CHAPTER THREE

Laminated Composites

Composites are frequently made of layers (plies) bonded together to form a laminate (Fig. 3.1). A layer may consist of short fibers, unidirectional continuous fibers, or woven or braided fibers embedded in a matrix (Figs. 1.1 and 1.2). A layer containing woven or braided fibers is referred to as fabric.

Adjacent plies having the same material and the same orientation are referred to as a ply group. Since the properties and the orientations are the same across the ply group, a ply group may be treated as one layer.

3.1 Laminate Code

An x, y, z orthogonal coordinate system is used in analyzing laminates with the z coordinate being perpendicular to the plane of the laminate (Fig. 3.2).

The orientations of continuous, unidirectional plies are specified by the angle Θ (in degree) with respect to the x-axis (Fig. 3.2). The angle Θ is positive in the counterclockwise direction. The number of plies within a ply group is specified by a numerical subscript. For example, the laminate consisting of unidirectional plies and shown in Figure 3.3 is designated as

$$[45_3/0_4/90_2/60]\,.$$

This laminate contains four ply groups, the first containing three plies in the 45-degree direction, the second containing four plies in the 0-degree direction, the third containing two plies in the 90-degree direction, the fourth containing one ply in the 60-degree direction.

Symmetrical laminate. When the laminate is symmetrical with respect to the midplane it is referred to as a symmetrical laminate. Examples of symmetrical laminates are shown in Figure 3.4. The laminates represented in Figure 3.4 are

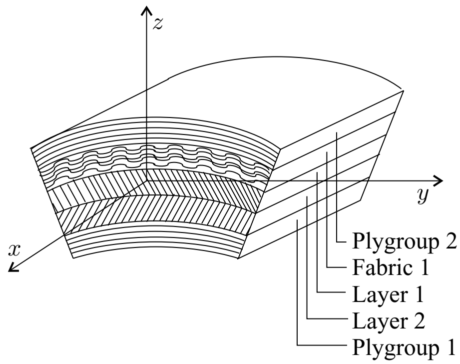


Figure 3.1: Laminated composite.

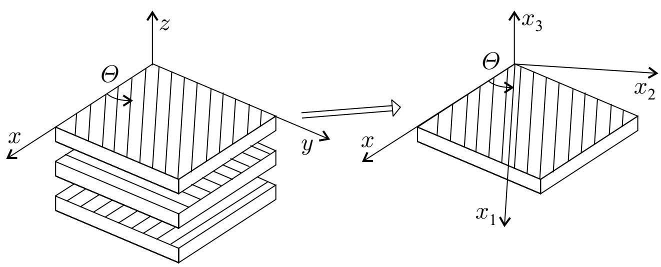
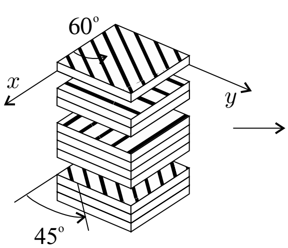


Figure 3.2: The x, y, z laminate coordinate system, the x_1, x_2, x_3 ply coordinate system, and the ply angle.



60
90
90
0
0
0
0
45
45
45

Figure 3.3: Description of the layup in a laminate consisting of unidirectional plies $[45_3/0_4/90_2/60]$.

–45
–45
0
0
0
0
–45
–45

$[-45_2/0_2]_s$

45
–45
–45
45
45
–45
–45
45

$[45/-45_2/45]_s$

Figure 3.4: Examples of symmetrical laminates.

$[45_2/90_2/30/-30/-45_2]$	$[45/-45/30/-30]_s$
–45	45
–45	–45
–30	30
30	–30
90	–30
90	30
45	–45
45	45

Figure 3.5: Examples of balanced laminates.

specified as

$$[-45_2/0_4/-45_2] \equiv [-45_2/0_2]_s$$
$$[45/-45_2/45_2/-45_2/45] \equiv [45/-45_2/45]_s \,.$$

The subscript s indicates symmetry about the midplane.

Balanced laminate. In balanced laminates, for every ply in the $+\Theta$ direction there is an identical ply in the $-\Theta$ direction. Examples of balanced laminates are shown in Figure 3.5.

Cross-ply laminates. In cross-ply laminates fibers are only in the 0- and 90-degree directions (Fig. 3.6). Cross-ply laminates may be symmetrical or unsymmetrical. Since there is no distinction between the +0 and −0 and between the +90- and −90-degree directions, cross-ply laminates are balanced.

Angle-ply laminate. Angle-ply laminates consist of plies in the $+\Theta$ and $-\Theta$ directions. Angle-ply laminates may by symmetrical or unsymmetrical, balanced or unbalanced. Examples of angle-ply laminates are shown in Figure 3.7.

$\pi/4$ laminate. $\pi/4$ laminates consist of plies in which the fibers are in the 0-, 45-, 90-, and −45-degree directions. The number of plies in each direction is the same (balanced laminate). In addition, the layup is also symmetrical.

3.2 Stiffness Matrices of Thin Laminates

Thin laminates are characterized by three stiffness matrices denoted by $[A]$, $[B]$, and $[D]$. In this section we determine these matrices for thin, flat laminates undergoing small deformations. The analyses are based on the laminate plate theory and are formulated using the approximations that the strains vary linearly across the

$[90_2/0_2]$	$[0/90]_s$
0	0
0	90
90	90
90	0

Figure 3.6: Examples of cross-ply laminates.

–45
–45
45
45
–45
–45

$[-45_2/45_2/-45_2]$

30
30
30
–30
–30
–30

$[-30_3/30_3]$

Figure 3.7: Examples of angle-ply laminates.

laminate, (out-of-plane) shear deformations are negligible, and the out-of-plane normal stress σ_z and the shear stresses τ_{xz}, τ_{yz} are small compared with the in-plane σ_x, σ_y, and τ_{xy} stresses. These approximations imply that the stress–strain relationships under plane-stress conditions may be applied. The x, y, z refer to a coordinate system with the x and y coordinates in a suitably chosen reference plane, and z is perpendicular to this reference plane (Fig. 3.8).

Frequently, though not always, for convenience the reference plane is taken to be the midplane of the laminate. Unless the laminate is symmetrical with respect to the reference plane, the reference plane is not a neutral plane, and the strains in the reference plane are not zero under pure bending. The strains in the reference plane are (see Eqs. 2.2, 2.3, and 2.11)

$$\epsilon_x^{\mathrm{o}} = \frac{\partial u^{\mathrm{o}}}{\partial x} \qquad \epsilon_y^{\mathrm{o}} = \frac{\partial v^{\mathrm{o}}}{\partial y} \qquad \gamma_{xy}^{\mathrm{o}} = \frac{\partial u^{\mathrm{o}}}{\partial y} + \frac{\partial v^{\mathrm{o}}}{\partial x}, \tag{3.1}$$

where u and v are the x, y components of the displacement and the superscript 0 refers to the reference plane.

We adopt the Kirchhoff hypothesis, namely, that normals to the reference surface remain normal and straight (Fig. 3.9). Accordingly, for small deflections the angle of rotation of the normal of the reference plane χ_{xz} is

$$\chi_{xz} = \frac{\partial w^{\mathrm{o}}}{\partial x}, \tag{3.2}$$

where w^{o} is the out-of-plane displacement of the reference plane. The total displacement in the x direction is

$$u = u^{\mathrm{o}} - z\chi_{xz} = u^{\mathrm{o}} - z\frac{\partial w^{\mathrm{o}}}{\partial x}. \tag{3.3}$$

Similarly, the total displacement in the y direction is

$$v = v^{\mathrm{o}} - z\frac{\partial w^{\mathrm{o}}}{\partial y}. \tag{3.4}$$

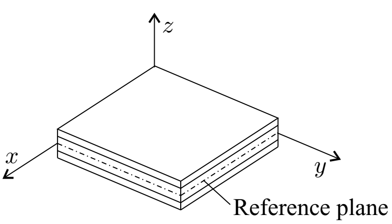


Figure 3.8: The coordinate system.

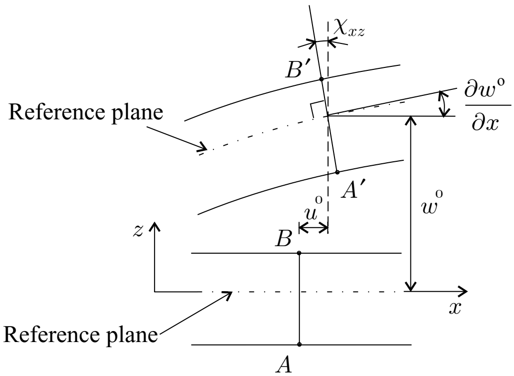


Figure 3.9: Deformation of a plate in the x–z plane.

