CHAPTER FIVE

Sandwich Plates

Sandwich plates, consisting of a core covered by facesheets, are frequently used instead of solid plates because of their high bending stiffness-to-weight ratio. The high bending stiffness is the result of the distance between the facesheets, which carry the load, and the light weight is due to the light weight of the core.

Here, we consider rectangular sandwich plates with facesheets on both sides of the core (Figs. 5.1 and 5.2). Each facesheet may be an isotropic material or a fiber-reinforced composite laminate but must be thin compared with the core. The core may be foam or honeycomb (Fig. 5.1) and must have a material symmetry plane parallel to its midplane; the core's in-plane stiffnesses must be small compared with the in-plane stiffnesses of the facesheets.

The behavior of thin plates undergoing small deformations may be analyzed by the Kirchhoff hypothesis, namely, by the assumptions that normals remain straight and perpendicular to the deformed reference plane. For a sandwich plate, consisting of a core covered on both sides by facesheets, the first assumption (normals remain straight) is reasonable. However, the second assumption may no longer be valid, because normals do not necessarily remain perpendicular to the reference plane (Fig. 5.3). In this case the x and y displacements of a point located at a distance z from an arbitrarily chosen reference plane are

$$u = u^{o} - z\chi_{xz} \qquad v = v^{o} - z\chi_{yz}, \tag{5.1}$$

where u^{o} and v^{o} are the x and y displacements at the reference plane (where $z = 0$) and χ_{xz}, χ_{yz} are the rotations of the normal in the x–z and y–z planes. The angle χ_{xz} is illustrated in Figure 5.3.

As shown in Figure 5.3, the first derivative of the deflection w^{o} of the reference plane with respect to x is

$$\frac{\partial w^{o}}{\partial x} = \chi_{xz} + \gamma_{xz}. \tag{5.2}$$

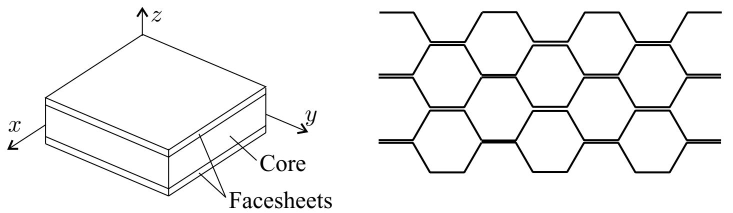

Figure 5.1: Illustration of the sandwich plate and the honeycomb core.

Similarly, the first derivative of the deflection w^o of the reference plane with respect to y is

$$\frac{\partial w^o}{\partial y} = \chi_{yz} + \gamma_{yz}. \tag{5.3}$$

5.1 Governing Equations

The strains at the reference plane are (Eq. 4.2)

$$\epsilon_x^o = \frac{\partial u^o}{\partial x} \qquad \epsilon_y^o = \frac{\partial v^o}{\partial y} \qquad \gamma_{xy}^o = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x}. \tag{5.4}$$

The transverse shear strains are (Eqs. 5.2 and 5.3)

$$\gamma_{xz} = \frac{\partial w^o}{\partial x} - \chi_{xz} \qquad \gamma_{yz} = \frac{\partial w^o}{\partial y} - \chi_{yz}. \tag{5.5}$$

For convenience we define κ_x, κ_y, and κ_{xy} as

$$\kappa_x = -\frac{\partial \chi_{xz}}{\partial x} \qquad \kappa_y = -\frac{\partial \chi_{yz}}{\partial y} \qquad \kappa_{xy} = -\frac{\partial \chi_{xz}}{\partial y} - \frac{\partial \chi_{yz}}{\partial x}. \tag{5.6}$$

We note that κ_x, κ_y, and κ_{xy} are not the curvatures of the reference plane. They are the reference plane's curvatures only in the absence of shear deformation.

The three equations above represent the strain–displacement relationships for a sandwich plate.

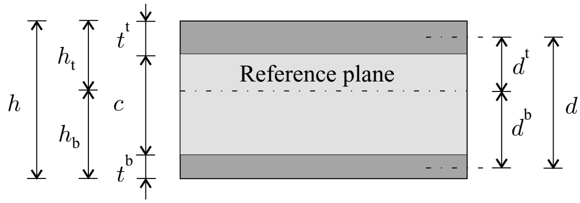

Figure 5.2: Sandwich-plate geometry.

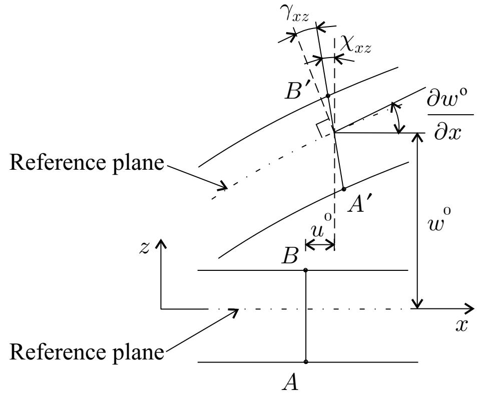

Figure 5.3: Deformation of a sandwich plate in the x–z plane.

Next we derive the force–strain relationships. The starting point of the analysis is the expressions for the forces and moments given by Eqs. (3.9) and (3.10)

$$
\begin{aligned}
N_x &= \int_{-h_b}^{h_t} \sigma_x dz \qquad N_y = \int_{-h_b}^{h_t} \sigma_y dz \qquad N_{xy} = \int_{-h_b}^{h_t} \tau_{xy} dz \\
M_x &= \int_{-h_b}^{h_t} z\sigma_x dz \qquad M_y = \int_{-h_b}^{h_t} z\sigma_y dz \qquad M_{xy} = \int_{-h_b}^{h_t} z\tau_{xy} dz
\end{aligned}
\tag{5.7}
$$

$$
V_x = \int_{-h_b}^{h_t} \tau_{xz} dz \qquad V_y = \int_{-h_b}^{h_t} \tau_{yz} dz, \tag{5.8}
$$

where N_i, M_i, and V_i are the in-plane forces, the moments, and the transverse shear forces per unit length (Fig. 3.11, page 68), respectively, and h_t and h_b are the distances from the arbitrarily chosen reference plane to the plate's surfaces (Fig. 5.2). The stresses (plane–stress condition) are (Eq. 2.126)

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}. \tag{5.9}
$$

From Eqs. (2.2), (2.3), and (2.11) together with Eq. (5.1) the strains at a distance z from the reference plane are

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u^o}{\partial x} - z\frac{\partial \chi_{xz}}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v^o}{\partial y} - z\frac{\partial \chi_{yz}}{\partial y} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} - z\left(\frac{\partial \chi_{xz}}{\partial y} + \frac{\partial \chi_{yz}}{\partial x}\right).
\end{aligned}
\tag{5.10}
$$

By combining Eqs. (5.4), (5.7), (5.9), and (5.10) and by utilizing the definitions of the [*A*], [*B*], [*D*] matrices (Eq. 3.18), we obtain

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [B] \begin{Bmatrix} -\frac{\partial \chi_{xz}}{\partial x} \\ -\frac{\partial \chi_{yz}}{\partial y} \\ -\frac{\partial \chi_{xz}}{\partial y} - \frac{\partial \chi_{yz}}{\partial x} \end{Bmatrix} \tag{5.11}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [D] \begin{Bmatrix} -\frac{\partial \chi_{xz}}{\partial x} \\ -\frac{\partial \chi_{yz}}{\partial y} \\ -\frac{\partial \chi_{xz}}{\partial y} - \frac{\partial \chi_{yz}}{\partial x} \end{Bmatrix}. \tag{5.12}$$

With the definitions in Eq. (5.6), these equations may be written as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [B] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{5.13}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + [D] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}. \tag{5.14}$$

In addition we need the relationships between the transverse shear forces and the transverse shear strains. The relevant expressions are derived in Section 5.1.3. Here we quote the resulting expression, which is

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{12} & \widetilde{S}_{22} \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{5.15}$$

where $[\widetilde{S}]$ is the sandwich plate's shear stiffness matrix.

In the analyses we may employ either the equilibrium equations or the strain energy. The equilibrium equations are identical to those given for a thin plate (Eqs. 4.4 and 4.5).

5.1.1 Boundary Conditions

In order to determine the deflection, the conditions along the four edges of the plate must be specified. An edge may be built-in, free, or simply supported. Boundary conditions for an edge parallel with the *y*-axis (Fig. 5.4) are given below.

Along a built-in edge, the deflection w^o, the in-plane displacements u^o, v^o, and the rotations of normals χ_{xz}, χ_{yz} are zero:

$$w^o = 0 \qquad u^o = v^o = 0 \qquad \chi_{xz} = \chi_{yz} = 0. \tag{5.16}$$

Along a free edge, where no external loads are applied, the bending M_x and twist M_{xy} moments, the transverse shear force V_x, and the in-plane forces N_x,

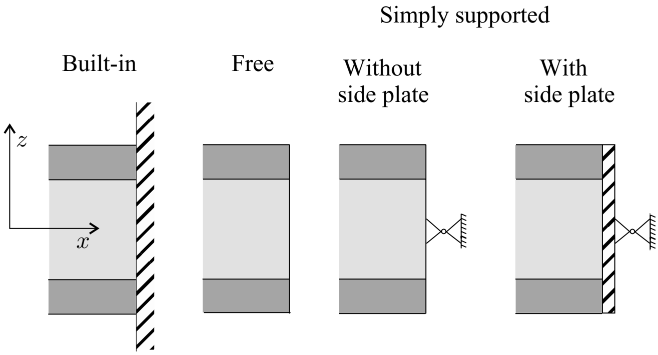

Figure 5.4: Boundary conditions for an edge parallel to the y-axis.

