CHAPTER ELEVEN

Micromechanics

Micromechanics is used to estimate the mechanical and hygrothermal properties of composite materials from the known values of the properties of the fiber and the matrix. There are three major categories of micromechanical approaches: (i) mechanics of material models based on simplifying assumptions that make it unnecessary to specify in detail the stress–strain distributions, (ii) elasticity models requiring that the stresses and strains be determined at the micromechanical level, and (iii) empirical expressions resulting from curve-fitting elasticity solutions or data.

It is not our intent to discuss the numerous available models. Instead, we focus on two mechanics of materials models, namely on the "rule of mixtures" and the "modified rule of mixtures." The rule of mixtures is the simplest and most intuitive approach and is useful for introducing concepts. However, it fails to represent some of the properties with reasonable accuracy. The modified rule of mixtures is an improvement over the rule of mixtures, and predicts the properties with better accuracies.

Methods to predict strength – though available – are not presented because they are less accurate than the models predicting elastic and hygrothermal properties.

11.1 Rule of Mixtures

We consider the volume V of an element shown in Figure 11.1. The volume of this element is

$$V = V_{\rm f} + V_{\rm m} + V_{\rm v}, \tag{11.1}$$

where the subscripts f, m and v refer to the fiber, the matrix, and the void. It is convenient to introduce the volume fractions as follows:

$$v_{\rm f} \equiv \frac{V_{\rm f}}{V} \qquad v_{\rm m} \equiv \frac{V_{\rm m}}{V} \qquad v_{\rm v} \equiv \frac{V_{\rm v}}{V}. \tag{11.2}$$

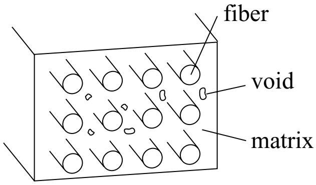

Figure 11.1: Illustration of the matrix, fiber, and void volumes.

Equations (11.1) and (11.2) give

$$v_f + v_m + v_v = 1. \tag{11.3}$$

When the void fraction is negligible ($v_v = 0$), we have

$$v_m = 1 - v_f. \tag{11.4}$$

The mass of the element in Figure 11.1 is

$$M = M_f + M_m + M_v. \tag{11.5}$$

By neglecting the mass of the void, we can write this equation as

$$M = \rho_f V_f + \rho_m V_m, \tag{11.6}$$

where ρ_f and ρ_m are the fiber and the matrix densities, respectively. The density of the composite is

$$\rho_{comp} = \frac{M}{V} = v_f \rho_f + v_m \rho_m. \tag{11.7}$$

In the following, we treat unidirectional, fiber-reinforced composites without voids. We derive the properties using two types of elements (Fig. 11.2). Element 1 contains a single fiber bundle of circular cross section. Element 2 consists of a fiber layer sandwiched between two layers of matrix material. In Element 2, the

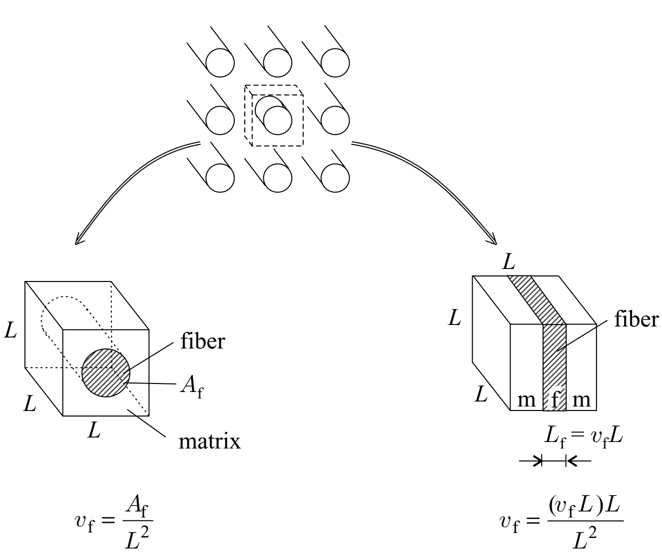

Figure 11.2: Representative elements. Element 1 (left) and Element 2 (right).

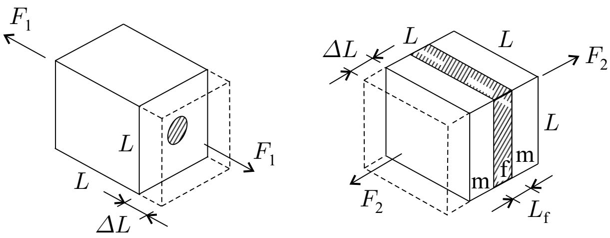

Figure 11.3: Element 1 subjected to a force in the fiber direction (left), and Element 2 subjected to a force in the transverse direction (right).

fiber and matrix volumes are the same as in Element 1. Hence, the thickness of the fiber layer in Element 2 is $v_f L$.

11.1.1 Longitudinal Young Modulus E_1

Element 1 is subjected to a force F_1 in the fiber direction. The force, distributed over the surface, is (Fig. 11.3)

$$F_1 = \sigma_1 A, \tag{11.8}$$

where σ_1 is the average normal stress across the entire cross-sectional area A ($A = L^2$). Part of the force is carried by the fibers and part by the matrix. Thus, we write

$$\sigma_1 A = A_f \sigma_{f1} + A_m \sigma_{m1}, \tag{11.9}$$

A_f and A_m are the cross-sectional areas of the fiber bundle and the matrix, respectively. When the Poisson effect is neglected, the normal stresses in the composite σ_1, in the fiber bundle σ_{f1}, and in the matrix σ_{m1} are

$$\sigma_1 = \epsilon_1 E_1 \qquad \sigma_{f1} = \epsilon_{f1} E_{f1} \qquad \sigma_{m1} = \epsilon_{m1} E_m, \tag{11.10}$$

where E_1 and E_{f1} are the composite and the fiber longitudinal Young moduli and E_m is the matrix Young modulus, respectively. Equations (11.9) and (11.10) give

$$\epsilon_1 E_1 = \frac{A_f}{A} \epsilon_{f1} E_{f1} + \frac{A_m}{A} \epsilon_{m1} E_m. \tag{11.11}$$

