CHAPTER TWO

Displacements, Strains, and Stresses

We consider composite materials consisting of continuous or discontinuous fibers embedded in a matrix. Such a composite is heterogeneous, and the properties vary from point to point. On a scale that is large with respect to the fiber diameter, the fiber and matrix properties may be averaged, and the material may be treated as homogeneous. This assumption, commonly employed in macromechanical analyses of composites, is adopted here. Hence, the material is considered to be quasi-homogeneous, which implies that the properties are taken to be the same at every point. These properties are not the same as the properties of either the fiber or the matrix but are a combination of the properties of the constituents.

In this chapter, equations are presented for calculating the displacements, stresses, and strains when the structure undergoes only small deformations and the material behaves in a linearly elastic manner.

Continuous fiber-reinforced composite materials (and structures made of such materials) often have easily identifiable preferred directions associated with fiber orientations or symmetry planes. It is therefore convenient to employ two coordinate systems: a local coordinate system aligned, at a point, either with the fibers or with axes of symmetry, and a global coordinate system attached to a fixed reference point (Fig. 2.1). In this book the local and global Cartesian coordinate systems are designated respectively by x_1, x_2, x_3 and the x, y, z axes. In the x, y, z directions the displacements at a point A are denoted by u, v, w, and in the x_1, x_2, x_3 directions by u_1, u_2, u_3 (Fig. 2.2).

In the x, y, z coordinate system the normal stresses are denoted by σ_x, σ_y, and σ_z and the shear stresses by τ_{yz}, τ_{xz}, and τ_{xy} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_x, \epsilon_y, \epsilon_z$ and $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$, respectively.

In the x_1, x_2, x_3 coordinate system the normal stresses are denoted by σ_1, σ_2, and σ_3 and the shear stresses by τ_{23}, τ_{13}, and τ_{12} (Fig. 2.3). The corresponding normal and shear strains are $\epsilon_1, \epsilon_2, \epsilon_3$, and $\gamma_{23}, \gamma_{13}, \gamma_{12}$, respectively. The symbol γ represents engineering shear strain that is twice the tensorial shear strain, $\gamma_{ij} = 2\epsilon_{ij}$ ($i, j = x, y, z$ or $i, j = 1, 2, 3$).

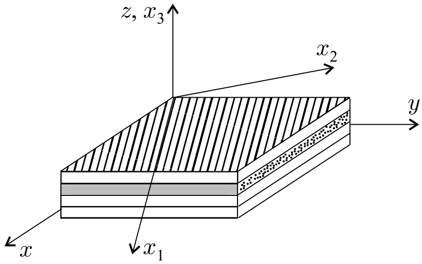


Figure 2.1: The global x, y, z and local x_1, x_2, x_3 coordinate systems.

A stress is taken to be positive when it acts on a positive face in the positive direction. According to this definition, all the stresses shown in Figure 2.3 are positive.

The preceding stress and strain notations, referred to as engineering notations, are used throughout this book. Other notations, most notably tensorial and contracted notations, can frequently be found in the literature. The stresses and strains in different notations are summarized in Tables 2.1 and 2.2.

2.1 Strain–Displacement Relations

We consider a Δx long segment that undergoes a change in length, the new length being denoted by $\Delta x'$. From Figure 2.4 it is seen that

$$u + \Delta x' = \Delta x + \left(u + \frac{\partial u}{\partial x} \Delta x \right), \tag{2.1}$$

where u and $u + \frac{\partial u}{\partial x} \Delta x$ are the displacements of points A and B, respectively, in the x direction. Accordingly, the normal strain in the x direction is

$$\epsilon_x = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{\partial u}{\partial x}. \tag{2.2}$$

Similarly, in the y and z directions the normal strains are

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{2.3}$$

$$\epsilon_z = \frac{\partial w}{\partial z}, \tag{2.4}$$

where v and w are the displacements in the y and z directions, respectively.

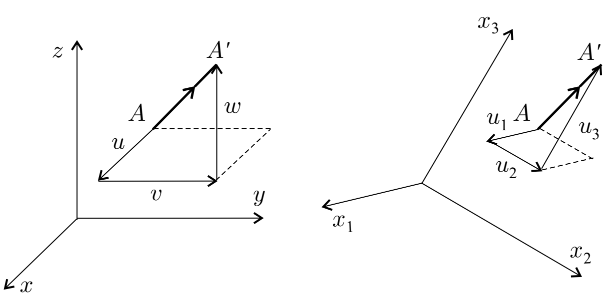


Figure 2.2: The x, y, z and x_1, x_2, x_3 coordinate systems and the corresponding displacements.

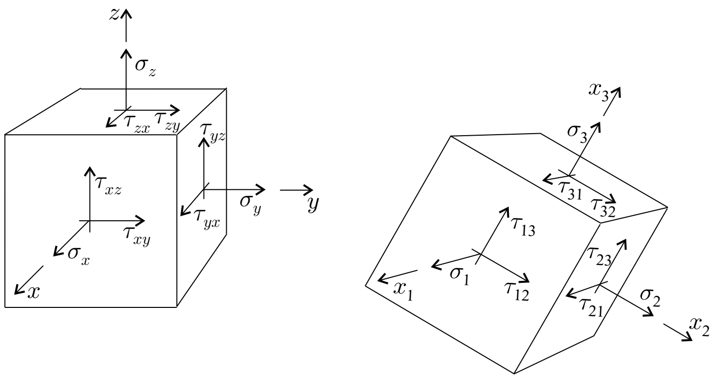


Figure 2.3: The stresses in the global x, y, z and the local x_1, x_2, x_3 coordinate systems.

For angular (shear) deformation the tensorial shear strain is the average change in the angle between two mutually perpendicular lines (Fig. 2.5)

$$\epsilon_{xy} = \frac{\alpha + \beta}{2}. \tag{2.5}$$

For small deformations we have

$$\alpha \approx \tan\alpha = \frac{\left(v + \frac{\partial v}{\partial x}\Delta x\right) - v}{\Delta x} = \frac{\partial v}{\partial x}. \tag{2.6}$$

Similarly $\beta = \partial u/\partial y$, and the xy component of the tensorial shear strain is

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.7}$$

In a similar manner we obtain the following expressions for the ϵ_{yz} and ϵ_{xz} components of the tensorial shear strains:

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{2.8}$$

Table 2.1. Stress notations

	Normal stress			Shear stress		
x, y, z coordinate system						
Tensorial stress	σ_{xx}	σ_{yy}	σ_{zz}	σ_{yz}	σ_{xz}	σ_{xy}
Engineering stress	σ_x	σ_y	σ_z	τ_{yz}	τ_{xz}	τ_{xy}
Contracted notation	σ_x	σ_y	σ_z	σ_q	σ_r	σ_s
x_1, x_2, x_3 coordinate system						
Tensorial stress	σ_{11}	σ_{22}	σ_{33}	σ_{23}	σ_{13}	σ_{12}
Engineering stress	σ_1	σ_2	σ_3	τ_{23}	τ_{13}	τ_{12}
Contracted notation	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6

Table 2.2. Strain notations (the engineering and contracted notation shear strains are twice the tensorial shear strain)

	Normal strain			Shear strain		
x, y, z coordinate system						
Tensorial strain	ϵ_{xx}	ϵ_{yy}	ϵ_{zz}	ϵ_{yz}	ϵ_{xz}	ϵ_{xy}
Engineering strain	ϵ_x	ϵ_y	ϵ_z	γ_{yz}	γ_{xz}	γ_{xy}
Contracted notation	ϵ_x	ϵ_y	ϵ_z	ϵ_q	ϵ_r	ϵ_s
x_1, x_2, x_3 coordinate system						
Tensorial strain	ϵ_{11}	ϵ_{22}	ϵ_{33}	ϵ_{23}	ϵ_{13}	ϵ_{12}
Engineering strain	ϵ_1	ϵ_2	ϵ_3	γ_{23}	γ_{13}	γ_{12}
Contracted notation	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6

The engineering shear strains are twice the tensorial shear strains:

$$\gamma_{yz} = 2\epsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{2.9}$$

$$\gamma_{xz} = 2\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2.10}$$

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{2.11}$$

In the x_1, x_2, x_3 coordinate system the strain–displacement relationships are also given by Eqs. (2.2)–(2.4) and (2.9)–(2.11) with x, y, z replaced by x_1, x_2, x_3, the subscripts x, y, z by 1, 2, 3, and u, v, w by u_1, u_2, u_3.

2.2 Equilibrium Equations

The equilibrium equations at a point O are obtained by considering force and moment balances on a small $\Delta x \Delta y \Delta z$ cubic element located at point O. (The point O is at the center of the element, Fig. 2.6.) We relate the stresses at one face to those at the opposite face by the Taylor series. By using only the first term of the Taylor series, force balance in the x direction gives

$$-\sigma_x \Delta z \Delta y - \tau_{zx} \Delta x \Delta y - \tau_{yx} \Delta x \Delta z + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \Delta x\right) \Delta z \Delta y$$
$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \Delta z\right) \Delta x \Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \Delta y\right) \Delta x \Delta z + f_x \Delta x \Delta y \Delta z = 0, \tag{2.12}$$

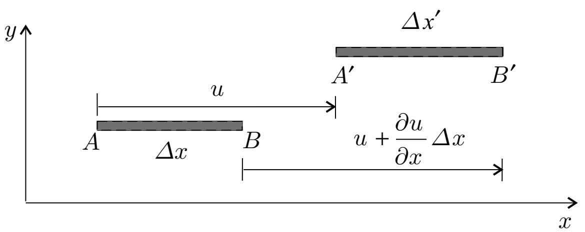


Figure 2.4: Displacement of the AB line segment.

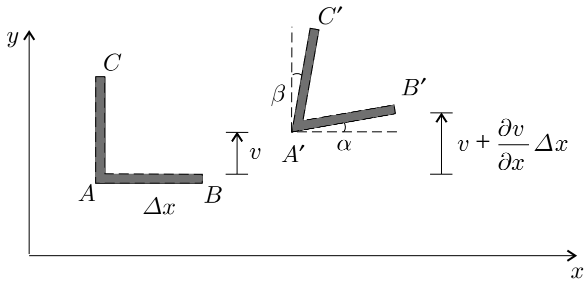


Figure 2.5: Displacement of the *ABC* segment.

where f_x is the body force per unit volume in the x direction. After simplification, this equation becomes

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0. \tag{2.13}$$

By similar arguments, the equilibrium equations in the y and z directions are

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0, \tag{2.14}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0, \tag{2.15}$$

where f_y and f_z are the body forces per unit volume in the y and z directions.

A moment balance about an axis parallel to x and passing through the center (point O) gives (Fig. 2.7)

$$\begin{aligned} &\tau_{yz}\Delta x\Delta z\frac{\Delta y}{2} - \tau_{zy}\Delta x\Delta y\frac{\Delta z}{2} \\ &+\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\Delta y\right)\Delta x\Delta z\frac{\Delta y}{2} - \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\Delta z\right)\Delta x\Delta y\frac{\Delta z}{2} = 0. \end{aligned} \tag{2.16}$$

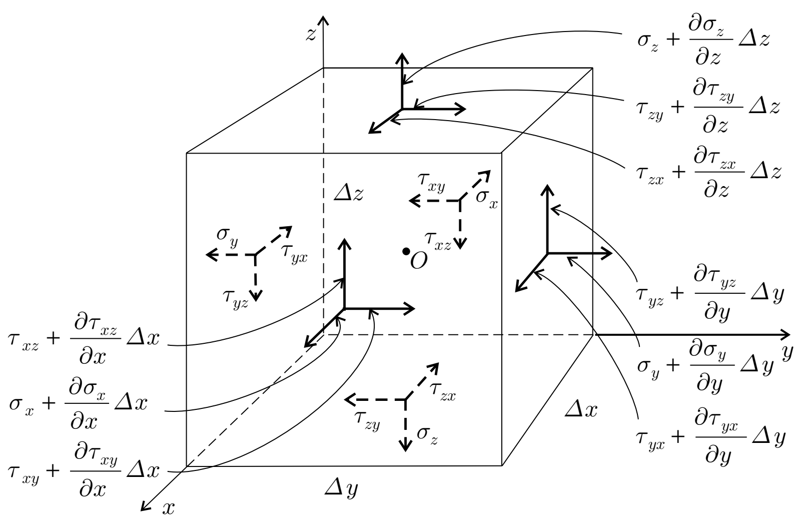


Figure 2.6: Stresses on the $\Delta x\Delta y\Delta z$ cubic element.

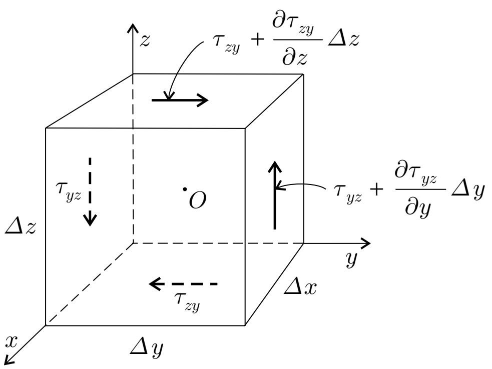


Figure 2.7: Stresses on the $\Delta x \Delta y \Delta z$ cubic element that appear in the moment balance about an axis parallel to x and passing through the center (point O).

By omitting higher order terms, which vanish in the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, this equation becomes

$$\tau_{yz} = \tau_{zy}. \tag{2.17}$$

Similarly, we obtain the following equalities:

$$\tau_{xz} = \tau_{zx} \qquad \tau_{xy} = \tau_{yx}. \tag{2.18}$$

By virtue of Eqs. (2.17) and (2.18), the three equilibrium equations (Eqs. 2.13–2.15) contain six unknowns, namely, the three normal stresses (σ_x, σ_y, σ_z) and the three shear stresses (τ_{yz}, τ_{xz}, τ_{xy}).

In the x_1, x_2, x_3 coordinate system the equilibrium equations are also given by Eqs. (2.13)–(2.15) with x, y, z replaced by x_1, x_2, x_3 and the subscripts x, y, z by $1, 2, 3$.

2.3 Stress–Strain Relationships

In a composite material the fibers may be oriented in an arbitrary manner. Depending on the arrangements of the fibers, the material may behave differently in different directions. According to their behavior, composites may be characterized as generally anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic. In the following, we present the stress–strain relationships for these types of materials under linearly elastic conditions.

2.3.1 Generally Anisotropic Material

When there are no symmetry planes with respect to the alignment of the fibers the material is referred to as generally anisotropic. A fiber-reinforced composite material is, for example, generally anisotropic when the fibers are aligned in three nonorthogonal directions (Fig. 2.8).

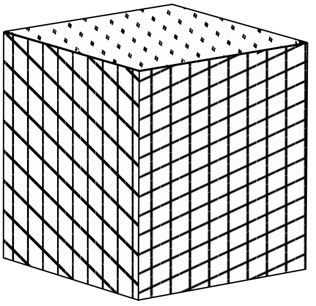


Figure 2.8: Example of a generally anisotropic material.

For a generally anisotropic linearly elastic material, in the x, y, z global coordinate system, the stress–strain relationships are

$$\begin{aligned}
\sigma_x &= \overline{C}_{11}\epsilon_x + \overline{C}_{12}\epsilon_y + \overline{C}_{13}\epsilon_z + \overline{C}_{14}\gamma_{yz} + \overline{C}_{15}\gamma_{xz} + \overline{C}_{16}\gamma_{xy} \\
\sigma_y &= \overline{C}_{21}\epsilon_x + \overline{C}_{22}\epsilon_y + \overline{C}_{23}\epsilon_z + \overline{C}_{24}\gamma_{yz} + \overline{C}_{25}\gamma_{xz} + \overline{C}_{26}\gamma_{xy} \\
\sigma_z &= \overline{C}_{31}\epsilon_x + \overline{C}_{32}\epsilon_y + \overline{C}_{33}\epsilon_z + \overline{C}_{34}\gamma_{yz} + \overline{C}_{35}\gamma_{xz} + \overline{C}_{36}\gamma_{xy} \\
\tau_{yz} &= \overline{C}_{41}\epsilon_x + \overline{C}_{42}\epsilon_y + \overline{C}_{43}\epsilon_z + \overline{C}_{44}\gamma_{yz} + \overline{C}_{45}\gamma_{xz} + \overline{C}_{46}\gamma_{xy} \\
\tau_{xz} &= \overline{C}_{51}\epsilon_x + \overline{C}_{52}\epsilon_y + \overline{C}_{53}\epsilon_z + \overline{C}_{54}\gamma_{yz} + \overline{C}_{55}\gamma_{xz} + \overline{C}_{56}\gamma_{xy} \\
\tau_{xy} &= \overline{C}_{61}\epsilon_x + \overline{C}_{62}\epsilon_y + \overline{C}_{63}\epsilon_z + \overline{C}_{64}\gamma_{yz} + \overline{C}_{65}\gamma_{xz} + \overline{C}_{66}\gamma_{xy}.
\end{aligned} \tag{2.19}$$

Equation (2.19) may be written in the form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} & \overline{C}_{34} & \overline{C}_{35} & \overline{C}_{36} \\ \overline{C}_{41} & \overline{C}_{42} & \overline{C}_{43} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{51} & \overline{C}_{52} & \overline{C}_{53} & \overline{C}_{54} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{61} & \overline{C}_{62} & \overline{C}_{63} & \overline{C}_{64} & \overline{C}_{65} & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}, \tag{2.20}$$

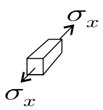
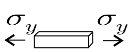
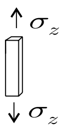
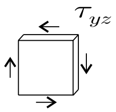
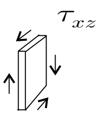
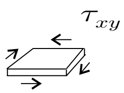
where $\overline{C}_{ij}$ are the elements of the stiffness matrix $[\overline{C}]$ in the x, y, z coordinate system.

Inversion of Eq. (2.20) results in the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} & \overline{S}_{15} & \overline{S}_{16} \\ \overline{S}_{21} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} & \overline{S}_{25} & \overline{S}_{26} \\ \overline{S}_{31} & \overline{S}_{32} & \overline{S}_{33} & \overline{S}_{34} & \overline{S}_{35} & \overline{S}_{36} \\ \overline{S}_{41} & \overline{S}_{42} & \overline{S}_{43} & \overline{S}_{44} & \overline{S}_{45} & \overline{S}_{46} \\ \overline{S}_{51} & \overline{S}_{52} & \overline{S}_{53} & \overline{S}_{54} & \overline{S}_{55} & \overline{S}_{56} \\ \overline{S}_{61} & \overline{S}_{62} & \overline{S}_{63} & \overline{S}_{64} & \overline{S}_{65} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}, \tag{2.21}$$

where $\overline{S}_{ij}$ are the elements of the compliance matrix $[\overline{S}]$ in the x, y, z coordinate system and are defined in Table 2.3 (page 10). In this table tests are illustrated that, in principle, could provide means of determining the different compliance matrix elements.

Table 2.3. The elements of the compliance matrix $[\overline{S}]$ in the x, y, z coordinate system. The elements S_{ij} (without bar) in the x_1, x_2, x_3 coordinate system are obtained by replacing x, y, z by 1, 2, 3 on the right-hand sides of the expressions.

Test	Elements of the compliance matrix	
	$\overline{S}_{11} = \epsilon_x/\sigma_x$	$\overline{S}_{41} = \gamma_{yz}/\sigma_x$
	$\overline{S}_{21} = \epsilon_y/\sigma_x$	$\overline{S}_{51} = \gamma_{xz}/\sigma_x$
	$\overline{S}_{31} = \epsilon_z/\sigma_x$	$\overline{S}_{61} = \gamma_{xy}/\sigma_x$
	$\overline{S}_{12} = \epsilon_x/\sigma_y$	$\overline{S}_{42} = \gamma_{yz}/\sigma_y$
	$\overline{S}_{22} = \epsilon_y/\sigma_y$	$\overline{S}_{52} = \gamma_{xz}/\sigma_y$
	$\overline{S}_{32} = \epsilon_z/\sigma_y$	$\overline{S}_{62} = \gamma_{xy}/\sigma_y$
	$\overline{S}_{13} = \epsilon_x/\sigma_z$	$\overline{S}_{43} = \gamma_{yz}/\sigma_z$
	$\overline{S}_{23} = \epsilon_y/\sigma_z$	$\overline{S}_{53} = \gamma_{xz}/\sigma_z$
	$\overline{S}_{33} = \epsilon_z/\sigma_z$	$\overline{S}_{63} = \gamma_{xy}/\sigma_z$
	$\overline{S}_{14} = \epsilon_x/\tau_{yz}$	$\overline{S}_{44} = \gamma_{yz}/\tau_{yz}$
	$\overline{S}_{24} = \epsilon_y/\tau_{yz}$	$\overline{S}_{54} = \gamma_{xz}/\tau_{yz}$
	$\overline{S}_{34} = \epsilon_z/\tau_{yx}$	$\overline{S}_{64} = \gamma_{xy}/\tau_{yz}$
	$\overline{S}_{15} = \epsilon_x/\tau_{xz}$	$\overline{S}_{45} = \gamma_{yz}/\tau_{xz}$
	$\overline{S}_{25} = \epsilon_y/\tau_{xz}$	$\overline{S}_{55} = \gamma_{xz}/\tau_{xz}$
	$\overline{S}_{35} = \epsilon_z/\tau_{xz}$	$\overline{S}_{65} = \gamma_{xy}/\tau_{xz}$
	$\overline{S}_{16} = \epsilon_x/\tau_{xy}$	$\overline{S}_{46} = \gamma_{yz}/\tau_{xy}$
	$\overline{S}_{26} = \epsilon_y/\tau_{xy}$	$\overline{S}_{56} = \gamma_{xz}/\tau_{xy}$
	$\overline{S}_{36} = \epsilon_z/\tau_{xy}$	$\overline{S}_{66} = \gamma_{xy}/\tau_{xy}$

In the x_1, x_2, x_3 coordinate system the stress–strain relationships are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}, \tag{2.22}$$

where C_{ij} are the elements of the stiffness matrix $[C]$ in the x_1, x_2, x_3 coordinate system.