N_{xy} are zero:

$$M_x = M_{xy} = 0 \qquad V_x = 0 \qquad N_x = N_{xy} = 0. \tag{5.17}$$

Along a simply supported edge, the deflection w^{o}, the bending M_x and twist M_{xy} moments, and the in-plane forces N_x, N_{xy} are zero:

$$w^{\mathrm{o}} = 0 \qquad M_x = M_{xy} = 0 \qquad N_x = N_{xy} = 0. \tag{5.18}$$

When in-plane motions are prevented by the support, the in-plane forces are not zero ($N_x \neq 0$, $N_{xy} \neq 0$), whereas the in-plane displacements are zero:

$$u^{\mathrm{o}} = 0 \qquad v^{\mathrm{o}} = 0. \tag{5.19}$$

When there is a rigid plate covering the side of the sandwich plate the normal cannot rotate in the y–z plane, and we have

$$\chi_{yz} = 0. \tag{5.20}$$

However, the twist moment is not zero ($M_{xy} \neq 0$).

For an edge parallel with the x-axis, the equations above hold with x and y interchanged.

5.1.2 Strain Energy

As we noted previously, solutions to plate problems may be obtained by the equations described above or via energy methods. The strain energy (for a linearly elastic material) is given by Eq. (2.200). The thickness of the sandwich plate is assumed to remain unchanged and, accordingly, $\epsilon_z = 0$. The expression for the strain energy (Eq. 2.200) simplifies to

$$U = \frac{1}{2} \int_0^{L_x} \int_0^{L_y} \int_{-h_{\mathrm{b}}}^{h_{\mathrm{t}}} \left(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) dz dy dx. \tag{5.21}$$

Substitution of Eqs. (5.4)–(5.15) and Eqs. (5.26)–(5.32) (derived on pages 175–176) into Eq. (5.21) gives

$$U = \frac{1}{2}\int_0^{L_x}\int_0^{L_y}\left\{\begin{Bmatrix}\epsilon_x^o\\ \epsilon_y^o\\ \gamma_{xy}^o\\ \kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}^T\begin{bmatrix}A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16}\\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26}\\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66}\\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16}\\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26}\\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}\end{bmatrix}\begin{Bmatrix}\epsilon_x^o\\ \epsilon_y^o\\ \gamma_{xy}^o\\ \kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}\right.$$

$$\left. + \{\gamma_{xz} \quad \gamma_{yz}\}\begin{bmatrix}\widetilde{S}_{11} & \widetilde{S}_{12}\\ \widetilde{S}_{12} & \widetilde{S}_{22}\end{bmatrix}\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}\right\} dydx, \qquad (5.22)$$

where the superscript T denotes transpose of the vector.

5.1.3 Stiffness Matrices of Sandwich Plates

The stiffness matrices are evaluated by assuming that the thickness of the core remains constant under loading and the in-plane stiffnesses of the core are negligible. Under these assumptions the $[A]$, $[B]$, and $[D]$ stiffness matrices of a sandwich plate are governed by the stiffnesses of the facesheets and may be obtained by the parallel axes theorem (Eq. 3.47, page 80). The resulting expressions are given in Table 5.1. In this table the $[A]^t$, $[B]^t$, $[D]^t$ and $[A]^b$, $[B]^b$, $[D]^b$ are to be evaluated in a coordinate system whose origin is at each facesheet's reference plane. When the top and bottom facesheets are identical and their layup is symmetrical with respect to each facesheet's midplane, the $[B]$ matrix is zero and the $[A]$, $[D]$ matrices simplify, as shown in Table 5.1. (When the layup of each facesheet is symmetrical, the reference plane may conveniently be taken at the facesheets'

Table 5.1. The [*A*], [*B*], [*D*] stiffness matrices of sandwich plates. The supersripts t and b refer to the top and bottom facesheets. The distances d, d^t, and d^b are shown in Figure 5.2.

	Layup of each facesheet with respect to the facesheet's midplane	
	Unsymmetrical	Symmetrical (identical facesheets)
$[A]$	$[A]^t + [A]^b$	$2[A]^t$
$[B]$	$d^t[A]^t - d^b[A]^b + [B]^t + [B]^b$	0
$[D]$	$(d^t)^2[A]^t + (d^b)^2[A]^b + [D]^t + [D]^b + 2d^t[B]^t - 2d^b[B]^b$	$\frac{1}{2}d^2[A]^t + 2[D]^t$

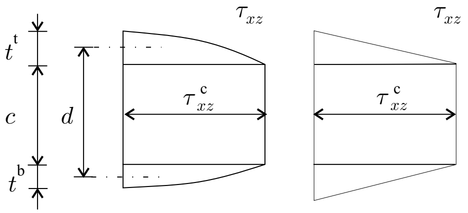

Figure 5.5: Shear stress distribution τ_{xz} (left) in a sandwich plate and the approximate distribution (right).

midplane.) When the top and bottom facesheets are unsymmetrical with respect to the facesheets' midplane but are symmetrical with respect to the midplane of the sandwich plate, then $[A]^{\mathrm{t}} = [A]^{\mathrm{b}}$, $[B]^{\mathrm{t}} = -[B]^{\mathrm{b}}$, $[D]^{\mathrm{t}} = [D]^{\mathrm{b}}$, and the $[A]$, $[B]$, $[D]$ matrices of the sandwich plate become

$$[A] = 2\,[A]^{\mathrm{t}} \tag{5.23}$$

$$[B] = 0 \tag{5.24}$$

$$[D] = \frac{1}{2} d^2\,[A]^{\mathrm{t}} + 2\,[D]^{\mathrm{t}} + 2d\,[B]^{\mathrm{t}}\,. \tag{5.25}$$

The shear stiffness matrix $[\widetilde{S}]$ is determined as follows. In the core, as a consequence of the assumption that the in-plane stiffnesses are negligible, the transverse shear stress τ_{xz} is uniform. In general, in the facesheets the shear stress distribution is as shown in Figure 5.5 (left). We approximate this distribution by the linear shear stress distribution shown in Figure 5.5 (right). Accordingly, the transverse shear force V_x is

$$V_x = \int_{-h_{\mathrm{b}}}^{h_{\mathrm{t}}} \tau_{xz} dz = \tau_{xz}^{\mathrm{c}} c + \tau_{xz}^{\mathrm{c}} \frac{t^{\mathrm{t}}}{2} + \tau_{xz}^{\mathrm{c}} \frac{t^{\mathrm{b}}}{2} = \tau_{xz}^{\mathrm{c}} d, \tag{5.26}$$

where the superscripts c, t, and b refer to the core, the top, and the bottom facesheets, respectively. The distance $d = c + t^{\mathrm{t}}/2 + t^{\mathrm{b}}/2$ is shown in Figure 5.5.

Similarly, we have

$$V_y = \tau_{yz}^{\mathrm{c}} d. \tag{5.27}$$

The stress–strain relationship for the core material is given by Eqs. (2.20) and (2.27). With the superscript c identifying the core, these equations give

$$\begin{Bmatrix} \tau_{xz}^{\mathrm{c}} \\ \tau_{yz}^{\mathrm{c}} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{55}^{\mathrm{c}} & \overline{C}_{45}^{\mathrm{c}} \\ \overline{C}_{45}^{\mathrm{c}} & \overline{C}_{44}^{\mathrm{c}} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^{\mathrm{c}} \\ \gamma_{yz}^{\mathrm{c}} \end{Bmatrix}, \tag{5.28}$$

where $\overline{C}_{ij}^{\mathrm{c}}$ are the elements of the core stiffnesses matrix.

We neglect the shear deformation of the thin facesheets. With this approximation the shear deformation γ_{xz}^{c} of the cross section is as shown in Figure 5.6 (left). We approximate this deformation by the average shear deformation γ_{xz} shown in

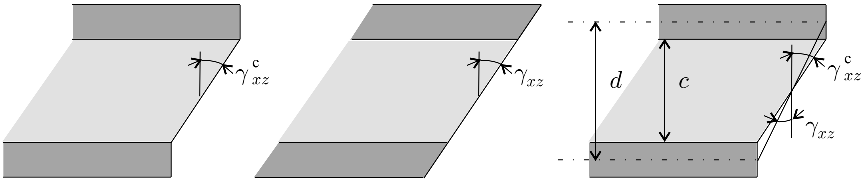

Figure 5.6: Shear deformation of a sandwich plate.

Figure 5.6 (middle). The relationship between this average shear deformation and the core deformation is given by (see Fig. 5.6, right)

$$\gamma_{xz}^{c} = \frac{d}{c}\gamma_{xz}. \tag{5.29}$$

Similarly, we have

$$\gamma_{yz}^{c} = \frac{d}{c}\gamma_{yz}. \tag{5.30}$$

Equations (5.26)–(5.30) yield the relationship between the transverse shear forces and the average shear deformation:

$$\begin{Bmatrix} V_x \\ V_y \end{Bmatrix} = \frac{d^2}{c}\begin{bmatrix} \overline{C}_{55}^{c} & \overline{C}_{45}^{c} \\ \overline{C}_{45}^{c} & \overline{C}_{44}^{c} \end{bmatrix}\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}. \tag{5.31}$$

By comparing this equation with Eq. (5.15), we obtain

$$\begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{12} & \widetilde{S}_{22} \end{bmatrix} = \frac{d^2}{c}\begin{bmatrix} \overline{C}_{55}^{c} & \overline{C}_{45}^{c} \\ \overline{C}_{45}^{c} & \overline{C}_{44}^{c} \end{bmatrix}. \tag{5.32}$$

The preceding four elements of the matrix $[\overline{C}^{c}]$ characterize the core material, whereas $[\widetilde{S}]$ is the shear stiffness matrix of the sandwich plate. We point out that $[\widetilde{S}]$ is *not* the inverse of the $[\overline{C}]$ matrix.