The elongations ΔL and, hence, the longitudinal normal strains of the composite ϵ_1, the matrix ϵ_{m1}, and the fiber ϵ_{f1} are equal:

$$\epsilon_1 = \epsilon_{f1} = \epsilon_{m1}. \tag{11.12}$$

The volume fractions are

$$v_f = \frac{A_f}{A} \qquad v_m = \frac{A_m}{A}. \tag{11.13}$$

Equations (11.11)–(11.13) result in the following expression for the longitudinal Young modulus of the composite in terms of the fiber and matrix moduli:

$$E_1 = v_f E_{f1} + v_m E_m = v_f E_{f1} + (1 - v_f) E_m. \tag{11.14}$$

11.1.2 Transverse Young Modulus E_2

We consider a rectangular element with sides L made up of three layers (Element 2, Fig. 11.3). The inner layer is a sheet of fiber with the same volume as the fiber bundle in Element 1. The length and the transverse (x_2 direction) Young modulus of this inner layer are L_{f} and E_{f2}, respectively, and the length and the Young modulus of the outer layers are $L_{\mathrm{m}} = (L - L_{\mathrm{f}})/2$ and E_{m}. The element is subjected to a force F_2. The force is distributed uniformly across the surface $A\,(A = L^2)$. The normal stress in the transverse direction is $\sigma_2 = F_2/A$, which can be expressed as

$$\sigma_2 = \epsilon_2 E_2, \tag{11.15}$$

where E_2 is the transverse Young modulus of the element and ϵ_2 is the average transverse normal strain

$$\epsilon_2 = \frac{\Delta L}{L}. \tag{11.16}$$

The change in the length of the element is

$$\Delta L = 2L_{\mathrm{m}}\epsilon_{\mathrm{m2}} + L_{\mathrm{f}}\epsilon_{\mathrm{f2}}, \tag{11.17}$$

where ϵ_{m2} and ϵ_{f2} are the transverse normal strains in the matrix and fiber layers. By neglecting the Poisson effect, we have

$$\epsilon_{\mathrm{m2}} = \frac{\sigma_{\mathrm{m2}}}{E_{\mathrm{m}}} \qquad \epsilon_{\mathrm{f2}} = \frac{\sigma_{\mathrm{f2}}}{E_{\mathrm{f2}}}. \tag{11.18}$$

By introducing Eqs. (11.17) and (11.18) into Eq. (11.16), we obtain

$$\epsilon_2 = \frac{\Delta L}{L} = \frac{2L_{\mathrm{m}}}{L}\frac{\sigma_{\mathrm{m2}}}{E_{\mathrm{m}}} + \frac{L_{\mathrm{f}}}{L}\frac{\sigma_{\mathrm{f2}}}{E_{\mathrm{f2}}}. \tag{11.19}$$

The transverse normal stresses in the composite σ_2, the matrix σ_{m2}, and the fiber σ_{f2} layers are equal as follows:

$$\sigma_2 = \sigma_{\mathrm{m2}} = \sigma_{\mathrm{f2}}. \tag{11.20}$$

The matrix and fiber volume fractions are

$$v_{\mathrm{f}} = \frac{L_{\mathrm{f}}}{L} \qquad v_{\mathrm{m}} = \frac{2L_{\mathrm{m}}}{L}. \tag{11.21}$$

By substituting Eqs. (11.20), (11.21), and $\epsilon_2 = \sigma_2/E_2$ into Eq. (11.19), we obtain the transverse Young modulus

$$E_2 = \left(\frac{v_{\mathrm{f}}}{E_{\mathrm{f2}}} + \frac{v_{\mathrm{m}}}{E_{\mathrm{m}}}\right)^{-1} = \left(\frac{v_{\mathrm{f}}}{E_{\mathrm{f2}}} + \frac{1 - v_{\mathrm{f}}}{E_{\mathrm{m}}}\right)^{-1}. \tag{11.22}$$

11.1.3 Longitudinal Shear Modulus G_{12}

We consider Element 2. The length and the longitudinal shear modulus of the inner layer are L_{f} and G_{f12} respectively, and the length and the shear modulus of the outer layers are $L_{\mathrm{m}} = (L - L_{\mathrm{f}})/2$ and G_{m}. The element is subjected to a

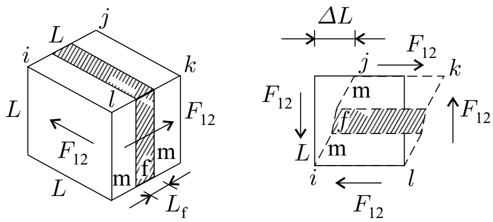

Figure 11.4: Element 2 subjected to a shear force (left) and deformation of the "top" (*ijkl*) surface (right).

shear force F_{12} (Fig. 11.4) distributed uniformly across the surface A $(A = L^2)$. The shear stress $\tau_{12} = F_{12}/A$ is

$$\tau_{12} = \gamma_{12} G_{12}, \tag{11.23}$$

where G_{12} is the longitudinal shear modulus of the element and γ_{12} is the average shear strain,

$$\gamma_{12} \cong \tan\,\gamma_{12} = \frac{\Delta L}{L}, \tag{11.24}$$

where ΔL is due to the shear deformations of the matrix and fiber layers,

$$\Delta L = 2L_{\rm m}\gamma_{\rm m12} + L_{\rm f}\gamma_{\rm f12}, \tag{11.25}$$

where $\gamma_{\rm m12}$ and $\gamma_{\rm f12}$ are the shear strains in the matrix and fiber layers as follows:

$$\gamma_{\rm m12} = \frac{\tau_{\rm m12}}{G_{\rm m}} \qquad \gamma_{\rm f12} = \frac{\tau_{\rm f12}}{G_{\rm f12}}. \tag{11.26}$$