By inverting Eq. (2.22) we obtain the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}, \tag{2.23}$$

where S_{ij} are the elements of the compliance matrix $[S]$ in the x_1, x_2, x_3 coordinate system.

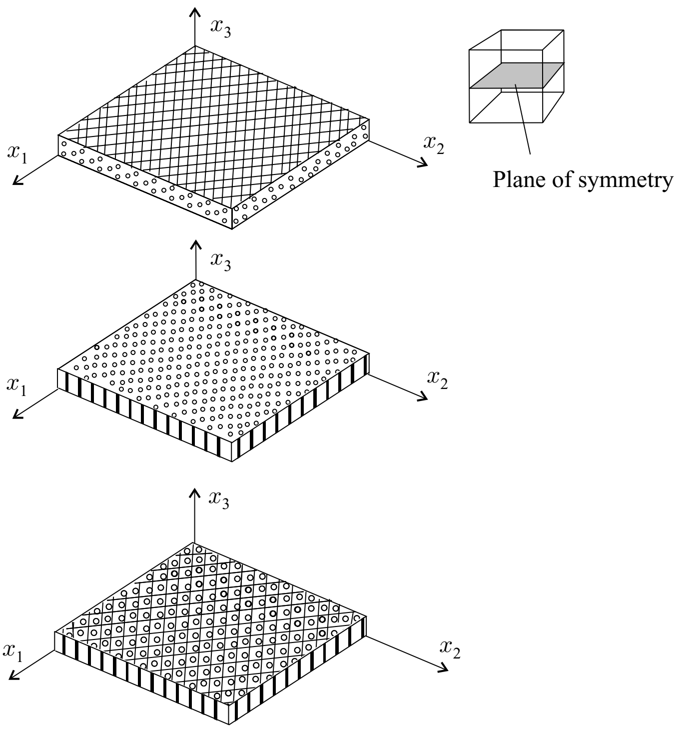


Figure 2.9: Illustrations of fiber-reinforced monoclinic materials. The fibers are only in planes parallel to the x_1–x_2 plane of symmetry (top), only perpendicular to the plane of symmetry (middle), and in the plane of symmetry and perpendicular to the plane of symmetry (bottom).

It is evident from Eqs. (2.20)–(2.23) that the compliance matrix $[S]$ is the inverse of the stiffness matrix $[C]$:

$$[\overline{S}] = [\overline{C}]^{-1} \qquad [S] = [C]^{-1}. \tag{2.24}$$

It can be shown (see Section 2.11.1) that for an elastic material the stiffness and compliance matrices are symmetrical in both the x, y, z and x_1, x_2, x_3 coordinate systems as follows:

$$\overline{S}_{ij} = \overline{S}_{ji} \qquad S_{ij} = S_{ji} \qquad \overline{C}_{ij} = \overline{C}_{ji} \qquad C_{ij} = C_{ji} \quad i, j = 1, 2, \ldots, 6. \tag{2.25}$$

Because of this symmetry, in both the $[\overline{S}]$ and the $[\overline{C}]$ matrices only 21 of the 36 elements are independent.

2.3.2 Monoclinic Material

When there is a symmetry plane with respect to the alignment of the fibers, the material is referred to as monoclinic. Examples of monoclinic fiber-reinforced composites are shown in Figure 2.9.

For a monoclinic material we specify the compliance $[S]$ and stiffness $[C]$ matrices in an x_1, x_2, x_3 coordinate system chosen in such a way that x_1 and x_2 are in the plane of symmetry, whereas x_3 is perpendicular to this plane.

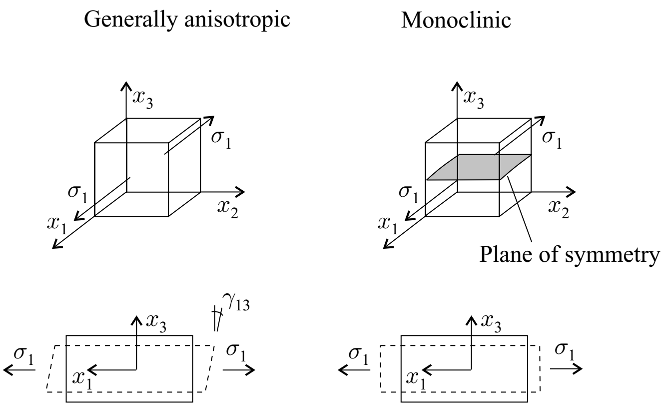
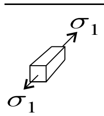
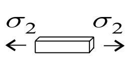
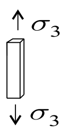
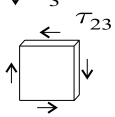
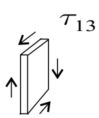
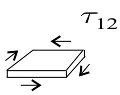


Figure 2.10: The normal stress σ_1 causes shear strain γ_{13} in a generally anisotropic material (left) and no shear strain in a monoclinic material (right).

The elements of the compliance matrix for a monoclinic material are obtained by modifying the compliance matrix of a generally anisotropic material. We observe that in a generally anisotropic material a normal stress σ_1 causes an out-of-plane shear strain γ_{13} (Fig. 2.10 left), but in a monoclinic material subjected to a normal stress σ_1 (σ_1 being in the plane of symmetry) the out-of-plane shear strain γ_{13} is zero (Fig. 2.10 right). Consequently, for a monoclinic material the S_{51} element of the compliance matrix is zero. By similar arguments it can be shown that

Table 2.4. The elements of the compliance matrix for monoclinic materials. For orthotropic, transversely isotropic, and isotropic materials $S_{16} = S_{61} = 0$, $S_{26} = S_{62} = 0$, $S_{36} = S_{63} = 0$, $S_{45} = S_{54} = 0$

Test	Elements of the compliance matrix	
 (σ_1)	$S_{11} = \epsilon_1/\sigma_1$ $S_{21} = \epsilon_2/\sigma_1$ $S_{31} = \epsilon_3/\sigma_1$	$S_{41} = \gamma_{23}/\sigma_1 = 0$ $S_{51} = \gamma_{13}/\sigma_1 = 0$ $S_{61} = \gamma_{12}/\sigma_1$
 (σ_2)	$S_{12} = \epsilon_1/\sigma_2$ $S_{22} = \epsilon_2/\sigma_2$ $S_{32} = \epsilon_3/\sigma_2$	$S_{42} = \gamma_{23}/\sigma_2 = 0$ $S_{52} = \gamma_{13}/\sigma_2 = 0$ $S_{62} = \gamma_{12}/\sigma_2$
 (σ_3)	$S_{13} = \epsilon_1/\sigma_3$ $S_{23} = \epsilon_2/\sigma_3$ $S_{33} = \epsilon_3/\sigma_3$	$S_{43} = \gamma_{23}/\sigma_3 = 0$ $S_{53} = \gamma_{13}/\sigma_3 = 0$ $S_{63} = \gamma_{12}/\sigma_3$
 (τ_{23})	$S_{14} = \epsilon_1/\tau_{23} = 0$ $S_{24} = \epsilon_2/\tau_{23} = 0$ $S_{34} = \epsilon_3/\tau_{23} = 0$	$S_{44} = \gamma_{23}/\tau_{23}$ $S_{54} = \gamma_{13}/\tau_{23}$ $S_{64} = \gamma_{12}/\tau_{23} = 0$
 (τ_{13})	$S_{15} = \epsilon_1/\tau_{13} = 0$ $S_{25} = \epsilon_2/\tau_{13} = 0$ $S_{35} = \epsilon_3/\tau_{13} = 0$	$S_{45} = \gamma_{23}/\tau_{13}$ $S_{55} = \gamma_{13}/\tau_{13}$ $S_{65} = \gamma_{12}/\tau_{13} = 0$
 (τ_{12})	$S_{16} = \epsilon_1/\tau_{12}$ $S_{26} = \epsilon_2/\tau_{12}$ $S_{36} = \epsilon_3/\tau_{12}$	$S_{46} = \gamma_{23}/\tau_{12} = 0$ $S_{56} = \gamma_{13}/\tau_{12} = 0$ $S_{66} = \gamma_{12}/\tau_{12}$

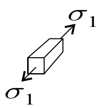
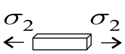
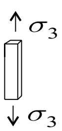
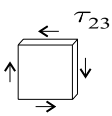
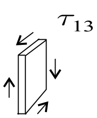
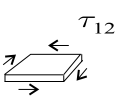
for a monoclinic material the S_{41}, S_{42}, S_{52}, S_{43}, S_{53}, S_{64}, S_{65} elements are also zero. (Since the compliance matrix is symmetrical the elements S_{14}, S_{24}, S_{25}, S_{34}, S_{35}, S_{46}, S_{56} are also zero.) The elements of the compliance matrix are listed in Table 2.4.

The elements of the compliance matrix may be expressed in terms of the engineering constants defined in Table 2.5. In Tables 2.4 and 2.5 the types of tests are also illustrated that, at least in principle, could provide the elements of the compliance matrix and the engineering constants. The relationships between the elements of the compliance matrix and the engineering constants are shown in Tables 2.6 and 2.7.

The nonzero and zero elements of the compliance matrix can best be seen when the matrix is written in the form

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}. \tag{2.26}$$

Table 2.5. The engineering constants for monoclinic materials. For orthotropic, transversely isotropic, and isotropic materials $\nu_{16} = \nu_{61} = 0$, $\nu_{26} = \nu_{62} = 0$, $\nu_{36} = \nu_{63} = 0$, $\nu_{45} = \nu_{54} = 0$

Test	Engineering constants	
	Young's modulus in the x_1 direction	$E_1 = \sigma_1/\epsilon_1$
	Poisson's ratio in the x_1–x_2 plane	$\nu_{12} = -\epsilon_2/\epsilon_1$
	Poisson's ratio in the x_1–x_3 plane	$\nu_{13} = -\epsilon_3/\epsilon_1$
	Poisson parameter	$\nu_{16} = \gamma_{12}/\epsilon_1$
	Young's modulus in the x_2 direction	$E_2 = \sigma_2/\epsilon_2$
	Poisson's ratio in the x_2–x_1 plane	$\nu_{21} = -\epsilon_1/\epsilon_2$
	Poisson's ratio in the x_2–x_3 plane	$\nu_{23} = -\epsilon_3/\epsilon_2$
	Poisson parameter	$\nu_{26} = \gamma_{12}/\epsilon_2$
	Young's modulus in the x_3 direction	$E_3 = \sigma_3/\epsilon_3$
	Poisson's ratio in the x_3–x_1 plane	$\nu_{31} = -\epsilon_1/\epsilon_3$
	Poisson's ratio in the x_3–x_2 plane	$\nu_{32} = -\epsilon_2/\epsilon_3$
	Poisson parameter	$\nu_{36} = \gamma_{12}/\epsilon_3$
	Shear modulus in the x_2–x_3 plane	$G_{23} = \tau_{23}/\gamma_{23}$
	Poisson parameter	$\nu_{45} = \gamma_{13}/\gamma_{23}$
	Shear modulus in the x_1–x_3 plane	$G_{13} = \tau_{13}/\gamma_{13}$
	Poisson parameter	$\nu_{54} = \gamma_{23}/\gamma_{13}$
	Shear modulus in the x_1–x_2 plane	$G_{12} = \tau_{12}/\gamma_{12}$
	Poisson parameter	$\nu_{61} = \epsilon_1/\gamma_{12}$
	Poisson parameter	$\nu_{62} = \epsilon_2/\gamma_{12}$
	Poisson parameter	$\nu_{63} = \epsilon_3/\gamma_{12}$

$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad \frac{\nu_{45}}{G_{23}} = \frac{\nu_{54}}{G_{13}}, \quad \frac{\nu_{i6}}{E_i} = \frac{\nu_{6i}}{G_{12}} \quad (i, j = 1, 2, 3)$

Table 2.6. Elements of the compliance matrix in terms of the engineering constants for monoclinic materials. The expressions are also valid for orthotropic, transversely isotropic, and isotropic materials with $S_{16} = S_{61} = 0$, $S_{26} = S_{62} = 0$, $S_{36} = S_{63} = 0$, $S_{45} = S_{54} = 0$.

$S_{11} = \epsilon_1/\sigma_1 = \epsilon_1/(E_1\epsilon_1) = \frac{1}{E_1}$	$S_{41} = 0$
$S_{21} = \epsilon_2/\sigma_1 = \epsilon_2/(E_1\epsilon_1) = -\frac{\nu_{12}}{E_1}$	$S_{51} = 0$
$S_{31} = \epsilon_3/\sigma_1 = \epsilon_3/(E_1\epsilon_1) = -\frac{\nu_{13}}{E_1}$	$S_{61} = \gamma_{12}/\sigma_1 = \gamma_{12}/(E_1\epsilon_1) = \frac{\nu_{16}}{E_1}$
$S_{12} = \epsilon_1/\sigma_2 = \epsilon_1/(E_2\epsilon_2) = -\frac{\nu_{21}}{E_2}$	$S_{42} = 0$
$S_{22} = \epsilon_2/\sigma_2 = \epsilon_2/(E_2\epsilon_2) = \frac{1}{E_2}$	$S_{52} = 0$
$S_{32} = \epsilon_3/\sigma_2 = \epsilon_3/(E_2\epsilon_2) = -\frac{\nu_{23}}{E_2}$	$S_{62} = \gamma_{12}/\sigma_2 = \gamma_{12}/(E_2\epsilon_2) = \frac{\nu_{26}}{E_2}$
$S_{13} = \epsilon_1/\sigma_3 = \epsilon_1/(E_3\epsilon_3) = -\frac{\nu_{31}}{E_3}$	$S_{43} = 0$
$S_{23} = \epsilon_2/\sigma_3 = \epsilon_2/(E_3\epsilon_3) = -\frac{\nu_{32}}{E_3}$	$S_{53} = 0$
$S_{33} = \epsilon_3/\sigma_3 = \epsilon_3/(E_3\epsilon_3) = \frac{1}{E_3}$	$S_{63} = \gamma_{12}/\sigma_3 = \gamma_{12}/(E_3\epsilon_3) = \frac{\nu_{36}}{E_3}$
$S_{14} = 0$	$S_{44} = \gamma_{23}/\tau_{23} = \gamma_{23}/(G_{23}\gamma_{23}) = \frac{1}{G_{23}}$
$S_{24} = 0$	$S_{54} = \gamma_{13}/\tau_{23} = \gamma_{13}/(G_{23}\gamma_{23}) = \frac{\nu_{45}}{G_{23}}$
$S_{34} = 0$	$S_{64} = 0$
$S_{15} = 0$	$S_{45} = \gamma_{23}/\tau_{13} = \gamma_{23}/(G_{13}\gamma_{13}) = \frac{\nu_{54}}{G_{13}}$
$S_{25} = 0$	$S_{55} = \gamma_{13}/\tau_{13} = \gamma_{13}/(G_{13}\gamma_{13}) = \frac{1}{G_{13}}$
$S_{35} = 0$	$S_{65} = 0$
$S_{16} = \epsilon_1/\tau_{12} = \epsilon_1/(G_{12}\gamma_{12}) = \frac{\nu_{61}}{G_{12}}$	$S_{46} = 0$
$S_{26} = \epsilon_2/\tau_{12} = \epsilon_2/(G_{12}\gamma_{12}) = \frac{\nu_{62}}{G_{12}}$	$S_{56} = 0$
$S_{36} = \epsilon_3/\tau_{12} = \epsilon_3/(G_{12}\gamma_{12}) = \frac{\nu_{63}}{G_{12}}$	$S_{66} = \gamma_{12}/\tau_{12} = \gamma_{12}/(G_{12}\gamma_{12}) = \frac{1}{G_{12}}$

The stiffness matrix is obtained by inverting the compliance matrix as follows:

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}. \quad (2.27)$$

Because the $[S]$ and $[C]$ matrices are symmetrical (Eq. 2.25) only 13 of the elements are independent (Table 2.8).

2.3.3 Orthotropic Material

When there are three mutually perpendicular symmetry planes with respect to the alignment of the fibers the material is referred to as orthotropic (Fig. 2.11). Examples of orthotropic fiber-reinforced composites are shown in Figure 2.12. For an orthotropic material we specify the stiffness and compliance matrices in the x_1, x_2, x_3 coordinate system defined in such a way that the axes are perpendicular to the three planes of symmetry (Fig. 2.11).

We apply a normal stress σ_1 (Fig. 2.13). Because σ_1 is in the x_1–x_2 symmetry (orthotropy) plane the out-of-plane shear strains are zero, ($\gamma_{13} = \gamma_{23} = 0$); and

Table 2.7. The compliance matrices in terms of the engineering constants for monoclinic, orthotropic, transversely isotropic, and isotropic materials

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \frac{\nu_{61}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \frac{\nu_{62}}{G_{12}} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \frac{\nu_{63}}{G_{12}} \\ 0 & 0 & 0 & \frac{1}{G_{23}} & \frac{\nu_{54}}{G_{13}} & 0 \\ 0 & 0 & 0 & \frac{\nu_{45}}{G_{23}} & \frac{1}{G_{13}} & 0 \\ \frac{\nu_{16}}{E_1} & \frac{\nu_{26}}{E_2} & \frac{\nu_{36}}{E_3} & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad \text{monoclinic}$$

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad \text{orthotropic}$$

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \quad \text{transversely isotropic}$$

$$[S]=\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \quad \text{isotropic}$$

because σ_1 is also in the x_1–x_3 symmetry plane, the $\gamma_{12} = 0$ shear strain is zero. This implies that S_{14}, S_{15}, S_{16} are zero. By similar arguments it can be shown that for an orthotropic material the S_{24}, S_{25}, S_{26}, S_{34}, S_{35}, S_{36}, S_{45}, S_{46}, S_{56} elements are also zero. Accordingly, the compliance matrix is

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}. \tag{2.28}$$

The elements of the compliance matrix are listed in Table 2.4. In terms of the engineering constants, the compliance matrix is given in Table 2.7. The stiffness

Table 2.8. The nonzero engineering constants for monoclinic, orthotropic, transversely isotropic, and isotropic materials ($\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}$, see Table 2.7)

	Nonzero engineering constants	
Material	Independent	Dependent
Monoclinic	E_1, E_2, E_3 G_{23}, G_{13}, G_{12} $\nu_{12}, \nu_{13}, \nu_{23}$ $\nu_{16}, \nu_{26}, \nu_{45}, \nu_{36}$	
Orthotropic	E_1, E_2, E_3 G_{23}, G_{13}, G_{12} $\nu_{12}, \nu_{13}, \nu_{23}$	
Transversely isotropic	E_1, E_2 G_{12} ν_{12}, ν_{23}	$E_3 = E_2, G_{13} = G_{12}$ $G_{23} = \frac{E_2}{2(1+\nu_{23})}$ $\nu_{13} = \nu_{12}$
Isotropic	$E_1 \quad (= E)$ $\nu_{12} \quad (= \nu)$	$E_2 = E_3 = E, \nu_{13} = \nu_{23} = \nu$ $G_{23} = G_{13} = G_{12} = \frac{E}{2(1+\nu)}$

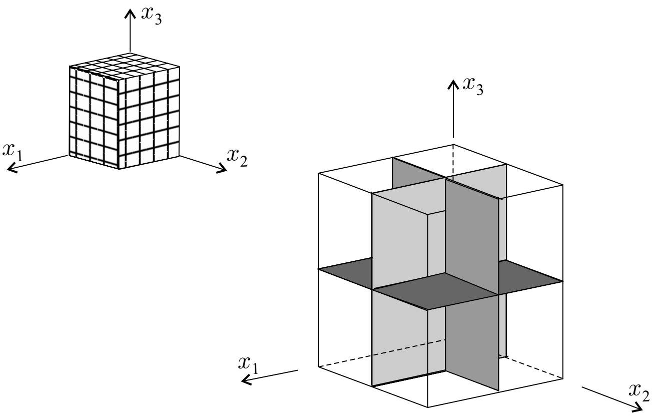


Figure 2.11: Material with three planes of symmetry.

