CHAPTER NINE

Finite Element Analysis

The finite element method offers a practical means of calculating the deformations of, and stresses and strains in, complex structures. A detailed description of the finite element method is beyond the scope of this book. Instead, we focus on those features specific to composite materials.

Finite element analysis consists of the following major steps:

1. A mesh encompassing the structure is generated (Fig. 9.1).
2. The stiffness matrix $[k]$ of each element is determined.
3. The stiffness matrix $[K]$ of the structure is determined by assembling the element stiffness matrices.
4. The loads applied to the structure are replaced by an equivalent force system such that the forces act at the nodal points.
5. The displacements of the nodal points $\mathbf{d}$ are calculated by

$$[K]\,\mathbf{d} = \mathbf{f}, \tag{9.1}$$

where $\mathbf{f}$ is the force vector representing the equivalent applied nodal forces (Fig. 9.1).
6. The vector $\mathbf{d}$ is subdivided into subvectors $\boldsymbol{\delta}$, each $\boldsymbol{\delta}$ representing the displacements of the nodal points of a particular element.
7. The displacements at a point inside the element are calculated by

$$\mathbf{u} = [N]\,\boldsymbol{\delta}, \tag{9.2}$$

where the vector $\mathbf{u}$ represents the displacements and $[N]$ is the matrix of the shape vectors.
8. The strains at a point inside the elements are calculated by

$$\varepsilon = [B]\,\boldsymbol{\delta}, \tag{9.3}$$

where $[B]$ is the strain–displacement matrix.

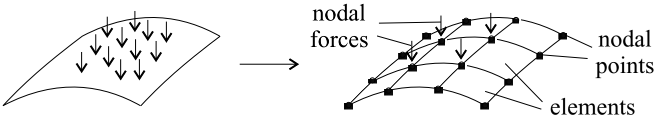

Figure 9.1: Structure and its finite element mesh.

9. The stresses at a point inside the element are calculated by

$$\boldsymbol{\sigma} = [E]\,\boldsymbol{\varepsilon}, \tag{9.4}$$

where $[E]$ is the stiffness matrix characterizing the material.

The element stiffness matrix, referred to in Step 2, is defined as

$$[k]\,\boldsymbol{\delta} = \mathbf{f}_{\mathrm{e}}, \tag{9.5}$$

where $\mathbf{f}_{\mathrm{e}}$ represents the forces acting at the nodal points of the element. The element stiffness matrix is[1]

$$[k] = \int_{(V)} [B]^T\,[E]\,[B]\,dV, \tag{9.6}$$

where V is the volume of the element.

The preceding steps apply to structures made of either isotropic or composite materials. The only difference between isotropic and composite structures is in the material stiffness matrix $[E]$. In the following we present expressions for $[E]$.

9.1 Three-Dimensional Element

The stress–strain relationships for a three-dimensional element are (Eq. 2.20)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} & \overline{C}_{34} & \overline{C}_{35} & \overline{C}_{36} \\ \overline{C}_{41} & \overline{C}_{42} & \overline{C}_{43} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{51} & \overline{C}_{52} & \overline{C}_{53} & \overline{C}_{54} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{61} & \overline{C}_{62} & \overline{C}_{63} & \overline{C}_{64} & \overline{C}_{65} & \overline{C}_{66} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \tag{9.7}$$

where $[E]$ is the stiffness matrix for a three-dimensional element.

[1] R. D. Cook, D. S. Malkus, and M. E. Plesha, *Concepts and Applications of Finite Element Analysis*. 3rd edition. John Wiley and Sons, New York, 1989, p. 110.

9.2 Plate Element

In the absence of shear deformation, the force–strain relationships for a thin-plate element are (Eq. 3.21)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \tag{9.8}$$

where $[E]$ is the stiffness matrix for a plate element without shear deformations. In the presence of shear deformation, the force–strain relationships are (Eqs. 5.13–5.15)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ V_x \\ V_y \end{Bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{S}_{11} & \widetilde{S}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & \widetilde{S}_{12} & \widetilde{S}_{22} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{9.9}$$

where $[E]$ is the stiffness matrix for a plate element with shear deformations.

Frequently, the behavior of shells can be described by replacing the curved surface with small, flat elements. The stiffness matrices above are applicable to such flat shell elements.