The shear stresses in the composite, the matrix, and the fiber layers are equal:

$$\tau_{12} = \tau_{\rm m12} = \tau_{\rm f12}. \tag{11.27}$$

By combining Eqs. (11.21) and (11.23)–(11.27), we obtain the longitudinal shear modulus:

$$G_{12} = \left(\frac{v_{\rm f}}{G_{\rm f12}} + \frac{v_{\rm m}}{G_{\rm m}}\right)^{-1} = \left(\frac{v_{\rm f}}{G_{\rm f12}} + \frac{1 - v_{\rm f}}{G_{\rm m}}\right)^{-1}. \tag{11.28}$$

11.1.4 Transverse Shear Modulus G_{23}

We again consider Element 2 made up of three layers (Fig. 11.4). The transverse shear modulus G_{23} of this element is derived in the same way as the longitudinal shear modulus G_{12}. The result is

$$G_{23} = \left(\frac{v_{\rm f}}{G_{\rm f23}} + \frac{1 - v_{\rm f}}{G_{\rm m}}\right)^{-1}. \tag{11.29}$$

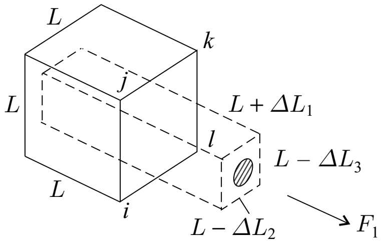

Figure 11.5: Deformation of Element 1 subjected to an axial force in the x_1 direction.

11.1.5 Longitudinal Poisson Ratio ν_{12}

Element 1, shown in Figure 11.2, is subjected to a force F_1 in the longitudinal x_1 direction. Because of this force the element deforms, as illustrated in Figure 11.5. The sides of the deformed element are $(L + \Delta L_1)$, $(L - \Delta L_2)$, and $(L - \Delta L_3)$. The change in the cross-sectional area A of the face *ijkl* is

$$\Delta A = A - (L - \Delta L_2)(L - \Delta L_3). \tag{11.30}$$

By definition, the normal strains in the x_2 and x_3 directions are

$$\epsilon_2 = -\frac{\Delta L_2}{L} \qquad \epsilon_3 = -\frac{\Delta L_3}{L}. \tag{11.31}$$

Owing to symmetry, $\Delta L_2 = \Delta L_3$ and, consequently, we have

$$\epsilon_2 = \epsilon_3. \tag{11.32}$$

By neglecting higher-order terms, we find that Eqs. (11.30)–(11.32) give

$$\Delta A = A(\epsilon_2 + \epsilon_3) = 2A\epsilon_2. \tag{11.33}$$

By similar arguments, the changes in the matrix and fiber areas are

$$\Delta A_{\rm f} = 2A_{\rm f}\epsilon_{\rm f2} \qquad \Delta A_{\rm m} = 2A_{\rm m}\epsilon_{\rm m2}. \tag{11.34}$$

The change in the total area is the sum of the changes of the fiber and matrix areas:

$$\Delta A = \Delta A_{\rm f} + \Delta A_{\rm m}. \tag{11.35}$$

Under the action of the longitudinal normal stress $\sigma_1 = F_1/A$, the transverse normal strains are related to the longitudinal normal strain by

$$\epsilon_2 = -\nu_{12}\epsilon_1 \qquad \epsilon_3 = -\nu_{13}\epsilon_1. \tag{11.36}$$

By virtue of Eqs. (11.32) and (11.36), we have

$$\nu_{12} = \nu_{13}. \tag{11.37}$$

From Eqs. (11.36) and (11.33) we obtain

$$\nu_{12} = -\frac{\epsilon_2}{\epsilon_1} = -\frac{\Delta A}{2A}\frac{1}{\epsilon_1}. \tag{11.38}$$

By combining Eqs. (11.35) and (11.38) and introducing Eq. (11.34) into the resulting expression, we obtain

$$\nu_{12} = -\frac{2A_f\epsilon_{f2} + 2A_m\epsilon_{m2}}{2A}\frac{1}{\epsilon_1} = -\frac{A_f}{A}\frac{\epsilon_{f2}}{\epsilon_1} - \frac{A_m}{A}\frac{\epsilon_{m2}}{\epsilon_1}. \quad (11.39)$$

The changes in the lengths of the fiber, the matrix, and the composite are equal. Consequently, in the x_1 direction the normal strains in the composite, in the fiber, and in the matrix are equal as follows:

$$\epsilon_1 = \epsilon_{f1} = \epsilon_{m1}. \quad (11.40)$$

With the definitions

$$v_f = \frac{A_f}{A} \qquad v_m = \frac{A_m}{A} \qquad \nu_{f12} = -\frac{\epsilon_{f2}}{\epsilon_{f1}} \qquad \nu_m = -\frac{\epsilon_{m2}}{\epsilon_{m1}}, \quad (11.41)$$

Eqs. (11.39)–(11.41) give the following expression for the longitudinal Poisson ratio:

$$\nu_{12} = v_f\nu_{f12} + v_m\nu_m. \quad (11.42)$$

11.1.6 Transverse Poisson Ratio ν_{23}

For a transversely isotropic material (which is under consideration here), the transverse shear and the Young moduli are related by the following expression (Eq. 2.34):

$$G_{23} = \frac{E_2}{2(1+\nu_{23})}. \quad (11.43)$$

By rearranging this expression, we obtain the transverse Poisson ratio

$$\nu_{23} = \frac{E_2}{2G_{23}} - 1. \quad (11.44)$$

More accurate expressions for E_2 and G_{23} are obtained by the modified rule of mixtures (Section 11.2). These expressions are included in Table 11.1, below.