Orthotropic sandwich plate. A sandwich plate is orthotropic when both facesheets as well as the core are orthotropic and the orthotropy directions are parallel to the edges. The facesheets may be different, and their layups may be unsymmetrical. For such an orthotropic sandwich plate there are no extension–shear, bending–twist, and extension–twist couplings. Accordingly, the following elements of the stiffness matrices are zero:

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0. \tag{5.33}$$

Furthermore, for an orthotropic sandwich plate the transverse shear force V_x acting in the x–z plane does not cause a shear strain γ_{yz} in the y–z plane. This condition gives

$$\widetilde{S}_{12} = 0. \tag{5.34}$$

Isotropic sandwich plate. A sandwich plate is isotropic when the core of the sandwich plate is made of an isotropic (such as foam) or transversely isotropic (such as honeycomb) material and the top and bottom facesheets are made of

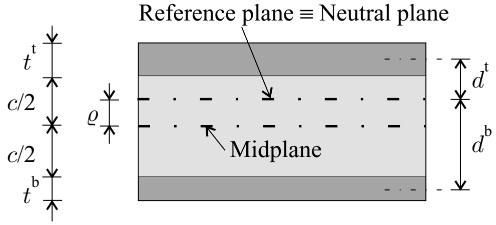

Figure 5.7: Neutral plane of an isotropic sandwich plate.

identical isotropic materials or are identical quasi-isotropic laminates. The thicknesses of the top and bottom facesheets may be different.

For isotropic facesheets the $[B]$ matrix is zero ($[B]^i = 0$). The $[A]$ and $[D]$ matrices for the isotropic facesheets are (Eqs. 3.41 and 3.42)

$$[A]^i = \frac{t^i E^{\rm f}}{1-(\nu^{\rm f})^2}\begin{bmatrix} 1 & \nu^{\rm f} & 0 \\ \nu^{\rm f} & 1 & 0 \\ 0 & 0 & \frac{1-\nu^{\rm f}}{2} \end{bmatrix} \qquad [D]^i = \frac{(t^i)^3 E^{\rm f}}{12(1-(\nu^{\rm f})^2)}\begin{bmatrix} 1 & \nu^{\rm f} & 0 \\ \nu^{\rm f} & 1 & 0 \\ 0 & 0 & \frac{1-\nu^{\rm f}}{2} \end{bmatrix}, \tag{5.35}$$

where the superscript i refers to the top ($i = {\rm t}$) or to the bottom ($i = {\rm b}$) facesheet (Fig. 5.7) and $E^{\rm f}$ and $\nu^{\rm f}$ are the Young modulus and the Poisson ratio of the facesheets.

We now proceed to evaluate the $[A]$, $[B]$, $[D]$ matrices for the entire sandwich plate. To this end, we choose a reference plane located at the center of gravity of the two facesheets. The distance ϱ from the midplane of the core to the center of gravity is (Fig. 5.7)

$$\varrho = \frac{t^{\rm t}(c+t^{\rm t}) - t^{\rm b}(c+t^{\rm b})}{2(t^{\rm t}+t^{\rm b})}. \tag{5.36}$$

The distances $d^{\rm t}$ and $d^{\rm b}$ between the reference plane (passing through the center of gravity) and the midplanes of the facesheets are

$$d^{\rm t} = \frac{c}{2} + \frac{t^{\rm t}}{2} - \varrho \qquad d^{\rm b} = \frac{c}{2} + \frac{t^{\rm b}}{2} + \varrho\,. \tag{5.37}$$

By substituting Eqs. (5.35)–(5.37) into the expression for the $[B]$ matrix given in Table 5.1 (page 174) we obtain that for the entire sandwich plate the $[B]$ matrix is zero with reference to the ϱ reference plane. This means that for a sandwich plate with isotropic core and isotropic facesheets bending does not cause strains in this plane. Therefore, this reference plane is a "neutral plane."

By substituting the expressions of $d^{\rm t}$ and $d^{\rm b}$ (Eq. 5.37) into the expressions given in Table 5.1, we obtain the following $[A]$ and $[D]$ matrices for the sandwich

Table 5.2. The stiffnesses and the Poisson ratios of isotropic solid plates and isotropic sandwich plates; R is defined in Eq. (3.46).

		Isotropic sandwich plate	
	Isotropic solid plate	**Isotropic facesheets**	**Quasi-isotropic facesheets**
A^{iso}	$\frac{Eh}{1-\nu^2}$	$(t^{\text{t}}+t^{\text{b}})\frac{E^{\text{f}}}{1-(\nu^{\text{f}})^2}$	$(t^{\text{t}}+t^{\text{b}})R$
D^{iso}	$\frac{Eh^3}{12(1-\nu^2)}$	$\frac{(d^{\text{t}})^2t^{\text{t}}+(d^{\text{b}})^2t^{\text{b}}+\frac{(t^{\text{t}})^3+(t^{\text{b}})^3}{12}}{1-(\nu^{\text{f}})^2}E^{\text{f}}$	$\left[t^{\text{t}}(d^{\text{t}})^2+t^{\text{b}}(d^{\text{b}})^2\right]R$
ν^{iso}	ν	ν^{f}	$\frac{Q_{11}+Q_{22}+6Q_{12}-4Q_{66}}{8R}$

plate:

$$[A]=A^{\text{iso}}\begin{bmatrix}1 & \nu^{\text{f}} & \\ \nu^{\text{f}} & 1 & \\ & & \frac{1-\nu^{\text{f}}}{2}\end{bmatrix}\qquad [D]=D^{\text{iso}}\begin{bmatrix}1 & \nu^{\text{f}} & \\ \nu^{\text{f}} & 1 & \\ & & \frac{1-\nu^{\text{f}}}{2}\end{bmatrix},\tag{5.38}$$

where A^{iso} and D^{iso} are defined in Table 5.2.

When the core is isotropic in the plane parallel to the facesheets from Eq. (2.40) we have $C_{45}=0$, $C_{44}=(C_{11}-C_{12})/2$, and the shear stiffnesses are (Eq. 5.32)

$$\widetilde{S}_{11}=\widetilde{S}_{22}=\widetilde{S}=\frac{d^2}{c}\frac{C_{11}^{\text{c}}-C_{12}^{\text{c}}}{2}\qquad \widetilde{S}_{12}=0.\tag{5.39}$$

The sandwich plate may also be treated as isotropic when the top and bottom facesheets are quasi-isotropic laminates (page 79) consisting of unidirectional plies made of the same material. For such sandwich plates the $[B]$ matrix is negligible, the $[A]$ and $[D]$ matrices are approximated by Eq. (5.38) (with the terms A^{iso} and D^{iso} defined in Table 5.2), and the elements of the shear stiffness matrix are given by Eq. (5.39).

5.2 Deflection of Rectangular Sandwich Plates

5.2.1 Long Plates

We consider a long rectangular sandwich plate whose length is large compared with its width ($L_y \gg L_x$). The long edges may be built-in, simply supported, or free, as shown in Figure 5.8. The sandwich plate is subjected to a transverse load p (per unit area). This load, as well as the edge supports, does not vary along the longitudinal y direction.

The deflected surface of the sandwich plate may be assumed to be cylindrical at a considerable distance from the short ends (Fig. 4.4). The generator of this cylindrical surface is parallel to the longitudinal y-axis of the plate, and hence the

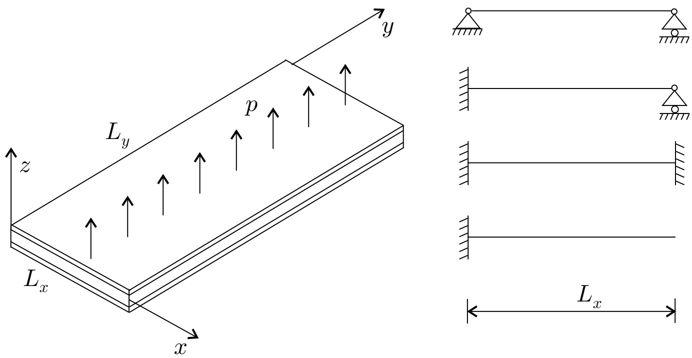

Figure 5.8: The different types of supports along the long edges of a long sandwich plate.

deflection of the plate w^{o} and the rotation χ_{xz} do not vary along y:

$$\frac{\partial w^{\text{o}}}{\partial y} = 0 \qquad \frac{\partial \chi_{xz}}{\partial y} = 0\,. \tag{5.40}$$

We neglect the shear deformation in the y–z plane ($\gamma_{yz} = 0$). Consequently, the rotation of the normal is zero (Eq. 5.3):

$$\chi_{yz} = 0. \tag{5.41}$$

The equilibrium equations are (Eqs. 4.22 and 4.23)

$$\frac{dV_x}{dx} + p = 0 \tag{5.42}$$

$$\frac{dM_x}{dx} - V_x = 0. \tag{5.43}$$

When the sandwich plate is symmetrical with respect to the midplane ($[B] = 0$) from Eqs. (5.12), (5.15), (5.40), and (5.41), we have

$$M_x = -D_{11}\frac{\partial \chi_{xz}}{\partial x} \qquad V_x = \widetilde{S}_{11}\gamma_{xz}\,. \tag{5.44}$$

Equations (5.42), (5.43), and (5.44), together with Eq. (5.2), give
sandwich plate, symmetrical layup:

$$-D_{11}\frac{d^3\chi_{xz}}{dx^3} + p = 0 \tag{5.45}$$

$$D_{11}\frac{d^2\chi_{xz}}{dx^2} + \widetilde{S}_{11}\left(\frac{dw^{\text{o}}}{dx} - \chi_{xz}\right) = 0. \tag{5.46}$$

For a transversely loaded isotropic sandwich beam the corresponding equations are (Eqs. 7.83 and 7.84)
isotropic sandwich beam:

$$-\widehat{EI}\frac{d^3\chi}{dx^3} + p' = 0 \tag{5.47}$$

$$\widehat{EI}\frac{d^2\chi}{dx^2} + \widehat{S}\left(\frac{dw}{dx} - \chi\right) = 0, \tag{5.48}$$

where $\widehat{EI}$ and $\widehat{S}$ are the bending and shear stiffnesses of the isotropic sandwich beam, respectively, and p' is the load per unit length.

