\Theta}{\rho} \end{bmatrix} \right\| = 0. \tag{7.239}$$

Solution of this equation yields

$$\omega_2 = 5\,428\frac{1}{\mathrm{s}} \qquad \omega_3 = 2\,563\frac{1}{\mathrm{s}}\cdot \tag{7.240}$$

The natural frequencies are ($f = \omega/2\pi$)

$$f_1 = 788\ \mathrm{Hz} \qquad f_2 = 864\ \mathrm{Hz} \qquad f_3 = 408\ \mathrm{Hz}. \tag{7.241}$$

Calculations that neglect the shear deformation give the natural frequencies as (Eq. 6.420)

$$f_1 = 1\,300\ \mathrm{Hz} \qquad f_2 = 959\ \mathrm{Hz} \qquad f_3 = 452\ \mathrm{Hz}. \tag{7.242}$$

We can compare these with the circular frequencies calculated above taking shear deformation into account. The circular frequencies calculated with and without shear deformation differ significantly.

7.5.3 Summary

In summary, when the cross section is such that the off-diagonal elements of the stiffness matrix are negligible ($\widehat{S}_{ij} = 0,\ i \neq j$), the circular frequencies may be approximated by the expressions derived for beams without shear deformation by the following substitutions:

$$
\begin{array}{ll}
\text{No shear deformation} & \text{With shear deformation} \\
\left(\omega_y^{\mathrm{B}}\right)^2 \Longrightarrow & \left(\dfrac{1}{\left(\omega_y^{\mathrm{B}}\right)^2} + \dfrac{1}{\frac{\widehat{S}_{zz}}{\rho}\frac{\mu_{\mathrm{S}i}^2}{L^2}} \right)^{-1} \\
\left(\omega_z^{\mathrm{B}}\right)^2 \Longrightarrow & \left(\dfrac{1}{\left(\omega_z^{\mathrm{B}}\right)^2} + \dfrac{1}{\frac{\widehat{S}_{yy}}{\rho}\frac{\mu_{\mathrm{S}i}^2}{L^2}} \right)^{-1} \\
\left(\omega_\omega^{\mathrm{B}}\right)^2 \Longrightarrow & \left(\dfrac{1}{\left(\omega_\omega^{\mathrm{B}}\right)^2} + \dfrac{1}{\frac{\widehat{S}_{\omega\omega}}{\Theta}\frac{\mu_{\mathrm{S}i}^2}{L^2}} \right)^{-1},
\end{array}
\tag{7.243}
$$

where ω_y^{B}, ω_z^{B}, and $\omega_\omega^{\mathrm{B}}$ are given by Eqs. (6.398), (6.399), and (6.405).

Numerical examples[18] show that the substitutions above are reasonable even for the cases when $\widehat{S}_{ij} \neq 0$.

Circular frequencies of beams with shear deformation are summarized in Appendix B.

7.6 Effect of Shear Deformation

In this section, we show some of the conditions under which shear deformation need to be considered in calculating the maximum deflection, maximum twist, buckling load, and natural frequency. We take into account shear deformation through a parameter α by expressing $\widetilde{w}$, ψ, $\widehat{N}_{\mathrm{cr}}$, and ω in the forms

$$
\begin{aligned}
\widetilde{w} &= \frac{\widetilde{w}^{\mathrm{B}}}{1-\alpha_{\mathrm{w}}} & \qquad \psi &= \frac{\psi^{\mathrm{B}}}{1-\alpha_\psi} \\
\widehat{N}_{\mathrm{cr}} &= \widehat{N}_{\mathrm{cr}}^{\mathrm{B}}(1-\alpha_N) & \qquad \omega^2 &= (\omega^{\mathrm{B}})^2(1-\alpha_\omega).
\end{aligned}
\tag{7.244}
$$

As before, $\widetilde{w}^{\mathrm{B}}$, ψ^{B}, $\widehat{N}_{\mathrm{cr}}^{\mathrm{B}}$, and ω^{B} are the values with no shear deformation. The effect of shear deformation is indicated by the magnitude of α. When α is small compared with unity ($\alpha \ll 1$) shear deformation may be neglected, and $\widetilde{w}$, ψ, $\widehat{N}_{\mathrm{cr}}$, and ω may, respectively, be approximated by $\widetilde{w}^{\mathrm{B}}$, ψ^{B}, $\widehat{N}_{\mathrm{cr}}^{\mathrm{B}}$, and ω^{B}. In the following we present expressions for estimating α_{w}, α_ψ, α_N, and α_ω.