Table 11.1. Expressions for the engineering constants

Longitudinal Young modulus	$E_1 = v_f E_{f1} + v_m E_m$
Transverse Young modulus	$E_2 = \left(\frac{\sqrt{v_f}}{E_{b2}} + \frac{1-\sqrt{v_f}}{E_m}\right)^{-1}$ where $E_{b2} = \sqrt{v_f}E_{f2} + (1-\sqrt{v_f})E_m$
Longitudinal shear modulus	$G_{12} = \left(\frac{\sqrt{v_f}}{G_{b12}} + \frac{1-\sqrt{v_f}}{G_m}\right)^{-1}$ where $G_{b12} = \sqrt{v_f}G_{f12} + (1-\sqrt{v_f})G_m$
Transverse shear modulus	$G_{23} = \left(\frac{\sqrt{v_f}}{G_{b23}} + \frac{1-\sqrt{v_f}}{G_m}\right)^{-1}$ where $G_{b23} = \sqrt{v_f}G_{f23} + (1-\sqrt{v_f})G_m$
Longitudinal Poisson ratio	$\nu_{12} = v_f\nu_{f12} + v_m\nu_m$
Transverse Poisson ratio	$\nu_{23} = \frac{E_2}{2G_{23}} - 1$

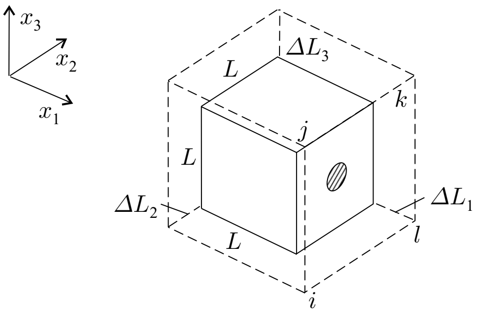

Figure 11.6: Deformation of Element 1 subjected to a uniform temperature change ΔT.

11.1.7 Thermal Expansion Coefficients

When the unrestrained Element 1 is subjected to a ΔT temperature change, its volume changes (Fig. 11.6). The new length, width, and height are $(L + \Delta L_1)$, $(L + \Delta L_2)$, and $(L + \Delta L_3)$, and the corresponding normal strains are

$$\epsilon_1 = \frac{\Delta L_1}{L} \qquad \epsilon_2 = \frac{\Delta L_2}{L} \qquad \epsilon_3 = \frac{\Delta L_3}{L}. \tag{11.45}$$

The longitudinal $\widetilde{\alpha}_1$ and transverse $\widetilde{\alpha}_2, \widetilde{\alpha}_3$ thermal expansion coefficients are defined as

$$\widetilde{\alpha}_1 = \frac{\epsilon_1}{\Delta T} \qquad \widetilde{\alpha}_2 = \frac{\epsilon_2}{\Delta T} \qquad \widetilde{\alpha}_3 = \frac{\epsilon_3}{\Delta T}. \tag{11.46}$$

Owing to symmetry, the transverse thermal expansion coefficients are equal $(\widetilde{\alpha}_2 = \widetilde{\alpha}_3)$, and the transverse normal strains are equal $(\epsilon_2 = \epsilon_3)$.

Equations (11.45) and (11.46) refer to the element.

For the fiber and the matrix, the strain–stress relationships are (Eqs. 2.162 and 2.163 and Table 2.7, page 15)

$$\epsilon_{f1} - \widetilde{\alpha}_{f1}\Delta T = \frac{\sigma_{f1}}{E_{f1}} \qquad \epsilon_{m1} - \widetilde{\alpha}_{m}\Delta T = \frac{\sigma_{m1}}{E_{m}} \tag{11.47}$$

$$\epsilon_{f2} - \widetilde{\alpha}_{f2}\Delta T = \frac{-\nu_{f12}}{E_{f1}}\sigma_{f1} \qquad \epsilon_{m2} - \widetilde{\alpha}_{m}\Delta T = \frac{-\nu_{m}}{E_{m}}\sigma_{m1}. \tag{11.48}$$

These equations are written with the assumptions that the longitudinal normal stress σ_1 dominates and is the only stress needed to be considered.

Longitudinal thermal expansion coefficient. When Element 1 is subjected to a temperature change ΔT, in the x direction the fiber and the matrix elongate the same amount as the composite. Hence, the normal strains in the x_1 direction are equal

$$\epsilon_1 = \epsilon_{f1} = \epsilon_{m1}. \tag{11.49}$$

From Eq. (11.47) the longitudinal normal stress (in the x_1 direction) in the fiber and the matrix may be written as (Eqs. 11.47 and 11.49)

$$\sigma_{f1} = (\epsilon_1 - \widetilde{\alpha}_{f1}\Delta T)E_{f1} \qquad \sigma_{m1} = (\epsilon_1 - \widetilde{\alpha}_{m}\Delta T)E_{m}. \tag{11.50}$$

The total force acting on the *ijkl* face (Fig. 11.6) is the sum of the forces acting on the fiber and the matrix:

$$F_1 = F_f + F_m. \tag{11.51}$$

Since there is no net force on the *ijkl* face, we write

$$F_f + F_m = \sigma_{f1} A_f + \sigma_{m1} A_m = 0. \tag{11.52}$$

By introducing Eq. (11.50) into (11.52) and by noting that $v_f = A_f/A$ and $v_m = A_m/A$, we obtain

$$\epsilon_1 \left(E_{f1} v_f + E_m v_m\right) - \left(\widetilde{\alpha}_{f1} E_{f1} v_f + \widetilde{\alpha}_m E_m v_m\right) \Delta T = 0. \tag{11.53}$$

By combining Eqs. (11.46, left) and (11.53), and by noting that $E_{f1} v_f + E_m v_m = E_1$ (Eq. 11.14), we obtain

$$\widetilde{\alpha}_1 = \frac{v_f E_{f1}}{E_1} \widetilde{\alpha}_{f1} + \frac{v_m E_m}{E_1} \widetilde{\alpha}_m. \tag{11.54}$$

Transverse thermal expansion coefficient. When the temperature of Element 1 is changed by ΔT, the cross-sectional area of the face *ijkl* as well as the fiber and matrix areas change (Fig. 11.5). The changes in these areas are (Eqs. 11.33–11.34)

$$\Delta A = 2A\epsilon_2 \qquad \Delta A_f = 2A_f \epsilon_{f2} \qquad \Delta A_m = 2A_m \epsilon_{m2}. \tag{11.55}$$

The change in the total area is the sum of the changes of the fiber and matrix areas:

$$\Delta A = \Delta A_f + \Delta A_m. \tag{11.56}$$

By combining Eqs. (11.55) and (11.56) and by noting that $v_f = A_f/A$ and $v_m = A_m/A$, we obtain

$$\epsilon_2 = \frac{\Delta A}{2A} = v_f \epsilon_{f2} + v_m \epsilon_{m2}. \tag{11.57}$$