By definition, the strains are (Eqs. 2.2, 2.3, 2.11)

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{3.5}$$

Substituting Eqs. (3.3) and (3.4) into these expressions, we obtain

$$\begin{aligned} \epsilon_x &= \frac{\partial u^o}{\partial x} - z\frac{\partial^2 w^o}{\partial x^2} \\ \epsilon_y &= \frac{\partial v^o}{\partial y} - z\frac{\partial^2 w^o}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} - z\frac{2\partial^2 w^o}{\partial x \partial y}. \end{aligned} \tag{3.6}$$

These equations can be written in the following form:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{3.7}$$

where ϵ_x^o, ϵ_y^o, γ_{xy}^o are the strains in the reference plane (Eq. 3.1), and κ_x, κ_y, and κ_{xy} are the curvatures of the reference plane of the plate (Fig. 3.10) defined as

$$\kappa_x = -\frac{\partial^2 w^o}{\partial x^2} \qquad \kappa_y = -\frac{\partial^2 w^o}{\partial y^2} \qquad \kappa_{xy} = -\frac{2\partial^2 w^o}{\partial x \partial y}. \tag{3.8}$$

The in-plane forces and moments acting on a small element are (Fig. 3.11)

$$\begin{aligned} N_x &= \int_{-h_b}^{h_t} \sigma_x dz \qquad N_y = \int_{-h_b}^{h_t} \sigma_y dz \qquad N_{xy} = \int_{-h_b}^{h_t} \tau_{xy} dz \\ M_x &= \int_{-h_b}^{h_t} z\sigma_x dz \qquad M_y = \int_{-h_b}^{h_t} z\sigma_y dz \qquad M_{xy} = \int_{-h_b}^{h_t} z\tau_{xy} dz, \end{aligned} \tag{3.9}$$

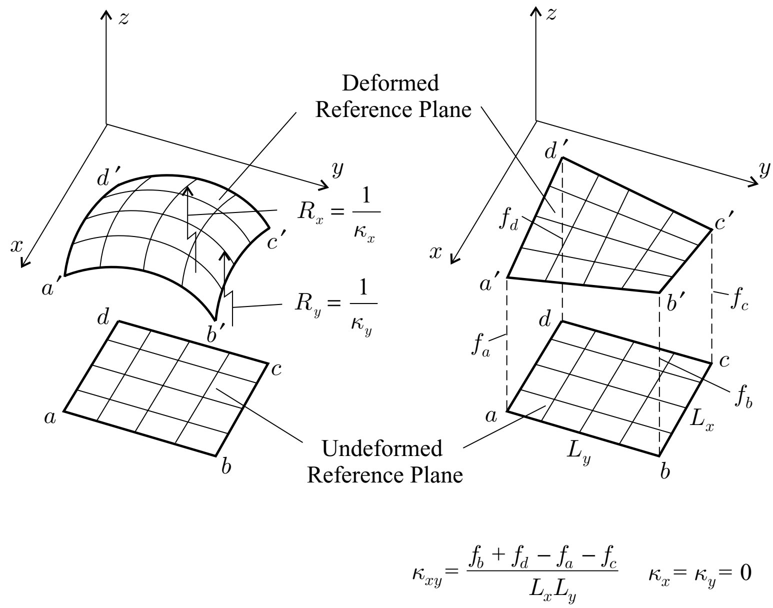


Figure 3.10: The curvatures κ_x, κ_y, and κ_{xy} of the reference plane.

where N and M are the in-plane forces and moments (per unit length), and h_{t} and h_{b} are the distances from the reference plane to the plate's surfaces (Fig. 3.12). The transverse shear forces (per unit length) are (Fig. 3.11, right)

$$V_x = \int_{-h_{\mathrm{b}}}^{h_{\mathrm{t}}} \tau_{xz} dz \qquad V_y = \int_{-h_{\mathrm{b}}}^{h_{\mathrm{t}}} \tau_{yz} dz. \tag{3.10}$$

We now recall that for plane-stress condition the stress–strain relationships *for each ply* are (Eq. 2.126)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{3.11}$$

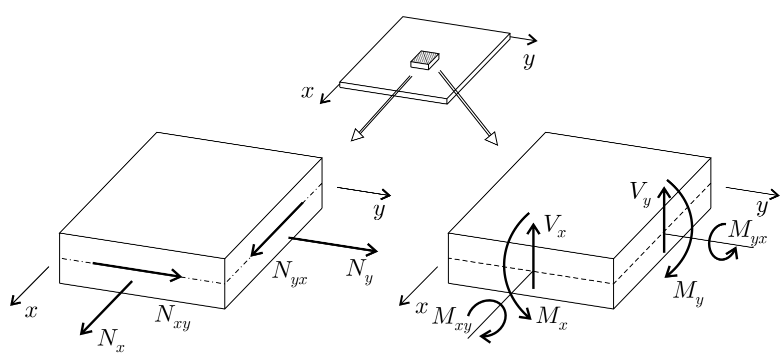


Figure 3.11: The in-plane forces acting at the reference plane (left) and the moments and the transverse shear forces (right).

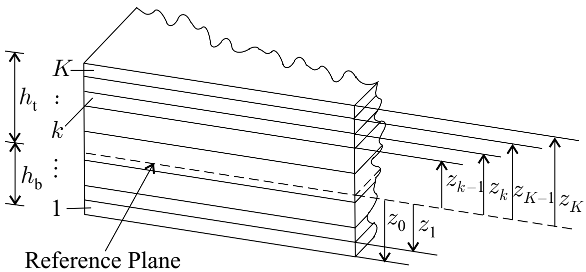


Figure 3.12: Distances from the reference plane.

By introducing the notation

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}, \tag{3.12}$$

we write the stress–strain relationships for a ply as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\overline{Q}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{3.13}$$

where $[\overline{Q}]$ is the stiffness matrix of the ply in the x–y coordinate system. The elements of this stiffness matrix are obtained from the elements of the stiffness matrix $[Q]$ in the $x_1 - x_2$ coordinate system by the transformation given by Eq. (2.195). By replacing $[Q]$ and $[Q']$ by $[\overline{Q}]$ and $[Q]$, respectively, Eq. (2.195) yields

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} = [T_\sigma]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} [T_\epsilon], \tag{3.14}$$

where $[T_\sigma]$ and $[T_\epsilon]$ are given by Eq. (2.196) and are reiterated below,

$$[T_\sigma] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \qquad [T_\epsilon] = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}, \tag{3.15}$$

and $c = \cos\Theta$, $s = \sin\Theta$ with Θ defined in Figure 3.2. For an orthotropic ply the local coordinates x_1, x_2 are in the orthotropy directions. For transversely isotropic plies these local coordinates are parallel and perpendicular to the fibers (Fig. 2.15). For orthotropic and transversely isotropic materials the elements of the stiffness matrix in the global coordinate system are given in Table 3.1 in terms of the elements of the stiffness matrix in the local coordinate system.

Table 3.1. The elements of the $[\overline{Q}]$ matrix for an othotropic or transversely isotropic ply oriented in the $+\Theta$ direction (Fig 3.2)

$$\overline{Q}_{11} = c^4 Q_{11} + s^4 Q_{22} + 2c^2 s^2 (Q_{12} + 2Q_{66})$$
$$\overline{Q}_{22} = s^4 Q_{11} + c^4 Q_{22} + 2c^2 s^2 (Q_{12} + 2Q_{66})$$
$$\overline{Q}_{12} = c^2 s^2 (Q_{11} + Q_{22} - 4Q_{66}) + (c^4 + s^4) Q_{12}$$
$$\overline{Q}_{66} = c^2 s^2 (Q_{11} + Q_{22} - 2Q_{12}) + (c^2 - s^2)^2 Q_{66}$$
$$\overline{Q}_{16} = cs(c^2 Q_{11} - s^2 Q_{22} - (c^2 - s^2)(Q_{12} + 2Q_{66}))$$
$$\overline{Q}_{26} = cs(s^2 Q_{11} - c^2 Q_{22} + (c^2 - s^2)(Q_{12} + 2Q_{66}))$$
$$c = \cos\,\Theta \qquad s = \sin\,\Theta$$

By substituting Eqs. (3.7) and (3.13) into Eq. (3.9), we obtain

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h_b}^{h_t} \left\{ [\overline{Q}] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [\overline{Q}] z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right\} dz$$
$$= \int_{-h_b}^{h_t} [\overline{Q}] dz \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \int_{-h_b}^{h_t} z [\overline{Q}] dz \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3.16}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h_b}^{h_t} z \left\{ [\overline{Q}] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [\overline{Q}] z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right\} dz$$
$$= \int_{-h_b}^{h_t} z [\overline{Q}] dz \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + \int_{-h_b}^{h_t} z^2 [\overline{Q}] dz \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{3.17}$$

The stiffness matrices of the laminate are defined as

$$[A] = \int_{-h_b}^{h_t} [\overline{Q}] dz$$
$$[B] = \int_{-h_b}^{h_t} z [\overline{Q}] dz \tag{3.18}$$
$$[D] = \int_{-h_b}^{h_t} z^2 [\overline{Q}] dz.$$