The equations describing the deflections of long sandwich plates and isotropic sandwich beams are identical when in Eqs. (5.45) and (5.46), D_{11}, $\widetilde{S}_{11}$, and p are replaced, respectively, by $\widehat{EI}$, $\widehat{S}$, and p'. Therefore, the deflection of a long sandwich plate (symmetrical layup) may be obtained by substituting the values of D_{11}, $\widetilde{S}_{11}$, and p for $\widehat{EI}$, $\widehat{S}$, and p' in the expression, given in Section 7.3, for the deflection of the corresponding isotropic beam.

When the layup is unsymmetrical, the expression for the moment M_x can be derived analogously to the equation of a solid composite plate (Section 4.2.2). Here we only quote the result, which for sandwich plates is

$$M_x = -\underbrace{\left(D_{11}^{\varrho} - \frac{B_{16}^{\varrho 2}}{A_{66}^{\varrho}}\right)}_{\Psi} \frac{\partial \chi_{xz}}{\partial x}, \tag{5.49}$$

where χ_{xz} is shown in Figure 5.3. The term in parentheses is the bending stiffness parameter defined by Eq. (4.52). Equations (5.42), (5.43), (5.44, right), and (5.49), together with Eq. (5.2), give

sandwich plate, unsymmetrical layup:

$$-\Psi \frac{d^3 \chi_{xz}}{dx^3} 0 + p = 0 \tag{5.50}$$

$$\Psi \frac{d^2 \chi_{xz}}{dx^2} + \widetilde{S}_{11}\left(\frac{dw^{\mathrm{o}}}{dx} - \chi_{xz}\right) = 0. \tag{5.51}$$

The preceding equations describing deflections of sandwich plates (unsymmetrical layup) become identical to the equations of sandwich beams (Eqs. 5.47 and 5.48) when Ψ, $\widetilde{S}_{11}$, and p are replaced, respectively, by $\widehat{EI}$, $\widehat{S}$, and p'. Therefore, the deflection of a long sandwich plate (unsymmetrical layup) may be obtained by substituting the values of Ψ, $\widetilde{S}_{11}$, and p for $\widehat{EI}$, $\widehat{S}$, and p' in the expression for the deflection of the corresponding isotropic beam.

5.1 Example. *A* 0.9*-m-long and* 0.2*-m-wide rectangular sandwich plate is made of a* 0.02*-m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is* $[\pm 45_2^{\mathrm{f}}/0_{12}/\pm 45_2^{\mathrm{f}}]$*, and the thickness of each facesheet is* 0.002 *m (Fig. 5.9). The* 0*-degree plies are parallel to the short edge of the plate. The plate is either simply supported or built-in along all four edges (Fig. 5.10). The plate is subjected to a*

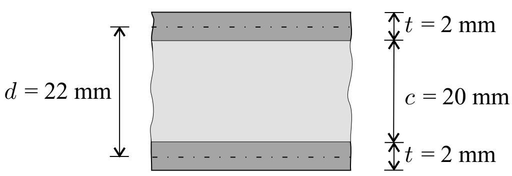

Figure 5.9: The cross section of the sandwich plate in Example 5.1.

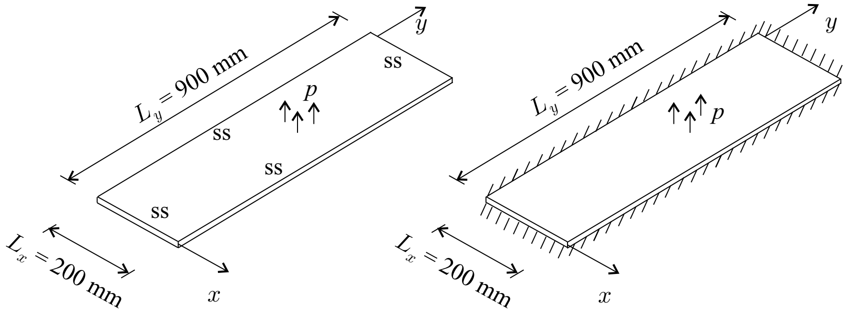

Figure 5.10: The sandwich plates in Example 5.1.

uniformly distributed transverse load 500 *kN/m*^{2}. *Calculate the maximum deflection. The core is isotropic (*$E_c = 2 \times 10^6$ *kN/m*^{2}, $\nu_c = 0.3$*).*

Solution. The tensile and bending stiffnesses are calculated from Table 5.1 (page 174) as follows:

$$[A] = 2\,[A]^t = \begin{bmatrix} 430.34 & 65.47 & 0 \\ 65.47 & 96.34 & 0 \\ 0 & 0 & 72.02 \end{bmatrix} 10^3 \frac{\text{kN}}{\text{m}} \tag{5.52}$$

$$[D] = \frac{1}{2} d^2\,[A]^t + 2\,[D]^t = \begin{bmatrix} 52.16 & 7.96 & 0 \\ 7.96 & 11.71 & 0 \\ 0 & 0 & 8.76 \end{bmatrix} \text{kN} \cdot \text{m}, \tag{5.53}$$

where $[A]^t$ and $[D]^t$ are given in Table 3.7 (page 84) and $d = c + t = 0.022$ m. The shear stiffness matrix is (Eq. 5.32)

$$\begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{12} & \widetilde{S}_{22} \end{bmatrix} = \frac{d^2}{c} \begin{bmatrix} \overline{C}^c_{55} & \overline{C}^c_{45} \\ \overline{C}^c_{45} & \overline{C}^c_{44} \end{bmatrix} = \begin{bmatrix} 18\,615 & 0 \\ 0 & 18\,615 \end{bmatrix} \frac{\text{kN}}{\text{m}}, \tag{5.54}$$

where (see Eq. 2.30 and Table 2.10, page 18) $\overline{C}^c_{55} = \overline{C}^c_{44} = \frac{E_c}{2(1+\nu_c)} = 769\,231$ kN/m^2, $\overline{C}^c_{45} = 0$.

We may treat this plate as long when (Eq. 4.19)

$$\frac{L_y}{L_x} > 3\sqrt[4]{\frac{D_{11}}{D_{22}}}. \tag{5.55}$$

In the present problem, $L_y/L_x = 4.5$ and $3\sqrt[4]{D_{11}/D_{22}} = 4.36$. Thus, the preceding condition is satisfied and the long plate expressions may be used. The maximum deflections of the corresponding beam are (Table 7.3, page 332)

$$\widetilde{w} = \frac{5}{384} \frac{p' L^4}{EI} + \frac{p' L^2}{8\widehat{S}} \qquad \text{(ss)} \tag{5.56}$$

$$\widetilde{w} = \frac{1}{384} \frac{p' L^4}{EI} + \frac{p' L^2}{8\widehat{S}} \qquad \text{(built-in).} \tag{5.57}$$

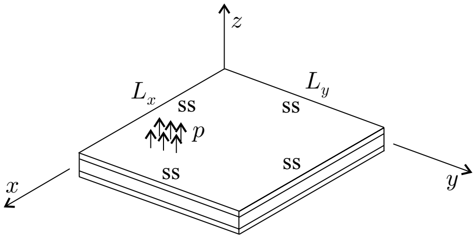

Figure 5.11: Rectangular simply supported (ss) sandwich plate subjected to transverse load.

The maximum deflections of the plate are obtained by replacing EI, $\widehat{S}$, p' by D_{11}, $\widetilde{S}_{11}$, p (see page 180)

$$\widetilde{w} = \frac{5}{384}\frac{pL_x^4}{D_{11}} + \frac{pL_x^2}{8\widetilde{S}_{11}} \qquad \text{(ss)} \tag{5.58}$$

$$\widetilde{w} = \frac{1}{384}\frac{pL_x^4}{D_{11}} + \frac{pL_x^2}{8\widetilde{S}_{11}} \qquad \text{(built-in).} \tag{5.59}$$

With the values of $D_{11} = 52.16\,\text{kN·m}$ and $\widetilde{S}_{11} = 18\,615\frac{\text{kN}}{\text{m}}$, and with $L_x = 0.2$ m, the maximum deflections are

$$\widetilde{w} = 0.000\,200 + 0.000\,134 = 0.000\,334\text{ m} = 0.334\text{ mm} \qquad \text{(ss)} \tag{5.60}$$

$$\widetilde{w} = 0.000\,040 + 0.000\,134 = 0.000\,174\text{ m} = 0.174\text{ mm} \qquad \text{(built-in).} \tag{5.61}$$

5.2.2 Simply Supported Sandwich Plates – Orthotropic and Symmetrical Layup

A simply supported rectangular sandwich plate with dimensions L_x and L_y is subjected to a uniformly distributed load p (Fig. 5.11). The layup of the plate is orthotropic (page 176) and symmetrical with respect to the plate's midplane.

For a simply supported symmetrical plate subjected to out-of-plane loads only, the in-plane strains in the midplane are zero (see Eq. 3.31) as follows:

$$\epsilon_x^o = 0 \qquad \epsilon_y^o = 0 \qquad \gamma_{xy}^o = 0. \tag{5.62}$$

Substitution of Eq. (5.62) into the expression of the strain energy (Eq. 5.22) gives

$$U = \frac{1}{2}\int_0^{L_x}\int_0^{L_y}\left\{\{\kappa_x\kappa_y\kappa_{xy}\}[D]\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix} + \{\gamma_{xz}\gamma_{yz}\}\begin{bmatrix}\widetilde{S}_{11} & \widetilde{S}_{12}\\ \widetilde{S}_{12} & \widetilde{S}_{22}\end{bmatrix}\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}\right\}dydx. \tag{5.63}$$

For orthotropic sandwich plates $D_{16} = D_{26} = \widetilde{S}_{12} = 0$ (Eqs. 5.33 and 5.34). With these values and the expressions in Eqs. (5.5) and (5.6), the strain energy becomes

$$U = \frac{1}{2}\int_0^{L_x}\int_0^{L_y}\Bigg[\left(\frac{\partial \chi_{xz}}{\partial x}\right)^2 D_{11} + \left(\frac{\partial \chi_{yz}}{\partial y}\right)^2 D_{22} + 2\frac{\partial \chi_{xz}}{\partial x}\frac{\partial \chi_{yz}}{\partial y}D_{12}$$
$$+\left(\frac{\partial \chi_{xz}}{\partial y} + \frac{\partial \chi_{yz}}{\partial x}\right)^2 D_{66} + \left(\frac{\partial w^{o}}{\partial x} - \chi_{xz}\right)^2 \widetilde{S}_{11} + \left(\frac{\partial w^{o}}{\partial y} - \chi_{yz}\right)^2 \widetilde{S}_{22}\Bigg] dydx. \tag{5.64}$$