From Eq. (11.48), the transverse normal strains in the fiber and the matrix may be written as (x_2 direction)

$$\epsilon_{f2} = -\nu_{f12} \frac{\sigma_{f1}}{E_{f1}} + \widetilde{\alpha}_{f2} \Delta T \qquad \epsilon_{m2} = -\nu_m \frac{\sigma_{m1}}{E_m} + \widetilde{\alpha}_m \Delta T. \tag{11.58}$$

By introducing Eqs. (11.47) into these expressions and observing that $\epsilon_1 = \epsilon_{f1} = \epsilon_{m1} = \widetilde{\alpha}_1 \Delta T$, we obtain

$$\epsilon_{f2} = -\nu_{f12} (\widetilde{\alpha}_1 - \widetilde{\alpha}_{f1}) \Delta T + \widetilde{\alpha}_{f2} \Delta T \qquad \epsilon_{m2} = -\nu_m (\widetilde{\alpha}_1 - \widetilde{\alpha}_m) \Delta T + \widetilde{\alpha}_m \Delta T. \tag{11.59}$$

Equations (11.46, middle), (11.57), and (11.59) yield

$$\widetilde{\alpha}_2 = v_f \widetilde{\alpha}_{f2} + v_m \widetilde{\alpha}_m + v_f \nu_{f12} \left(\widetilde{\alpha}_{f1} - \widetilde{\alpha}_1\right) + v_m \nu_m \left(\widetilde{\alpha}_m - \widetilde{\alpha}_1\right), \tag{11.60}$$

where $\widetilde{\alpha}_1$ is given by Eq. (11.54).

11.1.8 Moisture Expansion Coefficients

We assume that the moisture content of the interface is negligible. Further, we consider composites in which the fibers do not absorb moisture and the moisture is uniformly distributed throughout the matrix. The moisture content of Element 1 is the sum of the moisture contents of the fiber and the matrix

$$cV = c_f V_f + c_m V_m, \tag{11.61}$$

where V is the volume, c the moisture concentration, and, as before, f and m refer to the fiber and the matrix. Since here we treat composites in which the fiber moisture content is zero ($c_f = 0$), we have

$$c = \frac{c_m V_m}{V} = c_m v_m. \tag{11.62}$$

The moisture-induced strains of Element 1 are

$$\epsilon_1 = \widetilde{\beta}_1 c \qquad \epsilon_2 = \widetilde{\beta}_2 c \qquad \epsilon_3 = \widetilde{\beta}_3 c, \tag{11.63}$$

where $\widetilde{\beta}_1$ is the longitudinal and $\widetilde{\beta}_2$ and $\widetilde{\beta}_3$ are the transverse moisture expression coefficients. Owing to symmetry, the transverse moisture expansion coefficients are equal ($\widetilde{\beta}_2 = \widetilde{\beta}_3$).

The moisture expansion coefficients of the fiber and the matrix are ($i = 1, 2, 3$)

$$\widetilde{\beta}_{fi} = \frac{\epsilon_{fi}}{c_f} = \frac{c}{c_f}\frac{\epsilon_{fi}}{c} \qquad \widetilde{\beta}_{mi} = \frac{\epsilon_{mi}}{c_m} = \frac{c}{c_m}\frac{\epsilon_{mi}}{c}. \tag{11.64}$$

We now compare the definitions of the thermal and moisture expansion coefficients as follows:

$$\begin{aligned} \widetilde{\alpha}_i &= \frac{\epsilon_i}{\Delta T} & \widetilde{\beta}_i &= \frac{\epsilon_i}{c} \\ \widetilde{\alpha}_{fi} &= \frac{\epsilon_{fi}}{\Delta T} & \frac{c_f}{c}\widetilde{\beta}_{fi} &= \frac{\epsilon_{fi}}{c} \\ \widetilde{\alpha}_m &= \frac{\epsilon_{mi}}{\Delta T} & \frac{c_m}{c}\widetilde{\beta}_m &= \frac{\epsilon_{mi}}{c}. \end{aligned} \tag{11.65}$$

From these equations we note the analogy between the thermal and moisture expansion coefficients. The temperature change ΔT corresponds to the moisture concentration c, and the thermal expansion coefficients correspond to the moisture expansion coefficients as follows:

$$\begin{aligned} \widetilde{\alpha}_i &\Longrightarrow \widetilde{\beta}_i \\ \widetilde{\alpha}_{fi} &\Longrightarrow \frac{c_f}{c}\widetilde{\beta}_{fi} \\ \widetilde{\alpha}_m &\Longrightarrow \frac{c_m}{c}\widetilde{\beta}_m. \end{aligned} \tag{11.66}$$

Since $c_{\mathrm{f}} = 0$, Eqs. (11.66) and (11.62) give the following correspondence between the thermal and moisture expansion coefficients:

$$\begin{aligned} \widetilde{\alpha}_i &\Longrightarrow \widetilde{\beta}_i \\ \widetilde{\alpha}_{\mathrm{f}i} &\Longrightarrow 0 \\ \widetilde{\alpha}_{\mathrm{m}} &\Longrightarrow \frac{\widetilde{\beta}_{\mathrm{m}}}{v_{\mathrm{m}}}. \end{aligned} \tag{11.67}$$

Longitudinal moisture expansion coefficient. The longitudinal moisture expansion coefficient is obtained by replacing in Eq. (11.54) $\widetilde{\alpha}_1$ by $\widetilde{\beta}_1$, and $\widetilde{\alpha}_{\mathrm{m}}$ by $\widetilde{\beta}_{\mathrm{m}}/v_{\mathrm{m}}$, and by setting $\widetilde{\alpha}_{\mathrm{f}1} = 0$ as follows:

$$\widetilde{\beta}_1 = \frac{E_{\mathrm{m}}}{E_1}\widetilde{\beta}_{\mathrm{m}}. \tag{11.68}$$