[18] Ibid.

The maximum deflection of a beam with shear deformation is

$$\widetilde{w} = \widetilde{w}^{\mathrm{B}} + \widetilde{w}^{\mathrm{S}} = \frac{\widetilde{w}^{\mathrm{B}}}{1 - \frac{1}{1+\frac{\widetilde{w}^{\mathrm{B}}}{\widetilde{w}^{\mathrm{S}}}}}. \tag{7.245}$$

By comparing Eqs. (7.244) and (7.245), we observe that

$$\alpha_{\mathrm{w}} = \frac{1}{1 + \frac{\widetilde{w}^{\mathrm{B}}}{\widetilde{w}^{\mathrm{S}}}}. \tag{7.246}$$

The ratio $\widetilde{w}^{\mathrm{B}}/\widetilde{w}^{\mathrm{S}}$ is given in Table 7.3 (page 332) for different types of beams. By using the results in this table we may write α_{w} as

$$\alpha_{\mathrm{w}} = \frac{1}{1 + \left[\sqrt{C}\frac{\pi}{k}\right]^2 \frac{(kL)^2}{\pi^2}\frac{\widehat{S}}{\widehat{EI}}}, \tag{7.247}$$

where $\widehat{EI}$ and $\widehat{S}$ represent $\widehat{EI}_{yy}$ and $\widehat{S}_{zz}$ in the x–z plane and $\widehat{EI}_{zz}$ and $\widehat{S}_{yy}$ in the x–y plane and k is the equivalent length factor that depends only on the type of support. Values of k are given in Table 6.11 (page 293); C is a constant in front of $L^2\widehat{S}/\widehat{EI}$ in Table 7.3 (page 332). For the six cases listed in Table 7.3, $\sqrt{C}\pi/k$ ranges from 0.79 to 1.01. For other types of supports and loading, this ratio varies over a similar range. We are interested only in the magnitude of α_{w}. Therefore, we set $\sqrt{C}\pi/k$ equal to unity, and write

$$\alpha_{\mathrm{w}} = \frac{1}{1 + \frac{(kL)^2}{\pi^2}\frac{\widehat{S}}{\widehat{EI}}}. \tag{7.248}$$

The lateral buckling load of a beam is given by Eqs. (7.175) and (6.337). These equation may be rearranged to yield

$$\widehat{N}_{\mathrm{cr}} = \left(\frac{1}{\widehat{N}_{\mathrm{cr}}^{\mathrm{B}}} + \frac{1}{\widehat{S}}\right)^{-1} = \widehat{N}_{\mathrm{cr}}^{\mathrm{B}}\left(1 - \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}}{\widehat{EI}}}\right). \tag{7.249}$$

Equations (7.244) and (7.249) show that α_N is

$$\alpha_N = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}}{\widehat{EI}}}. \tag{7.250}$$

The circular frequency of a laterally vibrating beam is given by Eq. (7.207). This equation may be written as

$$\omega^2 = \left[\left(\frac{1}{\omega^{\mathrm{B}}}\right)^2 + \left(\frac{1}{\omega^{\mathrm{S}}}\right)^2\right]^{-1} = (\omega^{\mathrm{B}})^2\left(1 - \frac{1}{1 + \left(\frac{\pi\mu_{\mathrm{s}}}{k\mu_{\mathrm{B}}^2}\right)^2\left(\frac{kL}{\pi}\right)^2\frac{\widehat{S}}{\widehat{EI}}}\right), \tag{7.251}$$

where μ_{B} and μ_{s} are defined in Tables 6.13 and B.2 (pages 308 and 463). For the types of supports given in Table 6.11 (page 293), the ratio $\pi\mu_{\mathrm{s}}/k\mu_{\mathrm{B}}^2$ ranges from 0.7 to 1.0. We set this ratio equal to unity and approximate α_ω by (see Eqs. 7.244 and 7.251)

$$\alpha_\omega = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}}{\widehat{EI}}}. \tag{7.252}$$