For an applied transverse load p (per unit area), the potential of the external forces is (Eq. 4.56)

$$\Omega = -\int_0^{L_x}\int_0^{L_y} (pw^{o})\, dxdy. \tag{5.65}$$

For a simply supported sandwich plate the deflection, bending moments, and rotations of the normals along the edges are zero, resulting in the following boundary conditions:

$$w^{o} = 0 \quad \text{at} \quad \begin{cases} x = 0 & \text{and} \quad 0 \le y \le L_y \\ x = L_x & \text{and} \quad 0 \le y \le L_y \\ 0 \le x \le L_x & \text{and} \quad y = 0 \\ 0 \le x \le L_x & \text{and} \quad y = L_y \end{cases} \tag{5.66}$$

$$M_x = 0 \quad \text{at} \quad \begin{cases} x = 0 & \text{and} \quad 0 \le y \le L_y \\ x = L_x & \text{and} \quad 0 \le y \le L_y \end{cases} \tag{5.67}$$

$$M_y = 0 \quad \text{at} \quad \begin{cases} 0 \le x \le L_x & \text{and} \quad y = 0 \\ 0 \le x \le L_x & \text{and} \quad y = L_y \end{cases} \tag{5.68}$$

$$\chi_{yz} = 0 \quad \text{at} \quad \begin{cases} x = 0 & \text{and} \quad 0 \le y \le L_y \\ x = L_x & \text{and} \quad 0 \le y \le L_y \end{cases} \tag{5.69}$$

$$\chi_{xz} = 0 \quad \text{at} \quad \begin{cases} 0 \le x \le L_x & \text{and} \quad y = 0 \\ 0 \le x \le L_x & \text{and} \quad y = L_y\ . \end{cases} \tag{5.70}$$

The following deflection and rotations satisfy these conditions:

$$\begin{aligned} w^{o} &= \sum_{i=1}^{I}\sum_{j=1}^{J} w_{ij} \sin\frac{i\pi x}{L_x} \sin\frac{j\pi y}{L_y} \\ \chi_{xz} &= \sum_{i=1}^{I}\sum_{j=1}^{J} (\chi_{xz})_{ij} \cos\frac{i\pi x}{L_x} \sin\frac{j\pi y}{L_y} \\ \chi_{yz} &= \sum_{i=1}^{I}\sum_{j=1}^{J} (\chi_{yz})_{ij} \sin\frac{i\pi x}{L_x} \cos\frac{j\pi y}{L_y}, \end{aligned} \tag{5.71}$$

Table 5.3. Elements of the coefficient matrix in Eq. (5.73)

$$F_{33} = D_{11}\left(\frac{i\pi}{L_x}\right)^4 + 2\left(D_{12}+2D_{66}\right)\left(\frac{i\pi}{L_x}\right)^2\left(\frac{j\pi}{L_y}\right)^2 + D_{22}\left(\frac{j\pi}{L_y}\right)^4$$

$$F_{34} = -D_{11}\left(\frac{i\pi}{L_x}\right)^3 - \left(D_{12}+2D_{66}\right)\frac{i\pi}{L_x}\left(\frac{j\pi}{L_y}\right)^2$$

$$F_{35} = -D_{22}\left(\frac{j\pi}{L_y}\right)^3 - \left(D_{12}+2D_{66}\right)\left(\frac{i\pi}{L_x}\right)^2\frac{j\pi}{L_y}$$

$$F_{44} = D_{11}\left(\frac{i\pi}{L_x}\right)^2 + D_{66}\left(\frac{j\pi}{L_y}\right)^2 + \widetilde{S}_{11}$$

$$F_{45} = \left(D_{12}+2D_{66}\right)\left(\frac{i\pi}{L_x}\right)\left(\frac{j\pi}{L_y}\right)$$

$$F_{55} = D_{22}\left(\frac{j\pi}{L_y}\right)^2 + D_{66}\left(\frac{i\pi}{L_x}\right)^2 + \widetilde{S}_{22}$$

where I and J are the number of terms, chosen arbitrarily, for the summations and w_{ij}, $(\chi_{xz})_{ij}$, and $(\chi_{yz})_{ij}$ are unknowns and are evaluated by the principle of stationary potential energy expressed as

$$\begin{aligned} \frac{\partial\left(U+\Omega\right)}{\partial\left(\chi_{xz}\right)_{ij}} &= 0 \\ \frac{\partial\left(U+\Omega\right)}{\partial\left(\chi_{yz}\right)_{ij}} &= 0 \\ \frac{\partial\left(U+\Omega\right)}{\partial w_{ij}} &= 0. \end{aligned} \tag{5.72}$$

We substitute w^{o}, χ_{xz}, χ_{yz} (from Eq. 5.71) into the expressions of U (Eq. 5.64) and Ω (Eq. 5.65) and perform the differentiations indicated above. Algebraic manipulations yield the following system of simultaneous algebraic equations:

$$\frac{L_x L_y}{4}\begin{bmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{bmatrix}\begin{Bmatrix} w_{ij} \\ (\gamma_{xz})_{ij} \\ (\gamma_{yz})_{ij} \end{Bmatrix} = \begin{Bmatrix} \frac{4pL_xL_y}{\pi^2 ij} \\ 0 \\ 0 \end{Bmatrix}, \tag{5.73}$$

where we have $i, j = 1, 3, 5, \ldots (w_{ij} = (\gamma_{xz})_{ij} = (\gamma_{yz})_{ij} = 0$ when i or $j = 2, 4, 6. \ldots)$. The elements of the coefficient matrix are given in Table 5.3 and $(\gamma_{xz})_{ij}$ and $(\gamma_{yz})_{ij}$ are defined as

$$(\gamma_{xz})_{ij} = \frac{i\pi}{L_x}w_{ij} - (\chi_{xz})_{ij} \qquad (\gamma_{yz})_{ij} = \frac{j\pi}{L_y}w_{ij} - (\chi_{yz})_{ij}\,. \tag{5.74}$$

For each set of i, j values the three equations in Eq. (5.73) are solved simultaneously for the three unknowns w_{ij}, $(\gamma_{xz})_{ij}$, $(\gamma_{yz})_{ij}$. The deflection and the rotations are then calculated by Eqs. (5.71) and (5.74).

5.3 Buckling of Rectangular Sandwich Plates

5.3.1 Long Plates

We consider a long rectangular sandwich plate whose length L_y is large compared with its width L_x. The edges may be built-in, simply supported, or free, as shown in Figure 5.12. A uniform compressive force N_{x0} is applied along one of the long edges of the plate. This force, as well as the edge supports, does not vary along the longitudinal y direction. We are interested in the load at which the plate buckles.

The deflected surface of the plate may be assumed to be cylindrical at a considerable distance from the short edges (Fig. 4.4). The equilibrium equations are given by Eqs. (4.160) and (4.161) and for convenience are repeated here as follows:

$$\frac{dV_x}{dx} - N_{x0}\frac{d^2 w^{\text{o}}}{dx^2} = 0 \tag{5.75}$$

$$\frac{dM_x}{dx} - V_x = 0. \tag{5.76}$$

We now consider a sandwich plate that is symmetrical with respect to the midplane. For this plate the bending moment and the transverse shear force are (Eq. 5.44)

$$M_x = -D_{11}\frac{\partial \chi_{xz}}{\partial x} \qquad V_x = \widetilde{S}_{11}\gamma_{xz}. \tag{5.77}$$

Equations (5.75), (5.76), (5.77), together with Eq. (5.2), give
sandwich plate, symmetrical layup:

$$-D_{11}\frac{d^3\chi}{dx^3} - N_{x0}\frac{d^2 w^{\text{o}}}{dx^2} = 0 \tag{5.78}$$

$$D_{11}\frac{d^2\chi}{dx^2} + \widetilde{S}_{11}\left(\frac{dw^{\text{o}}}{dx} - \chi\right) = 0. \tag{5.79}$$

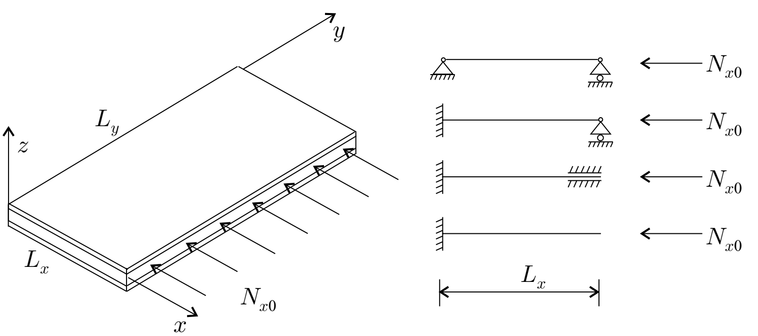

Figure 5.12: Long rectangular sandwich plate subjected to a uniform compressive edge load and the different types of supports along the long edges.

For an isotropic sandwich beam the corresponding equations are (Eqs. 7.113 and 7.114)
isotropic sandwich beam:

$$-\widehat{EI}\frac{d^3\chi}{dx^3} - \widehat{N}_{x0}\frac{d^2w}{dx^2} = 0 \tag{5.80}$$

$$\widehat{EI}\frac{d^2\chi}{dx^2} + \widehat{S}\left(\frac{dw}{dx} - \chi\right) = 0, \tag{5.81}$$

where $\widehat{EI}$ and $\widehat{S}$ are the bending and shear stiffnesses of the sandwich beam and $\widehat{N}_{x0}$ is the compressive load (per unit length).