Transverse moisture expansion coefficient. The transverse moisture expansion coefficient is obtained by replacing in Eq. (11.60) $\widetilde{\alpha}_i$ by $\widetilde{\beta}_i$, $\widetilde{\alpha}_{\mathrm{m}}$ by $\widetilde{\beta}_{\mathrm{m}}/v_{\mathrm{m}}$, and by setting $\widetilde{\alpha}_{\mathrm{f}i} = 0$:

$$\widetilde{\beta}_2 = \widetilde{\beta}_{\mathrm{m}} + v_{\mathrm{f}}\nu_{\mathrm{f}12}(-\widetilde{\beta}_1) + \nu_{\mathrm{m}}(\widetilde{\beta}_{\mathrm{m}} - v_{\mathrm{m}}\widetilde{\beta}_1). \tag{11.69}$$

11.1.9 Thermal Conductivity

The heat conducted (per unit time) in the longitudinal and transverse directions is given by Fourier's law

$$q_1 = A(K_1)\left(-\frac{\partial T}{\partial x_1}\right) \qquad q_2 = A(K_2)\left(-\frac{\partial T}{\partial x_2}\right), \tag{11.70}$$

where A is the cross-sectional area, $\partial T/\partial x$ is the temperature gradient, and K_1, K_2 are the longitudinal and transverse heat conduction coefficients, respectively.

The longitudinal and transverse normal forces are related to the longitudinal and transverse normal strains by Hooke's law as follows:

$$F_1 = A(E_1)(\epsilon_1) \qquad F_2 = A(E_2)(\epsilon_2). \tag{11.71}$$

Equations (11.70) and (11.71) show the analogy between Fourier's and Hooke's laws. This analogy is used to obtain the micromechanical expressions for the thermal conductivities. The heat-conducted q corresponds to the force F, the thermal conductivity K to the Young modulus E and the temperature gradient $\partial T/\partial x$ to the strain ϵ.

$$\begin{aligned} q &\Longrightarrow F \\ K &\Longrightarrow E \\ \frac{\partial T}{\partial x} &\Longrightarrow \epsilon. \end{aligned} \tag{11.72}$$

Longitudinal thermal conductivity. The longitudinal thermal conductivity is obtained by replacing in Eq. (11.14) E_{f1} by K_{f1} and E_m by K_m. The resulting expression is

$$K_1 = v_f K_{f1} + v_m K_m \tag{11.73}$$

Transverse thermal conductivity. The transverse thermal conductivity is obtained by replacing in Eq. (11.22) E_{f2} by K_{f2} and E_m by K_m. The resulting expression is

$$K_2 = \left(\frac{v_f}{K_{f2}} + \frac{v_m}{K_m} \right)^{-1}. \tag{11.74}$$

11.1.10 Moisture Diffusivity

Moisture diffusion in the longitudinal and transverse directions are assumed to follow Fick's law

$$\dot{m}_1 = A(D_1)\left(-\frac{\partial c}{\partial x_1}\right) \qquad \dot{m}_2 = A(D_2)\left(-\frac{\partial c}{\partial x_2}\right), \tag{11.75}$$

where $\dot{m}$ is the mass of moisture being transported (per unit time) across the surface A, $\partial c/\partial x$ is the moisture concentration gradient, and D_1, D_2 are the longitudinal and transverse diffusivities, respectively.

Equations (11.70) and (11.75) show the analogy between Fourier's and Fick's laws. This analogy is used to obtain the micromechanical expressions for the moisture diffusivities. The heat conduction q corresponds to the moisture transport $\dot{m}$, the thermal conductivity K to the diffusivity D, and the temperature gradient $\partial T/\partial x$ to the moisture gradient $\partial c/\partial x$ as follows:

$$\begin{aligned} q &\Longrightarrow m \\ K &\Longrightarrow D \\ \frac{\partial T}{\partial x} &\Longrightarrow \frac{\partial c}{\partial x}. \end{aligned} \tag{11.76}$$

Longitudinal moisture diffusivity. The longitudinal moisture diffusivity is obtained by replacing in Eq. (11.73) K_{f1} by D_{f1} and K_m by D_m. The resulting expression is

$$D_1 = v_f D_{f1} + v_m D_m. \tag{11.77}$$

When the diffusivity of the fibers is negligible, we have

$$D_1 = v_m D_m. \tag{11.78}$$

Transverse moisture diffusivity. The transverse moisture diffusivity is obtained by replacing in Eq. (11.74) K_{f2} by D_{f2} and K_m by D_m. The resulting expression is

$$D_2 = \left(\frac{v_f}{D_{f2}} + \frac{v_m}{D_m} \right)^{-1}. \tag{11.79}$$

When the diffusivity of the fibers is small ($D_{f2} \to 0$), this expression gives that $D_2 \to 0$, and Eq. (11.79) should not be used.

11.1.11 Specific Heat

When the temperature of Element 1 is raised from a specific datum T_0 to T, the heat contents of the composite, the fiber, and the matrix are

$$H = \rho C V (T - T_0) \qquad H_f = \rho_f C_f V_f (T - T_0) \qquad H_m = \rho_m C_m V_m (T - T_0), \tag{11.80}$$

where C, C_f, C_m are the specific heats of the composite, the fiber, and the matrix, respectively. The total heat content of the composite is

$$H = H_f + H_m. \tag{11.81}$$

Thus, the specific heat is

$$C = C_f \frac{\rho_f v_f}{\rho} + C_m \frac{\rho_m v_m}{\rho}. \tag{11.82}$$

By substituting the expression for the density ρ given by Eq. (11.7), we obtain

$$C = \frac{C_f \rho_f v_f + C_m \rho_m v_m}{\rho_f v_f + \rho_m v_m}. \tag{11.83}$$

11.2 Modified Rule of Mixtures

The modified rule of mixtures is somewhat more complex than the rule of mixtures, but it gives the transverse properties with better accuracy. We illustrate the main features of the model through the derivation of the transverse Young modulus.

The basic building block is an element in which the fiber bundle is taken to be of rectangular cross section (Fig. 11.7).[1] The volume fraction of this rectangular fiber bundle is the same as the volume fraction of the corresponding circular fiber bundle.