Equations (7.248), (7.250), and (7.252) show that the effect of shear deformation on the maximum deflection, lateral buckling, and lateral vibration may be estimated by the parameter α ($\alpha = \alpha_{\mathrm{w}} = \alpha_N = \alpha_\omega$):

$$\alpha = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}}{\widehat{EI}}} \qquad \begin{array}{l}\text{deflection}\\ \text{lateral buckling}\\ \text{lateral vibration.}\end{array} \tag{7.253}$$

where $\widehat{EI}$ and $\widehat{S}$ are

$$\begin{array}{lll} \widehat{EI} = \widehat{EI}_{yy} & \widehat{S} = \widehat{S}_{zz} & \text{in the } x\text{–}z \text{ plane} \\ \widehat{EI} = \widehat{EI}_{zz} & \widehat{S} = \widehat{S}_{yy} & \text{in the } x\text{–}y \text{ plane.} \end{array} \tag{7.254}$$

The preceding expression for α applies to solid cross sections as well as to thin-walled, open- and closed-section beams. Next, we present α for torsional buckling, torsional vibration and twist of short, thin-walled open-section beams. We only consider short open-section beams because twist and torsion are significantly larger for open-section beams than for closed-section beams, and shear deformation is more pronounced for short than for long beams.

The torsional buckling load is given by Eqs. (7.154)–(7.156). For short, open-section beams, $\widehat{GI}_{\mathrm{t}} \ll \widehat{EI}_\omega / L^2$, and these equations yield

$$\widehat{N}_{\mathrm{cr}} = \left(\frac{1}{\widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega}} + \frac{1}{\frac{1}{i_\omega^2}\widehat{S}_{\omega\omega}} \right)^{-1} = \widehat{N}^{\mathrm{B}}_{\mathrm{cr}\omega} \left(1 - \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}_{\omega\omega}}{\widehat{EI}_\omega}} \right). \tag{7.255}$$

Equations (7.244) and (7.255) show that α_N is

$$\alpha_N = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}_{\omega\omega}}{\widehat{EI}_\omega}}. \tag{7.256}$$

The parameter α can be derived similarly for torsional vibration and for the rate of twist of short, open-section beams. For such beams the following inequality holds: $\widehat{GI}_{\mathrm{t}} \ll \widehat{EI}_\omega / L^2$. With this inequality, and with the approximations $\sqrt{C}\pi/k = 1$, and $\pi \mu_{\mathrm{s}} / k \mu_{\mathrm{B}}^2 = 1$ the expressions for α_ψ and α_ω are identical to that given by Eq. (7.256). Thus, for torsional buckling, torsional vibration and twist of short, open-section beams, α is ($\alpha = \alpha_\psi = \alpha_N = \alpha_\omega$):

$$\alpha = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}_{\omega\omega}}{\widehat{EI}_\omega}} \qquad \begin{array}{l}\text{twist}\\ \text{torsional buckling}\\ \text{torsional vibration.}\end{array} \tag{7.257}$$

We note the similarity in α given by Eqs. (7.253) and (7.257) and use this similarity to establish the conditions under which shear deformation may be neglected in calculating the deflection, twist, buckling load, and frequency. The error introduced into $\widetilde{w}$, ψ, $\widehat{N}_{\mathrm{cr}}$, and ω by neglecting shear deformation is less than 5 percent when α is smaller than 0.05. We now set $\alpha = 0.05$ in Eqs. (7.253) and (7.257). This

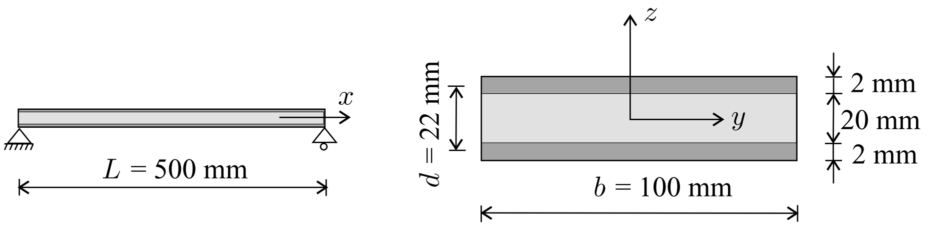


Figure 7.17: The sandwich beam in Example 7.6.

results in the expression

$$0.05 = \frac{1}{1 + \left(\frac{kL}{\pi}\right)^2 \frac{\widehat{S}}{\widehat{EI}}}. \tag{7.258}$$

By rearranging this expression, we obtain ($\alpha = 0.05$):

$$L \gtrless \frac{14}{k}\sqrt{\frac{\widehat{EI}}{\widehat{S}}} \qquad \begin{array}{l}\text{shear deformation negligible}\\ \text{shear deformation not negligible.}\end{array} \tag{7.259}$$