9.3 Beam Element

For a beam element, the force–strain relationships are

arbitrary layup, no shear deformation, no restrained warping (Eq. 6.2)

$$\begin{Bmatrix} \widehat{N} \\ \widehat{M}_y \\ \widehat{M}_z \\ \widehat{T} \end{Bmatrix} = \underbrace{\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12} & P_{22} & P_{23} & P_{24} \\ P_{13} & P_{23} & P_{33} & P_{34} \\ P_{14} & P_{24} & P_{34} & P_{44} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^o \\ \frac{1}{\rho_y} \\ \frac{1}{\rho_z} \\ \vartheta \end{Bmatrix} \tag{9.10}$$

orthotropic, no shear deformation, no restrained warping (Eq. 6.8)

$$\begin{Bmatrix} \widehat{N} \\ \widehat{M}_y \\ \widehat{M}_z \\ \widehat{T} \end{Bmatrix} = \underbrace{\begin{bmatrix} \widehat{EA} & 0 & 0 & 0 \\ 0 & \widehat{EI}_{yy} & \widehat{EI}_{yz} & 0 \\ 0 & \widehat{EI}_{yz} & \widehat{EI}_{zz} & 0 \\ 0 & 0 & 0 & \widehat{GI}_{\mathrm{t}} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^{\mathrm{o}} \\ \frac{1}{\rho_y} \\ \frac{1}{\rho_z} \\ \vartheta \end{Bmatrix} \tag{9.11}$$

orthotropic, no shear deformation, restrained warping (Eqs. 6.8 and 6.233)

$$\begin{Bmatrix} \widehat{N} \\ \widehat{M}_y \\ \widehat{M}_z \\ \widehat{M}_\omega \\ \widehat{T}_{\mathrm{sv}} \end{Bmatrix} = \underbrace{\begin{bmatrix} \widehat{EA} & 0 & 0 & 0 & 0 \\ 0 & \widehat{EI}_{yy} & \widehat{EI}_{yz} & 0 & 0 \\ 0 & \widehat{EI}_{yz} & \widehat{EI}_{zz} & 0 & 0 \\ 0 & 0 & 0 & \widehat{EI}_\omega & 0 \\ 0 & 0 & 0 & 0 & \widehat{GI}_{\mathrm{t}} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^{\mathrm{o}} \\ \frac{1}{\rho_y} \\ \frac{1}{\rho_z} \\ -\frac{d\vartheta}{dx} \\ \vartheta \end{Bmatrix} \tag{9.12}$$

orthotropic, shear deformation, restrained warping (Eqs. 7.30, 7.32, 7.34, 7.36)

$$\begin{Bmatrix} \widehat{N} \\ \widehat{M}_y \\ \widehat{M}_z \\ \widehat{M}_\omega \\ \widehat{T}_{\mathrm{sv}} \\ \widehat{V}_y \\ \widehat{V}_z \\ \widehat{T}_\omega \end{Bmatrix} = \underbrace{\begin{bmatrix} \widehat{EA} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{EI}_{yy} & \widehat{EI}_{yz} & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{EI}_{yz} & \widehat{EI}_{zz} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{EI}_\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{GI}_{\mathrm{t}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{S}_{yy} & \widehat{S}_{yz} & \widehat{S}_{y\omega} \\ 0 & 0 & 0 & 0 & 0 & \widehat{S}_{yz} & \widehat{S}_{zz} & \widehat{S}_{z\omega} \\ 0 & 0 & 0 & 0 & 0 & \widehat{S}_{y\omega} & \widehat{S}_{z\omega} & \widehat{S}_{\omega\omega} \end{bmatrix}}_{[E]} \begin{Bmatrix} \epsilon_x^{\mathrm{o}} \\ -\frac{d\chi_z}{dx} \\ -\frac{d\chi_y}{dx} \\ -\frac{d\vartheta^{\mathrm{B}}}{dx} \\ \vartheta \\ \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{Bmatrix}, \tag{9.13}$$

where $[E]$ is the stiffness matrix for a beam element.

9.4 Sublaminate

A laminate consisting of several plies may be analyzed by either plate (flat shell) or three-dimensional elements (Fig. 9.2). For thick laminates neither of these

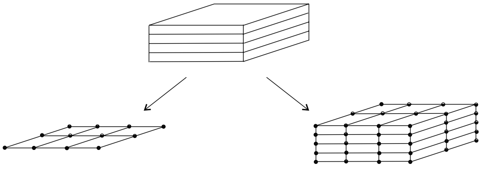

Figure 9.2: Thick laminate (top), analysis with plate elements (left), analysis with three-dimensional elements (right).

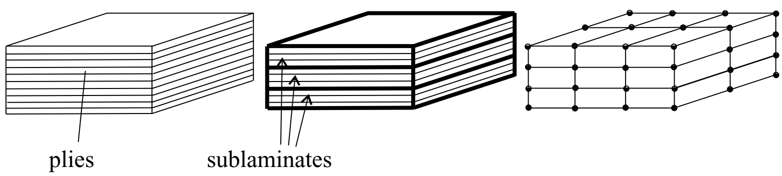

Figure 9.3: Thick laminate (left), sublaminates (middle), and the finite element mesh (right).

is practical. Plate elements give inaccurate results. Three-dimensional elements require that the material be uniform throughout the element, and, hence, an element must contain a single layer or adjacent identical layers. This may result in a very large number of elements, making the numerical computation difficult and often infeasible.