The elements of these matrices are $(i, j = 1, 2, 6)$

$$A_{ij} = \int_{-h_b}^{h_t} \overline{Q}_{ij} dz \qquad B_{ij} = \int_{-h_b}^{h_t} z\overline{Q}_{ij} dz$$
$$D_{ij} = \int_{-h_b}^{h_t} z^2 \overline{Q}_{ij} dz. \tag{3.19}$$

The $[A]$, $[B]$, and $[D]$ matrices are the stiffness matrices of the laminate, and $[\overline{Q}]$ is the stiffness matrix of the ply. Since $[\overline{Q}]$ is constant across each ply, the integrals in the equations above (Eq. 3.19) may be replaced by summations (Fig. 3.12) as follows $(i, j = 1, 2, 6)$:

$$A_{ij} = \sum_{k=1}^{K} (\overline{Q}_{ij})_k (z_k - z_{k-1})$$
$$B_{ij} = \frac{1}{2} \sum_{k=1}^{K} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2) \tag{3.20}$$
$$D_{ij} = \frac{1}{3} \sum_{k=1}^{K} (\overline{Q}_{ij})_k (z_k^3 - z_{k-1}^3),$$

where K is the total number of plies (or ply groups) in the laminate; z_k, z_{k-1} are the distances from the reference plane to the two surfaces of the kth ply; and $(\overline{Q}_{ij})_k$ are the elements of the stiffness matrix of the kth ply.

With the preceding definitions of the stiffness matrices, the expressions for the in-plane forces and moments (Eqs. 3.16 and 3.17) become

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{3.21}$$

The vectors on the left and right hand side represent generalized forces and strains. Hereafter, we simply refer to these as forces and strains.

By inverting Eqs. (3.21), we obtain the strains and curvatures in terms of the in-plane forces and moments:

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{16} & \beta_{11} & \beta_{12} & \beta_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{26} & \beta_{21} & \beta_{22} & \beta_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{66} & \beta_{61} & \beta_{62} & \beta_{66} \\ \beta_{11} & \beta_{21} & \beta_{61} & \delta_{11} & \delta_{12} & \delta_{16} \\ \beta_{12} & \beta_{22} & \beta_{62} & \delta_{12} & \delta_{22} & \delta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} & \delta_{16} & \delta_{26} & \delta_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix}. \tag{3.22}$$

The $[\alpha]$, $[\beta]$, and $[\delta]$ matrices are related to the $[A]$, $[B]$, and $[D]$ matrices by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{16} & \beta_{11} & \beta_{12} & \beta_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{26} & \beta_{21} & \beta_{22} & \beta_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{66} & \beta_{61} & \beta_{62} & \beta_{66} \\ \beta_{11} & \beta_{21} & \beta_{61} & \delta_{11} & \delta_{12} & \delta_{16} \\ \beta_{12} & \beta_{22} & \beta_{62} & \delta_{12} & \delta_{22} & \delta_{26} \\ \beta_{16} & \beta_{26} & \beta_{66} & \delta_{16} & \delta_{26} & \delta_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}^{-1}. \tag{3.23}$$

3.2.1 The Significance of the $[A]$, $[B]$, and $[D]$ Stiffness Matrices

The $[A]$, $[B]$, and $[D]$ matrices represent the stiffnesses of a laminate and describe the response of the laminate to in-plane forces and moments.

A_{ij} are the in-plane stiffnesses that relate the in-plane forces N_x, N_y, N_{xy} to the in-plane deformations ϵ_x^o, ϵ_y^o, γ_{xy}^o.

D_{ij} are the bending stiffnesses that relate the moments M_x, M_y, M_{xy} to the curvatures κ_x, κ_y, κ_{xy}.

Table 3.2. Illustration of the coupling terms A_{16}, D_{16}, B_{16}, B_{11}, B_{12}, B_{66} for composite materials. When the element shown in the last column is zero, there is no coupling. (The coupling terms A_{26}, D_{26}, B_{26}, B_{22} can be illustrated in a similar manner by applying a force N_y and a moment M_y in the y–z plane.)

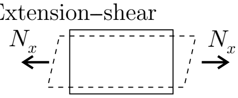
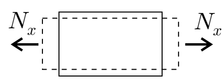
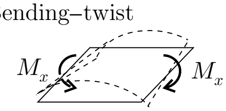
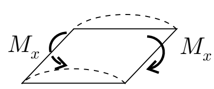
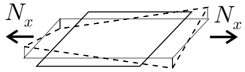
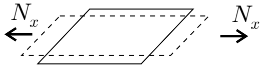
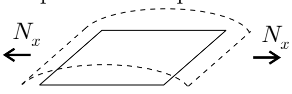
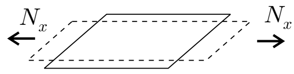
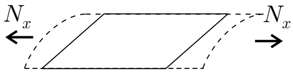
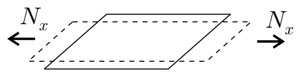
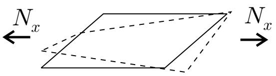
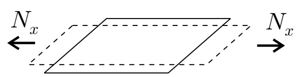
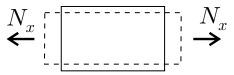
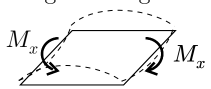
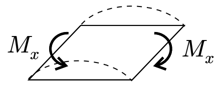
Coupling	No Coupling	Element
Extension–shear 		A_{16}
Bending–twist 		D_{16}
Extension–twist 		B_{16}
In-plane–out-of-plane 		B_{11}
		B_{12}
		B_{66}

Table 3.3. Illustration of the coupling terms A_{12}, D_{12} that may be present both in composite and in isotropic materials. When the element shown in the last column is zero, there is no coupling.

Coupling	No Coupling	Element
Extension–extension		
		A_{12}
Bending–bending		
		D_{12}

B_{ij} are the in-plane–out-of-plane coupling stiffnesses that relate the in-plane forces N_x, N_y, N_{xy} to the curvatures κ_x, κ_y, κ_{xy} and the moments M_x, M_y, M_{xy} to the in-plane deformations ϵ_x^o, ϵ_y^o, γ_{xy}^o.

Examination of the $[A]$, $[B]$, and $[D]$ matrices shows that different types of couplings may occur as discussed below and illustrated in Tables 3.2 and 3.3.

Extension–shear coupling. When the elements A_{16}, A_{26} are not zero, in-plane normal forces N_x, N_y cause shear deformation γ_{xy}^o, and a twist force N_{xy} causes elongations in the x and y directions.

Bending–twist coupling. When the elements D_{16}, D_{26} are not zero, bending moments M_x, M_y cause twist of the laminate κ_{xy}, and a twist moment M_{xy} causes curvatures in the x–z and y–z planes.

Extension–twist and bending–shear coupling. When the elements B_{16}, B_{26} are not zero, in-plane normal forces N_x, N_y cause twist κ_{xy}, and bending moments M_x, M_y result in shear deformation γ_{xy}^o.

In-plane–out-of-plane coupling. When the elements B_{ij} are not zero, in-plane forces N_x, N_y, N_{xy} cause out-of-plane deformations (curvatures) of the laminate, and moments M_x, M_y, M_{xy} cause in-plane deformations in the x–y plane.

The preceding four types of coupling are characteristic of composite materials and do not occur in homogeneous isotropic materials. The following two couplings occur in both composite and isotropic materials (Table 3.3):

Extension–extension coupling. When the element A_{12} is not zero, a normal force N_x causes elongation in the y direction ϵ_y^o, and a normal force N_y causes elongation in the x direction ϵ_x^o.

Bending–bending coupling. When the element D_{12} is not zero, a bending moment M_x causes curvature of the laminate in the y–z plane κ_y, and a bending moment M_y causes curvature of the laminate in the x–z plane κ_x.

3.2.2 Stiffness Matrices for Selected Laminates

For certain ply arrangements (layups), some of the couplings described do not occur, and the $[A]$, $[B]$, $[D]$ matrices become simpler.