7.6 Example. *We consider an $L = 0.5$-m-long simply supported sandwich beam. The core is isotropic ($E_c = 2 \times 10^6$ kN/m^2, $\nu_c = 0.3$), and the facesheets are made of graphite epoxy. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is $[\pm 45^f_2/0_{12}/\pm 45^f_2]$. The dimensions are shown in (Fig. 7.17). Transverse or axial load may be applied in the x–z plane. Determine whether or not shear deformation needs be taken into account when calculating the maximum deflection, buckling load, and circular frequency.*

Solution. The beam is orthotropic, the cross section is symmetrical about the z-axis, and the load is in the x–z symmetry plane. Therefore, we only consider α in the x–z plane. The bending stiffness is $\widehat{EI}_{yy} = \frac{b}{\delta_{11}} \approx \frac{bd^2}{2a^f_{11}}$ (see Eq. 7.68 and Table 5.1, page 174), and the shear stiffness is $\widehat{S}_{zz} = \widetilde{S}_{11} b \approx \overline{C}^c_{55} bd$ (see Eqs. 7.62 and 5.32); a^f_{11} is the 11 element of the facesheet's compliance matrix and $\overline{C}^c_{55} = G$ is the shear modulus of the isotropic core. With these expressions, Eq. (7.253) gives ($\alpha = \alpha_w = \alpha_N = \alpha_\omega$):

$$\alpha = \left(1 + \frac{2}{\pi^2}\frac{(kL)^2 a^f_{11} G}{d}\right)^{-1} \qquad \text{in the } x\text{–}z \text{ plane.} \tag{7.260}$$

For the $[\pm 45^f_2/0_{12}/\pm 45^f_2]$ facesheets we have $a^f_{11} = 5.18 \times 10^{-9}$m/N (Table 3.8, page 85), and for the isotropic core $G = E_c/2\,(1 + \nu_c) = 769 \times 10^3$ kN/m^2. For a simply supported beam, we have $k = 1$ (Table 6.11, page 293). With these values, Eq. (7.260) becomes

$$\alpha_w = \alpha_N = \alpha_\omega = 0.10 \qquad \text{in the } x\text{–}z \text{ plane.} \tag{7.261}$$

For the sandwich beam in this problem we have

$$\frac{14}{k}\sqrt{\frac{\widehat{EI}}{\widehat{S}}} = 10\sqrt{\frac{d}{a^f_{11} G}} = 0.74 \text{ m} \qquad \text{in the } x\text{–}z \text{ plane.}$$

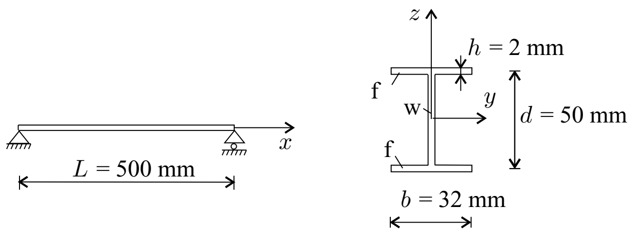


Figure 7.18: The I-beam in Example 7.7.

Thus, shear deformation is negligible when L is greater than 0.74 m. The length of the beam is 0.5 m. Therefore, shear deformation must be considered in calculating the deflection, buckling load, and circular frequency.

7.7 Example. *An $L = 0.5$-m-long doubly symmetrical I-beam is made of graphite epoxy. The material properties are given in Table 3.6 (page 81). The dimensions of the cross section are $b = 32$ mm and $d = 50$ mm (Fig 7.18).The layup of both the flange and the web is* $[\pm 45_2^{\mathrm{f}}/0_{12}/\pm 45_2^{\mathrm{f}}]$. *Determine whether or not shear deformation needs to be considered in calculating the maximum deflection, twist, buckling load, and circular frequency.*

Solution. The cross section of the I-beam is doubly symmetrical, and the layup of each flange and each web is orthotropic and symmetrical with respect to the midplane of the flange or the web. We approximate the replacement bending and shear stiffnesses by (Tables A.1, A.5, A.8)

$$\begin{aligned}
\widehat{EI}_{yy} &\approx \frac{b}{(a_{11})_{\mathrm{f}}}\frac{d^2}{2} & \widehat{S}_{zz} &\approx d\frac{1}{(a_{66})_{\mathrm{w}}} & &\text{in the } x\text{–}z \text{ plane} \\
\widehat{EI}_{zz} &\approx \frac{b^3}{6(a_{11})_{\mathrm{f}}} & \widehat{S}_{yy} &= \frac{2b}{1.2(a_{66})_{\mathrm{f}}} & &\text{in the } x\text{–}y \text{ plane} \\
\widehat{EI}_{\omega} &= \frac{1}{(a_{11})_{\mathrm{f}}}\frac{d^2b^3}{24} & \widehat{S}_{\omega\omega} &= \frac{d^2b}{2\times 1.2(a_{66})_{\mathrm{f}}} & &\text{in torsion.}
\end{aligned} \tag{7.262}$$