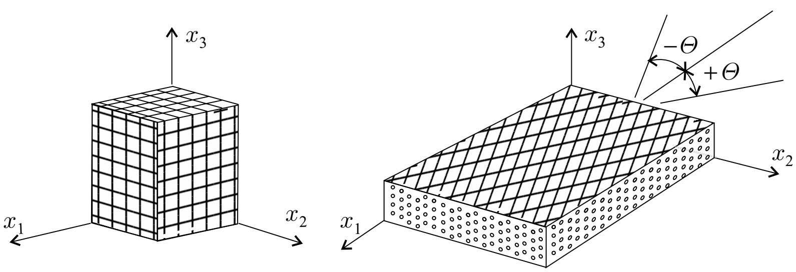


Figure 2.12: Illustrations of fiber-reinforced orthotropic composites.The fibers are oriented in three mutually perpendicular directions (left); the fibers are distributed equally in the $+\Theta$ and $-\Theta$ directions in planes parallel to the x_1–x_2 plane (right).

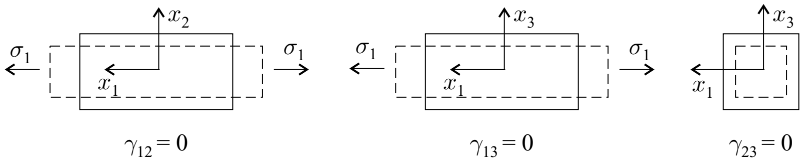


Figure 2.13: A normal stress σ_1 applied in the x_1–x_2 and x_1–x_3 symmetry planes of an orthotropic material.

matrix is obtained by inverting the compliance matrix. The nonzero terms of the stiffness matrix are

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.29}$$

In the $[S]$ and $[C]$ matrices of the 12 nonzero elements only 9 are independent (Table 2.8). Equation (2.29) can be written in the form

$$[C] = \begin{bmatrix} [L] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & [M] \end{bmatrix}. \tag{2.30}$$

The submatrices $[L]$ and $[M]$ are given in Tables 2.9 and 2.10 in terms of the engineering constants.

With the compliance matrix given by Eq. (2.28), the strain–stress relationships (Eq. 2.23) become

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}. \tag{2.31}$$

This equation shows an important feature of orthotropic materials, namely, that normal stresses do not produce shear deformations when these stresses are in the x_1, x_2, x_3 orthotropy directions. Note, however, that normal stresses applied in the x, y, z directions (which do not coincide with the x_1, x_2, x_3 orthotropy directions) result in shear deformations, as illustrated in Figure 2.14. In this case (in the x, y, z coordinate system) none of the elements of the compliance and stiffness matrices is zero.

Table 2.9. The [L] submatrix in Eq. (2.30) for orthotropic, transversely isotropic, and isotropic materials

Orthotropic

$$[L] = \frac{1}{D}\begin{bmatrix} E_1\left(1-\frac{E_3}{E_2}\nu_{23}^2\right) & E_2\left(\nu_{12}+\frac{E_3}{E_2}\nu_{13}\nu_{23}\right) & E_3\left(\nu_{13}+\nu_{12}\nu_{23}\right) \\ E_2\left(\nu_{12}+\frac{E_3}{E_2}\nu_{13}\nu_{23}\right) & E_2\left(1-\frac{E_3}{E_1}\nu_{13}^2\right) & E_3\left(\nu_{23}+\frac{E_2}{E_1}\nu_{12}\nu_{13}\right) \\ E_3\left(\nu_{13}+\nu_{12}\nu_{23}\right) & E_3\left(\nu_{23}+\frac{E_2}{E_1}\nu_{12}\nu_{13}\right) & E_3\left(1-\frac{E_2}{E_1}\nu_{12}^2\right) \end{bmatrix}$$

$$D = \frac{E_1E_2E_3-\nu_{23}^2E_1E_3^2-\nu_{12}^2E_2^2E_3-2\nu_{12}\nu_{13}\nu_{23}E_2E_3^2-\nu_{13}^2E_2E_3^2}{E_1E_2E_3}$$

Transversely isotropic

$$[L] = \frac{1}{D}\begin{bmatrix} E_1\left(1-\nu_{23}^2\right) & E_2\nu_{12}\left(1+\nu_{23}\right) & E_2\nu_{12}\left(1+\nu_{23}\right) \\ E_2\nu_{12}\left(1+\nu_{23}\right) & E_2\left(1-\frac{E_2}{E_1}\nu_{12}^2\right) & E_2\left(\nu_{23}+\frac{E_2}{E_1}\nu_{12}^2\right) \\ E_2\nu_{12}\left(1+\nu_{23}\right) & E_2\left(\nu_{23}+\frac{E_2}{E_1}\nu_{12}^2\right) & E_2\left(1-\frac{E_2}{E_1}\nu_{12}^2\right) \end{bmatrix}$$

$$D = 1-\nu_{23}^2-2\left(1+\nu_{23}\right)\frac{E_2}{E_1}\nu_{12}^2$$

Isotropic

$$[L] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}$$

Table 2.10. The [M] submatrix in Eq. (2.30) for orthotropic, transversely isotropic, and isotropic materials

Orthotropic

$$[M] = \begin{bmatrix} G_{23} & 0 & 0 \\ 0 & G_{13} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Transversely isotropic

$$[M] = \begin{bmatrix} \frac{E_2}{2(1+\nu_{23})} & 0 & 0 \\ 0 & G_{12} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Isotropic

$$[M] = \begin{bmatrix} \frac{E}{2(1+\nu)} & 0 & 0 \\ 0 & \frac{E}{2(1+\nu)} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$

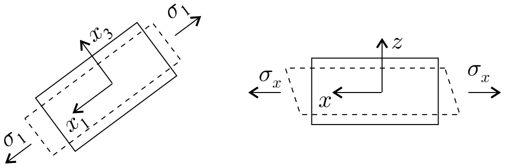


Figure 2.14: Orthotropic material subjected to a normal stress. There is no shear strain when the stress is applied in one of the orthotropy directions (left), but there is shear strain when the stress is not along an orthotropy direction (right).

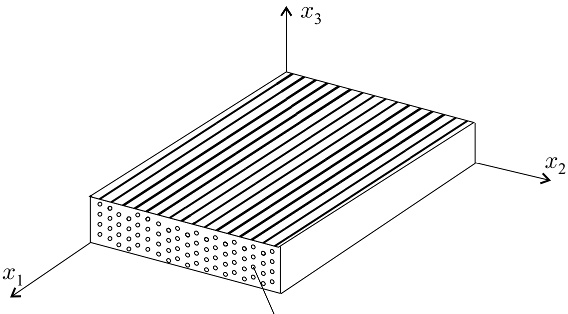


Figure 2.15: Example of a fiber-reinforced, transversely isotropic composite material.

2.3.4 Transversely Isotropic Material

A transversely isotropic material has three planes of symmetry (Fig. 2.11) and, as such, it is orthotropic. In one of the planes of symmetry the material is treated as isotropic. An example of transversely isotropic material is a composite reinforced with continuous unidirectional fibers with all the fibers aligned in the x_1 direction (Fig. 2.15). In this case the material in the plane perpendicular to the fibers (x_2–x_3 plane) is treated as isotropic.

For a transversely isotropic material we specify the stiffness and compliance matrices in an x_1, x_2, x_3 coordinate system chosen in such a way that the axes are perpendicular to the planes of symmetry and x_1 is perpendicular to the plane of isotropy (Fig. 2.15). In this coordinate system, because of material symmetry four of the Poisson ratios are zero ($\nu_{16} = \nu_{26} = \nu_{36} = \nu_{45} = 0$). Furthermore, because of isotropy the following engineering constants are related:

$$E_3 = E_2, \qquad G_{13} = G_{12}, \qquad \nu_{13} = \nu_{12}. \tag{2.32}$$

For an isotropic material the shear modulus is[1]

$$G = \frac{E}{2(1+\nu)}. \tag{2.33}$$

Correspondingly, for a material that is isotropic in the x_2–x_3 plane we write

$$G_{23} = \frac{E_2}{2(1+\nu_{23})}. \tag{2.34}$$

Equations (2.32) and (2.34), together with the expressions in Table 2.6 (page 14), yield the compliance matrix in terms of the engineering constants. The

[1] E. P. Popov, *Engineering Mechanics of Solids*. Prentice-Hall, Englewood Cliffs, New Jersey, 1990, p. 151.

results are given in Table 2.7 (page 15). The zero and nonzero elements of the compliance matrix are

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,(S_{22} - S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}. \tag{2.35}$$

The stiffness matrix is obtained by inverting the compliance matrix. The zero and nonzero elements of the stiffness matrix are

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \tag{2.36}$$

In terms of the engineering constants, the elements of the stiffness matrix are given by Eq. (2.30).

In both the compliance and stiffness matrices, of the 12 nonzero elements only 5 are independent (Table 2.8, page 16).

2.3.5 Isotropic Material

In an isotropic material there are no preferred directions and every plane is a plane of symmetry. For example, a composite containing a large number of randomly oriented fibers behaves in an isotropic manner. For an isotropic material the coordinate system may be chosen arbitrarily. Here, we present the compliance and the stiffness matrices in the x_1, x_2, and x_3 coordinate system.

Because of material symmetry four of the Poisson ratios are zero ($\nu_{16} = \nu_{26} = \nu_{36} = \nu_{45} = 0$). Also, because of isotropy some of the engineering constants are related as follows:

$$\begin{aligned} &E_1 = E_2 = E_3 = E \qquad G_{23} = G_{13} = G_{12} = G \\ &\nu_{23} = \nu_{13} = \nu_{12} = \nu \end{aligned} \tag{2.37}$$

$$G = \frac{E}{2\,(1+\nu)}. \tag{2.38}$$

Equations (2.37) and (2.38), together with the expressions in Table 2.6 (page 14), give the compliance matrix in terms of the engineering constants. The results are in Table 2.7. The elements of the compliance matrix are

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\left(S_{11} - S_{12}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\left(S_{11} - S_{12}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\left(S_{11} - S_{12}\right) \end{bmatrix}. \tag{2.39}$$

The stiffness matrix is obtained by inverting the compliance matrix. The elements of the stiffness matrix are

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix}. \tag{2.40}$$

In terms of the engineering constants, the elements of the stiffness matrix are given by Eq. (2.30).

In both the compliance and stiffness matrices, of the 12 nonzero elements only 2 are independent (Table 2.8, page 16).

2.1 Example. *Calculate the elements of the stiffness and compliance matrices of a graphite epoxy unidirectional ply. The engineering constants are given as* $E_1 = 148 \times 10^9$ *N/m*^{2}*,* $E_2 = 9.65 \times 10^9$ *N/m*^{2}*,* $G_{12} = 4.55 \times 10^9$ *N/m*^{2}*,* $\nu_{12} = 0.3$*, and* $\nu_{23} = 0.6$.

Solution. For a transversely isotropic material the compliance matrix is given in Table 2.7 (page 15, third row). By substituting the engineering constants into the expression in Table 2.7, and by using the condition that $\nu_{ij}/E_i = \nu_{ji}/E_j$ (see Table 2.8, page 16) we obtain

$$[S] = \begin{bmatrix} 6.76 & -2.03 & -2.03 & 0 & 0 & 0 \\ -2.03 & 103.63 & -62.18 & 0 & 0 & 0 \\ -2.03 & -62.18 & 103.63 & 0 & 0 & 0 \\ 0 & 0 & 0 & 331.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 219.78 & 0 \\ 0 & 0 & 0 & 0 & 0 & 219.78 \end{bmatrix} 10^{-12}\,\frac{\text{m}^2}{\text{N}}. \tag{2.41}$$

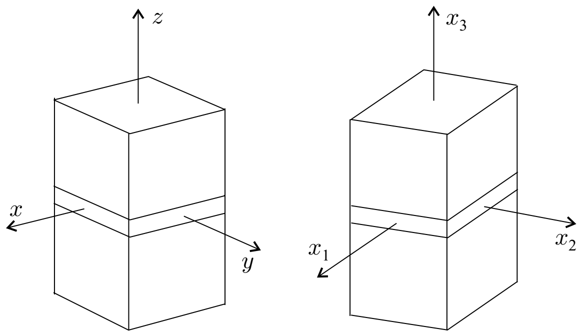


Figure 2.16: The x, y, z and the x_1, x_2, x_3 coordinate systems.

The elements of the stiffness matrix are obtained by inverting the compliance matrix

$$[C] = [S]^{-1} = \begin{bmatrix} 152.47 & 7.46 & 7.46 & 0 & 0 & 0 \\ 7.46 & 15.44 & 9.41 & 0 & 0 & 0 \\ 7.46 & 9.41 & 15.44 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.016 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.55 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.55 \end{bmatrix} 10^9 \frac{\mathrm{N}}{\mathrm{m}^2}. \tag{2.42}$$

2.4 Plane-Strain Condition

There are circumstances when the stresses and strains do not vary in a certain direction. This direction is designated by either the x_3 or the z axis (Fig. 2.16). Although the stresses and strains do not vary along x_3 (or z), they may vary in planes perpendicular to the x_3 (or z) axis. This condition is referred to as plane–strain condition.

When plane-strain condition exists in a body made of an isotropic material the x_1–x_2 (or x–y) planes of the cross section remain plane and perpendicular to the x_3 (or z) axis. In a body made of an anisotropic material these planes do not necessarily remain plane.

Plane-strain condition may exist far from the edges in a long body with constant cross section when both the material properties and the applied loads are uniform

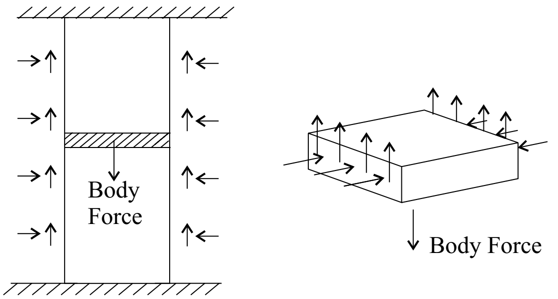


Figure 2.17: Surface and body forces that may be applied under plane-strain condition. The applied forces must be uniform along the longitudinal axis and must be in equilibrium for each segment.

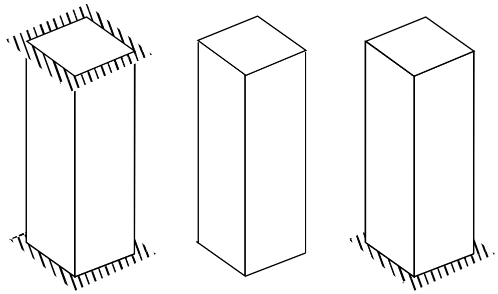


Figure 2.18: Possible end supports for structures analyzed by plane-strain condition.

along the longitudinal axis and, in addition, the loads are in equilibrium on any plane segment (Fig. 2.17).

When the aforementioned plane-strain condition exists, the three-dimensional analysis simplifies considerably. For an isotropic material, the normal strain ϵ_3 (or ϵ_z) in the axial direction (x_3 or z) and the out-of-plane shear strains γ_{13} and γ_{23} (or γ_{xz} and γ_{yz}) are zero. For fiber-reinforced composites these strains are not necessarily zero. Nonetheless, as is discussed in this chapter, plane-strain condition introduces simplifications that facilitate the analysis.

Geometry. The cross section perpendicular to the axis and the material properties must not vary along the length. Both ends of the body may be built-in or may be free, or one end may be built-in while the other one is free (Fig. 2.18). When both ends are built-in, the longitudinal axis (x_3 or z) remains straight and its length remains constant. When one or both ends are free, the longitudinal axis may become curved and its length may change.

Fiber orientation. On the basis of fiber orientation, the body is analyzed as generally anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic (Fig. 2.19).

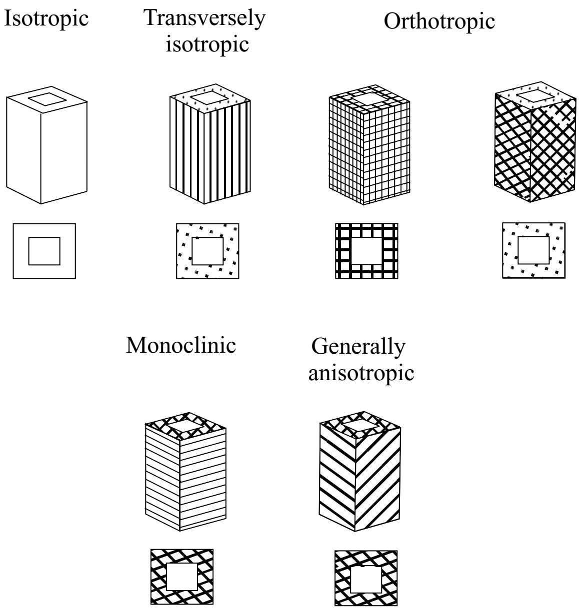


Figure 2.19: Illustrations of possible fiber orientations for plane-strain condition.

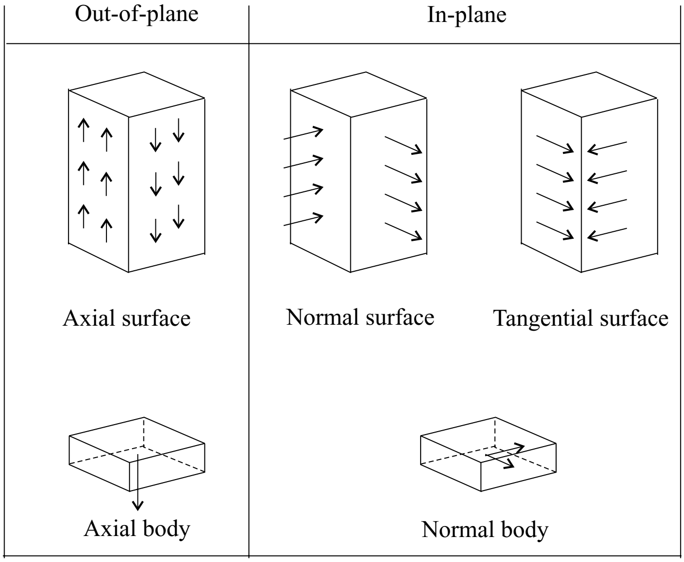


Figure 2.20: Illustrations of surface and body forces that may be applied under plane-strain condition.

In addition to the fiber arrangements shown in Figure 2.19, the composite may contain any combination of the fiber arrangements shown in this figure. The important fact is that the analysis must be performed according to the most complex fiber arrangement inside the body. A body that contains at least one generally anisotropic fiber arrangement must be treated as generally anisotropic; a body that consists of monoclinic, orthotropic, and transversely isotropic fiber arrangements must be treated as monoclinic; a body that consists of orthotropic and transversely isotropic fiber arrangements must be treated as orthotropic.

For monoclinic, orthotropic, and transversely isotropic materials the analysis simplifies considerably when one of the material symmetry planes coincides with the plane of plane-strain.

Loads. Normal, tangential, and axial forces may act on the surface, and body forces may act parallel and normal to the longitudinal axis (Fig. 2.20). These

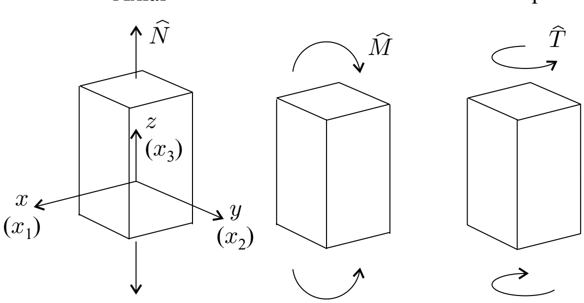


Figure 2.21: Possible end loads for plane-strain condition.

Table 2.11. Relationships between the stresses and the end loads

Load	x, y, z coordinate system	x_1, x_2, x_3 coordinate system
End axial	$\widehat{N} = \int_A \sigma_z dA$	$\widehat{N} = \int_A \sigma_3 dA$
End moment	$\widehat{M}_y = \int_A x\sigma_z dA$	$\widehat{M}_2 = \int_A x_1\sigma_3 dA$
End moment	$\widehat{M}_x = \int_A y\sigma_z dA$	$\widehat{M}_1 = \int_A x_2\sigma_3 dA$
Torque	$\widehat{T} = \int_A (y\tau_{xz} + x\tau_{yz})\, dA$	$\widehat{T} = \int_A (x_2\tau_{13} + x_1\tau_{23})\, dA$

applied forces (loads) may not vary along the length. In addition, an axial force, moment, and torque may be applied at the ends (Fig. 2.21). The forces acting perpendicular and parallel to the longitudinal axis are referred to as in-plane and out-of-plane.

Each of the loads shown in Figures 2.20 and 2.21 may act in combination. As stated previously, the only requirement is that the loads be in equilibrium on each segment (Fig. 2.17) of the body.

The end loads shown in Figure 2.21 are related the to the stresses by the expressions given in Table 2.11.

When two or more types of loads are applied, the stresses and strains can independently be calculated for each type of load. The stresses and strains thus obtained are then superimposed to obtain the final results.