As a first step we replace this element with a rectangular "box" with three layers. The two outer layers consist of the matrix whereas, for now, the middle layer is taken to consist of a fictitious homogeneous material. This "box" is compared with Element 2 shown in Figure 11.2. For the present "box" the middle layer (designated by b) corresponds to the middle fiber sheet in Element 2. Thus, the transverse Young modulus of the "box" is given by Eq. (11.22) with the subscript f replaced by b

$$E_2 = \left(\frac{v_b}{E_{b2}} + \frac{1 - v_b}{E_m} \right)^{-1}. \tag{11.84}$$

[1] D. A. Hopkins and C. C. Chamis, A Unique Set of Micromechanical Equations for High Temperature Metal Matrix Composites. In: *Testing Technology of Metal Matrix Composites,* ASTM STP 964, American Society for Testing and Materials, Philadelphia, 1988, pp. 159–176.

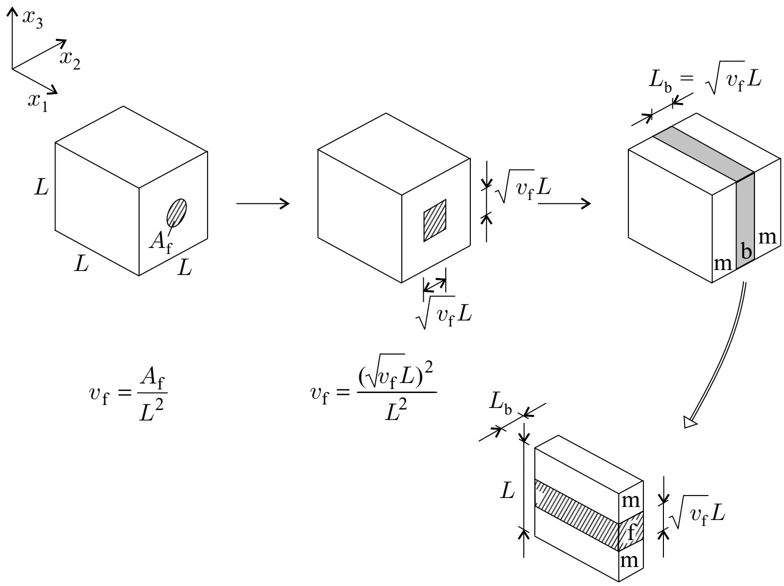

Figure 11.7: The model for the modified rule of mixtures (top) and the representation of the middle "b" layer (bottom).

The volume fraction of the middle layer v_b is determined as follows. The width of the middle layer L_b is the same as the width of the rectangular fiber bundle. Thus, v_b is

$$v_b = \frac{(L)(L_b)}{L^2} = \frac{L\sqrt{v_f}}{L} = \sqrt{v_f}, \tag{11.85}$$

where v_f is the fiber volume fraction of the entire box ($v_f = (\sqrt{v_f}L)^2/L^2$, Fig. 11.7, middle).

The Young modulus E_{b2} of the middle layer is obtained by replacing this homogenized fictional layer by a layer consisting of the rectangular fiber bundle surrounded by matrix (Fig. 11.7, bottom). The volume fraction of the fiber bundle in this middle layer is $\frac{(\sqrt{v_f}L)L_b}{LL_b} = \sqrt{v_f}$. We now apply the longitudinal rule of mixtures (Eq. 11.14) to the middle layer. The result is

$$E_{b2} = \sqrt{v_f}E_{f2} + (1 - \sqrt{v_f})\,E_m. \tag{11.86}$$

Equations (11.84) and (11.86) give the transverse Young modulus. Other properties may be obtained similarly. Details of the derivations are not given here. The results are included in Tables 11.1 and 11.2.

11.3 Note on the Micromechanics Models

The various available micromechanics approaches yield similar results, except for the rule of mixtures, which underestimates the shear moduli and the transverse properties. This is illustrated by the examples in Figure 11.8, where the transverse Young modulus and the longitudinal shear modulus are shown as calculated by

Table 11.2. Expressions for the hygrothermal properties

Thermal expansion coefficients	$\widetilde{\alpha}_1 = \frac{v_f E_{f1}}{E_1}\widetilde{\alpha}_{f1} + \frac{v_m E_m}{E_1}\widetilde{\alpha}_m$ $\widetilde{\alpha}_2 = v_f\widetilde{\alpha}_{f2} + v_m\widetilde{\alpha}_m$ $+v_f\nu_{f12}(\widetilde{\alpha}_{f1} - \widetilde{\alpha}_1) + v_m\nu_m(\widetilde{\alpha}_m - \widetilde{\alpha}_1)$
Moisture expansion coefficients ($c_f = 0$)	$\widetilde{\beta}_1 = \frac{E_m}{E_1}\widetilde{\beta}_m$ $\widetilde{\beta}_2 = \widetilde{\beta}_m + v_f\nu_{f12}(-\widetilde{\beta}_1) + \nu_m(\widetilde{\beta}_m - v_m\widetilde{\beta}_1)$
Longitudinal thermal conductivity	$K_1 = v_f K_{f1} + v_m K_m$
Transverse thermal conductivity	$K_2 = \left(\frac{\sqrt{v_f}}{K_{b2}} + \frac{1-\sqrt{v_f}}{K_m}\right)^{-1}$ where $K_{b2} = \sqrt{v_f}K_{f2} + (1-\sqrt{v_f})K_m$
Longitudinal moisture diffusivity	$D_1 = v_f D_{f1} + v_m D_m$ $D_1 = v_m D_m$ when $D_{f1} = 0$
Transverse moisture diffusivity	$D_2 = \left(\frac{\sqrt{v_f}}{D_{b2}} + \frac{1-\sqrt{v_f}}{D_m}\right)^{-1}$ where $D_{b2} = \sqrt{v_f}D_{f2} + (1-\sqrt{v_f})D_m$ $D_2 = D_m\left(\frac{1}{1-\sqrt{v_f}} - \sqrt{v_f}\right)^{-1}$ when $D_{f2} = 0$
Specific heat	$C = C_f\frac{\rho_f v_f}{\rho} + C_m\frac{\rho_m v_m}{\rho} = \frac{C_f\rho_f v_f + C_m\rho_m v_m}{\rho_f v_f + \rho_m v_m}$