The dimension b and d are shown in Figure 7.18, and a_{11} and a_{66} are the elements of the compliance matrices given in Eq. (3.29). The subscripts f and w refer to the flange and the web. We obtain α by substituting the preceding stiffnesses into Eq. (7.253). For maximum deflection, buckling, and vibration in the x–y and x–z planes, α ($\alpha = \alpha_{\mathrm{w}} = \alpha_N = \alpha_{\omega}$) is

$$\alpha = \left(1 + \frac{2}{\pi^2}\frac{(kL)^2}{bd}\frac{(a_{11})_{\mathrm{f}}}{(a_{66})_{\mathrm{w}}}\right)^{-1} \qquad \text{in the } x\text{–}z \text{ plane} \tag{7.263}$$

$$\alpha = \left(1 + \frac{10}{\pi^2}\frac{(kL)^2}{b^2}\frac{(a_{11})_{\mathrm{f}}}{(a_{66})_{\mathrm{f}}}\right)^{-1} \qquad \text{in the } x\text{–}y \text{ plane,} \tag{7.264}$$

For the twist, torsional buckling, and torsional vibration α ($\alpha = \alpha_{\psi} = \alpha_N = \alpha_{\omega}$) is (see Eqs. 7.257 and 7.262)

$$\alpha = \left(1 + \frac{10}{\pi^2}\frac{(kL)^2}{b^2}\frac{(a_{11})_{\mathrm{f}}}{(a_{66})_{\mathrm{f}}}\right)^{-1} \qquad \text{about the } x\text{-axis,} \tag{7.265}$$

From Table 3.8 (page 85) the relevant elements of the compliance matrix are $(a_{11})_{\mathrm{f}} = (a_{11})_{\mathrm{w}} = 5.18 \times 10^{-9}$m/N, $(a_{66})_{\mathrm{f}} = (a_{66})_{\mathrm{w}} = 27.77 \times 10^{-9}$m/N. For a simply supported beam we have $k = 1$ (Table 6.11, page 293). With these values Eqs. (7.263)–(7.265) give

$$\begin{aligned} &\alpha_{\mathrm{w}} = \alpha_N = \alpha_\omega = 0.14 && \text{in the } x\text{–}z \text{ plane} \\ &\alpha_{\mathrm{w}} = \alpha_N = \alpha_\omega = 0.02 && \text{in the } x\text{–}y \text{ plane} \\ &\alpha_\psi = \alpha_N = \alpha_\omega = 0.02 && \text{about the } x\text{-axis.} \end{aligned} \tag{7.266}$$

For the I-beam in this problem we have

$$\begin{aligned} &\frac{14}{k}\sqrt{\frac{\widehat{EI}}{\widehat{S}}} = 14\sqrt{bd}\sqrt{\frac{(a_{66})_{\mathrm{w}}}{(a_{11})_{\mathrm{f}}}} = 1.3 \text{ m} && \text{in the } x\text{–}z \text{ plane} \\ &\frac{14}{k}\sqrt{\frac{\widehat{EI}}{\widehat{S}}} = 4.4b\sqrt{\frac{(a_{66})_{\mathrm{f}}}{(a_{11})_{\mathrm{f}}}} = 0.33\,\text{m} && \text{in the } x\text{–}y \text{ plane} \\ &\frac{14}{k}\sqrt{\frac{\widehat{EI}}{\widehat{S}}} = 4.4b\sqrt{\frac{(a_{66})_{\mathrm{f}}}{(a_{11})_{\mathrm{f}}}} = 0.33\,\text{m} && \text{about the } x\text{-axis.} \end{aligned} \tag{7.267}$$

Thus, shear deformation is negligible when L is greater than 1.3 m. The length of the beam is 0.5 m. Therefore, shear deformation must be taken into account when calculating the deflection, twist, buckling load, and circular frequency.