Symmetrical laminate. In a symmetrical laminate the ply located at a position $+z$ is identical to the ply at $-z$. Correspondingly, the stiffness matrix $[\overline{Q}]$ of the ply at $+z$ is identical to the stiffness matrix of the ply at $-z$:

$$[\overline{Q}]\,(z) = [\overline{Q}]\,(-z)\,. \tag{3.24}$$

By substituting these stiffnesses into Eq. (3.18), we find that the $[B]$ matrix is zero:

$$[B] = 0 \qquad \text{(symmetrical)}. \tag{3.25}$$

Thus, for a symmetrical laminate there is no in-plane–out-of-plane coupling, and Eq. (3.21) reduces to

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{\text{o}} \\ \epsilon_y^{\text{o}} \\ \gamma_{xy}^{\text{o}} \end{Bmatrix} \tag{3.26}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{3.27}$$

When the laminate is symmetrical, the compliance matrix is generally expressed as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{16} \\ \alpha_{12} & \alpha_{22} & \alpha_{26} \\ \alpha_{16} & \alpha_{26} & \alpha_{66} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \quad \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{16} \\ \delta_{12} & \delta_{22} & \delta_{26} \\ \delta_{16} & \delta_{26} & \delta_{66} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix}, \tag{3.28}$$

where

$$\begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \tag{3.29}$$

$$\begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}^{-1}. \tag{3.30}$$

The relationships between the strains and curvatures and the forces and moments (Eq. 3.22) now simplify to

$$\begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \tag{3.31}$$

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}. \tag{3.32}$$

Balanced laminate. In a balanced laminate, for every unidirectional ply in the $+\Theta$ direction (measured counterclockwise from the x coordinate) there is an identical ply in the $-\Theta$ direction. The elements of the stiffness matrix $[\overline{Q}]$ are given in Table 3.1 (page 70). From this table we deduce that the elements of the stiffness matrices of plies in the $+\Theta$ and $-\Theta$ directions are related as follows:

$$\begin{aligned} \overline{Q}_{11(+\Theta)} &= \overline{Q}_{11(-\Theta)} & \overline{Q}_{22(+\Theta)} &= \overline{Q}_{22(-\Theta)} \\ \overline{Q}_{12(+\Theta)} &= \overline{Q}_{12(-\Theta)} & \overline{Q}_{66(+\Theta)} &= \overline{Q}_{66(-\Theta)} \\ \overline{Q}_{16(+\Theta)} &= -\overline{Q}_{16(-\Theta)} & \overline{Q}_{26(+\Theta)} &= -\overline{Q}_{26(-\Theta)}. \end{aligned} \tag{3.33}$$

By substituting these elements into the expression of the stiffness matrix in Eq. (3.20), we find that

$$A_{16} = A_{26} = 0. \tag{3.34}$$

The structure of the stiffness matrix given in Table 3.4 shows that there is no extension–shear coupling in a balanced laminate (Table 3.5). (Note that A_{16} and A_{26} are zero only in the x–y coordinate system.)

Elements A_{16} and A_{26} are zero for symmetrical and unsymmetrical laminates. Correspondingly, for symmetrical balanced laminates the a_{16} and a_{26} elements of the compliance matrix are zero ($a_{16} = 0$ and $a_{26} = 0$). However, for unsymmetrical balanced laminates none of the elements of the compliance matrix is zero (see Eq. 3.22).

Orthotropic laminate. In orthotropic laminates we are interested in two mutually perpendicular directions, called orthotropy directions, in the plane of the laminate. Normal forces and bending moments applied in these directions do not cause shear or twist of the laminate. Hence, there are no extension–shear, bending–twist, and extension–twist couplings.

A laminate is orthotropic when every ply is orthotropic and the orthotropy directions coincide with the x and y directions. Fiber-reinforced plies are orthotropic under the following conditions:

- when the ply is made of unidirectional fibers and all the fibers are aligned with one of the laminate's orthotropy directions (Fig. 3.13);

Table 3.4. The [*A*], [*B*], [*D*] matrices for laminates. When the laminate is symmetrical, the [*B*] matrix is zero. Cross-ply laminates are orthotropic.

	[*A*]	**[*B*]**	**[*D*]**
Symmetrical	$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Balanced	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$
Orthotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$
Isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{11} & 0 \\ 0 & 0 & \frac{B_{11}-B_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{11} & 0 \\ 0 & 0 & \frac{D_{11}-D_{12}}{2} \end{bmatrix}$
Quasi-isotropic	$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}-A_{12}}{2} \end{bmatrix}$	$\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$	$\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$

Table 3.5. Couplings in selected laminates ("no" means that the indicated element of the stiffness matrix is zero and the corresponding coupling does not occur)

	Extension–shear A_{16}	**Bending–twist** D_{16}	**Extension–twist** B_{16}	**In-plane–out-of-plane** B_{ij}
Symmetrical			no	no
Balanced	no			
Orthotropic	no	no	no	
Quasi-isotropic	no			
Isotropic layered	no	no	no	
Isotropic single layer	no	no	no	no

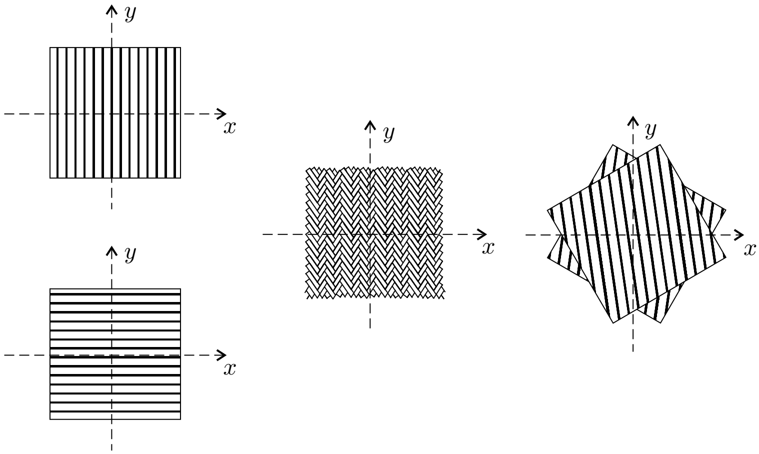


Figure 3.13: Ply arrangements in orthotropic laminates. The ply's symmetry axes (dashed lines) must coincide with the laminate's orthotropy x, y axes.

- when the ply is a woven fabric and the ply's symmetry axes are aligned with the laminates orthotropy directions;
- when two adjacent unidirectional plies (oriented in different directions) are treated as a single layer and the symmetry axes of this layer are aligned with the laminate's orthotropy directions.

For the orthotropic plies described above, the $\overline{Q}_{16}$ and $\overline{Q}_{26}$ elements of the ply stiffness matrix are zero (Eq. 2.138):

$$\overline{Q}_{16} = \overline{Q}_{26} = 0. \tag{3.35}$$

With these values, Eq. (3.20) gives that the 16 and 26 elements of the $[A]$, $[B]$, and $[D]$ matrices are zero:

$$A_{16} = A_{26} = 0 \qquad B_{16} = B_{26} = 0 \qquad D_{16} = D_{26} = 0. \tag{3.36}$$

Accordingly, there is no extension–shear, bending–twist, or extension–twist coupling in an orthotropic laminate (Table 3.5). On the other hand, when the laminate is not orthotropic, these couplings are present and result in unexpected deformations.

We observe that the 16 and 26 elements of the $[A]$, $[B]$, and $[D]$ matrices are zero only in the x–y coordinate system, where x and y are the orthotropy directions (Table 3.4, page 76).

For unsymmetrical orthotropic laminates the compliance matrix becomes (see Eq. 3.23)

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & \beta_{11} & \beta_{12} & 0 \\ \alpha_{12} & \alpha_{22} & 0 & \beta_{21} & \beta_{22} & 0 \\ 0 & 0 & \alpha_{66} & 0 & 0 & \beta_{66} \\ \beta_{11} & \beta_{21} & 0 & \delta_{11} & \delta_{12} & 0 \\ \beta_{12} & \beta_{22} & 0 & \delta_{12} & \delta_{22} & 0 \\ 0 & 0 & \beta_{66} & 0 & 0 & \delta_{66} \end{bmatrix}. \tag{3.37}$$

When the layup is orthotropic and symmetrical, the elements of the compliance matrices are (see Eqs. 3.29 and 3.30)

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \quad \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{66} \end{bmatrix}. \tag{3.38}$$

Isotropic laminate. We consider a laminate in which each ply is isotropic. (The material may be different in each ply.) Since in isotropic materials there is no preferred direction, the $[\overline{Q}]$ matrix in the x–y coordinate system is the same as the $[Q]$ matrix in the x_1–x_2 coordinate system:

$$[\overline{Q}] = [Q]. \tag{3.39}$$

Consequently, the $[A]$, $[B]$, $[D]$ matrices are independent of the coordinate directions. By introducing the elements of the $[Q]$ matrix given by Eq. (2.145) into Eq. (3.20), we obtain the following elements of the $[A]$, $[B]$, and $[D]$ matrices:

$$\begin{array}{lll} A_{11} & A_{22} = A_{11} & A_{12} \\ A_{66} = \dfrac{A_{11} - A_{12}}{2} & A_{16} = 0 & A_{26} = 0 \\ B_{11} & B_{22} = B_{11} & B_{12} \\ B_{66} = \dfrac{B_{11} - B_{12}}{2} & B_{16} = 0 & B_{26} = 0 \\ D_{11} & D_{22} = D_{11} & D_{12} \\ D_{66} = \dfrac{D_{11} - D_{12}}{2} & D_{16} = 0 & D_{26} = 0. \end{array} \tag{3.40}$$

In isotropic laminates there are no extension–shear, bending–twist, or extension–twist couplings (Table 3.5, page 76), but there may be in-plane–out-of-plane coupling.