We can overcome these difficulties by dividing the laminate into sublaminates (Fig. 9.3). Each layer in the sublaminate may be monoclinic, orthotropic, transversely isotropic, or isotropic. The thickness of each element is the same as the thickness of the corresponding sublaminate. The stiffness matrix $[E]$ of such a sublaminate is defined by the relationship

$$\begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\sigma}_z \\ \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \\ \overline{\tau}_{xy} \end{Bmatrix} = [E] \begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\epsilon}_z \\ \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \\ \overline{\gamma}_{xy} \end{Bmatrix}. \tag{9.14}$$

The bar denotes average stresses and strains. It is convenient to represent this expression in terms of the compliance matrix $[J]$

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\epsilon}_z \\ \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \\ \overline{\gamma}_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{bmatrix}}_{[J]} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\sigma}_z \\ \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \\ \overline{\tau}_{xy} \end{Bmatrix}, \tag{9.15}$$

where

$$[E] = [J]^{-1}. \tag{9.16}$$

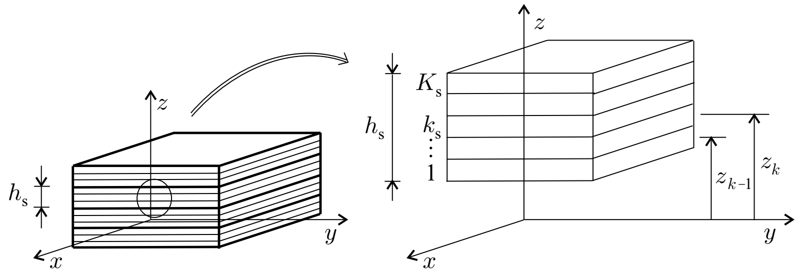

Figure 9.4: Illustration of a sublaminate.

The x,y,z coordinate system is shown in Figure 9.4. The average stresses are defined as

$$\begin{aligned}
\overline{\sigma}_x &= \frac{1}{h_s}\int_{(h_s)} \sigma_x dz = \frac{1}{h_s} N_x & \overline{\sigma}_z &= \sigma_z \\
\overline{\sigma}_y &= \frac{1}{h_s}\int_{(h_s)} \sigma_y dz = \frac{1}{h_s} N_y & \overline{\tau}_{yz} &= \tau_{yz} \\
\overline{\tau}_{xy} &= \frac{1}{h_s}\int_{(h_s)} \tau_{xy} dz = \frac{1}{h_s} N_{xy} & \overline{\tau}_{xz} &= \tau_{xz}.
\end{aligned} \tag{9.17}$$

The second equalities in the left-hand column are written by virtue of Eq. (3.9). The average strains are

$$\begin{aligned}
\overline{\epsilon}_z &= \frac{1}{h_s}\int_{(h_s)} \epsilon_z dz & \overline{\epsilon}_x &= \epsilon_x \\
\overline{\gamma}_{yz} &= \frac{1}{h_s}\int_{(h_s)} \gamma_{yz} dz & \overline{\epsilon}_y &= \epsilon_y \\
\overline{\gamma}_{xz} &= \frac{1}{h_s}\int_{(h_s)} \gamma_{xz} dz & \overline{\gamma}_{xy} &= \gamma_{xy},
\end{aligned} \tag{9.18}$$

where h_s is the thickness of the sublaminate (Fig. 9.4). The terms in the right-hand columns of Eqs. (9.17) and (9.18) show that the stresses σ_z, τ_{yz}, τ_{xz} and the strains ϵ_x, ϵ_y, γ_{xy} do not vary across the thickness.

In the following we derive the elements of the compliance matrix.

9.4.1 Step 1. Elements of [J] due to In-Plane Stresses

In this step we determine the elements in the first, second, and sixth columns of the matrix [J]. To this end, we impose the average in-plane stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$, and $\overline{\tau}_{xy}$ on the sublaminate (Fig. 9.5, top). Since $\overline{\sigma}_z$, $\overline{\tau}_{yz}$, $\overline{\tau}_{xz}$ are zero, the stress–strain

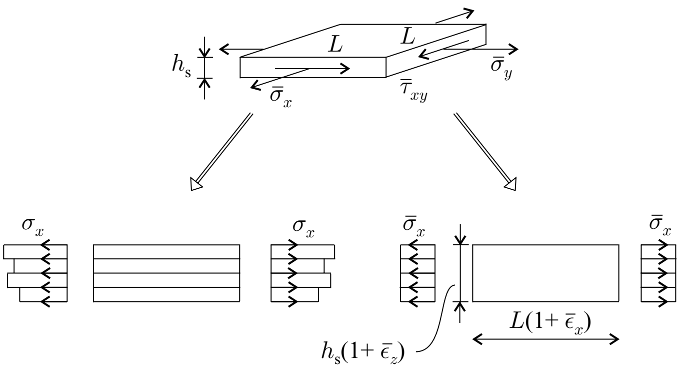

Figure 9.5: Illustration of Step 1. The ply stress and the corresponding average stress on the sublaminate.