Displacements. Plane-strain condition requires that the strains do not vary along the longitudinal axis. Thus, in the x, y, z coordinate system we have

$$\frac{\partial \epsilon_x}{\partial z} = 0 \qquad \frac{\partial \epsilon_y}{\partial z} = 0 \qquad \frac{\partial \epsilon_z}{\partial z} = 0$$
$$\frac{\partial \gamma_{yz}}{\partial z} = 0 \qquad \frac{\partial \gamma_{xz}}{\partial z} = 0 \qquad \frac{\partial \gamma_{xy}}{\partial z} = 0. \tag{2.43}$$

The following displacements satisfy these conditions[2]

$$\begin{aligned} u &= U(x, y) - C_1 yz - \frac{1}{2}C_2 z^2 \\ v &= V(x, y) + C_1 xz - \frac{1}{2}C_3 z^2 \\ w &= W(x, y) + (C_2 x + C_3 y + C_4)\, z, \end{aligned} \tag{2.44}$$

where U, V, and W are functions that depend only on x and y, and C_1, C_2, C_3, and C_4 are constants. For small displacements we have the following relationships[3]:

$$\epsilon_z^{o} = \frac{\partial w}{\partial z} \qquad \frac{1}{\rho_y} = -\frac{\partial^2 u}{\partial z^2} \qquad \frac{1}{\rho_x} = -\frac{\partial^2 v}{\partial z^2}. \tag{2.45}$$

[2] S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Body.* Mir Publishers, Moscow, 1981, p. 104.

[3] T. H. G. Megson, *Aircraft Structures for Engineering Students*. 3d edition. Halsted Press, John Wiley & Sons, New York, 1999, p. 284.

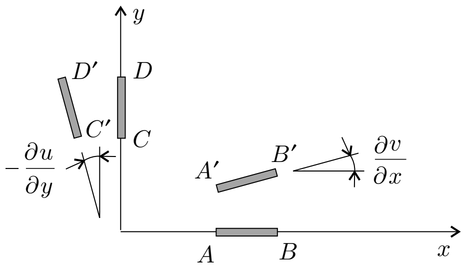


Figure 2.22: Illustration of the rotations of line elements parallel with the x- and y-axes.

where ϵ_z^{o} is the strain along the longitudinal axis; $1/\rho_y$, $1/\rho_x$ are the curvatures of the longitudinal axis in the x–z and y–z planes, respectively. By virtue of $\partial \gamma_{xy}/\partial z = 0$ (Eq. 2.43), we have (Eq. 2.11)

$$\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} = 0. \tag{2.46}$$

We define

$$\vartheta \equiv \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right). \tag{2.47}$$

As is illustrated in Figure 2.22, ϑ represents the rate of twist of the cross section.

The constants in Eq. (2.44) are determined as follows. From Eqs. (2.47) and (2.44) we have

$$\vartheta \equiv \frac{\partial^2 v}{\partial x \partial z} = -\frac{\partial^2 u}{\partial y \partial z} = C_1. \tag{2.48}$$

From Eqs. (2.45) and (2.44) we write

$$\begin{aligned} \epsilon_z^{\mathrm{o}} &= \frac{\partial w}{\partial z} = C_4 \\ \frac{1}{\rho_y} &= -\frac{\partial^2 u}{\partial z^2} = C_2 \\ \frac{1}{\rho_x} &= -\frac{\partial^2 v}{\partial z^2} = C_3. \end{aligned} \tag{2.49}$$

We can now write the displacements as

$$u = U(x, y) - \vartheta y z - \frac{1}{2}\frac{1}{\rho_y} z^2 \tag{2.50}$$

$$v = V(x, y) + \vartheta x z - \frac{1}{2}\frac{1}{\rho_x} z^2 \tag{2.51}$$

$$w = W(x, y) + \left(\frac{1}{\rho_y} x + \frac{1}{\rho_x} y + \epsilon_z^{\mathrm{o}}\right) z. \tag{2.52}$$

These represent the following five deformations (Fig. 2.23):

- Planar – planes perpendicular to the longitudinal axis remain plain and perpendicular to the axis
- Nonplanar – planes perpendicular to the longitudinal axis deform out of the plane

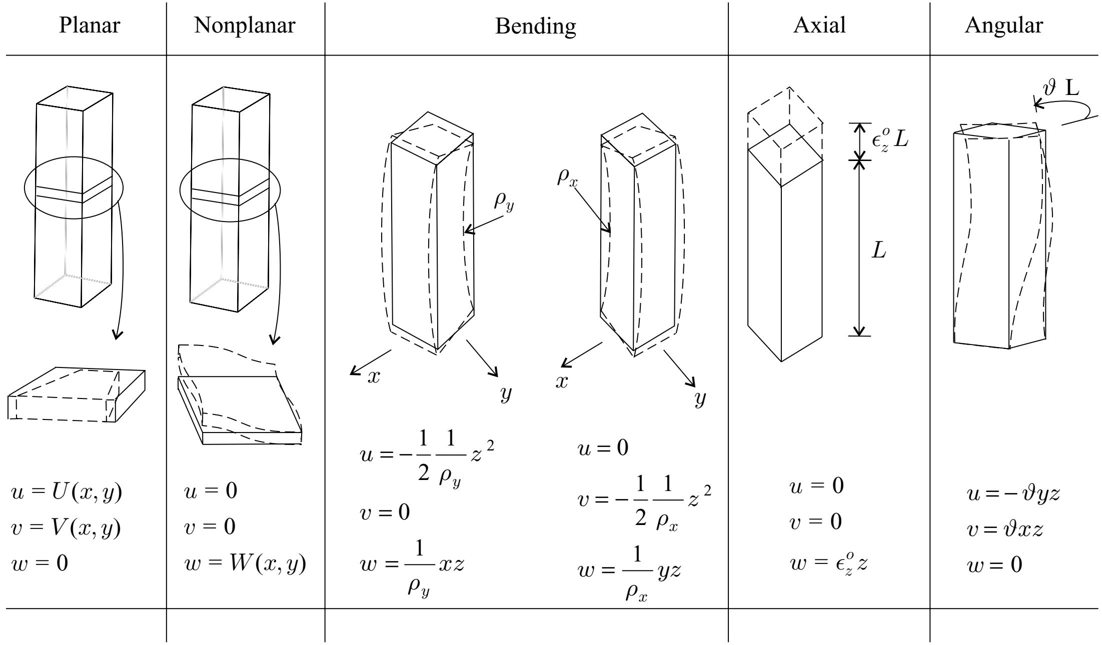


Figure 2.23: Deformations and displacements under plane-strain condition. In the x_1, x_2, x_3 coordinate system u, v, w are replaced by u_1, u_2, u_3; U, V, W by U_1, U_2, U_3; and ρ_x, ρ_y by ρ_1, ρ_2.

- Bending – the straight longitudinal axis becomes curved
- Axial – the straight longitudinal axis remains straight and undergoes axial elongation
- Angular – planes perpendicular to the longitudinal axis rotate about the axis.

The u, v, w components of the displacements corresponding to each of the deformations are shown in Figure 2.23.

We again use the x, y, z coordinate system for generally anisotropic materials. For monoclinic, orthotropic, transversely isotropic, and isotropic materials we use the x_1, x_2, x_3 coordinate system, with x_3 being along the longitudinal axis of the body (Fig. 2.16). In the x_1, x_2, x_3 coordinate system the displacements are

$$\begin{aligned} u_1 &= U_1(x_1, x_2) - \vartheta x_2 x_3 - \frac{1}{2}\frac{1}{\rho_2} x_3^2 \\ u_2 &= U_2(x_1, x_2) + \vartheta x_1 x_3 - \frac{1}{2}\frac{1}{\rho_1} x_3^2 \\ u_3 &= U_3(x_1, x_2) + \left(\frac{1}{\rho_2} x_1 + \frac{1}{\rho_1} x_2 + \epsilon_3^{\mathrm{o}}\right) x_3, \end{aligned} \tag{2.53}$$

where ϵ_3^{o} is the elongation of the longitudinal axis; $1/\rho_2, 1/\rho_1$ are the curvatures of this axis in the x_1–x_3 and x_2–x_3 planes, respectively; ϑ represents the rate of twist of the cross section.

In the following, we present the equilibrium equations, the strain–displacement relationships, and the stress–strain relationships when the aforementioned conditions of plane-strain are satisfied. The analyses are applicable in regions away from the two ends of the body.

We treat problems in two groups: (i) when one or both ends are free and (ii) when both ends are built-in (Fig. 2.18).

2.4.1 Free End – Generally Anisotropic Material

We consider a generally anisotropic body with one or both ends free. This body may undergo every deformation shown in Figure 2.23, and the displacements are the sum of all possible displacements shown in this figure. The strains are then obtained from the strain–displacement relationships given by Eqs. (2.2)–(2.4) and (2.9)–(2.11). By introducing Eqs. (2.50)–(2.52) we have

$$\epsilon_x = \frac{\partial U}{\partial x} \qquad \epsilon_y = \frac{\partial V}{\partial y} \tag{2.54}$$

$$\epsilon_z = \frac{1}{\rho_y} x + \frac{1}{\rho_x} y + \epsilon_z^{\mathrm{o}} \tag{2.55}$$

$$\gamma_{yz} = \frac{\partial W}{\partial y} + \vartheta x \qquad \gamma_{xz} = \frac{\partial W}{\partial x} - \vartheta y \tag{2.56}$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}. \tag{2.57}$$

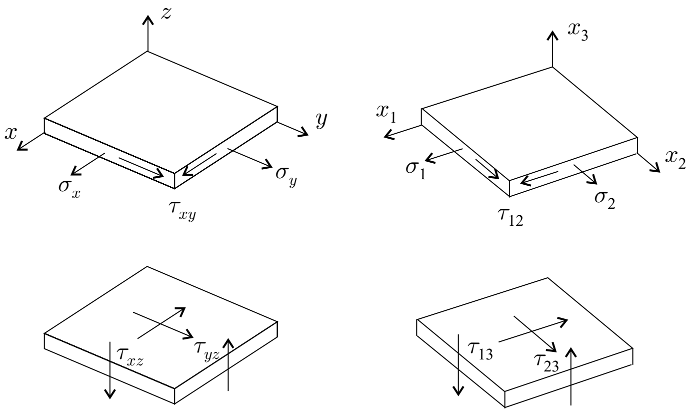


Figure 2.24: The stress components that appear in the equilibrium equations under plane-strain condition.

Since the strain components are independent of z, the stress components are also independent of z, and we have

$$\frac{\partial \sigma_z}{\partial z} = 0 \qquad \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad \frac{\partial \tau_{xz}}{\partial z} = 0. \tag{2.58}$$

By utilizing Eq. (2.58), the equilibrium equations (Eqs. 2.13–2.15) become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0 \tag{2.59}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \tag{2.60}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + f_z = 0\,. \tag{2.61}$$

The five stress components $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz}$ that appear in these equilibrium equations are illustrated in Figure 2.24.

For a generally anisotropic material the stress–strain relationships (Eq. 2.20) may be written in partitioned form, as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{14} & \overline{C}_{24} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{15} & \overline{C}_{25} & \overline{C}_{45} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{46} & \overline{C}_{56} & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} + \begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{34} \\ \overline{C}_{35} \\ \overline{C}_{36} \end{Bmatrix} \epsilon_z \tag{2.62}$$

$$\sigma_z = [\overline{C}_{13} \quad \overline{C}_{23} \quad \overline{C}_{34} \quad \overline{C}_{35} \quad \overline{C}_{36}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} + \overline{C}_{33}\epsilon_z, \tag{2.63}$$

where, we recall, ϵ_z is (Eq. 2.55)

$$\epsilon_z = \frac{1}{\rho_y}x + \frac{1}{\rho_x}y + \epsilon_z^{\mathrm{o}}. \tag{2.64}$$

The stiffness matrix $[\overline{C}]$ is the inverse of the compliance matrix $[\overline{S}]$ defined in Table 2.3 (page 10). The strain–stress relationships are obtained by inverting Eq. (2.62) as follows:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{R}_{11} & \overline{R}_{12} & \overline{R}_{14} & \overline{R}_{15} & \overline{R}_{16} \\ \overline{R}_{12} & \overline{R}_{22} & \overline{R}_{24} & \overline{R}_{25} & \overline{R}_{26} \\ \overline{R}_{14} & \overline{R}_{24} & \overline{R}_{44} & \overline{R}_{45} & \overline{R}_{46} \\ \overline{R}_{15} & \overline{R}_{25} & \overline{R}_{45} & \overline{R}_{55} & \overline{R}_{56} \\ \overline{R}_{16} & \overline{R}_{26} & \overline{R}_{46} & \overline{R}_{56} & \overline{R}_{66} \end{bmatrix} \left(\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} - \begin{Bmatrix} \overline{C}_{13} \\ \overline{C}_{23} \\ \overline{C}_{34} \\ \overline{C}_{35} \\ \overline{C}_{36} \end{Bmatrix} \epsilon_z \right), \tag{2.65}$$

where $\overline{R}_{ij}$ $(i, j = 1, 2, 4, 5, 6)$ are the in-plane elements of the compliance matrix under plane-strain condition calculated from the relationship

$$\begin{bmatrix} \overline{R}_{11} & \overline{R}_{12} & \overline{R}_{14} & \overline{R}_{15} & \overline{R}_{16} \\ \overline{R}_{12} & \overline{R}_{22} & \overline{R}_{24} & \overline{R}_{25} & \overline{R}_{26} \\ \overline{R}_{14} & \overline{R}_{24} & \overline{R}_{44} & \overline{R}_{45} & \overline{R}_{46} \\ \overline{R}_{15} & \overline{R}_{25} & \overline{R}_{45} & \overline{R}_{55} & \overline{R}_{56} \\ \overline{R}_{16} & \overline{R}_{26} & \overline{R}_{46} & \overline{R}_{56} & \overline{R}_{66} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{14} & \overline{C}_{24} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{15} & \overline{C}_{25} & \overline{C}_{45} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{46} & \overline{C}_{56} & \overline{C}_{66} \end{bmatrix}^{-1}. \tag{2.66}$$

2.4.2 Free End – Monoclinic Material

We consider a body made of a monoclinic material (Fig. 2.9) with the plane of material symmetry coinciding with the x_1–x_2 plane shown in Figure 2.16. One or both ends of the body are free. The displacements resulting from specified loads are summarized in Figure 2.25.

In-plane loads. The body is subjected to the in-plane loads shown in Figure 2.20. Under these loads the body may undergo only planar, bending, and axial deformations (Fig. 2.25). Accordingly, Eq. (2.53) reduces to (see Fig. 2.23)

$$\begin{aligned} u_1 &= U_1(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_2}x_3^2 \\ u_2 &= U_2(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_1}x_3^2 \\ u_3 &= \left(\frac{1}{\rho_2}x_1 + \frac{1}{\rho_1}x_2 + \epsilon_3^{\mathrm{o}}\right)x_3. \end{aligned} \tag{2.67}$$

The strain–displacement relationships (Eqs. 2.2–2.4, 2.9–2.11) show that for these displacements the strains are

$$\epsilon_1 = \frac{\partial U_1}{\partial x_1} \qquad \epsilon_2 = \frac{\partial U_2}{\partial x_2} \tag{2.68}$$

$$\epsilon_3 = \frac{1}{\rho_2}x_1 + \frac{1}{\rho_1}x_2 + \epsilon_3^{\mathrm{o}} \tag{2.69}$$

$$\gamma_{23} = 0 \qquad \gamma_{13} = 0 \tag{2.70}$$

$$\gamma_{12} = \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}. \tag{2.71}$$

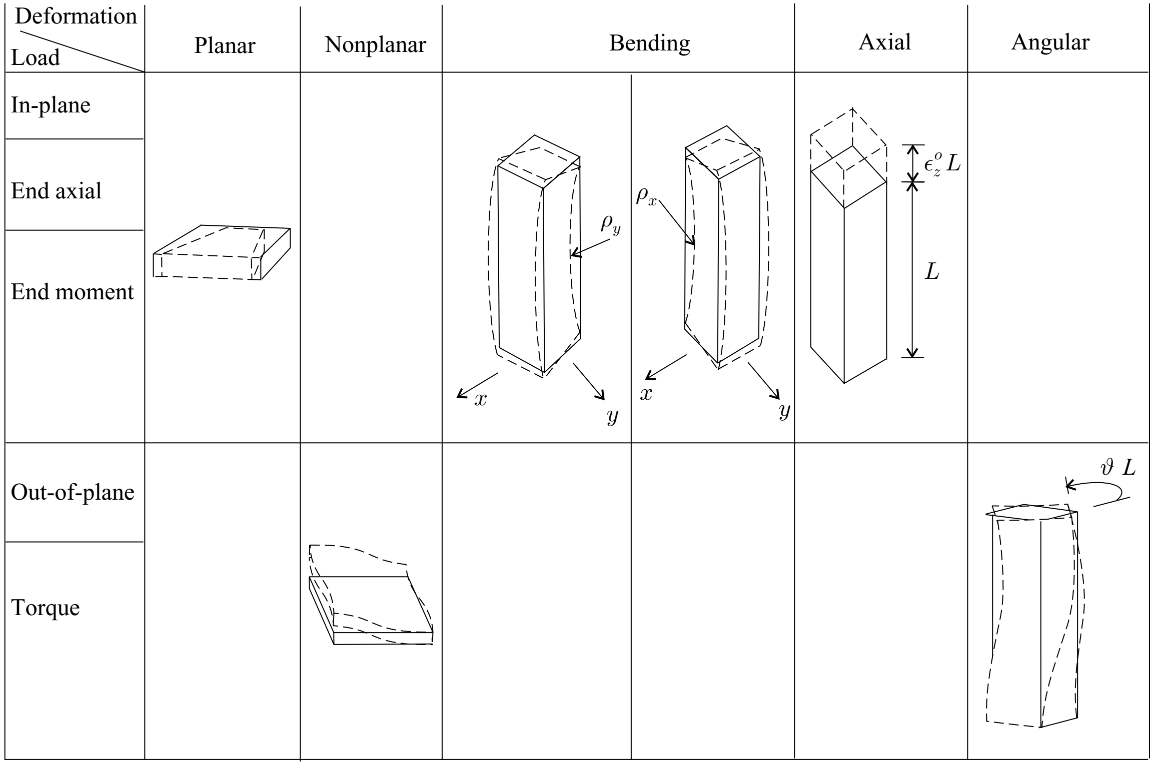


Figure 2.25: Deformations of a monoclinic body under plane-strain condition for different loads. One or both ends are free.

Since $\gamma_{23} = 0$ and $\gamma_{13} = 0$, Eqs. (2.22) and (2.27) give

$$\tau_{23} = 0 \qquad \tau_{13} = 0. \tag{2.72}$$

Furthermore, since the strain and stress components are independent of x_3, we have

$$\frac{\partial \sigma_3}{\partial x_3} = 0. \tag{2.73}$$

With these stresses the relevant equilibrium equations (Eqs. 2.13 and 2.14) – in the x_1, x_2, x_3 coordinate system – reduce to

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + f_1 = 0 \tag{2.74}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + f_2 = 0. \tag{2.75}$$

The stress components σ_1, σ_2, τ_{12}, which appear in the equilibrium equations, are illustrated in Figure 2.24 (top, right).

By substituting Eq. (2.70) into Eq. (2.22) and by utilizing Eq. (2.27) we obtain the following stress–strain relationships for in-plane loads:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \epsilon_3 \tag{2.76}$$

$$\sigma_3 = [C_{31} \quad C_{32} \quad C_{36}] \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} + C_{33}\epsilon_3, \tag{2.77}$$

where, we recall, ϵ_3 is (Eq. 2.69)

$$\epsilon_3 = \frac{1}{\rho_2} x_1 + \frac{1}{\rho_1} x_2 + \epsilon_3^{\text{o}}. \tag{2.78}$$

The strain–stress relationships are obtained from Eq. (2.76) by inversion

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{16} \\ R_{12} & R_{22} & R_{26} \\ R_{16} & R_{26} & R_{66} \end{bmatrix} \left(\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} - \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \epsilon_3 \right), \tag{2.79}$$

where R_{ij} $(i, j = 1, 2, 6)$ are the in-plane elements of the compliance matrix in the x_1, x_2 coordinate system calculated from the relationship

$$\begin{bmatrix} R_{11} & R_{12} & R_{16} \\ R_{12} & R_{22} & R_{26} \\ R_{16} & R_{26} & R_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1}. \tag{2.80}$$

Out-of-plane loads. The body is subjected to the out-of-plane loads shown in Figure 2.20. Under these loads the body may undergo only nonplanar and angular deformations (Fig. 2.25), and the displacements (in the x_1, x_2, x_3 coordinate system) are (see Eq. 2.53 and Fig. 2.23)

$$\begin{aligned} u_1 &= -\vartheta x_2 x_3 \\ u_2 &= \vartheta x_1 x_3 \\ u_3 &= U_3(x_1, x_2). \end{aligned} \tag{2.81}$$

For these displacements the strains are (see Eqs. 2.2–2.4, 2.9–2.11)

$$\epsilon_1 = 0 \qquad \epsilon_2 = 0 \tag{2.82}$$

$$\epsilon_3 = 0 \tag{2.83}$$

$$\gamma_{23} = \frac{\partial U_3}{\partial x_2} + \vartheta x_1 \qquad \gamma_{13} = \frac{\partial U_3}{\partial x_1} - \vartheta x_2 \tag{2.84}$$

$$\gamma_{12} = 0. \tag{2.85}$$

Only γ_{23} and γ_{13} are nonzero. Correspondingly, the only nonzero stresses are τ_{13} and τ_{23}. Thus, two of the equilibrium equations (Eqs. 2.13 and 2.14) become trivial, and the third (Eq. 2.15) – in the x_1, x_2, x_3 coordinate system – reduces to

$$\frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + f_3 = 0. \tag{2.86}$$

The τ_{23}, τ_{13} stress components, which appear in the equilibrium equation, are illustrated in Figure 2.24 (bottom, right).