The equations describing buckling of long sandwich plates (symmetrical layup) and isotropic sandwich beams are identical when in Eqs. (5.78) and (5.79) D_{11}, $\widetilde{S}_{11}$, and N_{x0} are replaced, respectively, by $\widehat{EI}$, $\widehat{S}$, and $\widehat{N}_{x0}$. Therefore, the buckling load (per unit length) of a long sandwich plate (symmetrical layup) may be obtained by substituting the values of D_{11} and $\widetilde{S}_{11}$ for $\widehat{EI}$ and $\widehat{S}$ in the expression for the buckling load of the corresponding isotropic sandwich beam (Section 7.4).

It was shown in Section 5.2.1 (page 180) that when the layup of the sandwich plate is unsymmetrical the deflection may be obtained by substituting the values of Ψ, $\widetilde{S}_{11}$, and p for $\widehat{EI}$, $\widehat{S}$, and p' in the expression for the deflection of the corresponding isotropic sandwich beam. Similarly, the buckling load of a long unsymmetrical sandwich plate may be obtained by substituting the values of Ψ and $\widetilde{S}_{11}$ for $\widehat{EI}$ and $\widehat{S}$ in the expression for the buckling load of the corresponding isotropic beam (where Ψ is given by Eq. 4.52).

5.2 Example. *A* 0.9-*m-long and* 0.2-*m-wide rectangular sandwich plate is made of a* 0.02-*m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is* $[\pm 45^{\mathrm{f}}_2/0_{12}/\pm 45^{\mathrm{f}}_2]$, *and the thickness of each facesheet is* 0.002 *m (Fig. 5.9). The* 0-*degree plies are parallel to the short edge of the plate. The plate is either simply supported or built-in along all four edges (Fig. 5.13). The plate is subjected to uniform compressive loads along the long edges. Calculate the buckling load. The core is isotropic* ($E_c = 2 \times 10^6$ *kN/m*^{2}, $\nu_c = 0.3$).

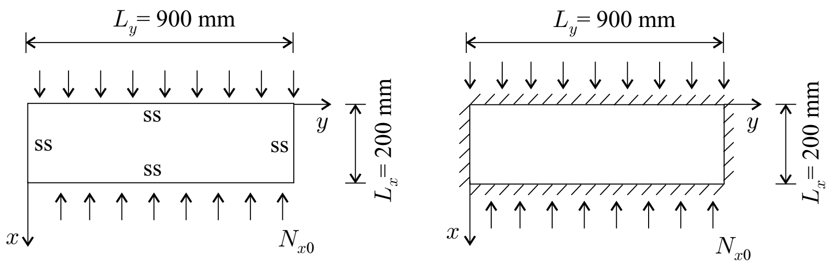

Figure 5.13: The sandwich plates in Example 5.2.

Solution. The plate may be treated as "long" (Example 5.1, page 180). The buckling loads of the corresponding beam are (Eqs. 7.175 and 6.337)

$$\widehat{N}_{\rm cr} = \left(\frac{L^2}{\pi^2 EI} + \frac{1}{\widehat{S}}\right)^{-1} \qquad \text{(ss)} \tag{5.82}$$

$$\widehat{N}_{\rm cr} = \left(\frac{4L^2}{\pi^2 EI} + \frac{1}{\widehat{S}}\right)^{-1} \qquad \text{(built-in).} \tag{5.83}$$

The buckling loads of the plate are obtained by replacing EI, $\widehat{S}$ by D_{11}, $\widetilde{S}_{11}$ (see page 186) as follows:

$$N_{x,\,\rm cr} = \left(\frac{L_x^2}{\pi^2 D_{11}} + \frac{1}{\widetilde{S}_{11}}\right)^{-1} \qquad \text{(ss)} \tag{5.84}$$

$$N_{x,\,\rm cr} = \left(\frac{4L_x^2}{\pi^2 D_{11}} + \frac{1}{\widetilde{S}_{11}}\right)^{-1} \qquad \text{(built-in).} \tag{5.85}$$

With the values of $D_{11} = 52.16\,\text{kN·m}$ and $\widetilde{S}_{11} = 18\,615\,\frac{\text{kN}}{\text{m}}$, (see Eqs. 5.53 and 5.54) and $L_x = 0.2\,\text{m}$, the buckling loads are

$$N_{x,\,\rm cr} = \left(\frac{1}{12\,870} + \frac{1}{18\,615}\right)^{-1} = 7\,609\,\text{kN/m} \qquad \text{(ss)} \tag{5.86}$$

$$N_{x,\,\rm cr} = \left(\frac{1}{51\,481} + \frac{1}{18\,615}\right)^{-1} = 13\,672\,\text{kN/m} \qquad \text{(built-in).} \tag{5.87}$$

5.3.2 Simply Supported Plates – Orthotropic and Symmetrical Layup

We consider a rectangular sandwich plate with dimensions L_x and L_y (Fig. 5.14). The layup of the plate is orthotropic (page 176) and symmetrical with respect to the plate's midplane. All four edges of the plate are simply supported. The sandwich plate is subjected to uniformly distributed compressive loads N_{x0} and N_{y0} along the edges. These loads are increased proportionally, that is, the loads are λN_{x0}, λN_{y0}, where λ is the load parameter. For a buckled plate the load parameter is denoted by $\lambda_{\rm cr}$.

The expression for the strain energy is given by Eq. (5.64).

For a plate subjected to in-plane loads only, the potential of the external forces is (Eq. 4.108)

$$\Omega = \frac{1}{2}\int_0^{L_x}\int_0^{L_y} N_x\left(\frac{\partial w^{\rm o}}{\partial x}\right)^2 + N_y\left(\frac{\partial w^{\rm o}}{\partial y}\right)^2 dydx, \tag{5.88}$$

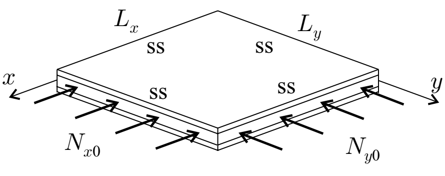

Figure 5.14: Rectangular sandwich plate subjected to biaxial compressive edge loads.

where N_x, N_y are the in-plane tensile forces related to the in-plane compressive forces λN_{x0}, λN_{y0} by

$$N_x = -\lambda N_{x0} \qquad N_y = -\lambda N_{y0}. \tag{5.89}$$

The deflection is assumed to be of the form given in Eq. (5.71). By substituting Eqs. (5.71), (5.64), (5.88), and (5.89) into Eqs. (5.72) and by performing the differentiation, after algebraic manipulations we obtain

$$\frac{L_x L_y}{4}\left(\begin{bmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{bmatrix} - \lambda \begin{bmatrix} N_{x0}\left(\frac{i\pi}{L_x}\right)^2 + N_{y0}\left(\frac{j\pi}{L_y}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \times \begin{Bmatrix} w_{ij} \\ (\gamma_{xz})_{ij} \\ (\gamma_{yz})_{ij} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{5.90}$$

where $(\gamma_{xz})_{ij}$ and $(\gamma_{yz})_{ij}$ are defined by Eq. (5.74) and F_{ij} are given in Table 5.3 (page 184). When the load set is under the critical value, the deflection of the plate is zero. When the plate is not buckled, the deflection of the plate is zero, whereas for a buckled plate it is nonzero. The values of λ for the buckled plate (denoted by λ_{cr}) are the eigenvalues of Eq. (5.90) and are obtained by setting the determinant of the coefficient matrix to zero. This gives

$$(\lambda_{ij})_{cr} = \frac{1}{N_{x0}\left(\frac{i\pi}{L_x}\right)^2 + N_{y0}\left(\frac{j\pi}{L_y}\right)^2} \frac{\begin{vmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{vmatrix}}{\begin{vmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{vmatrix}}, \tag{5.91}$$

where $| \quad |$ denotes the determinant. The values of $(\lambda_{ij})_{cr}$ are calculated for different sets of i and j, $(i, j = 1, 2, \dots)$. The lowest resulting value of $(\lambda_{ij})_{cr}$ is the value of interest.

When the sandwich plate is isotropic, we obtain $(\lambda_{ij})_{cr}$ by replacing D_{11}, D_{12}, D_{66} in Table 5.3 by D^{iso}, $\nu^{iso} D^{iso}$, and $(1 - \nu^{iso})\, D^{iso}/2$ (see Table 5.2, page 178) and $\widetilde{S}_{11}$, $\widetilde{S}_{22}$ by $\widetilde{S}$ (see Eq. 5.39). With these substitutions Eq. (5.91) simplifies to

$$(\lambda_{ij})_{cr}^{iso} = \frac{\left(\frac{i}{L_x}\right)^2 + \left(\frac{j}{L_y}\right)^2}{N_{x0}\left(\frac{i}{L_x}\right)^2 + N_{y0}\left(\frac{j}{L_y}\right)^2}\left(N_{D,cr}^{-1} + \widetilde{S}^{-1}\right)^{-1}, \tag{5.92}$$

where $N_{D,\,cr}$ is defined as

$$N_{D,\,cr} = \pi^2 D^{iso}\left[\left(\frac{i}{L_x}\right)^2 + \left(\frac{j}{L_y}\right)^2\right]. \tag{5.93}$$

5.3 Example. *A* 0.9-*m-long and* 0.2-*m-wide rectangular sandwich plate is made of a* 0.02-*m-thick core covered on both sides by graphite epoxy facesheets. The*

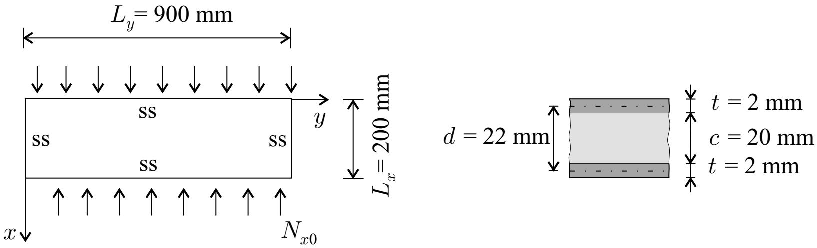

Figure 5.15: The sandwich plate in Example 5.3.