When the laminate consists of a single isotropic layer, the nonzero elements of the $[A]$, $[B]$, and $[D]$ matrices are (Eqs. 2.145 and 3.20)

$$\begin{array}{lll} A_{11} = A_{22} = A^{\mathrm{iso}} & A_{12} = \nu A^{\mathrm{iso}} & A_{66} = \frac{1-\nu}{2} A^{\mathrm{iso}} \\ D_{11} = D_{22} = D^{\mathrm{iso}} & D_{12} = \nu D^{\mathrm{iso}} & D_{66} = \frac{1-\nu}{2} D^{\mathrm{iso}}, \end{array} \tag{3.41}$$

where

$$A^{\mathrm{iso}} = \frac{Eh}{1-\nu^2}, \qquad D^{\mathrm{iso}} = \frac{Eh^3}{12(1-\nu^2)}, \tag{3.42}$$

E is the Young modulus, ν is the Poisson ratio, and h is the thickness. The preceding stiffnesses are identical to the stiffnesses of isotropic plates.[1]

[1] S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells.* 2nd edition. McGraw-Hill, New York, 1959, pp. 5 and 81.

Inversion of matrices $[A]$ and $[D]$ yields the compliance matrices $[a]$ and $[d]$ (see Eqs. 3.29 and 3.30). The nonzero elements are

$$\begin{aligned} a_{11} &= \frac{1}{Eh} \qquad a_{12} = -\nu a_{11} \qquad a_{66} = \frac{2(1+\nu)}{Eh} = \frac{1}{Gh} \\ d_{11} &= \frac{12}{Eh^3} \qquad d_{12} = -\nu d_{11} \qquad d_{66} = \frac{24(1+\nu)}{Eh^3} = \frac{12}{Gh^3}. \end{aligned} \tag{3.43}$$

Quasi-isotropic laminate. A laminate is quasi-isotropic when

- there are at least three fiber directions;
- the orientation (fiber angle Θ) of each ply is $\Theta = i\pi/I$, where i is an integer ($i = 1, 2, \ldots, I$), and I is the total number of fiber orientations ($I \geq 3$);
- the number of plies in each fiber direction is the same; and
- each ply is made of the same material and has the same thickness.

For example, in $\pi/4$ laminates (page 65) there are fibers in the 0°, 45°, −45°, and 90° directions, and $I = 4$.

For each ply, the elements of the $[\overline{Q}]$ matrix are obtained by substituting $\Theta = i180°/I$ into the expressions in Table 3.1 (page 70). Then, by substituting these elements into the expression for the $[A]$ matrix (see Eq. 3.20), we obtain the following nonzero elements of the $[A]$ matrix:

$$\begin{aligned} A_{11} &= \frac{3}{8}h(Q_{11} + Q_{22}) + \frac{1}{4}hQ_{12} + \frac{1}{2}hQ_{66} \\ A_{22} &= A_{11} \\ A_{12} &= \frac{1}{8}h(Q_{11} + Q_{22}) + \frac{3}{4}hQ_{12} - \frac{1}{2}hQ_{66} \\ A_{66} &= \frac{A_{11} - A_{12}}{2}. \end{aligned} \tag{3.44}$$

The stiffness matrix $[A]$ may be written as

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} = hR \begin{bmatrix} 1 & \nu^{\text{iso}} & \\ \nu^{\text{iso}} & 1 & \\ & & \frac{1-\nu^{\text{iso}}}{2} \end{bmatrix}, \tag{3.45}$$

where R and ν^{iso} are parameters defined as

$$\begin{aligned} R &= \frac{3}{8}(Q_{11} + Q_{22}) + \frac{1}{4}Q_{12} + \frac{1}{2}Q_{66} \\ \nu^{\text{iso}} &= \frac{1}{8R}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}), \end{aligned} \tag{3.46}$$

and h is the thickness of the laminate. It is stated here without proof that Eqs. (3.45) and (3.46) are valid[2] for all values of I as long as $I \geq 3$.

[2] S. W. Tsai and H. T. Hahn, *Introduction to Composite Materials*. Technomic, Lancaster, Pennsylvania, 1980, p. 145.

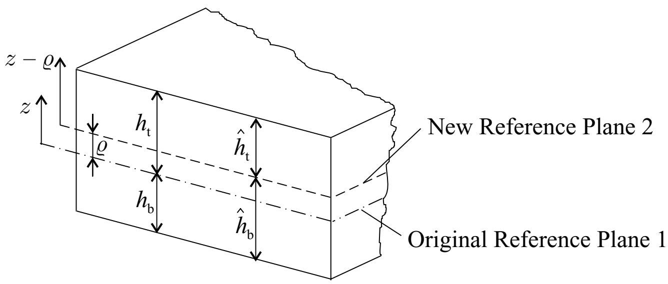


Figure 3.14: Definition of the new reference plane.

We observe that for both quasi-isotropic and isotropic laminates, A_{22} and A_{66} are $A_{22} = A_{11}$ and $A_{66} = (A_{11} - A_{12})/2$. Thus, under in-plane forces, quasi-isotropic laminates behave in the same way as isotropic laminates, that is, there is no extension–shear coupling (Table 3.5) and the $[A]$ matrix is independent of the coordinate directions. The $[B]$ and $[D]$ matrices do not simplify for quasi-isotropic laminates.

Elements A_{16} and A_{26} are zero and $A_{66} = (A_{11} - A_{12})/2$ for symmetrical and unsymmetrical quasi-isotropic laminates. Correspondingly, for symmetrical quasi-isotropic laminates, the a_{16} and a_{26} elements of the compliance matrix are zero ($a_{16} = 0$ and $a_{26} = 0$) and $a_{66} = 2(a_{11} - a_{12})$. For unsymmetrical quasi-isotropic laminates, none of the elements of the compliance matrix is zero (see Eq. 3.22).

Reference plane. The stiffness matrices $[A]$, $[B]$, and $[D]$ refer to Reference Plane 1. The stiffness matrices for Reference Plane 2 (located at a distance ϱ from Reference Plane 1, Fig. 3.14) are obtained from Eq. (3.20) by replacing z_k by $(z_k - \varrho)$ as follows:

$$
\begin{aligned}
A_{ij}^{\varrho} &= \sum_{k=1}^{K} (\overline{Q}_{ij})_k \left[(z_k - \varrho) - (z_{k-1} - \varrho)\right] \\
&= \sum_{k=1}^{K} (\overline{Q}_{ij})_k (z_k - z_{k-1}) = A_{ij} \\
B_{ij}^{\varrho} &= \frac{1}{2} \sum_{k=1}^{K} (\overline{Q}_{ij})_k \left[(z_k - \varrho)^2 - (z_{k-1} - \varrho)^2\right] \\
&= \sum_{k=1}^{K} (\overline{Q}_{ij})_k \left[\frac{z_k^2 - z_{k-1}^2}{2} - \varrho\,(z_k - z_{k-1})\right] = B_{ij} - \varrho A_{ij} \\
D_{ij}^{\varrho} &= \frac{1}{3} \sum_{k=1}^{K} (\overline{Q}_{ij})_k \left[(z_k - \varrho)^3 - (z_{k-1} - \varrho)^3\right] \\
&= \sum_{k=1}^{K} (\overline{Q}_{ij})_k \left[\frac{z_k^3 - z_{k-1}^3}{3} - 2\varrho \frac{z_k^2 - z_{k-1}^2}{2} + \varrho^2 (z_k - z_{k-1})\right] \\
&= D_{ij} - 2\varrho B_{ij} + \varrho^2 A_{ij}.
\end{aligned}
\tag{3.47}
$$

These equations correspond to the parallel axis theorem.

The superscript ϱ refers to Reference Plane 2. When the laminate is symmetrical and the reference plane coincides with the midplane, the matrix $[B]$ is zero. When the laminate is unsymmetrical, the matrix $[B]$ is not zero.

In general, there is no ϱ value that results in a nonzero $[B]$ matrix for an anisotropic composite laminate. In other words, for an unsymmetrical laminate there is no reference plane that is also a neutral plane.

The compliance matrices $[\alpha]$, $[\beta]$, and $[\delta]$ refer to Reference Plane 1. The elements of these matrices for Reference Plane 2 are obtained by introducing Eq. (3.47) into Eq. (3.23). After algebraic manipulations, we obtain

$$\begin{aligned} \alpha_{ij}^{\varrho} &= \alpha_{ij} + \varrho\,(\beta_{ij} + \beta_{ji}) + \varrho^2\delta_{ij} \\ \beta_{ij}^{\varrho} &= \beta_{ij} + \varrho\delta_{ij} \\ \delta_{ij}^{\varrho} &= \delta_{ij}. \end{aligned} \tag{3.48}$$

The third of these equations shows that the bending compliance matrix $[\delta]$ is independent of the choice of the reference surface.