From Eqs. (2.22) and (2.27) we obtain the following stress–strain relationships for out-of-plane loads:

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}. \tag{2.87}$$

The strain–stress relationships are obtained from Eq. (2.87) by inversion

$$\begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix} \begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}, \tag{2.88}$$

where the compliance matrix $[S]$ is the inverse of the stiffness matrix in Eq. (2.87)

$$\begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix}^{-1}. \tag{2.89}$$

End axial loads. Axial loads are applied at the ends (Fig. 2.21). Under such loads the body may undergo only planar, bending, and axial deformations

(Fig. 2.25), and the displacements are (see Eq. 2.53 and Fig. 2.23)

$$u_1 = U_1(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_2}x_3^2 \tag{2.90}$$

$$u_2 = U_2(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_1}x_3^2 \tag{2.91}$$

$$u_3 = \left(\frac{1}{\rho_2}x_1 + \frac{1}{\rho_1}x_2 + \epsilon_3^{\mathrm{o}}\right)x_3. \tag{2.92}$$

These displacements are the same as those for in-plane loads (Eq. 2.67). Hence, the strain–displacement, equilibrium, and stress–strain relationships are the same as those given by Eqs. (2.68)–(2.80).

End moment. A bending moment is applied at each end (Fig. 2.21). Under this loading the body may undergo only planar, bending, and axial deformations (Fig. 2.25), and the displacements are (see Eq. 2.53 and Fig. 2.23)

$$\begin{aligned} u_1 &= U_1(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_2}x_3^2 \\ u_2 &= U_2(x_1, x_2) - \frac{1}{2}\frac{1}{\rho_1}x_3^2 \\ u_3 &= \left(\frac{1}{\rho_2}x_1 + \frac{1}{\rho_1}x_2 + \epsilon_3^{\mathrm{o}}\right)x_3. \end{aligned} \tag{2.93}$$

These displacements are the same as those for in-plane loads (Eq. 2.67). Hence, the strain–displacement, equilibrium, and stress–strain relationships are the same as those given by Eqs. (2.68)–(2.80).

Torque. A torque is applied at each end (Fig. 2.21). Under this load the body may undergo only nonplanar and angular deformations (Fig. 2.25), and the displacements are (see Eq. 2.53 and Fig. 2.23)

$$u_1 = -\vartheta x_2 x_3 \qquad u_2 = \vartheta x_1 x_3 \qquad u_3 = U_3(x_1, x_2). \tag{2.94}$$

These displacements are the same as those for out-of-plane loads (Eq. 2.81). Hence, the strain–displacement, equilibrium, and stress–strain relationships are the same as those given by Eqs. (2.81)–(2.88).

2.4.3 Free End – Orthotropic, Transversely Isotropic, or Isotropic Material

When at least one end of the body is free, the expressions given in Section 2.4.2 are also applicable to bodies treated as orthotropic, transversely isotropic, or isotropic provided that one of the material symmetry planes coincides with the x_1–x_2 plane shown in Figure 2.16. For such bodies the following simplifications apply:

Orthotropic

$$C_{16} = C_{26} = C_{36} = 0 \qquad R_{16} = R_{26} = 0. \tag{2.95}$$

Transversely isotropic

$$
\begin{aligned}
&C_{16} = C_{26} = C_{36} = 0 \qquad R_{16} = R_{26} = 0 \\
&C_{13} = C_{12}.
\end{aligned}
\tag{2.96}
$$

Isotropic

$$
\begin{aligned}
&C_{16} = C_{26} = C_{36} = 0 \qquad && R_{16} = R_{26} = 0 \\
&C_{22} = C_{11} && R_{22} = R_{11} \\
&C_{66} = \tfrac{1}{2}\left(C_{11} - C_{12}\right) && R_{66} = 2\left(R_{11} - R_{12}\right) \\
&C_{13} = C_{23} = C_{12}.
\end{aligned}
\tag{2.97}
$$

2.4.4 Built-In Ends – Generally Anisotropic Material

When both ends of a generally anisotropic body are built-in (Fig. 2.18), in-plane as well as out-of-plane loads may be applied (Fig. 2.20). The body may undergo only planar and nonplanar deformations (Fig. 2.23). However, the longitudinal axis of the body remains straight ($1/\rho_y = 1/\rho_x = 0$), its length remains constant ($\epsilon_z^{\text{o}} = 0$), and the body does not twist ($\vartheta = 0$). The permissible deformations are summarized in Figure 2.26 and the displacements are (see Fig. 2.23)

$$
u = U(x, y) \qquad v = V(x, y) \qquad w = W(x, y). \tag{2.98}
$$

These displacements are the same as the displacements given by Eqs. (2.50)–(2.52) for a body with free ends, when $1/\rho_y$, $1/\rho_x$, ϵ_z^{o}, and ϑ are set equal to zero. Thus, the strain–displacement, the stress–strain, and the equilibrium equations are obtained by setting $1/\rho_y$, $1/\rho_x$, ϵ_z^{o}, and ϑ equal to zero in the equations of Section 2.4.1. The strain–displacement relationships (Eqs. 2.54–2.57) become

$$
\epsilon_x = \frac{\partial U}{\partial x} \qquad \epsilon_y = \frac{\partial V}{\partial y} \tag{2.99}
$$

$$
\epsilon_z = 0 \tag{2.100}
$$

$$
\gamma_{yz} = \frac{\partial W}{\partial y} \qquad \gamma_{xz} = \frac{\partial W}{\partial x} \tag{2.101}
$$

$$
\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}. \tag{2.102}
$$

The equilibrium equations are identical with Eqs. (2.59)–(2.61).

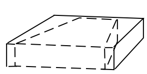
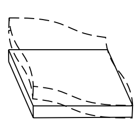
Deformation / Load	Planar	Nonplanar
In-plane		
Out-of-plane		

Figure 2.26: Deformations of a generally anisotropic body under plane-strain condition for different loads; built-in ends.

The stress–strain relationships (Eqs. 2.62 and 2.63) become

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{14} & \overline{C}_{24} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{15} & \overline{C}_{25} & \overline{C}_{45} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{46} & \overline{C}_{56} & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{2.103}$$

$$\sigma_z = [\overline{C}_{13} \quad \overline{C}_{23} \quad \overline{C}_{34} \quad \overline{C}_{35} \quad \overline{C}_{36}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}. \tag{2.104}$$

By inverting Eq. (2.103), we obtain the strain–stress relationships

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{R}_{11} & \overline{R}_{12} & \overline{R}_{14} & \overline{R}_{15} & \overline{R}_{16} \\ \overline{R}_{12} & \overline{R}_{22} & \overline{R}_{24} & \overline{R}_{25} & \overline{R}_{26} \\ \overline{R}_{14} & \overline{R}_{24} & \overline{R}_{44} & \overline{R}_{45} & \overline{R}_{46} \\ \overline{R}_{15} & \overline{R}_{25} & \overline{R}_{45} & \overline{R}_{55} & \overline{R}_{56} \\ \overline{R}_{16} & \overline{R}_{26} & \overline{R}_{46} & \overline{R}_{56} & \overline{R}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}, \tag{2.105}$$

where $[\overline{R}]$ is defined by Eq. (2.66).

2.4.5 Built-In Ends – Monoclinic Material

We consider a body made of a monoclinic material with the plane of material symmetry coinciding with the x_1–x_2 plane shown in Figure 2.16. Both ends of the body are built-in. The longitudinal axis of the body remains straight ($1/\rho_y = 1/\rho_x = 0$), its length remains constant ($\epsilon_z^o = 0$), and the body does not twist ($\vartheta = 0$); hence, the body may undergo only planar and nonplanar deformations (Fig. 2.23). The deformations resulting from specified loads are summarized in Figure 2.27.

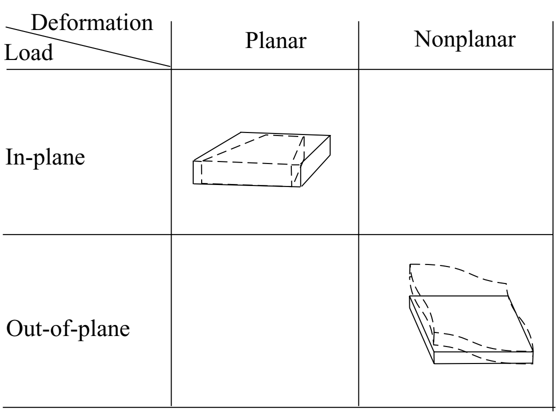


Figure 2.27: Deformations of a monoclinic body under plane-strain condition for different loads; built-in ends.

In-plane loads. The body is subjected to the in-plane loads shown in Figure 2.20. Under these loads the body undergoes planar deformations (Fig. 2.27), and the displacements are (see Eq. 2.53 and Fig. 2.23)

$$u_1 = U_1(x_1, x_2) \qquad u_2 = U_2(x_1, x_2) \qquad u_3 = 0. \tag{2.106}$$

These displacements are the same as the displacements of a body with free ends with $1/\rho_2$, $1/\rho_1$, and ϵ_3^o set equal to zero in Eq. (2.67). Thus, the strain–displacement, the stress–strain, and the equilibrium equations are obtained by setting $1/\rho_2$, $1/\rho_1$, and ϵ_3^o equal to zero in the equations of Section 2.4.2. The strain–displacement relationships (see Eqs. 2.68–2.71) are

$$\epsilon_1 = \frac{\partial U_1}{\partial x_1} \qquad \epsilon_2 = \frac{\partial U_2}{\partial x_2} \tag{2.107}$$

$$\epsilon_3 = 0 \tag{2.108}$$

$$\gamma_{23} = 0 \qquad \gamma_{13} = 0 \tag{2.109}$$

$$\gamma_{12} = \frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}. \tag{2.110}$$

The equilibrium equations are identical to Eqs. (2.74) and (2.75).

The stress–strain relationships (Eqs. 2.76 and 2.77) become

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{2.111}$$

$$\sigma_3 = [C_{31} \quad C_{32} \quad C_{36}] \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}. \tag{2.112}$$

By inverting Eq. (2.111), we obtain the strain–stress relationships

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{16} \\ R_{12} & R_{22} & R_{26} \\ R_{16} & R_{26} & R_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}. \tag{2.113}$$

The in-plane elements of the compliance matrix R_{ij} are defined in Eq. (2.80).

Out-of-plane loads. The body is subjected to out-of-plane loads shown in Figure 2.20. Under these loads the body may undergo only nonplanar deformations (Fig. 2.27), and the displacements (in the x_1, x_2, x_3 coordinate system) are (see Eq. 2.53 and Fig. 2.23)

$$u_1 = 0 \qquad u_2 = 0 \qquad u_3 = U_3(x_1, x_2). \tag{2.114}$$

These displacements are the same as the displacements of a body with free ends with ϑ set equal to zero in Eq. (2.81). Thus, the strain–displacement and the stress–strain relations as well as the equilibrium equations can be obtained by setting ϑ equal to zero in the equations of Section 2.4.2. The strain–displacement relationships (see Eqs. 2.82–2.85) are

$$\epsilon_1 = 0 \qquad \epsilon_2 = 0 \tag{2.115}$$

$$\epsilon_3 = 0 \tag{2.116}$$

$$\gamma_{23} = \frac{\partial U_3}{\partial x_2} \qquad \gamma_{13} = \frac{\partial U_3}{\partial x_1} \tag{2.117}$$

$$\gamma_{12} = 0. \tag{2.118}$$

The equilibrium equation is identical to Eq. (2.86).

The stress–strain relationships (see Eq. 2.87) are

$$\begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix}. \tag{2.119}$$

By inverting this equation, we obtain the strain–stress relationships

$$\begin{Bmatrix} \gamma_{23} \\ \gamma_{13} \end{Bmatrix} = \begin{bmatrix} S_{44} & S_{45} \\ S_{45} & S_{55} \end{bmatrix} \begin{Bmatrix} \tau_{23} \\ \tau_{13} \end{Bmatrix}, \tag{2.120}$$

where the compliance matrix is given by Eq. (2.89).

2.4.6 Built-In Ends – Orthotropic, Transversely Isotropic, or Isotropic Material

The expressions given in Section 2.4.5 for monoclinic bodies with both ends built-in are applicable to bodies treated as orthotropic, transversely isotropic, or isotropic provided that one of the material symmetry planes coincides with the x_1–x_2 plane shown in Figure 2.16. However, for such bodies the simplifications given by Eqs. (2.95)–(2.97) must be employed.

2.5 Plane-Stress Condition

Under plane-stress condition one of the normal stresses and both out-of-plane shear stresses are zero. We select the normal stress to be zero in the z (or x_3) coordinate direction and the out-of-plane shear stresses to be zero in the x–y (or x_1–x_2) plane (Fig. 2.28)

$$\sigma_z = 0 \qquad \tau_{yz} = 0 \qquad \tau_{xz} = 0. \tag{2.121}$$

Plane-stress condition may approximate the stresses in a thin fiber-reinforced composite plate when the fibers are parallel to the x–y plane and the plate is loaded by forces along the edges such that the forces are parallel to the plane of the plate and are distributed uniformly over the thickness (Fig. 2.29). The plane-stress condition does not provide the stresses exactly, not even for this thin-plate problem. Nevertheless, for many thin wall structures it is a useful approximation, yielding answers within reasonable accuracy.

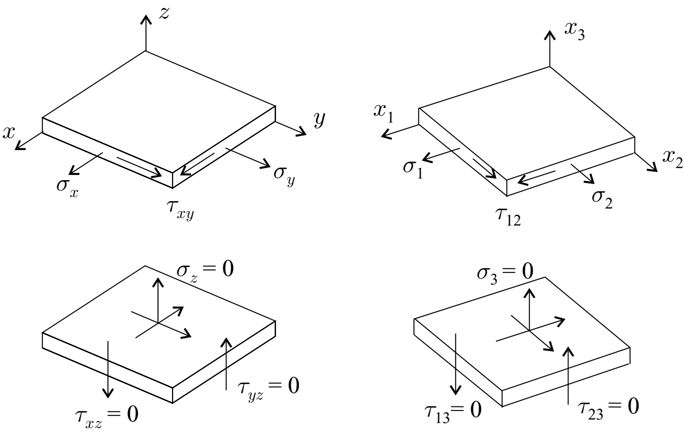


Figure 2.28: The stresses under plane-stress condition.

With the stipulations of Eq. (2.121), the equilibrium equations (Eqs. 2.13–2.15) become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0 \tag{2.122}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0. \tag{2.123}$$

Equations (2.121) and Eq. (2.21) give

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.124}$$

$$\begin{Bmatrix} \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{36} \\ \overline{S}_{14} & \overline{S}_{24} & \overline{S}_{46} \\ \overline{S}_{15} & \overline{S}_{25} & \overline{S}_{56} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}. \tag{2.125}$$

The elements of the compliance matrix $\overline{S}_{ij}$ are given in Table 2.3 (page 10). The stress–strain relationships are obtained from Eq. (2.124) by inversion

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}, \tag{2.126}$$

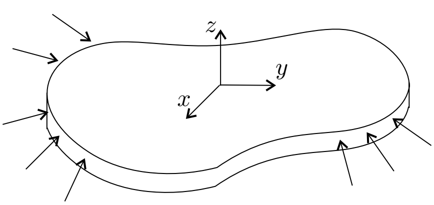


Figure 2.29: In-plane loads applied to a thin plate, resulting in plane-stress condition.

where $\overline{Q}_{ij}$ are the in-plane elements of the stiffness matrix in the x, y coordinate system under plane-stress condition:

$$\begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix}^{-1}. \tag{2.127}$$

The stress–strain and strain–stress relationships simplify considerably when the fiber orientations are such that the material can be treated as monoclinic (Fig. 2.9), orthotropic (Fig. 2.12), transversely isotropic (Fig. 2.15), or isotropic. In the following we consider problems involving these material types and use an x_1, x_2, x_3 coordinate system with the x_1, x_2 coordinates being in one of the symmetry planes and x_3 being perpendicular to this plane. In this coordinate system, under plane-stress condition, we have (Fig. 2.28, right)

$$\sigma_3 = 0 \qquad \tau_{23} = 0 \qquad \tau_{13} = 0. \tag{2.128}$$

Monoclinic materials. By substituting Eq. (2.128) into Eq. (2.23) and by employing the compliance matrix of a monoclinic material (Eq. 2.26), we obtain

$$\gamma_{23} = 0 \qquad \gamma_{13} = 0. \tag{2.129}$$

With the stipulations in Eq. (2.128), the equilibrium equations (Eqs. 2.13–2.15) in the x_1, x_2, x_3 coordinate system become

$$\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + f_1 = 0 \tag{2.130}$$

$$\frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + f_2 = 0. \tag{2.131}$$

By substituting Eq. (2.128) into Eq. (2.23), we obtain the following strain–stress relationships:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.132}$$

$$\epsilon_3 = [S_{13} \quad S_{23} \quad S_{36}] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}. \tag{2.133}$$

The elements of the compliance matrix S_{ij} are given in Table 2.7 (page 15). The stress–strain relationships are obtained from Eq. (2.132) by inversion

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}, \tag{2.134}$$

where Q_{ij} are the in-plane elements of the stiffness matrix in the x_1, x_2 coordinate system under plane-stress condition:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}. \tag{2.135}$$

The expressions presented in this section for monoclinic materials also apply to orthotropic, transversely isotropic, and isotropic materials. However, for these types of materials the stiffness and compliance matrices further simplify, as shown below.

Orthotropic. For an orthotropic material the following elements of the compliance matrix are zero (Table 2.7, page 15):

$$S_{16} = S_{26} = 0. \tag{2.136}$$

Also, from Table 2.7 we have

$$S_{11} = \frac{1}{E_1} \qquad S_{12} = -\frac{\nu_{12}}{E_1} \qquad S_{22} = \frac{1}{E_2} \qquad S_{66} = \frac{1}{G_{12}}. \tag{2.137}$$

From Eqs. (2.135) and (2.136) we obtain

$$Q_{16} = Q_{26} = 0. \tag{2.138}$$

Accordingly, the elements of the stiffness matrix $[Q]$ in terms of the engineering constants are (Eqs. 2.135, 2.137, and 2.138)

$$[Q] = \begin{bmatrix} \frac{E_1}{D} & \frac{\nu_{12} E_2}{D} & 0 \\ \frac{\nu_{12} E_2}{D} & \frac{E_2}{D} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}, \qquad \text{where} \quad D = 1 - \frac{E_2}{E_1}\nu_{12}^2 = 1 - \nu_{12}\nu_{21}. \tag{2.139}$$

A woven, or filament wound, layer is orthotropic when there is the same number of fibers in the $+\Theta$ and $-\Theta$ directions in the x_1–x_2 plane (Fig. 2.30). For such a layer the elements of the stiffness matrix are calculated from

$$Q_{ij}^{\text{woven}} = \frac{1}{2}\left[(\overline{Q}_{ij})_{\Theta} + (\overline{Q}_{ij})_{-\Theta}\right], \quad i, j = 1, 2, 6, \tag{2.140}$$

where $(\overline{Q}_{ij})_{\Theta}$ and $(\overline{Q}_{ij})_{-\Theta}$ are the elements of the stiffness matrices of plies oriented in the $+\Theta$ and $-\Theta$ directions, respectively. The elements of the stiffness

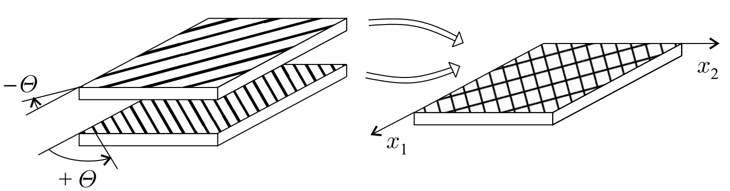


Figure 2.30: A layer consisting of fibers in the $+\Theta$ and $-\Theta$ directions.

Table 2.12. Elements of the stiffness matrix of a woven layer with the fibers oriented in the $\pm\Theta$ directions

$$
\begin{aligned}
Q_{11}^{\text{woven}} &= c^4 Q_{11} + s^4 Q_{22} + 2c^2 s^2 (Q_{12} + 2Q_{66}) \\
Q_{22}^{\text{woven}} &= s^4 Q_{11} + c^4 Q_{22} + 2c^2 s^2 (Q_{12} + 2Q_{66}) \\
Q_{12}^{\text{woven}} &= c^2 s^2 (Q_{11} + Q_{22} - 4Q_{66}) + \left(c^4 + s^4\right) Q_{12} \\
Q_{66}^{\text{woven}} &= c^2 s^2 (Q_{11} + Q_{22} - 2Q_{12}) + \left(c^2 - s^2\right)^2 Q_{66} \\
Q_{16}^{\text{woven}} &= 0 \\
Q_{26}^{\text{woven}} &= 0 \\
c = \cos\Theta &\qquad s = \sin\Theta
\end{aligned}
$$

matrix in the Θ direction $(\overline{Q}_{ij})_{\Theta}$ are obtained by the transformation described in Section 2.9.3 and summarized in Table 3.1 (page 70). The elements $(\overline{Q}_{ij})_{-\Theta}$ are calculated by replacing Θ by $-\Theta$ in Table 3.1. By performing these steps, we obtain the elements Q_{ij}^{woven} given in Table 2.12.