relationship (Eq. 9.15) may be written as

$$\begin{Bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} \tag{9.19}$$

$$\{\bar{\epsilon}_z\} = [J_{31} \quad J_{32} \quad J_{36}] \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} \tag{9.20}$$

$$\begin{Bmatrix} \bar{\gamma}_{yz} \\ \bar{\gamma}_{xz} \end{Bmatrix} = \begin{bmatrix} J_{41} & J_{42} & J_{46} \\ J_{51} & J_{52} & J_{56} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix}. \tag{9.21}$$

The strains are uniform across the thickness (Eq. 9.18). Under these conditions $\kappa_x, \kappa_y, \kappa_{xy}$ are zero, and we have (see Eqs. 3.21 and 3.7)

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{9.22}$$

where $[A]$ is the tensile stiffness matrix of the sublaminate. (The summation in Eq. 3.20 is performed from 1 to K_s, where K_s is the number of layers or ply groups in the sublaminate; see Fig. 9.4). For the sublaminate Eqs. (9.17), (9.18), and (9.22) yield

$$\begin{Bmatrix} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{Bmatrix} = \underbrace{h_s [A]^{-1}}_{\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix}} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix}. \tag{9.23}$$

By comparing Eqs. (9.19) and (9.23), we have

$$\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} = h_{\rm s}\,[A]^{-1}. \tag{9.24}$$

Owing to the Poisson effect, the in-plane forces introduce a normal strain ϵ_z in the z direction. For a single layer this strain is (see Eq. 2.133)

$$\epsilon_z = [\overline{S}_{13} \quad \overline{S}_{23} \quad \overline{S}_{36}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \tag{9.25}$$

where σ_x, σ_y, and τ_{xy} are the stresses in the plies. The average normal stress $\overline{\sigma}_x$ across the laminate and the normal ply stress σ_x are illustrated in Figure 9.5 (bottom). The stresses in a single layer are (Eq. 3.13)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\overline{Q}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{9.26}$$

Equations (9.25), (9.26), (9.23), and (9.18) give

$$\epsilon_z = [\overline{S}_{13} \quad \overline{S}_{23} \quad \overline{S}_{36}][\overline{Q}]h_{\rm s}\,[A]^{-1} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix}. \tag{9.27}$$

By combining Eqs. (9.18) and (9.27), and by replacing the integral with a summation, for the sublaminate we obtain

$$\overline{\epsilon}_z = \underbrace{\sum_{k=1}^{K_{\rm s}} ([\overline{S}_{13} \quad \overline{S}_{23} \quad \overline{S}_{36}]\,(z_k - z_{k-1})\,[\overline{Q}]_k)\,[A]^{-1}}_{[J_{31} \quad J_{32} \quad J_{36}]} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix}. \tag{9.28}$$

By comparing Eqs. (9.20) and (9.28), we have

$$[J_{31} \quad J_{32} \quad J_{36}] = \sum_{k=1}^{K_{\rm s}} ([\overline{S}_{13} \quad \overline{S}_{23} \quad \overline{S}_{36}]_k\,(z_k - z_{k-1})\,[\overline{Q}]_k)[A]^{-1}, \tag{9.29}$$

where z_k, z_{k-1} are illustrated in Figure 9.4. Since $\overline{\tau}_{yz} = \overline{\tau}_{xz} = 0$, we have that γ_{yz} and γ_{xz} are zero (Eqs. 2.26 and 2.21), and Eq. (9.21) now becomes

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} J_{41} & J_{42} & J_{46} \\ J_{51} & J_{52} & J_{56} \end{bmatrix} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix}. \tag{9.30}$$

To satisfy this equation, the preceding elements of the compliance matrix must be zero:

$$\begin{bmatrix} J_{41} & J_{42} & J_{46} \\ J_{51} & J_{52} & J_{56} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.31}$$

9.4.2 Step 2. Elements of [J] due to Out-of-Plane Normal Stresses

In this step we determine the elements in the third column of the matrix [J]. To accomplish this we consider a sublaminate in which there are only σ_z stresses. To form such a sublaminate, first we consider a sublaminate restrained along its edges ($\epsilon_x = \epsilon_y = \gamma_{xy} = 0$) and apply a uniform stress $\sigma_z = \overline{\sigma}_z$ (Stage 1, Fig. 9.6), which introduces in-plane stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$, $\overline{\tau}_{xy}$. Second, we apply in-plane stresses $-\overline{\sigma}_x$, $-\overline{\sigma}_y$, $-\overline{\tau}_{xy}$ (Stage 2, Fig. 9.6). Third, we superimpose Stages 1 and 2 and arrive at the sublaminate inside which the in-plane average stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$, $\overline{\tau}_{xy}$ and the transverse shear stresses $\overline{\tau}_{yz}$, $\overline{\tau}_{xz}$ are zero and the stress–strain relationships (Eq. 9.15) are

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} J_{13} \\ J_{23} \\ J_{63} \end{bmatrix} \overline{\sigma}_z \tag{9.32}$$

$$\overline{\epsilon}_z = J_{33}\overline{\sigma}_z \tag{9.33}$$

$$\begin{Bmatrix} \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \end{Bmatrix} = \begin{bmatrix} J_{43} \\ J_{53} \end{bmatrix} \overline{\sigma}_z. \tag{9.34}$$