Curved laminates. The stiffness and compliance matrices derived in this chapter for flat laminates may be applied to thin curved laminates when the radius of curvature is large compared with the thickness.

Numerical values of the stiffness and compliance matrices of selected laminates. Below, we present numerical values of the stiffness and compliance matrices of laminates with different lay-ups. The engineering constants used to calculate the laminate stiffnesses and compliances are listed in Table 3.6. While the properties in this table are not intended to depict a particular material, they are characteristic of many graphite-epoxy composites. Therefore, the properties in Table 3.6 are used in the examples in the book.

3.1 Example. *Calculate the stiffness* $[A]$, $[B]$, $[D]$ *and the compliance* $[\alpha]$, $[\beta]$, $[\delta]$ *matrices of a* $[0_{10}/45_{10}]$ *laminate made of graphite epoxy unidirectional plies. The ply properties are given in Table 3.6.*

Solution. The stiffness matrix of a unidirectional ply with the fibers in the 0-degree direction is $[\overline{Q}]^0 = [Q]$. The stiffness matrix $[Q]$ is given by Eq. (2.147),

Table 3.6. Properties of the material used in the examples

		[0]	$\pm45^{f}$
Longitudinal Young's modulus (GPa)	E_1	148	16.39
Transverse Young's modulus (GPa)	E_2	9.65	16.39
Longitudinal shear modulus (GPa)	G_{12}	4.55	38.19
Longitudinal Poission's ratio	ν_{12}	0.3	0.801
Thickness (mm)	h_0	0.1	0.2

and thus $[\overline{Q}]^0$ is

$$[\overline{Q}]^0 = [Q] = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9 \frac{\text{N}}{\text{m}^2}. \tag{3.49}$$

The stiffness matrix $[\overline{Q}]$ of a ply not in the 0-degree direction is (Eq. 3.14)

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} = [T_\sigma]^{-1} [Q] [T_\epsilon], \tag{3.50}$$

where $[T_\sigma]$ and $[T_\epsilon]$ are given by Eq. (3.15). For the 45-degree ply $c = \cos 45^\circ = 0.707$ and $s = \sin 45^\circ = 0.707$, and we have

$$[T_\sigma] = \begin{bmatrix} 0.5 & 0.5 & 1.0 \\ 0.5 & 0.5 & -1.0 \\ -0.5 & 0.5 & 0 \end{bmatrix} \qquad [T_\epsilon] = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -1.0 & 1.0 & 0 \end{bmatrix}. \tag{3.51}$$

By substituting Eqs. (3.49) and (3.51) into Eq. (3.50), we obtain the stiffness matrix of the 45-degree ply as follows:

$$[\overline{Q}]^{45} = \begin{bmatrix} 45.65 & 36.55 & 34.79 \\ 36.55 & 45.65 & 34.79 \\ 34.79 & 34.79 & 38.19 \end{bmatrix} 10^9 \frac{\text{N}}{\text{m}^2}. \tag{3.52}$$

The layup is shown in Fig 3.15. In calculating the $[A]$, $[B]$, $[D]$ matrices we treat the ten 0-degree plies as one layer and the ten 45-degree plies as another layer. The $[A]$, $[B]$, $[D]$ matrices are (Eq. 3.20)

$$\begin{aligned} [A] &= [\overline{Q}]^0 (z_1 - z_0) + [\overline{Q}]^{45} (z_2 - z_1) \\ [B] &= [\overline{Q}]^0 \frac{z_1^2 - z_0^2}{2} + [\overline{Q}]^{45} \frac{z_2^2 - z_1^2}{2} \\ [D] &= [\overline{Q}]^0 \frac{z_1^3 - z_0^3}{3} + [\overline{Q}]^{45} \frac{z_2^3 - z_1^3}{3}. \end{aligned} \tag{3.53}$$

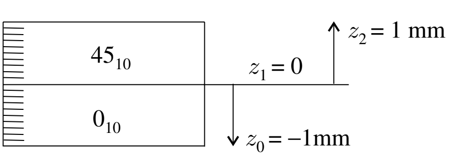


Figure 3.15: The $[0_{10}/45_{10}]$ laminate in Example 3.1.

The $[\overline{Q}]$ matrices are given by Eqs. (3.49) and (3.52). The distances (in meters) are $z_0 = -0.001$, $z_1 = 0$, $z_2 = 0.001$ (Fig. 3.15). With these values Eq. (3.53) yields

$$[A] = [\overline{Q}]^0[0 - (-0.001)] + [\overline{Q}]^{45}(0.001 - 0) = \begin{bmatrix} 194.52 & 39.46 & 34.79 \\ 39.46 & 55.36 & 34.79 \\ 34.79 & 34.79 & 42.74 \end{bmatrix} 10^6 \frac{\text{N}}{\text{m}}$$

$$[B] = [\overline{Q}]^0 \frac{0^2 - (-0.001)^2}{2} + [\overline{Q}]^{45} \frac{0.001^2 - 0^2}{2} = \begin{bmatrix} -51.61 & 16.82 & 17.40 \\ 16.82 & 17.97 & 17.40 \\ 17.40 & 17.40 & 16.82 \end{bmatrix} 10^3\,\text{N}$$

$$[D] = [\overline{Q}]^0 \frac{0^3 - (-0.001)^3}{3} + [\overline{Q}]^{45} \frac{0.001^3 - 0^3}{3} = \begin{bmatrix} 64.84 & 13.15 & 11.60 \\ 13.15 & 18.45 & 11.60 \\ 11.60 & 11.60 & 14.25 \end{bmatrix} \text{N} \cdot \text{m}. \tag{3.54}$$

The compliance matrices are (Eq. 3.23)

$$\begin{bmatrix} \alpha & \beta \\ \beta^T & \delta \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1}. \tag{3.55}$$

Hence, we have

$$[\alpha] = \begin{bmatrix} 13.44 & -4.85 & -7.14 \\ -4.85 & 41.81 & -21.23 \\ -7.14 & -21.23 & 64.95 \end{bmatrix} 10^{-9} \frac{\text{m}}{\text{N}}$$

$$[\beta] = \begin{bmatrix} 17.07 & -6.01 & -11.06 \\ -6.01 & -5.04 & -11.06 \\ -11.06 & -11.06 & -24.05 \end{bmatrix} 10^{-6} \frac{1}{\text{N}} \tag{3.56}$$

$$[\delta] = \begin{bmatrix} 40.32 & -14.56 & -21.41 \\ -14.56 & 125.42 & -63.68 \\ -21.41 & -63.68 & 194.86 \end{bmatrix} 10^{-3} \frac{1}{\text{N} \cdot \text{m}}.$$

The compliance and stiffness matrices of $[45_2/-45_2/0_{12}/-45_2/45_2]$, $[-30_4/15_4/0_2]_s$, $[0_2/45_2/90_2/-45_2]_s$, $[45_6/0_4]_s$, and $[0_2/45_2/0_2/45_2]$ laminates are calculated similarly. The results are in Tables 3.7, 3.8, and 3.9 (pages 84–86). Note that, for symmetrical laminates, the following simplifications apply: $[B] = [\beta] = 0$ and $[\alpha] = [a]$, $[\delta] = [d]$.

3.2 Example. *Calculate the stiffness* $[A]$, $[B]$, $[D]$ *and the compliance* $[\alpha]$, $[\beta]$, $[\delta]$ *matrices of a* $[0_{20}]$ *laminate made of graphite epoxy unidirectional plies. The ply properties are given in Table 3.6 (page 81).*

Solution. The unidirectional laminate is symmetrical, and the $[B]$ matrix is zero:

$$[B] = 0. \tag{3.57}$$

Table 3.7. The [*A*] and [*D*] matrices for symmetrical laminates. The unit of [*A*] is $10^6\,\frac{\text{N}}{\text{m}}$ and the unit of [*D*] is $\text{N}\cdot\text{m}$. The material properties are given in Table 3.6 (page 81).