Transversely isotropic. For a transversely isotropic material we have (Table 2.7, page 15)

$$S_{16} = S_{26} = 0 \tag{2.141}$$

$$S_{13} = S_{12}. \tag{2.142}$$

By substituting these values, together with the expressions in Table 2.7 (page 15), into Eq. (2.135), we obtain the $[Q]$ matrix in terms of the engineering constants. The results are identical to those of Eq. (2.139).

Isotropic. For an isotropic material the following relationships hold (Table 2.7, page 15):

$$
\begin{aligned}
S_{22} &= S_{11} \qquad & S_{16} &= S_{26} = 0 \\
S_{13} &= S_{23} = S_{12} \qquad & S_{66} &= 2(S_{11} - S_{12}).
\end{aligned}
\tag{2.143}
$$

From Eq. (2.135) we obtain

$$
\begin{aligned}
Q_{22} &= Q_{11} \qquad & Q_{16} &= Q_{26} = 0 \\
Q_{66} &= \frac{1}{2}(Q_{11} - Q_{12}).
\end{aligned}
\tag{2.144}
$$

The elements of the stiffness matrix in terms of the engineering constants are

$$[Q] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{2.145}$$

2.2 Example. *Calculate the elements of the stiffness matrix of a graphite epoxy unidirectional ply under plane-stress condition. The engineering constants are given as* $E_1 = 148 \times 10^9\,N/m^2$, $E_2 = 9.65 \times 10^9\,N/m^2$, $G_{12} = 4.55 \times 10^9\,N/m^2$, $\nu_{12} = 0.3$, *and* $\nu_{23} = 0.6$.

Solution. Under plane-stress condition the stiffness matrix is (Eq. 2.135)

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}. \tag{2.146}$$

The elements of the compliance matrix are given in Example 2.1 (page 21). By substituting these values into the expression above we obtain

$$[Q] = \begin{bmatrix} 6.76 & -2.03 & 0 \\ -2.03 & 103.63 & 0 \\ 0 & 0 & 219.78 \end{bmatrix}^{-1} = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9 \frac{\text{N}}{\text{m}^2}. \tag{2.147}$$

The stiffness matrix may also be calculated by Eq. (2.139) with the engineering constants and with $D = 1 - \frac{E_2}{E_1}\nu_{12}^2 = 0.994$. Equation (2.139) gives

$$[Q] = \begin{bmatrix} \frac{E_1}{D} & \frac{\nu_{12}E_2}{D} & 0 \\ \frac{\nu_{12}E_2}{D} & \frac{E_2}{D} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9 \frac{\text{N}}{\text{m}^2}. \tag{2.148}$$

This is the same result as that given by Eq. (2.147).

2.3 Example. *Estimate the plane-stress stiffness matrix and the engineering constants of a* (±45) *woven fabric layer made of graphite fibers and epoxy resin.*

Solution. We approximate the woven fabric by a layer made of one 45° and one −45° ply. The properties of these plies are taken to be those of the graphite epoxy unidirectional ply given in Example 2.2. The stiffness matrix of a ply made of this material under plane-stress condition is (Eq. 2.147)

$$[Q] = \begin{bmatrix} 148.87 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix} 10^9 \frac{\text{N}}{\text{m}^2}. \tag{2.149}$$

We obtain the stiffness matrix of the woven fabric by substituting the elements of this matrix into the expressions in Table 2.12 (page 42) with $\Theta = 45°$. The result is

$$\begin{aligned} Q_{11}^{\pm 45} &= Q_{22}^{\pm 45} = 45.65 \times 10^9 \text{ N/m}^2 & Q_{12}^{\pm 45} &= 36.55 \times 10^9 \text{ N/m}^2 \\ Q_{16}^{\pm 45} &= Q_{26}^{\pm 45} = 0 & Q_{66}^{\pm 45} &= 38.19 \times 10^9 \text{ N/m}^2. \end{aligned} \tag{2.150}$$

The compliance matrix is

$$\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}^{\pm 45} = \begin{bmatrix} Q_{11}^{\pm 45} & Q_{12}^{\pm 45} & 0 \\ Q_{12}^{\pm 45} & Q_{22}^{\pm 45} & 0 \\ 0 & 0 & Q_{66}^{\pm 45} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 61.03 & -48.86 & 0 \\ -48.86 & 61.03 & 0 \\ 0 & 0 & 26.19 \end{bmatrix} 10^{-12} \frac{\mathrm{m}^2}{\mathrm{N}}. \tag{2.151}$$

The engineering constants of the woven fabric are (see Eq. 2.137)

$$\begin{aligned} E_1^{\pm 45} &= \frac{1}{S_{11}} = 16.39 \times 10^9 \frac{\mathrm{N}}{\mathrm{m}}^2 & \nu_{12}^{\pm 45} &= -S_{12} E_1 = 0.801 \\ E_2^{\pm 45} &= \frac{1}{S_{22}} = 16.39 \times 10^9 \frac{\mathrm{N}}{\mathrm{m}}^2 & G_{12}^{\pm 45} &= \frac{1}{S_{66}} = 38.19 \times 10^9 \frac{\mathrm{N}}{\mathrm{m}}^2. \end{aligned} \tag{2.152}$$

2.6 Hygrothermal Strains and Stresses

An unrestrained composite may change both its size and shape when the temperature is increased or decreased uniformly by ΔT. The corresponding strains in the x, y, z coordinate system are

$$\begin{aligned} \epsilon_x^T &= \widetilde{\alpha}_x \Delta T & \epsilon_y^T &= \widetilde{\alpha}_y \Delta T & \epsilon_z^T &= \widetilde{\alpha}_z \Delta T \\ \gamma_{yz}^T &= \widetilde{\alpha}_{yz} \Delta T & \gamma_{xz}^T &= \widetilde{\alpha}_{xz} \Delta T & \gamma_{xy}^T &= \widetilde{\alpha}_{xy} \Delta T. \end{aligned} \tag{2.153}$$

Similarly, moisture inside an unrestrained composite causes a change in size and shape. A uniform moisture concentration c in the material results in the following strains:

$$\begin{aligned} \epsilon_x^{\mathrm{c}} &= \widetilde{\beta}_x c & \epsilon_y^{\mathrm{c}} &= \widetilde{\beta}_y c & \epsilon_z^{\mathrm{c}} &= \widetilde{\beta}_z c \\ \gamma_{yz}^{\mathrm{c}} &= \widetilde{\beta}_{yz} c & \gamma_{xz}^{\mathrm{c}} &= \widetilde{\beta}_{xz} c & \gamma_{xy}^{\mathrm{c}} &= \widetilde{\beta}_{xy} c. \end{aligned} \tag{2.154}$$

Inside a dry material the moisture concentration c is zero. In Eqs. (2.153) and (2.154) $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the temperature and moisture expansion coefficients, respectively. (Note that $\widetilde{\alpha}$ and $\widetilde{\beta}$ follow the same transformation rules as strains, Section 2.9.2.) In an unrestrained composite the strains induced by uniform temperature and moisture distributions (referred to as hygrothermal strains) are

$$\begin{Bmatrix} \epsilon_x^{\mathrm{ht}} \\ \epsilon_y^{\mathrm{ht}} \\ \epsilon_z^{\mathrm{ht}} \\ \gamma_{yz}^{\mathrm{ht}} \\ \gamma_{xz}^{\mathrm{ht}} \\ \gamma_{xy}^{\mathrm{ht}} \end{Bmatrix} = \begin{Bmatrix} \widetilde{\alpha}_x \\ \widetilde{\alpha}_y \\ \widetilde{\alpha}_z \\ \widetilde{\alpha}_{yz} \\ \widetilde{\alpha}_{xz} \\ \widetilde{\alpha}_{xy} \end{Bmatrix} \Delta T + \begin{Bmatrix} \widetilde{\beta}_x \\ \widetilde{\beta}_y \\ \widetilde{\beta}_z \\ \widetilde{\beta}_{yz} \\ \widetilde{\beta}_{xz} \\ \widetilde{\beta}_{xy} \end{Bmatrix} c. \tag{2.155}$$

For a generally anisotropic material, in the x, y, z coordinate system, the strain–stress relationships are obtained by combining the hygrothermal strains (Eq. 2.155) with the stress-induced strains (Eq. 2.21):

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} & \overline{S}_{15} & \overline{S}_{16} \\ \overline{S}_{21} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} & \overline{S}_{25} & \overline{S}_{26} \\ \overline{S}_{31} & \overline{S}_{32} & \overline{S}_{33} & \overline{S}_{34} & \overline{S}_{35} & \overline{S}_{36} \\ \overline{S}_{41} & \overline{S}_{42} & \overline{S}_{43} & \overline{S}_{44} & \overline{S}_{45} & \overline{S}_{46} \\ \overline{S}_{51} & \overline{S}_{52} & \overline{S}_{53} & \overline{S}_{54} & \overline{S}_{55} & \overline{S}_{56} \\ \overline{S}_{61} & \overline{S}_{62} & \overline{S}_{63} & \overline{S}_{64} & \overline{S}_{65} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \epsilon_x^{\mathrm{ht}} \\ \epsilon_y^{\mathrm{ht}} \\ \epsilon_z^{\mathrm{ht}} \\ \gamma_{yz}^{\mathrm{ht}} \\ \gamma_{xz}^{\mathrm{ht}} \\ \gamma_{xy}^{\mathrm{ht}} \end{Bmatrix}, \tag{2.156}$$

where $\epsilon_x, \ldots, \gamma_{xy}$ are the actual strains in the composite.

By inverting Eq. (2.156) we obtain the following stress–strain relationships:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} & \overline{C}_{34} & \overline{C}_{35} & \overline{C}_{36} \\ \overline{C}_{41} & \overline{C}_{42} & \overline{C}_{43} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{51} & \overline{C}_{52} & \overline{C}_{53} & \overline{C}_{54} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{61} & \overline{C}_{62} & \overline{C}_{63} & \overline{C}_{64} & \overline{C}_{65} & \overline{C}_{66} \end{bmatrix} \left(\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \epsilon_x^{\mathrm{ht}} \\ \epsilon_y^{\mathrm{ht}} \\ \epsilon_z^{\mathrm{ht}} \\ \gamma_{yz}^{\mathrm{ht}} \\ \gamma_{xz}^{\mathrm{ht}} \\ \gamma_{xy}^{\mathrm{ht}} \end{Bmatrix} \right). \tag{2.157}$$

For a monoclinic material, in the x_1, x_2, x_3 local coordinate system (Fig. 2.9) temperature and moisture do not induce out-of-plane shear strains. Thus, the temperature- and moisture-induced strains are

$$\begin{aligned} \epsilon_1^T &= \widetilde{\alpha}_1 \Delta T & \epsilon_2^T &= \widetilde{\alpha}_2 \Delta T & \epsilon_3^T &= \widetilde{\alpha}_3 \Delta T \\ \gamma_{23}^T &= 0 & \gamma_{13}^T &= 0 & \gamma_{12}^T &= \widetilde{\alpha}_{12} \Delta T \end{aligned} \tag{2.158}$$

$$\begin{aligned} \epsilon_1^{\mathrm{c}} &= \widetilde{\beta}_1 c & \epsilon_2^{\mathrm{c}} &= \widetilde{\beta}_2 c & \epsilon_3^{\mathrm{c}} &= \widetilde{\beta}_3 c \\ \gamma_{23}^{\mathrm{c}} &= 0 & \gamma_{13}^{\mathrm{c}} &= 0 & \gamma_{12}^{\mathrm{c}} &= \widetilde{\beta}_{12} c, \end{aligned} \tag{2.159}$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the temperature and moisture expansion coefficients, respectively, in the x_1, x_2, x_3 coordinate system ($\widetilde{\alpha}$ and $\widetilde{\beta}$ follow the same transformation rules as strains, Section 2.9.2). The hygrothermal strains are

$$\begin{Bmatrix} \epsilon_1^{\mathrm{ht}} \\ \epsilon_2^{\mathrm{ht}} \\ \epsilon_3^{\mathrm{ht}} \\ \gamma_{23}^{\mathrm{ht}} \\ \gamma_{13}^{\mathrm{ht}} \\ \gamma_{12}^{\mathrm{ht}} \end{Bmatrix} = \begin{Bmatrix} \widetilde{\alpha}_1 \\ \widetilde{\alpha}_2 \\ \widetilde{\alpha}_3 \\ 0 \\ 0 \\ \widetilde{\alpha}_{12} \end{Bmatrix} \Delta T + \begin{Bmatrix} \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \\ \widetilde{\beta}_3 \\ 0 \\ 0 \\ \widetilde{\beta}_{12} \end{Bmatrix} c. \tag{2.160}$$

The stress–strain relationships are (Eq. 2.22)

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix} = [C]\left(\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix} - \begin{Bmatrix}\epsilon_1^{\mathrm{ht}}\\ \epsilon_2^{\mathrm{ht}}\\ \epsilon_3^{\mathrm{ht}}\\ 0\\ 0\\ \gamma_{12}^{\mathrm{ht}}\end{Bmatrix}\right), \tag{2.161}$$

where $[C]$ is prescribed by Eq. (2.27). Note that $\epsilon_1, \ldots, \gamma_{12}$ are the *actual* strains in the composite.

By inverting Eq. (2.161), we obtain the following strain–stress relationships:

$$\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{Bmatrix} - \begin{Bmatrix}\epsilon_1^{\mathrm{ht}}\\ \epsilon_2^{\mathrm{ht}}\\ \epsilon_3^{\mathrm{ht}}\\ 0\\ 0\\ \gamma_{12}^{\mathrm{ht}}\end{Bmatrix} = [S]\begin{Bmatrix}\sigma_1\\ \sigma_1\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{Bmatrix}, \tag{2.162}$$

where $[S]$ is prescribed by Eq. (2.26).

In orthotropic, transversely isotropic, and isotropic materials temperature and moisture do not introduce γ_{23}, γ_{13}, and γ_{12} shear strains. Thus, for these types of materials the temperature- and moisture-induced strains are

$$\begin{array}{lll}\epsilon_1^T = \widetilde{\alpha}_1 \Delta T & \epsilon_2^T = \widetilde{\alpha}_2 \Delta T & \epsilon_3^T = \widetilde{\alpha}_3 \Delta T\\ \gamma_{23}^T = 0 & \gamma_{13}^T = 0 & \gamma_{12}^T = 0\end{array} \tag{2.163}$$

$$\begin{array}{lll}\epsilon_1^{\mathrm{c}} = \widetilde{\beta}_1 c & \epsilon_2^{\mathrm{c}} = \widetilde{\beta}_2 c & \epsilon_3^{\mathrm{c}} = \widetilde{\beta}_3 c\\ \gamma_{23}^{\mathrm{c}} = 0 & \gamma_{13}^{\mathrm{c}} = 0 & \gamma_{12}^{\mathrm{c}} = 0.\end{array} \tag{2.164}$$

For orthotropic, transversely isotropic, and isotropic materials the stress–strain relationships given above for monoclinic material are valid (Eqs. 2.161 and 2.162) with $\gamma_{12}^{\mathrm{ht}} = 0$ and the $[S]$ and $[C]$ matrices specified in Sections 2.3.4 and 2.3.5 (see also Table 2.13).

Table 2.13. The thermal and moisture expansion coefficients

Material	Thermal expansion coefficients	Moisture expansion coefficients
Generally anisotropic	$\widetilde{\alpha}_x, \widetilde{\alpha}_y, \widetilde{\alpha}_z, \widetilde{\alpha}_{xy}, \widetilde{\alpha}_{xz}, \widetilde{\alpha}_{xy}$	$\widetilde{\beta}_x, \widetilde{\beta}_y, \widetilde{\beta}_z, \widetilde{\beta}_{yz}, \widetilde{\beta}_{xz}, \widetilde{\beta}_{xy}$
Monoclinic	$\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\alpha}_{12}$	$\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \widetilde{\beta}_{12}$
Orthotropic	$\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3$	$\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3$
Transversely isotropic	$\widetilde{\alpha}_1, \widetilde{\alpha}_2 = \widetilde{\alpha}_3$	$\widetilde{\beta}_1, \widetilde{\beta}_2 = \widetilde{\beta}_3$
Isotropic	$\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = \widetilde{\alpha}_3$	$\widetilde{\beta}_1 = \widetilde{\beta}_2 = \widetilde{\beta}_3$

2.6.1 Plane-Strain Condition

The hygrothermal stress–strain relationships under plane-strain condition can be obtained by making the following substitutions in the stress–strain relationships (*without* hygrothermal effects) given in Section 2.3:

$$\begin{aligned} \epsilon_i &\Rightarrow \epsilon_i - \widetilde{\alpha}_i \Delta T - \widetilde{\beta}_i c \\ \gamma_{ij} &\Rightarrow \gamma_{ij} - \widetilde{\alpha}_{ij} \Delta T - \widetilde{\beta}_{ij} c, \quad i \neq j \end{aligned} \tag{2.165}$$

where the subscripts ij represent x, y, z and $1, 2, 3$ in the x, y, z and x_1, x_2, x_3 coordinate systems, respectively.

2.6.2 Plane-Stress Condition

The hygrothermal stress–strain relationships under plane-stress condition can be obtained by replacing the strains ϵ_i and γ_{ij} in Eqs. (2.124) and (2.145) by those given in Eq. (2.165).

2.7 Boundary Conditions

To obtain solutions to the equilibrium, stress–strain, and strain–displacement equations, either the displacement or the applied force must be specified at every point on the surface. The relationships between the displacements and the strains are given by Eqs. (2.2)–(2.4) and (2.9)–(2.11). The surface forces per unit area are related to the stresses at the surface as follows.

We consider a small volume element ΔV (Fig. 2.31). A surface force per unit area p (with components p_x, p_y, p_z) is applied on the ΔA surface of this element. Force balance in the x direction gives

$$\sigma_x \frac{1}{2} \Delta z \Delta y + \tau_{zx} \frac{1}{2} \Delta x \Delta y + \tau_{yx} \frac{1}{2} \Delta x \Delta z = p_x \Delta A. \tag{2.166}$$

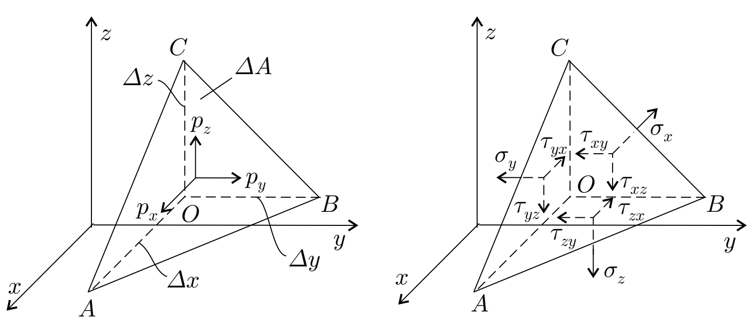


Figure 2.31: Components of the surface force p on the ΔA surface and the stresses on the OBC, OAC, and OAB surfaces.

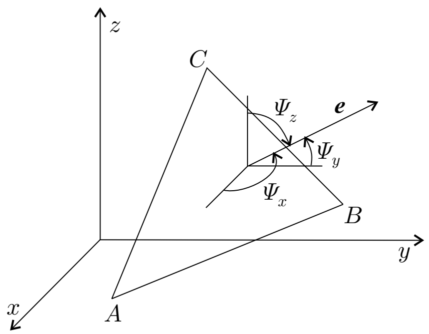


Figure 2.32: The angles Ψ_x, Ψ_y, Ψ_z.