material properties are given in Table 3.6 (page 81). The layup of each facesheet is $[\pm 45_2^{\mathrm{f}}/0_{12}/\pm 45_2^{\mathrm{f}}]$*, and the thickness of each facesheet is* $0.002\,m$*. The 0-degree plies are parallel to the short edge of the plate. The plate, simply supported along all four edges (Fig. 5.15), is subjected to uniform compressive loads along the long edges. Calculate the buckling load. The core is isotropic (*$E_c = 2 \times 10^6$ *kN/m*^{2}*,* $\nu_c = 0.3$*).*

Solution. We set $N_{y0} = 0$ in Eq. (5.91) and write

$$N_{x0}\,(\lambda_{ij})_{\mathrm{cr}} = \frac{1}{\left(\frac{i\pi}{L_x}\right)^2} \frac{\begin{vmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{vmatrix}}{\begin{vmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{vmatrix}}. \tag{5.94}$$

The parameters F_{ij} are given in Table 5.3 (page 184) as follows:

$$\begin{aligned}
F_{33} &= D_{11}\left(\frac{i\pi}{L_x}\right)^4 + 2\,(D_{12} + 2D_{66})\left(\frac{i\pi}{L_x}\right)^2\left(\frac{j\pi}{L_y}\right)^2 + D_{22}\left(\frac{j\pi}{L_y}\right)^4 \\
F_{34} &= -D_{11}\left(\frac{i\pi}{L_x}\right)^3 - (D_{12} + 2D_{66})\frac{i\pi}{L_x}\left(\frac{j\pi}{L_y}\right)^2 \\
F_{35} &= -D_{22}\left(\frac{j\pi}{L_y}\right)^3 - (D_{12} + 2D_{66})\left(\frac{i\pi}{L_x}\right)^2\frac{j\pi}{L_y} \\
F_{44} &= D_{11}\left(\frac{i\pi}{L_x}\right)^2 + D_{66}\left(\frac{j\pi}{L_y}\right)^2 + \widetilde{S}_{11} \\
F_{45} &= (D_{12} + 2D_{66})\left(\frac{i\pi}{L_x}\right)\left(\frac{j\pi}{L_y}\right) \\
F_{55} &= D_{22}\left(\frac{j\pi}{L_y}\right)^2 + D_{66}\left(\frac{i\pi}{L_x}\right)^2 + \widetilde{S}_{22}.
\end{aligned} \tag{5.95}$$

The elements of the stiffness matrices are (Eqs. 5.53 and 5.54)

$$D_{11} = 52.16\,\mathrm{kN\cdot m} \qquad D_{22} = 11.71\,\mathrm{kN\cdot m} \qquad D_{12} = 7.96\,\mathrm{kN\cdot m}$$

$$D_{66} = 8.76\,\mathrm{kN\cdot m} \qquad \widetilde{S}_{11} = 18\,615\frac{\mathrm{kN}}{\mathrm{m}} \qquad \widetilde{S}_{22} = 18\,615\frac{\mathrm{kN}}{\mathrm{m}}.$$

By substituting these stiffnesses and $L_x = 0.2\,\mathrm{m}$, $L_y = 0.9\,\mathrm{m}$ into Eq. (5.94), we obtain the following values of N_{x0}, $(\lambda_{ij})_{\mathrm{cr}}$:

$i \setminus j$	1	2	3
1	7 965	9 070	11 041
2	13 875	14 491	15 544
3	16 195	16 668	17 470.

(5.96)

The smallest value is $N_{x0}\,(\lambda_{\mathrm{cr}})_{ij} = 7\,965\,\mathrm{kN/m}$, which corresponds to $i = j = 1$, Thus, the buckling load is

$$N_{x,\mathrm{cr}} = (\lambda_{\mathrm{cr}})_{11}\,N_{x0} = 7\,965\,\mathrm{kN/m}. \tag{5.97}$$

In Example 5.2 we treated this sandwich as a long plate and obtained the buckling load $N_{x\mathrm{cr}} = 7\,609\,\mathrm{kN/m}$ (Eq. 5.86). This is within 5 percent of the value given by Eq. (5.97).

5.3.3 Face Wrinkling

We consider a sandwich plate. The top and bottom facesheets are identical, and each facesheet's layup is symmetrical with respect to the facesheet's midplane. The sandwich plate is subjected to in-plane forces N_x, N_y, N_{xy} (Fig. 5.16, left). Since the in-plane stiffnesses of the core are taken to be negligible, the in-plane stresses in the core may be neglected with respect to the in-plane stresses in the facesheets. Correspondingly, the in-plane forces (per unit length) in the facesheets are

$$N_x^{\mathrm{f}} = \frac{N_x}{2} \qquad N_y^{\mathrm{f}} = \frac{N_y}{2} \qquad N_{xy}^{\mathrm{f}} = \frac{N_{xy}}{2} \qquad \mathrm{f} = \mathrm{t, b}. \tag{5.98}$$

The superscript f denotes either the top or the bottom facesheet. Under these in-plane forces the facesheets may become wavy (Fig. 5.16, right). These waves are precursors of local buckling, and the loadset at which these waves first occur is taken as the buckling loads. The waves may propagate in two directions, although generally the waves in one direction dominate. In our analysis we consider only waves in one direction.

The wavelength $2l$ depends on the material and on the geometry of the sandwich plate. Here, we consider two cases: (i) the wavelength is "short," such that

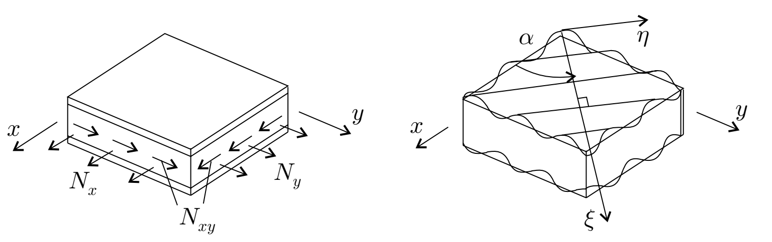

Figure 5.16: Face wrinkling of sandwich plates.

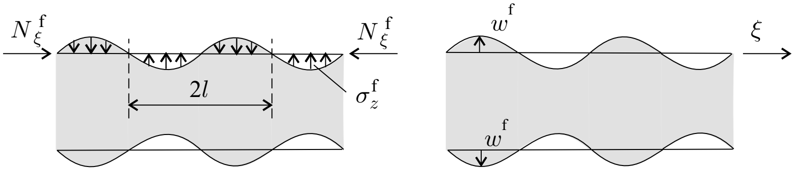

Figure 5.17: Forces and displacements of the buckled facesheets.

$l/h \ll 1$, and (ii) the wavelength is "long" such that $l/h \gg 1$ (h is the thickness of the plate as shown in Fig. 5.2).

The loads that contribute to the waviness of the facesheets are the in-plane load perpendicular to the wave N_ξ^f and the normal load σ_z^f (corresponding to the stress σ_z) exerted on the facesheet by the deformed core (Fig. 5.17, left). Under these loads the equilibrium equation of the facesheet is[1]

$$\Psi_\xi \frac{\partial^4 w^f}{\partial \xi^4} + N_\xi^f \frac{\partial^2 w^f}{\partial \xi^2} = \sigma_z^f, \tag{5.99}$$

where Ψ_ξ is the bending stiffness of the facesheet in the ξ direction – that is, the 11 element of the matrix $[D]$ in the ξ–η coordinate system. Transformation of the matrix $[D]$ follows the transformation rule of the matrix $[Q]$ given by Eq. (2.195). Thus, we have

$$\begin{aligned}\Psi_\xi = \left(D_{11}^f\right)_{\xi-\eta} &= D_{11}^f \cos^4\alpha + D_{22}^f \sin^4\alpha + \left(2D_{12}^f + 4D_{66}^f\right)\cos^2\alpha \sin^2\alpha \\ &+ 4\cos^3\alpha \sin\alpha D_{16}^f + 4\cos\alpha \sin^3\alpha D_{26}^f \qquad f = t, b,\end{aligned} \tag{5.100}$$

where α is the angle between the x- and ξ-axes and D_{ij}^f are the elements of the bending stiffness matrix of the facesheets in the x–y coordinate system.

The parameter w^f is the out-of-plane displacement (deflection) of the facesheet (Fig. 5.17, right),

$$w^f = w_0 \sin \frac{\pi \xi}{l}, \tag{5.101}$$

where w_0 is the amplitude of the deflection and l is the half buckling wavelength.

The parameter N_ξ^f is the in-plane force in the facesheet in the ξ direction and is obtained from the in-plane forces N_x^f, N_y^f, and N_{xy}^f by transformation. By using the stress transformation in Eq. (2.182), we can define N_ξ^f by the following equation:

$$N_\xi^f = -\left(N_x^f \cos^2\alpha + N_y^f \sin^2\alpha + 2N_{xy}^f \cos\alpha \sin\alpha\right) \qquad f = t, b. \tag{5.102}$$

We are interested in the value of N_ξ^f at which the waviness first arises.

Isotropic core – composite facesheets. When the core is isotropic, the out-of-plane stress in the core σ_z varies across the thickness as illustrated in Figure 5.18 (left). When the wavelength is small, the stress σ_z varies, as illustrated in Figure 5.18 (middle). Since the stresses vanish away from the facesheets, the problem may be

[1] S. P. Timoshenko and J. Gere, *Theory of Elastic Stability*. 2nd edition. McGraw-Hill, New York, 1961, p. 2.