[*A*]	[*D*]
$[0_{20}]$ (orthotropic, symmetrical)	
$\begin{bmatrix} 297.75 & 5.82 & 0 \\ 5.82 & 19.41 & 0 \\ 0 & 0 & 9.10 \end{bmatrix}$	$\begin{bmatrix} 99.25 & 1.94 & 0 \\ 1.94 & 6.47 & 0 \\ 0 & 0 & 3.03 \end{bmatrix}$
$[\pm 45^{\text{f}}_2/0_{12}/\pm 45^{\text{f}}_2]$ (orthotropic, symmetrical)	
$\begin{bmatrix} 215.17 & 32.74 & 0 \\ 32.74 & 48.17 & 0 \\ 0 & 0 & 36.01 \end{bmatrix}$	$\begin{bmatrix} 45.30 & 19.52 & 0 \\ 19.52 & 25.26 & 0 \\ 0 & 0 & 20.62 \end{bmatrix}$
$[45_2/-45_2/0_{12}/-45_2/45_2]$ (balanced, symmetrical)	
$\begin{bmatrix} 215.17 & 32.74 & 0 \\ 32.74 & 48.17 & 0 \\ 0 & 0 & 36.01 \end{bmatrix}$	$\begin{bmatrix} 45.30 & 19.52 & 4.45 \\ 19.52 & 25.26 & 4.45 \\ 4.45 & 4.45 & 20.62 \end{bmatrix}$
$[-30_4/15_4/0_2]_{\text{s}}$ (symmetrical)	
$\begin{bmatrix} 235.54 & 32.74 & -10.19 \\ 32.74 & 27.79 & -10.19 \\ -10.19 & -10.19 & 36.01 \end{bmatrix}$	$\begin{bmatrix} 65.42 & 16.29 & -18.93 \\ 16.29 & 11.60 & -7.74 \\ -18.93 & -7.74 & 17.39 \end{bmatrix}$
$[0_2/45_2/90_2/-45_2]_{\text{s}}$ (quasi-isotropic, symmetrical)	
$\begin{bmatrix} 99.95 & 31.57 & 0 \\ 31.57 & 99.95 & 0 \\ 0 & 0 & 34.19 \end{bmatrix}$	$\begin{bmatrix} 34.61 & 4.58 & 3.34 \\ 4.58 & 12.34 & 3.34 \\ 3.34 & 3.34 & 5.14 \end{bmatrix}$
$[45_6/0_4]_{\text{s}}$ (symmetrical)	
$\begin{bmatrix} 173.88 & 46.19 & 41.75 \\ 46.19 & 62.55 & 41.75 \\ 41.75 & 41.75 & 49.47 \end{bmatrix}$	$\begin{bmatrix} 34.84 & 22.93 & 21.71 \\ 22.93 & 28.90 & 21.71 \\ 21.71 & 21.71 & 24.02 \end{bmatrix}$

By treating the 20 plies as a single layer, the [*A*] and [*D*] matrices are (Eq. 3.20)

$$[A] = h[\overline{Q}] \qquad [D] = \frac{h^3}{12}[\overline{Q}], \tag{3.58}$$

where $h = 0.002$ m is the thickness of the laminate.

By substituting Eqs. (3.20) and (2.139) into Eq. (3.58) and by using the engineering constants in Table 3.6 (page 81) ($E_1 = 148 \times 10^9\ \text{N/m}^2$, $E_2 = 9.65 \times$

Table 3.8. The [*a*] and [*d*] matrices for symmetrical laminates. The unit of [*a*] is $10^{-9}\,\frac{\mathrm{m}}{\mathrm{N}}$ and the unit of [*d*] is $10^{-3}\,\frac{1}{\mathrm{N \cdot m}}$. The material properties are given in Table 3.6 (page 81).

[*a*]	[*d*]
$[0_{20}]$ (orthotropic, symmetrical)	
$\begin{bmatrix} 3.38 & -1.01 & 0 \\ -1.01 & 51.81 & 0 \\ 0 & 0 & 109.89 \end{bmatrix}$	$\begin{bmatrix} 10.14 & -3.04 & 0 \\ -3.04 & 155.44 & 0 \\ 0 & 0 & 329.67 \end{bmatrix}$
$[\pm 45^{\mathrm{f}}_2/0_{12}/\pm 45^{\mathrm{f}}_2]$ (orthotropic, symmetrical)	
$\begin{bmatrix} 5.18 & -3.52 & 0 \\ -3.52 & 23.15 & 0 \\ 0 & 0 & 27.77 \end{bmatrix}$	$\begin{bmatrix} 33.10 & -25.59 & 0 \\ -25.59 & 59.37 & 0 \\ 0 & 0 & 48.51 \end{bmatrix}$
$[45_2/-45_2/0_{12}/-45_2/45_2]$ (balanced, symmetrical)	
$\begin{bmatrix} 5.18 & -3.52 & 0 \\ -3.52 & 23.15 & 0 \\ 0 & 0 & 27.77 \end{bmatrix}$	$\begin{bmatrix} 33.16 & -25.33 & -1.69 \\ -25.33 & 60.51 & -7.60 \\ -1.69 & -7.60 & 50.51 \end{bmatrix}$
$[-30_4/15_4/0_2]_{\mathrm{s}}$ (symmetrical)	
$\begin{bmatrix} 5.08 & -6.09 & -0.29 \\ -6.09 & 47.44 & 11.70 \\ -0.29 & 11.70 & 31.00 \end{bmatrix}$	$\begin{bmatrix} 26.87 & -25.93 & 17.70 \\ -25.93 & 147.76 & 37.57 \\ 17.70 & 37.57 & 93.52 \end{bmatrix}$
$[0_2/45_2/90_2/-45_2]_{\mathrm{s}}$ (quasi-isotropic, symmetrical)	
$\begin{bmatrix} 11.11 & -3.51 & 0 \\ -3.51 & 11.11 & 0 \\ 0 & 0 & 29.25 \end{bmatrix}$	$\begin{bmatrix} 31.38 & -7.44 & -15.55 \\ -7.44 & 100.06 & -60.17 \\ -15.55 & -60.17 & 243.70 \end{bmatrix}$
$[45_6/0_4]_{\mathrm{s}}$ (symmetrical)	
$\begin{bmatrix} 7.45 & -2.99 & -3.77 \\ -2.99 & 37.81 & -29.39 \\ -3.77 & -29.39 & 48.20 \end{bmatrix}$	$\begin{bmatrix} 71.24 & -25.43 & -41.40 \\ -25.43 & 116.82 & -82.58 \\ -41.40 & -82.58 & 153.66 \end{bmatrix}$

10^9 N/m^2, $G_{12} = 4.55 \times 10^9$ N/m^2, $\nu_{12} = 0.3$), we obtain

$$A_{11} = \frac{E_1 h}{1 - \nu_{12}^2 \frac{E_2}{E_1}} = 297.75 \times 10^6 \,\frac{\mathrm{N}}{\mathrm{m}} \qquad A_{22} = \frac{E_2 h}{1 - \nu_{12}^2 \frac{E_2}{E_1}} = 19.41 \times 10^6 \,\frac{\mathrm{N}}{\mathrm{m}}$$
$$A_{12} = \nu_{12} A_{22} = 5.82 \times 10^6 \,\frac{\mathrm{N}}{\mathrm{m}} \qquad A_{66} = G_{12} h = 9.10 \times 10^6 \,\frac{\mathrm{N}}{\mathrm{m}} \tag{3.59}$$

$$D_{11} = \frac{E_1 h^3}{12\left(1 - \nu_{12}^2 \frac{E_2}{E_1}\right)} = 99.25\ \mathrm{N \cdot m} \qquad D_{22} = \frac{E_2 h^3}{12\left(1 - \nu_{12}^2 \frac{E_2}{E_1}\right)} = 6.47\ \mathrm{N \cdot m}$$
$$D_{12} = \nu_{12} D_{22} = 1.94\ \mathrm{N \cdot m} \qquad D_{66} = \frac{G_{12} h^3}{12} = 3.03\ \mathrm{N \cdot m}. \tag{3.60}$$

Table 3.9. The $[A]$, $[B]$, $[D]$ and the $[\alpha]$, $[\beta]$, and $[\delta]$ matrices for unsymmetrical laminates. $[A]$ is in $10^6\,\frac{\text{N}}{\text{m}}$, $[B]$ is in 10^3N, $[D]$ is in N · m, $[\alpha]$ is in $10^{-9}\,\frac{\text{m}}{\text{N}}$, $[\beta]$ is in $10^{-6}\,\frac{1}{\text{N}}$, and $[\delta]$ is in $10^{-3}\,\frac{1}{\text{N} \cdot \text{m}}$. The material properties are given in Table 3.6 (page 81).