By dividing both sides by ΔA, we obtain

$$p_x = \frac{1}{2\Delta A}\left(\sigma_x \Delta z \Delta y + \tau_{zx} \Delta x \Delta y + \tau_{yx} \Delta x \Delta z\right). \tag{2.167}$$

In the limit, when ΔA, Δx, Δy, Δz go to zero, Eq. (2.167) becomes

$$p_x = l\sigma_x + n\tau_{zx} + m\tau_{yx}, \tag{2.168}$$

where l, m, n are the direction cosines of the normal vector of the boundary surface (Fig. 2.32) as follows:

$$l = \frac{\Delta z \Delta y}{2\Delta A} = \cos(\Psi_x) \quad m = \frac{\Delta x \Delta z}{2\Delta A} = \cos(\Psi_y) \quad n = \frac{\Delta x \Delta y}{2\Delta A} = \cos(\Psi_z). \tag{2.169}$$

Similarly, we have

$$p_y = m\sigma_y + l\tau_{xy} + n\tau_{zy} \tag{2.170}$$

$$p_z = n\sigma_z + l\tau_{xz} + m\tau_{yz}. \tag{2.171}$$

2.8 Continuity Conditions

When composites are made of laminated layers, perfect bonding is assumed between each layer. Accordingly, at the interfaces of two adjacent layers the normal stresses are equal, the out-of-plane shear stresses are equal, and the displacements are equal. Thus, at two adjacent layers, denoted by k and $k+1$, we have (Fig. 2.33)

$$(\sigma_z)_{k,\mathrm{t}} = (\sigma_z)_{k+1,\mathrm{b}} \qquad (\tau_{xz})_{k,\mathrm{t}} = (\tau_{xz})_{k+1,\mathrm{b}} \qquad (\tau_{yz})_{k,\mathrm{t}} = (\tau_{yz})_{k+1,\mathrm{b}} \tag{2.172}$$

$$(u)_{k,\mathrm{t}} = (u)_{k+1,\mathrm{b}} \qquad (v)_{k,\mathrm{t}} = (v)_{k+1,\mathrm{b}} \qquad (w)_{k,\mathrm{t}} = (w)_{k+1,\mathrm{b}}. \tag{2.173}$$

The subscripts t and b refer to the top and bottom of a layer, respectively. The continuity conditions in terms of strains can be obtained from the preceding displacement continuity conditions (Eq. 2.173) together with Eqs. (2.2)–(2.4), (2.7), and (2.11) as follows:

$$(\epsilon_x)_{k,\mathrm{t}} = (\epsilon_x)_{k+1,\mathrm{b}} \qquad (\epsilon_y)_{k,\mathrm{t}} = (\epsilon_y)_{k+1,\mathrm{b}} \qquad (\gamma_{xy})_{k,\mathrm{t}} = (\gamma_{xy})_{k+1,\mathrm{b}}. \tag{2.174}$$

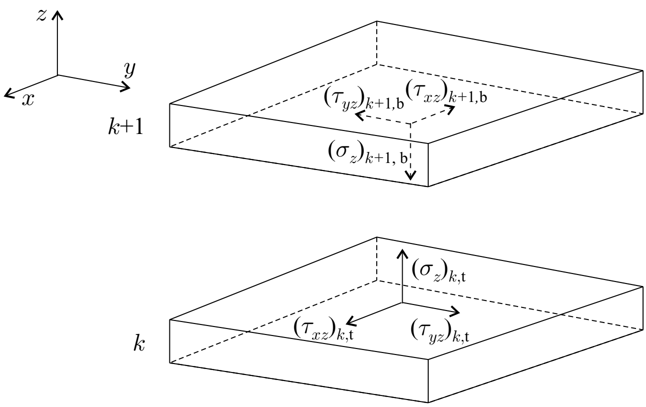


Figure 2.33: The stresses at the top and bottom surfaces of two adjacent layers that must match to satisfy continuity.

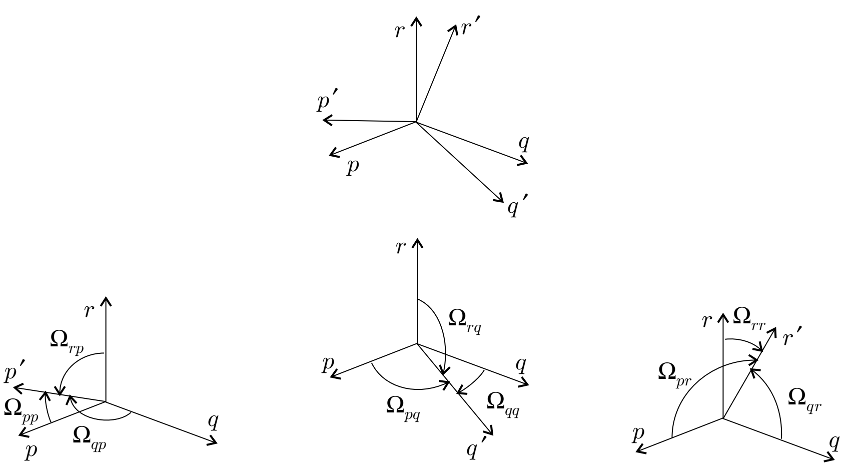
The σ_x, σ_y, τ_{xy} stress and the ϵ_z, γ_{xz}, γ_{yz} strain components are not necessarily continuous along two adjacent surfaces.

2.9 Stress and Strain Transformations

We consider two Cartesian coordinate systems with axes p, q, r and p', q', r'. The orientation of the primed coordinate system with respect to the unprimed coordinate system is given by the nine direction cosines (r_{11}, r_{21}, r_{31}), (r_{12}, r_{22}, r_{32}), (r_{13}, r_{23}, r_{33}) specified in Table 2.14.

The orientation of the primed p', q', r' coordinate system with respect to the unprimed coordinate system p, q, r can also be specified in terms of three

Table 2.14. Definitions of the direction cosines



$r_{11} = \cos\Omega_{pp}$	$r_{12} = \cos\Omega_{pq}$	$r_{13} = \cos\Omega_{pr}$
$r_{21} = \cos\Omega_{qp}$	$r_{22} = \cos\Omega_{qq}$	$r_{23} = \cos\Omega_{qr}$
$r_{31} = \cos\Omega_{rp}$	$r_{32} = \cos\Omega_{rq}$	$r_{33} = \cos\Omega_{rr}$

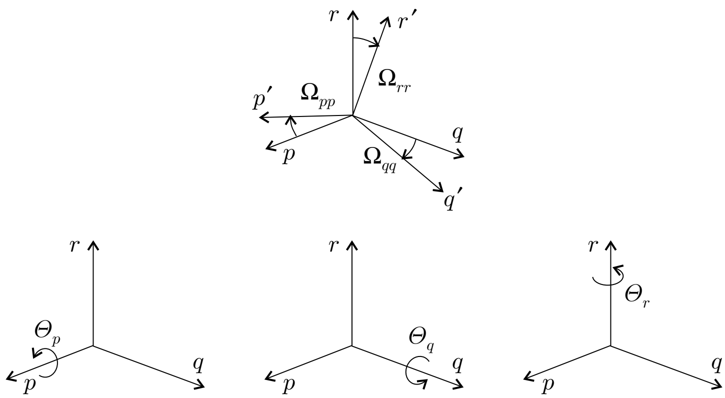


Figure 2.34: Consecutive rotations to arrive at the coordinate system p', q', and r'.

angles $\Theta_p, \Theta_q, \Theta_r$ (Fig. 2.34). These angles are consecutive rotations of the primed coordinate system about the p, q, r axes, as illustrated in Figure 2.34. The angles $\Theta_p, \Theta_q, \Theta_r$ are positive in the counterclockwise direction.

The Θ_p, Θ_q, Θ_r angles are related to the direction cosines by the following expressions[4]:

$$\begin{aligned}\Theta_q &= \mathrm{Atan2}\big(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\big)\\ \Theta_r &= \mathrm{Atan2}(r_{21}/\cos\Theta_q, r_{11}/\cos\Theta_q)\\ \Theta_p &= \mathrm{Atan2}(r_{32}/\cos\Theta_q, r_{33}/\cos\Theta_q),\end{aligned} \tag{2.175}$$

where $\mathrm{Atan2}(y, x)$ is a two-argument arc tangent function as follows:

$$\mathrm{Atan2}(y, x) = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{when} \quad x > 0 \tag{2.176}$$

$$\mathrm{Atan2}(y, x) = \tan^{-1}\left(\frac{y}{x}\right) + \pi \quad \text{when} \quad x < 0. \tag{2.177}$$

When $\Theta_q = 90^{\mathrm{o}}$, then Θ_r and Θ_p are

$$\Theta_r = 0 \qquad \Theta_p = \mathrm{Atan2}(r_{12}, r_{22}). \tag{2.178}$$

2.9.1 Stress Transformation

The stresses in the primed coordinate system are calculated from the stresses in the unprimed coordinate system by the transformation

$$\begin{Bmatrix}\sigma'_p\\ \sigma'_q\\ \sigma'_r\\ \tau'_{qr}\\ \tau'_{pr}\\ \tau'_{pq}\end{Bmatrix} = \begin{bmatrix} T_{\sigma 11} & T_{\sigma 12} & T_{\sigma 13} & T_{\sigma 14} & T_{\sigma 15} & T_{\sigma 16}\\ T_{\sigma 21} & T_{\sigma 22} & T_{\sigma 23} & T_{\sigma 24} & T_{\sigma 25} & T_{\sigma 26}\\ T_{\sigma 31} & T_{\sigma 32} & T_{\sigma 33} & T_{\sigma 34} & T_{\sigma 35} & T_{\sigma 36}\\ T_{\sigma 41} & T_{\sigma 42} & T_{\sigma 43} & T_{\sigma 44} & T_{\sigma 45} & T_{\sigma 46}\\ T_{\sigma 51} & T_{\sigma 52} & T_{\sigma 53} & T_{\sigma 54} & T_{\sigma 55} & T_{\sigma 56}\\ T_{\sigma 61} & T_{\sigma 62} & T_{\sigma 63} & T_{\sigma 64} & T_{\sigma 65} & T_{\sigma 66}\end{bmatrix} \begin{Bmatrix}\sigma_p\\ \sigma_q\\ \sigma_r\\ \tau_{qr}\\ \tau_{pr}\\ \tau_{pq}\end{Bmatrix}. \tag{2.179}$$

[4] J. J. Craig, *Intoduction to Robotics (Mechanics and Control)*. 2nd edition. Addison-Wesley, Reading, Massachusetts, 1989, pp. 43–56.

Table 2.15. The stress transformation matrices

$$[\hat{T}_\sigma^p] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_p^2 & s_p^2 & 2c_ps_p & 0 & 0 \\ 0 & s_p^2 & c_p^2 & -2c_ps_p & 0 & 0 \\ 0 & -c_ps_p & c_ps_p & c_p^2 - s_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_p & -s_p \\ 0 & 0 & 0 & 0 & s_p & c_p \end{bmatrix} \qquad \begin{array}{l} c_p = \cos\Theta_p \\ s_p = \sin\Theta_p \end{array}$$

$$[\hat{T}_\sigma^q] = \begin{bmatrix} c_q^2 & 0 & s_q^2 & 0 & 2c_qs_q & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s_q^2 & 0 & c_q^2 & 0 & -2c_qs_q & 0 \\ 0 & 0 & 0 & c_q & 0 & -s_q \\ -c_qs_q & 0 & c_qs_q & 0 & c_q^2 - s_q^2 & 0 \\ 0 & 0 & 0 & s_q & 0 & c_q \end{bmatrix} \qquad \begin{array}{l} c_q = \cos\Theta_q \\ s_q = \sin\Theta_q \end{array}$$

$$[\hat{T}_\sigma^r] = \begin{bmatrix} c_r^2 & s_r^2 & 0 & 0 & 0 & 2c_rs_r \\ s_r^2 & c_r^2 & 0 & 0 & 0 & -2c_rs_r \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_r & -s_r & 0 \\ 0 & 0 & 0 & s_r & c_r & 0 \\ -c_rs_r & c_rs_r & 0 & 0 & 0 & c_r^2 - s_r^2 \end{bmatrix} \qquad \begin{array}{l} c_r = \cos\Theta_r \\ s_r = \sin\Theta_r \end{array}$$

Equation (2.179) can also be written as

$$\boldsymbol{\sigma}' = [\hat{T}_\sigma]\,\boldsymbol{\sigma}, \tag{2.180}$$

where $[\hat{T}_\sigma]$ is the transformation matrix, which may be expressed as

$$[\hat{T}_\sigma] = [\hat{T}_\sigma^p]\,[\hat{T}_\sigma^q]\,[\hat{T}_\sigma^r], \tag{2.181}$$

where $[\hat{T}_\sigma^p]$, $[\hat{T}_\sigma^q]$, $[\hat{T}_\sigma^r]$ are given in Table 2.15. The subscript σ refers to the stress transformation, and the hat on T indicates that all six stress components are being transformed. The superscripts p, q, and r refer to the transformations about the p, q, and r axes, respectively.

Plane-strain and plane-stress. Under plane-strain and plane-stress conditions we are interested only in the stresses in the p–q and p'–q' planes. In this case the stresses in the primed coordinate system are obtained from the stresses in the unprimed coordinate system by rotation about the r-axis (see Fig. 2.35) as follows:

$$\begin{Bmatrix} \sigma_p' \\ \sigma_q' \\ \tau_{pq}' \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_p \\ \sigma_q \\ \tau_{pq} \end{Bmatrix} \tag{2.182}$$

$$c = \cos\Theta \qquad s = \sin\Theta. \tag{2.183}$$

Equation (2.182) may be written as

$$\boldsymbol{\sigma}' = [T_\sigma]\,\boldsymbol{\sigma}. \tag{2.184}$$

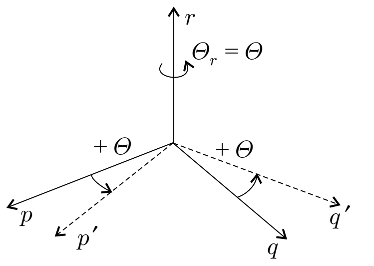


Figure 2.35: Rotation of the coordinate system around the r-axis.

The subscript σ refers to the stress transformation, and T without a hat indicates that only the three in-plane stress components are transformed.

2.9.2 Strain Transformation

The strains in the primed coordinate system are calculated from the strains in the unprimed coordinate system by

$$\begin{Bmatrix} \epsilon'_p \\ \epsilon'_q \\ \epsilon'_r \\ \gamma'_{qr} \\ \gamma'_{pr} \\ \gamma'_{pq} \end{Bmatrix} = \begin{bmatrix} T_{\epsilon 11} & T_{\epsilon 12} & T_{\epsilon 13} & T_{\epsilon 14} & T_{\epsilon 15} & T_{\epsilon 16} \\ T_{\epsilon 21} & T_{\epsilon 22} & T_{\epsilon 23} & T_{\epsilon 24} & T_{\epsilon 25} & T_{\epsilon 26} \\ T_{\epsilon 31} & T_{\epsilon 32} & T_{\epsilon 33} & T_{\epsilon 34} & T_{\epsilon 35} & T_{\epsilon 36} \\ T_{\epsilon 41} & T_{\epsilon 42} & T_{\epsilon 43} & T_{\epsilon 44} & T_{\epsilon 45} & T_{\epsilon 46} \\ T_{\epsilon 51} & T_{\epsilon 52} & T_{\epsilon 53} & T_{\epsilon 54} & T_{\epsilon 55} & T_{\epsilon 56} \\ T_{\epsilon 61} & T_{\epsilon 62} & T_{\epsilon 63} & T_{\epsilon 64} & T_{\epsilon 65} & T_{\epsilon 66} \end{bmatrix} \begin{Bmatrix} \epsilon_p \\ \epsilon_q \\ \epsilon_r \\ \gamma_{qr} \\ \gamma_{pr} \\ \gamma_{pq} \end{Bmatrix}. \tag{2.185}$$

Equation (2.185) may be written as

$$\epsilon' = [\hat{T}_\epsilon]\,\epsilon, \tag{2.186}$$

where the vector ϵ represents engineering strains. The strain transformation matrix $[\hat{T}_\epsilon]$ applies to engineering strains and thus is not the same as the stress transformation matrix $[\hat{T}_\sigma]$. (Tensorial strains transform by the same transformation matrix as the stresses.)

The transformation matrix $[\hat{T}_\epsilon]$ may be expressed as

$$[\hat{T}_\epsilon] = [\hat{T}_\epsilon^p]\,[\hat{T}_\epsilon^q]\,[\hat{T}_\epsilon^r], \tag{2.187}$$

where $[\hat{T}_\epsilon^p]$, $[\hat{T}_\epsilon^q]$, $[\hat{T}_\epsilon^r]$ are given in Table 2.16. The subscript ϵ refers to the strain transformation. The hat on T indicates that all six strain components are transformed. The superscripts p, q, and r refer to the transformations about the p-, q-, and r-axes, respectively.

Plane-strain and plane-stress. Under plane-strain and plane-stress conditions we are interested only in the strains in the p–q and p'–q' planes. In this case the strains in the primed coordinate system are obtained from the strains in the unprimed coordinate system by rotation about the r-axis (see Fig. 2.35) as follows:

$$\begin{Bmatrix} \epsilon'_p \\ \epsilon'_q \\ \gamma'_{pq} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \epsilon_p \\ \epsilon_q \\ \gamma_{pq} \end{Bmatrix}. \tag{2.188}$$

Table 2.16. The engineering strain transformation matrices

$$[\hat{T}^p_\epsilon] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_p^2 & s_p^2 & c_p s_p & 0 & 0 \\ 0 & s_p^2 & c_p^2 & -c_p s_p & 0 & 0 \\ 0 & -2c_p s_p & 2c_p s_p & c_p^2 - s_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_p & -s_p \\ 0 & 0 & 0 & 0 & s_p & c_p \end{bmatrix} \qquad \begin{matrix} c_p = \cos\Theta_p \\ s_p = \sin\Theta_p \end{matrix}$$

$$[\hat{T}^q_\epsilon] = \begin{bmatrix} c_q^2 & 0 & s_q^2 & 0 & c_q s_q & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s_q^2 & 0 & c_q^2 & 0 & -c_q s_q & 0 \\ 0 & 0 & 0 & c_q & 0 & -s_q \\ -2c_q s_q & 0 & 2c_q s_q & 0 & c_q^2 - s_q^2 & 0 \\ 0 & 0 & 0 & s_q & 0 & c_q \end{bmatrix} \qquad \begin{matrix} c_q = \cos\Theta_q \\ s_q = \sin\Theta_q \end{matrix}$$

$$[\hat{T}^r_\epsilon] = \begin{bmatrix} c_r^2 & s_r^2 & 0 & 0 & 0 & c_r s_r \\ s_r^2 & c_r^2 & 0 & 0 & 0 & -c_r s_r \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_r & -s_r & 0 \\ 0 & 0 & 0 & s_r & c_r & 0 \\ -2c_r s_r & 2c_r s_r & 0 & 0 & 0 & c_r^2 - s_r^2 \end{bmatrix} \qquad \begin{matrix} c_r = \cos\Theta_r \\ s_r = \sin\Theta_r \end{matrix}$$

Equation (2.188) may be written as

$$\boldsymbol{\epsilon}' = [T_\epsilon]\,\boldsymbol{\epsilon}. \tag{2.189}$$

The subscript ϵ refers to the strain transformation. Without a hat, T indicates that only the three in-plane strain components are transformed.

2.9.3 Transformation of the Stiffness and Compliance Matrices

The stress–strain relationships in the unprimed and primed coordinate systems are (Eq. 2.22)

$$\boldsymbol{\sigma} = [C]\,\boldsymbol{\epsilon} \qquad \boldsymbol{\sigma}' = [C']\,\boldsymbol{\epsilon}'. \tag{2.190}$$

To obtain the relation between the stiffness matrices in the unprimed $[C]$ and primed $[C']$ coordinate systems, we multiply both sides of the stress–strain equation in the unprimed coordinate system by $[\hat{T}_\sigma]$

$$[\hat{T}_\sigma]\,\boldsymbol{\sigma} = [\hat{T}_\sigma][C]\,\boldsymbol{\epsilon}. \tag{2.191}$$

A matrix multiplied by its inverse $[\hat{T}_\epsilon]^{-1}[\hat{T}_\epsilon]$ is a unit matrix, and we may write

$$\underbrace{[\hat{T}_\sigma]\,\boldsymbol{\sigma}}_{\sigma'} = \underbrace{[\hat{T}_\sigma]\,[C]\,[\hat{T}_\epsilon]^{-1}}_{[C']}\,\underbrace{[\hat{T}_\epsilon]\,\boldsymbol{\epsilon}}_{\epsilon'}. \tag{2.192}$$

By comparing Eqs. (2.190) and (2.192), we see that the elements of the stiffness matrix ($C'_{11}, C'_{12}, \ldots, C'_{66}$) in the primed ($p'$, q', r') coordinate system and the elements of the stiffness matrix ($C_{11}, C_{12}, \ldots, C_{66}$) in the unprimed ($p, q, r$) coordinate system are related by the expression

$$[C'] = [\hat{T}_\sigma][C][\hat{T}_\epsilon]^{-1}. \tag{2.193}$$

The elements of the compliance matrix ($S'_{11}, S'_{12}, \ldots, S'_{66}$) in the primed ($p', q', r'$) coordinate system and the elements of the compliance matrix ($S_{11}, S_{12}, \ldots, S_{66}$) in the unprimed ($p, q, r$) coordinate system are obtained by inverting Eq. (2.193) as follows:

$$[S'] = [\hat{T}_\epsilon][S][\hat{T}_\sigma]^{-1}. \tag{2.194}$$

Plane-stress and plane-strain conditions. Under plane-stress and plane-strain conditions we are interested in the stiffness and compliance matrices in the p–q and p'–q' planes.