Stage 1. Following the outline above, we apply σ_z to the sublaminate, which is restrained along the edges ($\epsilon_x = \epsilon_y = \gamma_{xy} = 0$). For a single layer, Eqs. (2.27) and

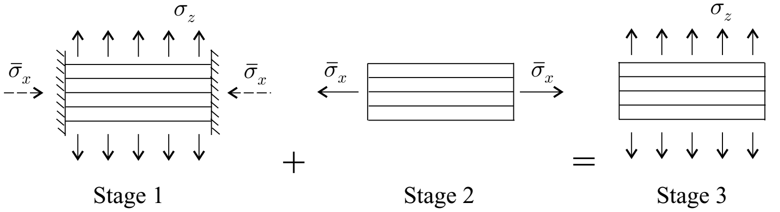

Figure 9.6: Illustration of Step 2. Sublaminate restrained along its edges subjected to σ_z and the resulting stress $\overline{\sigma}_x$ (Stage 1); unrestrained sublaminate subjected to $\overline{\sigma}_x$ (Stage 2); unrestrained sublaminate subjected to σ_z (Stage 3); ($\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ are not shown).

(2.20) give

$$\sigma_z = \overline{C}_{33}\epsilon_z \tag{9.35}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix} \epsilon_z. \tag{9.36}$$

These equations may be rearranged to yield

$$\epsilon_z = \frac{1}{\overline{C}_{33}}\sigma_z \tag{9.37}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix} \frac{1}{\overline{C}_{33}}\sigma_z. \tag{9.38}$$

By combining Eqs. (9.37) and (9.18), and by replacing the integrals by summations, for the sublaminate we obtain

$$\overline{\epsilon}_z = \frac{1}{h_s}\sum_{k=1}^{K_s}\left(\frac{z_k - z_{k-1}}{(\overline{C}_{33})_k}\right)\sigma_z. \tag{9.39}$$

Equations (9.38) and (9.18) yield

$$\begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix} = \frac{1}{h_s}\sum_{k=1}^{K_s}\left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k}\right)\sigma_z. \tag{9.40}$$

This equation gives the in-plane stresses in the restrained sublaminate.

Stage 2. We apply the equal and opposite of the stresses, calculated by Eq. (9.40), to the sublaminate. The corresponding strains are obtained from Eqs. (9.28), (9.23), and (9.40). Equations (9.28) and (9.40) result in

$$\begin{aligned} \overline{\epsilon}_z &= -[J_{31} \quad J_{32} \quad J_{36}]\begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix} \\ &= -[J_{31} \quad J_{32} \quad J_{36}]\frac{1}{h_s}\sum_{k=1}^{K_s}\left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k}\right)\sigma_z. \end{aligned} \tag{9.41}$$

Equations (9.23) and (9.40) give

$$\begin{aligned} \begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\gamma}_{xy} \end{Bmatrix} &= -\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix}\begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix} \\ &= -\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix}\frac{1}{h_s}\sum_{k=1}^{K_s}\left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k}\right)\sigma_z. \end{aligned} \tag{9.42}$$

Stage 3. We combine Eqs. (9.39), (9.41), and (9.42) to obtain the strains of a sublaminate subjected only to the out-of-plane stress σ_z

$$\overline{\epsilon}_z = \left(\frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) - [J_{31} \quad J_{32} \quad J_{36}] \frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) \right) \sigma_z \tag{9.43}$$

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\gamma}_{xy} \end{Bmatrix} = - \begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} \frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) \sigma_z. \tag{9.44}$$

Comparisons of Eqs. (9.32) and (9.33) with Eqs. (9.43) and (9.44) yield

$$\begin{bmatrix} J_{13} \\ J_{23} \\ J_{63} \end{bmatrix} = - \begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} \frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) \tag{9.45}$$

$$J_{33} = \frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) - [J_{31} \quad J_{32} \quad J_{36}] \frac{1}{h_\text{s}} \sum_{k=1}^{K_\text{s}} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right). \tag{9.46}$$

Since $\overline{\tau}_{yz} = \overline{\tau}_{xz} = 0$, we have that γ_{yz} and γ_{xz} are zero (Eqs. 2.26 and 2.21), and Eq. (9.34) becomes

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} J_{43} \\ J_{53} \end{bmatrix} \overline{\sigma}_z. \tag{9.47}$$

To satisfy this equation, the preceding elements of the compliance matrix must be zero

$$\begin{bmatrix} J_{43} \\ J_{53} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{9.48}$$

9.4.3 Step 3. Elements of [J] due to Out-of-Plane Shear Stresses

In this step we determine the elements in the fourth and fifth column of the matrix [J].