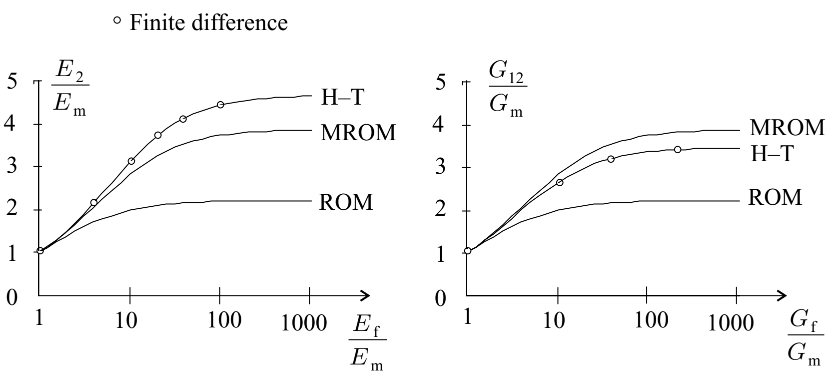

Figure 11.8: The transverse Young and shear moduli calculated by the rule of mixtures (ROM), the modified rule of mixtures (MROM), the Halpin–Tsai (H–T) equations, and the finite difference solutions (circles) of Adams and Doner ($v_f = 0.55$).

Table 11.3. The engineering constants in Example 11.1

	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}
Nominal ($\nu_f = 0.6$)	148	9.65	4.55	0.300
Back calculated ($\nu_f = 0.55$)				
Rule of mixtures	136	–	–	0.304
Modified rule of mixtures	–	9.00	4.14	–

(i) mechanics of materials models (rule of mixtures, modified rule of mixtures), (ii) the elasticity solutions of Adams and Doner,[2,3] and (iii) the Halpin–Tsai[4] semiempirical expressions.

Even when property values predicted by various micromechanics expressions are by and large similar, they should not be relied upon for design purposes because of inaccuracies introduced by approximations in the models, by uncertainties in constituents properties, and by manufacturing variances. Rather, it is recommended that the ply properties be determined by testing the ply itself.

Micromechanics expressions are useful for estimating the changes in ply properties with small changes in the volume fraction or in the properties of the fiber and the matrix. The procedure for estimating the changes in ply properties is referred to as "back calculation" and is illustrated in the following example.

11.1 Example. *The nominal engineering constants of a unidirectional graphite epoxy ply are given in Table 11.3 for a fiber volume fraction of 60 percent. The fiber volume is reduced to 55 percent. Estimate the engineering constants for this reduced fiber volume fraction.*

Solution. Generally, the matrix properties are known reasonably accurately, whereas the fiber properties are not. Therefore, the fiber properties are back calculated.

The ply's longitudinal Young modulus is (Eq. 11.14)

$$E_1 = v_f E_{f1} + v_m E_m. \tag{11.87}$$

The fiber's longitudinal Young modulus with the matrix properties in Table C.2 ($E_m = 4.1$ GPa, $G_m = 1.5$ GPa, $\nu_m = 0.35$) is

$$E_{f1} = \frac{1}{v_f}(E_1 - v_m E_m) = \frac{1}{0.6}(148 - 0.4 \times 4.1) = 244 \text{ GPa}.$$

With this value of E_{f1} the longitudinal Young modulus for $v_f = 0.55$ is (Eq. 11.87)

$$E_1^{0.55} = v_f E_{f1} + v_m E_m = 0.55 \times 244 + 0.45 \times 4.1 = 136 \text{ GPa}.$$

[2] D. F. Adams and D. R. Doner, Transverse Normal Loading of a Unidirectional Composite. *Journal of Composite Materials*, Vol. 1, 152–164, 1967.

[3] D. F. Adams and D. R. Doner, Longitudinal Shear Loading of a Unidirectional Composite. *Journal of Composite Materials*, Vol. 1, 4–17, 1967.

[4] J. C. Halpin and S. W. Tsai, *Effects of Environmental Factors on Composite Materials*. Air Force Material Laboratory, Wright–Patterson Air Force Base, Dayton, Ohio. TR–67–423, 1969.

The ply's longitudinal Poisson ratio υ_{f12} is obtained similarly by the rule of mixtures. The result is given in Table 11.3.

The ply's transverse Young modulus is (Eqs. 11.84 and 11.85)

$$E_2 = \left(\frac{\sqrt{\upsilon_f}}{E_{b2}} + \frac{1 - \sqrt{\upsilon_f}}{E_m} \right)^{-1}. \tag{11.88}$$

By rearranging this expression, we obtain ($\upsilon_f = 0.6$)

$$E_{b2} = \sqrt{\upsilon_f} \left(\frac{1}{E_2} - \frac{1 - \sqrt{\upsilon_f}}{E_m} \right)^{-1} = \sqrt{0.6} \left(\frac{1}{9.65} - \frac{1 - \sqrt{0.6}}{4.1} \right)^{-1} = 15.92 \text{ GPa}.$$

From (Eq. 11.86) we have

$$E_{b2} = \sqrt{\upsilon_f} E_{f2} + (1 - \sqrt{\upsilon_f})\, E_m. \tag{11.89}$$

From this expression the transverse fiber Young modulus is ($\upsilon_f = 0.6$)

$$\begin{aligned} E_{f2} &= \frac{1}{\sqrt{\upsilon_f}} (E_{b2} - (1 - \sqrt{\upsilon_f}) E_m) \\ &= \frac{1}{\sqrt{0.6}} (15.92 - (1 - \sqrt{0.6})(4.1)) = 19.36 \text{ GPa}. \end{aligned}$$

With this value of E_{f2}, Eq. (11.89) gives

$$E_{b2}^{0.55} = \sqrt{0.55}(19.36) + (1 - \sqrt{0.55})(4.1) = 10.73 \text{ GPa}.$$

The ply's transverse Young modulus for $\upsilon_f = 0.55$ is (Eq. 11.88)

$$E_2^{0.55} = \left(\frac{\sqrt{0.55}}{10.73} + \frac{1 - \sqrt{0.55}}{4.1} \right)^{-1} = 9.00 \text{ GPa}.$$

The shear modulus G_{12} is calculated by the modified rule of mixtures in a similar manner. The back-calculated engineering constants are tabulated in Table 11.3.