For plane-stress, the elements of the stiffness matrix in the primed and unprimed coordinate systems are Q'_{ij} and Q_{ij}, where $i, j = 1, 2, 6$. The relationship between Q'_{ij} and Q_{ij} are given by the transformation specified by Eq. (2.193). The result is

$$\begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{16} \\ Q'_{12} & Q'_{22} & Q'_{26} \\ Q'_{16} & Q'_{26} & Q'_{66} \end{bmatrix} = [T_\sigma] \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} [T_\epsilon]^{-1}, \tag{2.195}$$

where $[T_\sigma]$ and $[T_\epsilon]$ are (Eqs. 2.182 and 2.188)

$$[T_\sigma] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \qquad [T_\epsilon] = \begin{bmatrix} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix}. \tag{2.196}$$

The elements of the compliance matrix in the unprimed coordinate system are obtained by similar reasoning and are

$$\begin{bmatrix} S'_{11} & S'_{12} & S'_{16} \\ S'_{12} & S'_{22} & S'_{26} \\ S'_{16} & S'_{26} & S'_{66} \end{bmatrix} = [T_\epsilon] \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} [T_\sigma]^{-1}. \tag{2.197}$$

Similarly, under plane-strain conditions the stiffness and compliance matrices are

$$\begin{bmatrix} C'_{11} & C'_{12} & C'_{16} \\ C'_{12} & C'_{22} & C'_{26} \\ C'_{16} & C'_{26} & C'_{66} \end{bmatrix} = [T_\sigma] \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} [T_\epsilon]^{-1} \tag{2.198}$$

$$\begin{bmatrix} R'_{11} & R'_{12} & R'_{16} \\ R'_{12} & R'_{22} & R'_{26} \\ R'_{16} & R'_{26} & R'_{66} \end{bmatrix} = [T_\epsilon] \begin{bmatrix} R_{11} & R_{12} & R_{16} \\ R_{12} & R_{22} & R_{26} \\ R_{16} & R_{26} & R_{66} \end{bmatrix} [T_\sigma]^{-1}, \tag{2.199}$$

where $[T_\sigma]$ and $[T_\epsilon]$ are given by Eq. (2.196).

2.10 Strain Energy

In this section we define three parameters useful in the analyses of plates and beams. These are the strain energy, the potential of the external forces, and the total potential energy.

For a linearly elastic system, the strain energy of volume V is defined as

$$U = \frac{1}{2} \int\int\int (\epsilon_x \sigma_x + \epsilon_y \sigma_y + \epsilon_z \sigma_z + \gamma_{yz} \tau_{yz} + \gamma_{xz} \tau_{xz} + \gamma_{xy} \tau_{xy})\, dV. \tag{2.200}$$

This expression can be written as

$$U = \frac{1}{2} \int\int\int \{\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} dV. \tag{2.201}$$

With the use of the stress–strain relationships (Eq. 2.20) this expression can be expressed in terms of the strains, as follows:

$$U = \frac{1}{2} \int\int\int \{\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\} [\overline{C}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} dV, \tag{2.202}$$

where $[\overline{C}]$ is the stiffness matrix in the x, y, z coordinate system. The strain energy must be positive ($U > 0$) for any nonzero strain.

The potential of the external forces is defined as

$$\Omega = -\int\int\int (f_x u + f_y v + f_z w)\, dV - \int\int (p_x u + p_y v + p_z w)\, dA, \tag{2.203}$$

where, we recall, f_x, f_y, f_z are the body forces (per unit volume) and p_x, p_y, p_z are the surface forces (per unit area).

The total potential energy of the system is

$$\pi_p = U + \Omega. \tag{2.204}$$

2.10.1 The Ritz Method

In the Ritz method the displacements are assumed to be in the form

$$u = \sum_{i=1}^{I} A_i u_i \qquad v = \sum_{j=1}^{J} B_j v_j \qquad w = \sum_{k=1}^{K} C_k w_k. \tag{2.205}$$

The displacements u_i, v_j, w_k are conveniently chosen known functions that must satisfy the geometrical boundary conditions, whereas A_i, B_j, C_k are yet unknown constants. According to the principle of stationary potential energy, at equilibrium the potential energy (Eq. 2.204) must satisfy the conditions

$$\begin{aligned} \frac{\partial \pi_p}{\partial A_i} &= 0, \quad i = 1, \ldots, I \\ \frac{\partial \pi_p}{\partial B_j} &= 0, \quad j = 1, \ldots, J \\ \frac{\partial \pi_p}{\partial C_k} &= 0, \quad k = 1, \ldots, K. \end{aligned} \tag{2.206}$$

The constants are provided by the solution of these equations.

2.11 Summary

The equilibrium equations and the stress–strain and strain–displacement relationships presented in this chapter are summarized in Table 2.17 for generally anisotropic materials and in Table 2.18 for monoclinic materials. Equations for orthotropic, transversely isotropic, and isotropic materials are the same as for monoclinic materials (Table 2.18) with the elements of the stiffness and compliance matrices simplified according to Eqs. (2.95)–(2.97)

The unknowns of interest for generally anisotropic materials are summarized in Table 2.19 and for monoclinic, orthotropic, transversely isotropic, and isotropic materials in Table 2.20. It is evident that under plane-stress and plane-strain conditions the number of unknowns is reduced and the equations are simplified.

2.11.1 Note on the Compliance and Stiffness Matrices

In this section, we give proof of the important statement that the compliance matrix (and consequently its inverse the stiffness matrix) must be positive definite, and symmetrical.

We consider a small cube made of a linearly elastic material. This initially stress-free cube is deformed. During this deformation the internal energy (in this case, the strain energy) changes. Since initially the cube is stress free, the deformation results in a positive change in the strain energy[5] ($\Delta U \equiv U > 0$). This requirement may be expressed as (see Eq. 2.202)

$$\{\epsilon_x \quad \epsilon_y \quad \epsilon_z \quad \gamma_{yz} \quad \gamma_{xz} \quad \gamma_{xy}\}[\overline{C}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} > 0 \text{ when } \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \neq 0. \tag{2.207}$$

[5] L. E. Malvern, *Introduction to the Mechanics of a Continuous Medium.* Prentice-Hall, Englewood Cliffs, New Jersey, 1969, p. 292.

Table 2.17. The equations for a generally anisotropic material

	Equilibrium	Strain–displacement	Stress–strain
Three-dimensional	2.13, 2.14, 2.15	2.2, 2.3, 2.4, 2.9–2.11	2.20
Plane-strain			
Free end	2.59, 2.60, 2.61	2.54 2.56 2.57	2.62
Built-in ends	2.59, 2.60, 2.61	2.99 2.101, 2.102	2.103
Plane-stress	2.122, 2.123	2.2, 2.3, 2.11	2.126

Table 2.18. The equations for a monoclinic material. The equations for orthotropic, transversely isotropic, and isotropic materials are the same as for monoclinic materials with the stiffness and compliance matrix elements simplified as specified in Eqs. (2.95)–(2.97).

	Equilibrium	Strain–displacement	Stress–strain
Three-dimensional	2.13, 2.14, 2.15	2.2, 2.3, 2.4, 2.9–2.11	2.20
Plane-strain, free end			
In-plane load End axial load End moment	2.74, 2.75	2.68, 2.71	2.76
Out-of-plane load Torque	2.86	2.84	2.87
Plane-strain, built-in ends			
In-plane load	2.74, 2.75	2.107, 2.110	2.111
Out-of-plane load	2.86	2.117	2.119
Plane-stress	2.130, 2.131	2.2, 2.3, 2.11	2.134

Table 2.19. The unknowns in the equations for generally anisotropic materials.

	Displacements	Strains	Stresses
Three-dimensional	u, v, w	$\epsilon_x, \epsilon_y, \epsilon_z$ $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	$\sigma_x, \sigma_y, \sigma_z$ $\tau_{yz}, \tau_{xz}, \tau_{xy}$
Plane-strain			
Free end	u, v, w	ϵ_x, ϵ_y $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	σ_x, σ_y $\tau_{yz}, \tau_{xz}, \tau_{xy}$
Built-in ends	u, v, w	ϵ_x, ϵ_y $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	σ_x, σ_y $\tau_{yz}, \tau_{xz}, \tau_{xy}$
Plane-stress	u, v	$\epsilon_x, \epsilon_y, \gamma_{xy}$	$\sigma_x, \sigma_y, \tau_{xy}$

Table 2.20. The unknowns in the equations for monoclinic, orthotropic, transversely isotropic, and isotropic materials

	Displacements	Strains	Stresses
Three-dimensional	u_1, u_2, u_3	$\epsilon_1, \epsilon_2, \epsilon_3$ $\gamma_{23}, \gamma_{13}, \gamma_{12}$	$\sigma_1, \sigma_2, \sigma_3$ $\tau_{23}, \tau_{13}, \tau_{12}$
Plane-strain, free end			
In-plane load End axial load End moment	u_1, u_2	$\epsilon_1, \epsilon_2, \gamma_{12}$	$\sigma_1, \sigma_2, \tau_{12}$
Out-of-plane load Torque	u_3	γ_{23}, γ_{13}	τ_{23}, τ_{13}
Plane-strain, built-in ends			
In-plane load	u_1, u_2	$\epsilon_1, \epsilon_2, \gamma_{12}$	$\sigma_1, \sigma_2, \tau_{12}$
Out-of-plane load	u_3	γ_{23}, γ_{13}	τ_{23}, τ_{13}
Plane-stress	u_1, u_2	$\epsilon_1, \epsilon_2, \gamma_{12}$	$\sigma_1, \sigma_2, \tau_{12}$

The preceding inequality requires that $[\overline{C}]$ be positive definite.[6] (Correspondingly, $[\overline{S}]$, $[C]$ and $[S]$ must also be positive definite.) The manner in which we can determine whether or not $[\overline{C}]$ is positive definite is discussed subsequently (page 59).

First we show that the compliance matrix must be symmetrical. To this end we apply loads in four steps (Fig. 2.36) to our initially stress-free cube made of an elastic material.

(a) A tensile load p (per unit area) is applied in the x direction. As the result of this load $\sigma_x = p$ is the only nonzero stress in the material. The strains are (see Eq. 2.21)

$$\epsilon_x^a = \overline{S}_{11} p \qquad \epsilon_y^a = \overline{S}_{21} p. \tag{2.208}$$

The work is

$$W^a = \int_0^{\epsilon_x^a} \sigma_x d\epsilon_x + \int_0^{\epsilon_y^a} \sigma_y d\epsilon_y = \frac{1}{2}\overline{S}_{11} p^2. \tag{2.209}$$

(b) An additional tensile load p is applied in the y direction. The stresses in the material become $\sigma_x = \sigma_y = p$, and the corresponding strains are

$$\epsilon_x^{\mathrm{b}} = \overline{S}_{11} p + \overline{S}_{12} p \qquad \epsilon_y^{\mathrm{b}} = \overline{S}_{21} p + \overline{S}_{22} p. \tag{2.210}$$

The work is

$$W^{\mathrm{b}} = \int_{\epsilon_x^a}^{\epsilon_x^{\mathrm{b}}} \sigma_x d\epsilon_x + \int_{\epsilon_y^a}^{\epsilon_y^{\mathrm{b}}} \sigma_y d\epsilon_y = \overline{S}_{12} p^2 + \frac{1}{2}\overline{S}_{22} p^2. \tag{2.211}$$

[6] E. Kreyszig, *Advanced Engineering Mathematics.* 7th Edition. John Wiley & Sons, New York, 1993, p. 407.

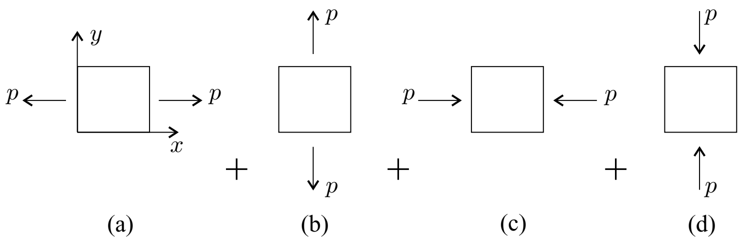


Figure 2.36: Illustration of the four load steps.

(c) An additional compressive load p is applied in the x direction. The stresses in the material become $\sigma_x = 0, \sigma_y = p$, and the corresponding strains are

$$\epsilon_x^{\mathrm{c}} = \overline{S}_{12} p \qquad \epsilon_y^{\mathrm{c}} = \overline{S}_{22} p. \tag{2.212}$$

The work is

$$W^{\mathrm{c}} = \int_{\epsilon_x^{\mathrm{b}}}^{\epsilon_x^{\mathrm{c}}} \sigma_x d\epsilon_x + \int_{\epsilon_y^{\mathrm{b}}}^{\epsilon_y^{\mathrm{c}}} \sigma_y d\epsilon_y = -\frac{1}{2}\overline{S}_{11} p^2 - \overline{S}_{21} p^2. \tag{2.213}$$

(d) An additional compressive load p is applied in the y direction. The stresses in the material become $\sigma_x = \sigma_y = 0$, and the corresponding strains are $\epsilon_x^d = \epsilon_y^d = 0$.
The work is

$$W^d = \int_{\epsilon_x^{\mathrm{c}}}^{0} \sigma_x d\epsilon_x + \int_{\epsilon_y^{\mathrm{c}}}^{0} \sigma_y d\epsilon_y = -\frac{1}{2}\overline{S}_{22} p^2. \tag{2.214}$$

The total work done on the cube is

$$W = W^a + W^{\mathrm{b}} + W^{\mathrm{c}} + W^d = (\overline{S}_{12} - \overline{S}_{21}) p^2. \tag{2.215}$$

For an elastic material the total work is zero. Accordingly, we must have

$$\overline{S}_{12} = \overline{S}_{21}. \tag{2.216}$$

By similar arguments it can be shown that

$$\overline{S}_{ij} = \overline{S}_{ji} \quad i, j = 1, 2, \ldots, 6. \tag{2.217}$$

Thus, the compliance matrix of an elastic material $[\overline{S}]$ (and correspondingly $[\overline{C}]$, $[C]$, and $[S]$) must be symmetrical.

Next we discuss the conditions that ensure a compliance matrix is positive definite. A symmetrical matrix is positive definite when each of its eigenvalues is positive.[7] Thus, to determine whether or not the matrix is positive definite we must examine the eigenvalues of the compliance (or stiffness) matrix.

For orthotropic, transversely isotropic, and isotropic materials there is a simpler method for determining whether or not the compliance (or stiffness) matrix

[7] F. B. Hildebrand, *Methods of Applied Mathematics.* 2nd edition. Prentice-Hall, Englewood Cliffs, New Jersey, 1965, p. 48.

is positive definite. The simpler method utilizes the condition that the matrix is positive definite when every subdeterminant of the main diagonal is positive.[8]

For an orthotropic material the compliance matrix is (Eq. 2.28)

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}. \tag{2.218}$$

The following subdeterminants must all be positive:

$$\begin{aligned} &D_1 = S_{11} > 0 \qquad D_2 = S_{22} > 0 \qquad D_3 = S_{33} > 0 \\ &D_4 = S_{44} > 0 \qquad D_5 = S_{55} > 0 \qquad D_6 = S_{66} > 0 \end{aligned} \tag{2.219}$$

$$D_{23}=\begin{vmatrix} S_{22} & S_{23} \\ S_{23} & S_{33} \end{vmatrix} > 0 \qquad D_{13}=\begin{vmatrix} S_{11} & S_{13} \\ S_{13} & S_{33} \end{vmatrix} > 0 \qquad D_{12}=\begin{vmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{vmatrix} > 0 \tag{2.220}$$

$$D_{123} = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix} > 0, \tag{2.221}$$

where $|\quad|$ denotes the determinant. Since the stiffness matrix is the inverse of the compliance matrix, according to the rules of matrix inversion the elements of the main diagonal of the stiffness matrix are

$$C_{11} = \frac{D_{23}}{D_{123}} \qquad C_{22} = \frac{D_{13}}{D_{123}} \qquad C_{33} = \frac{D_{12}}{D_{123}}, \tag{2.222}$$

where D_{23}, D_{13}, and D_{12} are positive when C_{11}, C_{22}, and C_{33} are positive,

$$C_{11} > 0 \qquad C_{22} > 0 \qquad C_{33} > 0, \tag{2.223}$$

and, at the same time, D_{123} is positive. Thus, the requirement for positive definiteness is met when all three of the following conditions are met:

- the elements of the main diagonal of the compliance matrix are positive (Eq. 2.219),
- the determinant of the compliance matrix is positive (Eq. 2.221), and
- the elements of the main diagonal of the stiffness matrix are positive (Eq. 2.223).

The same conditions apply for transversely isotropic and isotropic materials. If the relevant elements of the compliance and stiffness matrices are presented in

[8] E. Kreyszig, *Advanced Engineering Mathematics.* 7th edition. John Wiley & Sons, New York, 1993, p. 407.

Table 2.21. The constraints on the engineering constants.

Orthotropic	$E_1 > 0 \quad E_2 > 0 \quad E_3 > 0$
	$G_{23} > 0 \quad G_{13} > 0 \quad G_{12} > 0$
	$1 - \nu_{23}^2 \frac{E_3}{E_2} - \nu_{12}^2 \frac{E_2}{E_1} - 2\nu_{12}\nu_{13}\nu_{23}\frac{E_3}{E_1} - \nu_{13}^2 \frac{E_3}{E_1} > 0$
	$\nu_{23}^2 < \frac{E_2}{E_3}, \quad \nu_{13}^2 < \frac{E_1}{E_3}, \quad \nu_{12}^2 < \frac{E_1}{E_2}$
Transversely isotropic	$E_1 > 0, \quad E_2 > 0, \quad G_{12} > 0$
	$-1 < \nu_{23} < 1 - 2\frac{E_2}{E_1}\nu_{12}^2$
	$\nu_{12}^2 < \frac{E_1}{E_2}$
Isotropic	$E_1 > 0$
	$-1 < \nu_{12} < 0.5$

terms of the engineering constants (see Tables 2.7 and 2.9, pages 15 and 18) the preceding conditions can be expressed in terms of the engineering constants. Details of the algebraic manipulations are not given here. The results are summarized in Table 2.21.[9]

2.4 Example. *The engineering constants of a graphite epoxy unidirectional ply are given as* $E_1 = 148 \times 10^9$ *N/m²*, $E_2 = 9.65 \times 10^9$ *N/m²*, $G_{12} = 4.55 \times 10^9$ *N/m²*, $\nu_{12} = 0.3$, *and* $\nu_{23} = 0.6$. *Determine whether or not this set of constants is valid.*

Solution. There are two ways to find the answer to this problem.

Method 1. The compliance matrix is (Eq. 2.41)

$$[S] = \begin{bmatrix} 6.76 & -2.03 & -2.03 & 0 & 0 & 0 \\ -2.03 & 103.63 & -62.18 & 0 & 0 & 0 \\ -2.03 & -62.18 & 103.63 & 0 & 0 & 0 \\ 0 & 0 & 0 & 331.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 219.78 & 0 \\ 0 & 0 & 0 & 0 & 0 & 219.78 \end{bmatrix} 10^{-12}\frac{\text{m}^2}{\text{N}}. \tag{2.224}$$

The eigenvalues ($\lambda \times 10^{+9}$) of this matrix are

0.0065 0.0417 0.1658 0.2198 0.2198 0.3316.

Since every eigenvalue is positive, the specified set of engineering constants is valid.

Method 2. For a transversely isotropic material the engineering constants must satisfy the inequalities in Table 2.21. In terms of the engineering constants these inequalities are

$$-1 < \nu_{23} < 1 - 2\frac{E_2}{E_1}\nu_{12}^2 \qquad \nu_{12}^2 < \frac{E_1}{E_2} \tag{2.225}$$

[9] B. M. Lempriere, Poisson's Ratio in Orthotropic Materials. *AIAA Journal*, Vol. 7, 2226–2227, 1968.

or

$$-1 < 0.6 < 0.988 \qquad 0.09 < 15.3. \tag{2.226}$$

Since the inequalities are satisfied, the specified set of engineering constants is valid.

Let us now assume that $\nu_{21} = 0.3$. In this case $\nu_{12} = \frac{E_1}{E_2}\nu_{21} = 4.601$ (see Table 2.8, page 16), and the compliance matrix (Table 2.7, third row, page 15) is

$$[S] = \begin{bmatrix} 6.76 & -31.09 & -31.09 & 0 & 0 & 0 \\ -31.09 & 103.63 & -62.18 & 0 & 0 & 0 \\ -31.09 & -62.18 & 103.63 & 0 & 0 & 0 \\ 0 & 0 & 0 & 331.61 & 0 & 0 \\ 0 & 0 & 0 & 0 & 219.78 & 0 \\ 0 & 0 & 0 & 0 & 0 & 219.78 \end{bmatrix} 10^{-12}\frac{\text{m}^2}{\text{N}}. \tag{2.227}$$

The eigenvalues ($\lambda \times 10^{+9}$) are

$$-0.0232 \qquad 0.0714 \qquad 0.1658 \qquad 0.2198 \qquad 0.2198 \qquad 0.3316.$$

One of the eigenvalues is negative; hence, the specified set of engineering constants is invalid. We reach the same conclusion if we use the inequalities given by Eq. (2.225) since two of the inequalities are not satisfied:

$$-1 < 0.6 \not< -1.76 \qquad 21.2 \not< 15.3. \tag{2.228}$$

This example illustrates that care should be taken to use proper values of the Poisson ratios. The ν_{12} and ν_{21} Poisson ratios (referred to as the major and minor Poisson ratios) must not be interchanged.