To accomplish this we apply the shear stresses τ_{yz}, τ_{xz} on the sublaminate (Fig. 9.7). Since $\overline{\sigma}_x$, $\overline{\sigma}_y$, $\overline{\sigma}_z$, $\overline{\tau}_{xy}$ are zero, the stress–strain relationships are

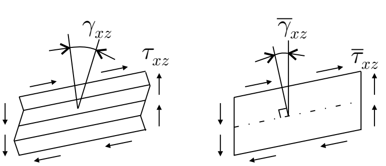

Figure 9.7: Illustration of Step 3. Stress τ_{xz} and strain γ_{xz} on a sublaminate subjected to transverse shear loads (left) and the corresponding average stress $\overline{\tau}_{xz}$ and average strain $\overline{\gamma}_{xz}$ (right).

(Eq. 9.15)

$$\begin{Bmatrix} \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \end{Bmatrix} = \begin{bmatrix} J_{44} & J_{45} \\ J_{54} & J_{55} \end{bmatrix} \begin{Bmatrix} \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \end{Bmatrix} \tag{9.49}$$

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\epsilon}_z \\ \overline{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} J_{14} & J_{15} \\ J_{24} & J_{25} \\ J_{34} & J_{35} \\ J_{64} & J_{65} \end{bmatrix} \begin{Bmatrix} \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \end{Bmatrix}. \tag{9.50}$$

For a single layer, we have (see Eqs. 2.26 and 2.21)

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{44} & \overline{S}_{45} \\ \overline{S}_{45} & \overline{S}_{55} \end{bmatrix} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}. \tag{9.51}$$

By combining Eqs. (9.18) and (9.51) and by replacing the integrals by summations, for the sublaminate we obtain

$$\begin{Bmatrix} \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \end{Bmatrix} = \frac{1}{h_s} \sum_{k=1}^{K_s} \left((z_k - z_{k-1}) \begin{bmatrix} \overline{S}_{44} & \overline{S}_{45} \\ \overline{S}_{45} & \overline{S}_{55} \end{bmatrix}_k \right) \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}. \tag{9.52}$$

By comparing Eqs. (9.49) and (9.52), we have

$$\begin{bmatrix} J_{44} & J_{45} \\ J_{54} & J_{55} \end{bmatrix} = \frac{1}{h_s} \sum_{k=1}^{K_s} \left((z_k - z_{k-1}) \begin{bmatrix} \overline{S}_{44} & \overline{S}_{45} \\ \overline{S}_{45} & \overline{S}_{55} \end{bmatrix}_k \right). \tag{9.53}$$

For the sublaminate subjected to transverse shear stresses resulting in $\overline{\gamma}_{yz}$ and $\overline{\gamma}_{xz}$, with all other strains being zero, Eq. (9.50) becomes

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} J_{14} & J_{15} \\ J_{24} & J_{25} \\ J_{34} & J_{35} \\ J_{64} & J_{65} \end{bmatrix} \begin{Bmatrix} \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \end{Bmatrix}. \tag{9.54}$$

To satisfy this equation, the preceding elements of the compliance matrix must be zero as follows:

$$\begin{bmatrix} J_{14} & J_{15} \\ J_{24} & J_{25} \\ J_{34} & J_{35} \\ J_{64} & J_{65} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9.55}$$

Table 9.1. The equations used for calculating the nonzero elements of the $[J]$ compliance matrix

Equation numbers

$$\begin{bmatrix} J_{11} \ldots J_{16} \\ \vdots \ \ddots \ \vdots \\ J_{61} \ldots J_{66} \end{bmatrix} \Longrightarrow \begin{bmatrix} 9.24 & 9.24 & 9.45 & - & - & 9.24 \\ 9.24 & 9.24 & 9.45 & - & - & 9.24 \\ 9.29 & 9.29 & 9.46 & - & - & 9.29 \\ - & - & - & 9.53 & 9.53 & - \\ - & - & - & 9.53 & 9.53 & - \\ 9.24 & 9.24 & 9.45 & - & - & 9.24 \end{bmatrix}$$

9.4.4 Step 4. The Stiffness Matrix

By combining the results of the preceding three steps, we obtain the following strain–stress relationship:

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\epsilon}_z \\ \overline{\gamma}_{yz} \\ \overline{\gamma}_{xz} \\ \overline{\gamma}_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} J_{11} & J_{12} & J_{13} & 0 & 0 & J_{16} \\ J_{21} & J_{22} & J_{23} & 0 & 0 & J_{26} \\ J_{31} & J_{32} & J_{33} & 0 & 0 & J_{36} \\ 0 & 0 & 0 & J_{44} & J_{45} & 0 \\ 0 & 0 & 0 & J_{54} & J_{55} & 0 \\ J_{61} & J_{62} & J_{63} & 0 & 0 & J_{66} \end{bmatrix}}_{[J]=[E]^{-1}} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\sigma}_z \\ \overline{\tau}_{yz} \\ \overline{\tau}_{xz} \\ \overline{\tau}_{xy} \end{Bmatrix}, \tag{9.56}$$

where $[J]$ is the compliance matrix of the sublaminate. The stiffness matrix is

$$[E] = [J]^{-1}. \tag{9.57}$$

We note that both $[J]$ and $[E]$ are symmetrical. The equations to be used for calculating the elements of $[J]$ are summarized in Table 9.1.