[A] **[B]** **[D]**

$[0_{10}/45_{10}]$

$$\begin{bmatrix} 194.52 & 39.46 & 34.79 \\ 39.46 & 55.36 & 34.79 \\ 34.79 & 34.79 & 42.74 \end{bmatrix} \quad \begin{bmatrix} -51.61 & 16.82 & 17.40 \\ 16.82 & 17.97 & 17.40 \\ 17.40 & 17.40 & 16.82 \end{bmatrix} \quad \begin{bmatrix} 64.84 & 13.15 & 11.60 \\ 13.15 & 18.45 & 11.60 \\ 11.60 & 11.60 & 14.25 \end{bmatrix}$$

$[0_2/45_2/0_2/45_2]$

$$\begin{bmatrix} 77.81 & 15.79 & 13.92 \\ 15.79 & 22.14 & 13.92 \\ 13.92 & 13.92 & 17.10 \end{bmatrix} \quad \begin{bmatrix} -4.129 & 1.346 & 1.392 \\ 1.346 & 1.438 & 1.392 \\ 1.392 & 1.392 & 1.346 \end{bmatrix} \quad \begin{bmatrix} 4.150 & 0.842 & 0.742 \\ 0.842 & 1.181 & 0.742 \\ 0.742 & 0.742 & 0.912 \end{bmatrix}$$

$[\pm 45^{\text{f}}_5/0_{10}]$

$$\begin{bmatrix} 194.52 & 39.46 & 0 \\ 39.46 & 55.36 & 0 \\ 0 & 0 & 42.74 \end{bmatrix} \quad \begin{bmatrix} 51.61 & -16.82 & 0 \\ -16.82 & -17.97 & 0 \\ 0 & 0 & -16.82 \end{bmatrix} \quad \begin{bmatrix} 64.84 & 13.15 & 0 \\ 13.15 & 18.45 & 0 \\ 0 & 0 & 14.25 \end{bmatrix}$$

$[\alpha]$ $[\beta]$ $[\delta]$

$[0_{10}/45_{10}]$

$$\begin{bmatrix} 13.44 & -4.85 & -7.14 \\ -4.85 & 41.81 & -21.23 \\ -7.14 & -21.23 & 64.95 \end{bmatrix} \quad \begin{bmatrix} 17.07 & -6.01 & -11.06 \\ -6.01 & -5.04 & -11.06 \\ -11.06 & -11.06 & -24.05 \end{bmatrix} \quad \begin{bmatrix} 40.32 & -14.56 & -21.41 \\ -14.56 & 125.42 & -63.68 \\ -21.41 & -63.68 & 194.86 \end{bmatrix}$$

$[0_2/45_2/0_2/45_2]$

$$\begin{bmatrix} 17.88 & -7.14 & -8.79 \\ -7.14 & 96.35 & -69.68 \\ -8.79 & -69.68 & 128.51 \end{bmatrix} \quad \begin{bmatrix} 28.37 & -9.99 & -18.38 \\ -9.99 & -8.38 & -18.38 \\ -18.38 & -18.38 & -39.96 \end{bmatrix} \quad \begin{bmatrix} 335 & -134 & -165 \\ -134 & 1\,807 & -1\,306 \\ -165 & -1\,306 & 2\,410 \end{bmatrix}$$

$[\pm 45^{\text{f}}_5/0_{10}]$

$$\begin{bmatrix} 11.65 & -8.58 & 0 \\ -8.58 & 32.94 & 0 \\ 0 & 0 & 43.70 \end{bmatrix} \quad \begin{bmatrix} -13.97 & 12.22 & 0 \\ 12.22 & 15.55 & 0 \\ 0 & 0 & 51.60 \end{bmatrix} \quad \begin{bmatrix} 34.94 & -25.74 & 0 \\ -25.74 & 98.83 & 0 \\ 0 & 0 & 131.11 \end{bmatrix}$$

The compliance matrices $[a]$ and $[d]$ are obtained by inverting the $[A]$ and $[D]$ stiffness matrices as follows:

$$[a] = [A]^{-1} \qquad [d] = [D]^{-1}. \tag{3.61}$$

Equations (3.59)–(3.61) give

$$a_{11} = \frac{1}{E_1 h} = 3.38 \times 10^{-9}\,\frac{\text{m}}{\text{N}} \qquad a_{22} = \frac{1}{E_2 h} = 51.81 \times 10^{-9}\,\frac{\text{m}}{\text{N}}$$
$$a_{12} = -\nu_{12} a_{11} = -1.01 \times 10^{-9}\,\frac{\text{m}}{\text{N}} \qquad a_{66} = \frac{1}{G_{12} h} = 109.89 \times 10^{-9}\,\frac{\text{m}}{\text{N}} \tag{3.62}$$

$$d_{11} = \frac{12}{E_1 h^3} = 10.14 \times 10^{-3}\,\frac{1}{\text{N}\cdot\text{m}} \qquad d_{22} = \frac{12}{E_2 h^3} = 155.44 \times 10^{-3}\,\frac{1}{\text{N}\cdot\text{m}}$$
$$d_{12} = -\nu_{12} d_{11} = -3.04 \times 10^{-3}\,\frac{1}{\text{N}\cdot\text{m}} \qquad d_{66} = \frac{12}{G_{12} h^3} = 329.67 \times 10^{-3}\,\frac{1}{\text{N}\cdot\text{m}}. \tag{3.63}$$

3.3 Example. *Calculate the stiffness and the compliance matrices of (i) a laminated composite consisting of two layers of ±45-degree woven fabric, twelve layers of 0-degree unidirectional plies, and two layers of ±45-degree woven fabric* ($[\pm 45^{\text{f}}_2/0_{12}/\pm 45^{\text{f}}_2]$)*; and (ii) a laminate consisting five layers of ±45-degree woven fabric and ten layers of 0-degree unidirectional plies* ($[\pm 45^{\text{f}}_5/0_{10}]$). *The material properties are given in Table 3.6 (page 81).*

Solution. First we consider the laminate with ($[\pm 45^{\text{f}}_2/0_{12}/\pm 45^{\text{f}}_2]$) layup (Fig. 3.16). The laminate is symmetrical, and the $[B]$ matrix is zero:

$$[B] = 0. \tag{3.64}$$

The compliance matrix of a unidirectional ply with the fibers in the 0-degree direction is $[\overline{Q}]^0 = [Q]$. The stiffness matrix $[Q]$ is given by Eq. (2.147), and thus $[\overline{Q}]^0$ is

$$[\overline{Q}]^0 = [Q] = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9\,\frac{\text{N}}{\text{m}^2}. \tag{3.65}$$

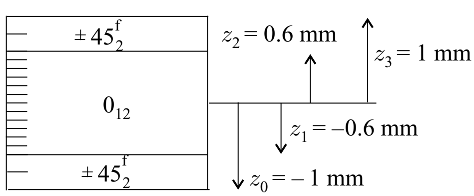


Figure 3.16: The layup of the laminate in Example 3.3. The supercript f denotes fabric.

For a ±45-degree woven fabric the stiffness matrix $[\overline{Q}]$ is (Eq. 2.150)

$$[\overline{Q}]^{\pm 45} = \begin{bmatrix} 45.65 & 36.55 & 0 \\ 36.55 & 45.65 & 0 \\ 0 & 0 & 38.19 \end{bmatrix} 10^9 \ \frac{\text{N}}{\text{m}^2} \cdot \tag{3.66}$$

In calculating the $[A]$, $[B]$, $[D]$ matrices we treat the twelve 0-degree plies as one layer and each adjacent woven fabric as one layer. The $[A]$ and $[D]$ matrices are

$$\begin{aligned} [A] &= [\overline{Q}]^{\pm 45}(z_1 - z_0) + [\overline{Q}]^0 (z_2 - z_1) + [\overline{Q}]^{\pm 45}(z_3 - z_2) \\ [D] &= [\overline{Q}]^{\pm 45}\frac{z_1^3 - z_0^3}{3} + [\overline{Q}]^0 \frac{z_2^3 - z_1^3}{3} + [\overline{Q}]^{\pm 45}\frac{z_3^3 - z_2^3}{3}. \end{aligned} \tag{3.67}$$

The $[\overline{Q}]$ matrices are given by Eqs. (3.65) and (3.66). The distances (in meters) are $z_0 = -0.001$, $z_1 = -0.0006$, $z_2 = 0.0006$, and $z_3 = 0.001$ (Fig. 3.16). With these values Eq. (3.67) yields

$$[A] = \begin{bmatrix} 215.17 & 32.74 & 0 \\ 32.74 & 48.17 & 0 \\ 0 & 0 & 36.01 \end{bmatrix} 10^6 \ \frac{\text{N}}{\text{m}} \tag{3.68}$$

$$[D] = \begin{bmatrix} 45.30 & 19.52 & 0 \\ 19.52 & 25.26 & 0 \\ 0 & 0 & 20.62 \end{bmatrix} \text{N} \cdot \text{m} . \tag{3.69}$$

The compliance matrices $[a]$ and $[d]$ are (Eqs. 3.29 and 3.30)

$$[a] = [A]^{-1} = \begin{bmatrix} 5.18 & -3.52 & 0 \\ -3.52 & 23.15 & 0 \\ 0 & 0 & 27.77 \end{bmatrix} 10^{-9} \ \frac{\text{m}}{\text{N}} \tag{3.70}$$

$$[d] = [D]^{-1} = \begin{bmatrix} 33.10 & -25.59 & 0 \\ -25.59 & 59.37 & 0 \\ 0 & 0 & 48.51 \end{bmatrix} 10^{-3} \ \frac{1}{\text{N} \cdot \text{m}} \cdot \tag{3.71}$$

The compliance and stiffness matrices of the $[\pm 45^{\text{f}}_5/0_{10}]$ laminate are calculated similarly. The results are given in Table 3.9.