Under plane-stress condition, the strain–stress relationship is

$$\begin{Bmatrix} \overline{\epsilon}_x \\ \overline{\epsilon}_y \\ \overline{\gamma}_{xy} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix}. \tag{9.58}$$

Therefore, under plane–stress condition we need to determine only those elements that appear in this expression.

9.1 Example. *Calculate the stiffness matrix* $[J]$ *of a sublaminate made of graphite epoxy unidirectional plies. The material properties are given in Table 3.6 (page 81). The layup is* $[0_2/45_2/0_2/45_2]$ *(Fig. 9.8). The thickness of the sublaminate is* $h_s = 0.0008$ *m.*

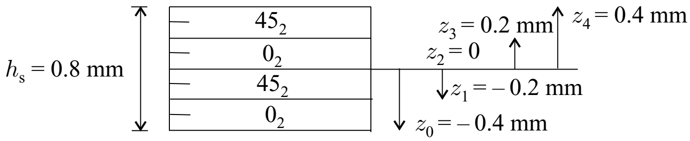

Figure 9.8: The sublaminate in Example 9.1.

Solution. First, we determine the stiffness and compliance matrices. The stiffness matrices $[\overline{Q}]^0$ and $[\overline{Q}]^{45}$ for the plies are given by Eqs. (3.49) and (3.52) as follows:

$$[\overline{Q}]^0 = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9 \, \frac{\text{N}}{\text{m}^2} \qquad [\overline{Q}]^{45} = \begin{bmatrix} 45.65 & 36.55 & 34.79 \\ 36.55 & 45.65 & 34.79 \\ 34.79 & 34.79 & 38.19 \end{bmatrix} 10^9 \, \frac{\text{N}}{\text{m}^2}. \tag{9.59}$$

The compliance matrix for the 0-degree ply is (Eq. 2.224)

$$[S] = [\overline{S}]^0 = \begin{bmatrix} 6.76 & -2.03 & -2.03 & 0 & 0 & 0 \\ -2.03 & 103.63 & -62.18 & 0 & 0 & 0 \\ -2.03 & -62.18 & 103.63 & 0 & 0 & 0 \\ 0 & 0 & 0 & 331.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 219.78 & 0 \\ 0 & 0 & 0 & 0 & 0 & 219.78 \end{bmatrix} 10^{-12} \, \frac{\text{m}^2}{\text{N}}. \tag{9.60}$$

The compliance matrix for the 45-degree ply is (Eq. 2.194, $[S']$ is replaced by $[\overline{S}]$) as follows:

$$[\overline{S}]^{45} = [\widehat{T}_\epsilon^r]\,[S]\,[\widehat{T}_\sigma^r]^{-1}. \tag{9.61}$$

The transformation matrices $[\widehat{T}_\sigma^r]$ and $[\widehat{T}_\epsilon^r]$ are given in Tables 2.15 (page 51) and 2.16 (page 53). The angle is 45°; hence, we have $c_r = \cos 45^\circ = s_r = \sin 45^\circ = 0.707$. The elements of the $[\overline{S}]^{45}$ matrix are

$$[\overline{S}]^{45} = \begin{bmatrix} 81.53 & -28.36 & -32.10 & 0 & 0 & -48.44 \\ -28.36 & 81.53 & -32.10 & 0 & 0 & -48.44 \\ -32.10 & -32.10 & 103.63 & 0 & 0 & 60.15 \\ 0 & 0 & 0 & 275.69 & -55.91 & 0 \\ 0 & 0 & 0 & -55.91 & 275.69 & 0 \\ -48.44 & -48.44 & 60.15 & 0 & 0 & 114.44 \end{bmatrix} 10^{-12} \, \frac{\text{m}^2}{\text{N}}. \tag{9.62}$$

The stiffness matrices are ($[\overline{C}] = [\overline{S}]^{-1}$)

$$[\overline{C}]^0 = \begin{bmatrix} 152.47 & 7.46 & 7.46 & 0 & 0 & 0 \\ 7.46 & 15.44 & 9.41 & 0 & 0 & 0 \\ 7.46 & 9.41 & 15.44 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.016 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.55 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.55 \end{bmatrix} 10^9 \, \frac{\text{N}}{\text{m}^2} \tag{9.63}$$

$$[\overline{C}]^{45} = \begin{bmatrix} 50.26 & 41.16 & 8.43 & 0 & 0 & 34.26 \\ 41.16 & 50.26 & 8.43 & 0 & 0 & 34.26 \\ 8.43 & 8.43 & 15.44 & 0 & 0 & -0.98 \\ 0 & 0 & 0 & 3.783 & 0.767 & 0 \\ 0 & 0 & 0 & 0.767 & 3.783 & 0 \\ 34.26 & 34.26 & -0.98 & 0 & 0 & 38.25 \end{bmatrix} 10^9 \, \frac{\text{N}}{\text{m}^2}. \tag{9.64}$$

The stiffness matrix $[A]$ is (Table 3.9, page 86)

$$[A] = \begin{bmatrix} 77.81 & 15.79 & 13.92 \\ 15.79 & 22.14 & 13.92 \\ 13.92 & 13.92 & 17.10 \end{bmatrix} 10^6 \, \frac{\text{N}}{\text{m}}. \tag{9.65}$$

We now proceed to determine the elements of the $[J]$ matrix. Equation (9.24) yields

$$\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} = h_s \, [A]^{-1} = \begin{bmatrix} 12.38 & -5.10 & -5.92 \\ -5.10 & 76.08 & -57.78 \\ -5.92 & -57.78 & 98.65 \end{bmatrix} 10^{-12} \, \frac{\text{m}^2}{\text{N}}. \tag{9.66}$$

From Eq. (9.29) we have

$$[J_{31} \quad J_{32} \quad J_{36}] = \sum_{k=1}^{K_s} ([\overline{S}_{13} \quad \overline{S}_{23} \quad \overline{S}_{36}]_k \, (z_k - z_{k-1}) \, [\overline{Q}]_k) [A]^{-1}, \tag{9.67}$$

where K_s is the total number of ply groups ($K_s = 4$) and z_k is the distance from the midplane to the ply (Fig. 9.8). Substitution of the numerical values into Eq. (9.67) yields

$$[J_{31} \quad J_{32} \quad J_{36}] = [-3.608 \quad -43.162 \quad 39.554] \times 10^{-12} \, \frac{\text{m}^2}{\text{N}}. \tag{9.68}$$

Equation (9.45) is

$$\begin{bmatrix} J_{13} \\ J_{23} \\ J_{63} \end{bmatrix} = -\begin{bmatrix} J_{11} & J_{12} & J_{16} \\ J_{21} & J_{22} & J_{26} \\ J_{61} & J_{62} & J_{66} \end{bmatrix} \frac{1}{h_s} \sum_{k=1}^{K_s} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right)$$

$$= \begin{bmatrix} -3.608 \\ -43.162 \\ 39.554 \end{bmatrix} 10^{-12} \frac{\text{m}^2}{\text{N}}. \tag{9.69}$$

Equation (9.46) gives

$$J_{33} = \frac{1}{h_s} \sum_{k=1}^{K_s} \left(\frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right) - [J_{31} \quad J_{32} \quad J_{36}] \frac{1}{h_s} \sum_{k=1}^{K_s} \left(\begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{63} \end{Bmatrix}_k \frac{z_k - z_{k-1}}{(\overline{C}_{33})_k} \right)$$

$$= 92.80 \times 10^{-12} \frac{\text{m}^2}{\text{N}}. \tag{9.70}$$

Equation (9.53) yields

$$\begin{bmatrix} J_{44} & J_{45} \\ J_{54} & J_{55} \end{bmatrix} = \frac{1}{h_s} \sum_{k=1}^{K_s} \left((z_k - z_{k-1}) \begin{bmatrix} \overline{S}_{44} & \overline{S}_{45} \\ \overline{S}_{45} & \overline{S}_{55} \end{bmatrix}_k \right) \tag{9.71}$$

$$= \begin{bmatrix} 303.65 & -27.96 \\ -27.96 & 247.74 \end{bmatrix} 10^{-12} \frac{\text{m}^2}{\text{N}}. \tag{9.72}$$

The $[J]$ matrix is

$$[J] = \begin{bmatrix} 12.38 & -5.10 & -3.61 & 0 & 0 & -5.92 \\ -5.10 & 76.08 & -43.16 & 0 & 0 & -57.78 \\ -3.61 & -43.16 & 92.80 & 0 & 0 & 39.55 \\ 0 & 0 & 0 & 303.65 & -27.96 & 0 \\ 0 & 0 & 0 & -27.96 & 247.74 & 0 \\ -5.92 & -57.78 & 39.55 & 0 & 0 & 98.65 \end{bmatrix} 10^{-12} \frac{\text{m}^2}{\text{N}}. \tag{9.73}$$

The stiffness matrix of the sublaminate is

$$[E] = [J]^{-1} = \begin{bmatrix} 101.35 & 24.32 & 7.95 & 0 & 0 & 17.14 \\ 24.32 & 32.84 & 8.92 & 0 & 0 & 17.11 \\ 7.95 & 8.92 & 15.44 & 0 & 0 & -0.489 \\ 0 & 0 & 0 & 3.33 & 0.376 & 0 \\ 0 & 0 & 0 & 0.376 & 4.08 & 0 \\ 17.14 & 17.11 & -0.489 & 0 & 0 & 21.39 \end{bmatrix} 10^{9} \frac{\text{N}}{\text{m}^2}. \tag{9.74}$$