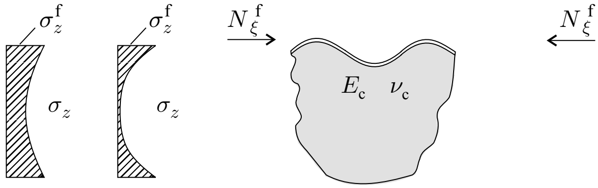

Figure 5.18: The stress σ_z distribution in an isotropic core (left) and in an isotropic core with short wavelength (middle); buckling of a plate on an elastic foundation (right).

treated as a plate on an infinite elastic foundation (Fig. 5.18, right). Then, the stress at the core facesheet interface σ_z^{f} is[2]

$$\sigma_z^{\mathrm{f}} = -\frac{a}{l} w_0 \sin \frac{\pi \xi}{l} \qquad \text{where} \qquad a = \frac{2\pi E_{\mathrm{c}}}{(3-\nu_{\mathrm{c}})(1+\nu_{\mathrm{c}})}, \tag{5.103}$$

where E_{c} and ν_{c} are the Young modulus and Poisson's ratio of the isotropic core.

Equations (5.99), (5.101), and (5.103) yield

$$\Psi_\xi w_0 \frac{\pi^4}{l^4} \sin \frac{\pi \xi}{l} - N_\xi^{\mathrm{f}} w_0 \frac{\pi^2}{l^2} \sin \frac{\pi \xi}{l} = -\frac{a}{l} w_0 \sin \frac{\pi \xi}{l}. \tag{5.104}$$

When the facesheet becomes wavy, w_0 is not zero. The values of N_ξ^{f} corresponding to $w_0 \neq 0$ are the buckling loads $(N_\xi^{\mathrm{f}})_{\mathrm{cr}}$. These buckling loads are given by the nontrivial solution of Eq. (5.104) and are

$$\left(N_\xi^{\mathrm{f}}\right)_{\mathrm{cr}} = \Psi_\xi \frac{\pi^2}{l^2} + \frac{a}{l}\frac{l^2}{\pi^2} \qquad \begin{array}{c}\text{isotropic core}\\ \text{composite facesheets}\\ \text{short wave.}\end{array} \tag{5.105}$$

When the wavelength is long, the solution is obtained by assuming that the stress distribution σ_z is uniform (Fig. 5.19, middle) and σ_z^{f} is approximated as

$$\sigma_z^{\mathrm{f}} = -\sigma_z = -E_{\mathrm{c}}\epsilon_z = -w^{\mathrm{f}} \frac{E_{\mathrm{c}}}{c/2}, \tag{5.106}$$

where $\epsilon_z = w^{\mathrm{f}}/(c/2)$ (Fig. 5.19, right). Substitution of Eqs. (5.101) and (5.106) into Eq. (5.99) gives

$$\Psi_\xi w_0 \frac{\pi^4}{l^4} \sin \frac{\pi \xi}{l} - N_\xi^{\mathrm{f}} w_0 \frac{\pi^2}{l^2} \sin \frac{\pi \xi}{l} = -\frac{E_{\mathrm{c}}}{c/2} w_0 \sin \frac{\pi \xi}{l}. \tag{5.107}$$

The nontrivial solution is

$$\left(N_\xi^{\mathrm{f}}\right)_{\mathrm{cr}} = \Psi_\xi \frac{\pi^2}{l^2} + \frac{E_{\mathrm{c}}}{c/2}\frac{l^2}{\pi^2} \qquad \begin{array}{c}\text{isotropic core}\\ \text{composite facesheets}\\ \text{long wave.}\end{array} \tag{5.108}$$

[2] H. G. Allen, *Analysis and Design of Structural Sandwich Panels*. Pergamon Press, Oxford, 1969, p. 158.

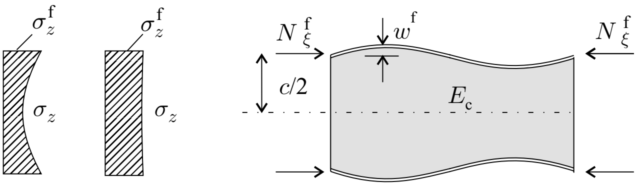

Figure 5.19: The stress σ_z distribution in an isotropic core (left); in an isotropic core with long wavelength (middle); buckled shape (right).

We are interested in the lowest value of $(N_\xi^{\rm f})_{\rm cr}$. This value is obtained by setting the derivative of $(N_\xi^{\rm f})_{\rm cr}$ with respect to l equal to zero as follows:

$$\frac{d\left(N_\xi^{\rm f}\right)_{\rm cr}}{dl} = 0. \tag{5.109}$$

Solution of this equation results in the half-wavelength $l_{\rm cr}$ corresponding to the lowest buckling load. Substitution of $l_{\rm cr}$ into the expressions for $(N_\xi^{\rm f})_{\rm cr}$ gives $(N_\xi^{\rm f})_{\rm cr,min}$. The results are summarized in the top half of Table 5.4.

The value of $(N_\xi^{\rm f})_{\rm cr,min}$ depends on the direction α in which the wave propagates because Ψ_ξ depends on α (Eq. 5.100). The wave direction is not known a priori and must be determined. This is accomplished by observing that $N_\xi^{\rm f}$ is also a function of α (Eq. 5.102). The wave will first appear in the direction in which the ratio $(N_\xi^{\rm f})_{\rm cr,min}/N_\xi^{\rm f}$ is the smallest. This ratio is calculated for different angles, and the angle that results in the smallest ratio is the desired α.

Honeycomb core – composite facesheets. When the sandwich plate consists of a honeycomb core with composite facesheets, the stress at the core facesheet interface may be approximated by Eq. (5.106) for both short and long wavelengths. The buckling loads are identical to those given by Eq. (5.108) and are

$$\left(N_\xi^{\rm f}\right)_{\rm cr} = \Psi_\xi \frac{\pi^2}{l^2} + \frac{E_{\rm c}}{c/2}\frac{l^2}{\pi^2} \qquad \begin{array}{c}\text{honeycomb core}\\ \text{composite facesheets}\\ \text{short or long waves,}\end{array} \tag{5.110}$$

Table 5.4. Face wrinkling of sandwich plates with either a honeycomb or an isotropic core. The constant a is given by Eq. (5.103) and $G_{\rm c} = \frac{E_{\rm c}}{2(1+\nu_{\rm c})}$.

	Isotropic core		Honeycomb core
	Short wave	Long wave	Short or long wave
Composite facesheets	$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 1.5\sqrt[3]{\frac{2\Psi_\xi a^2}{\pi^2}}$ $l_{\rm cr} = \sqrt[3]{2\frac{\pi^4\Psi_\xi}{a}}$	$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 2\sqrt{\Psi_\xi \frac{E_{\rm c}}{c/2}}$ $l_{\rm cr} = \pi\sqrt[4]{\frac{\Psi_\xi c/2}{E_{\rm c}}}$	
Isotropic facesheets	$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 1.5t\sqrt[3]{\frac{4E_{\rm f}E_{\rm c}G_{\rm c}}{3\left(1-\nu_{\rm f}^2\right)(3-\nu_{\rm c})^2(1+\nu_{\rm c})}}$ $\approx 0.79t\sqrt[3]{E_{\rm f}E_{\rm c}G_{\rm c}}$ $l_{\rm cr} = \sqrt[3]{\frac{\pi^4 E_{\rm f}t^3}{6\left(1-\nu_{\rm f}^2\right)a}}$	$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = t\sqrt{\frac{2}{3}\frac{tE_{\rm f}E_{\rm c}}{c\left(1-\nu_{\rm f}^2\right)}}$ $l_{\rm cr} = \pi\sqrt[4]{\frac{E_{\rm f}}{24E_{\rm c}\left(1-\nu_{\rm f}^2\right)}t^3c}$	

where E_c is the Young modulus of the core perpendicular to the plane of the sandwich plate.

The lowest value of the buckling load and the direction α in which the wave propagates is determined as previously above for plates with isotropic core and composite facesheets.

Isotropic core – isotropic facesheets. We now consider sandwich plates with isotropic core and isotropic facesheets. The bending stiffness of an isotropic facesheet is (see Eqs. 3.42 and 5.100)

$$\Psi_\xi = D^{\text{iso}} = \frac{E_f t^3}{12\left(1 - \nu_f^2\right)}, \tag{5.111}$$

where t is the thickness of the facesheet and the subscript f denotes the facesheet. When both the core and the facesheets are isotropic and the facesheets buckle with short waves, the lowest value of the buckling load is obtained by substituting Eq. (5.111) into Eq. (5.105) and by performing the differentiation indicated in Eq. (5.109). The result is

$$\left(N_\xi^f\right)_{\text{cr,min}} = t\left[1.5\sqrt[3]{\frac{4E_f E_c G_c}{3\left(1-\nu_f^2\right)(3-\nu_c)^2(1+\nu_c)}}\right] \quad \begin{matrix}\text{isotropic core}\\ \text{isotropic facesheets}\\ \text{short wave.}\end{matrix} \tag{5.112}$$

where G_c is the shear modulus of the core ($G_c = E_c/2\,(1+\nu_c)$), f and c refer to the facesheets and the core. By the definition of the in-plane force (Eq.5.7) the term in the bracket is the critical stress.

By neglecting the Poisson ratios ($\nu_c = \nu_f = 0$), Eq. (5.112) reduces to

$$\left(N_\xi^f\right)_{\text{cr,min}} = t\left[0.79\sqrt[3]{E_f E_c G_c}\right] \quad \begin{matrix}\text{isotropic core}\\ \text{isotropic facesheets}\\ \text{short wave.}\end{matrix} \tag{5.113}$$

Hoff and Mautner[3] obtained this expression with the value of the constant 0.91 instead of 0.79. However, for practical use they recommended the value 0.5.

When both the core and the facesheets are isotropic and the facesheets buckle with long waves, Eqs. (5.108), (5.111), and (5.109) give

$$\left(N_\xi^f\right)_{\text{cr,min}} = t\left[\sqrt{\frac{2}{3}\frac{E_f E_c}{c\left(1-\nu_f^2\right)}}\right] \quad \begin{matrix}\text{isotropic core}\\ \text{isotropic facesheets}\\ \text{long wave.}\end{matrix} \tag{5.114}$$

For an isotropic facesheet the bending stiffness does not depend on the direction. Consequently, buckling waves occur in the direction in which the compressive stress is maximum.

[3] N. J. Hoff and S. E. Mautner, Buckling of Sandwich Type Panels. *Journal of the Aeronautical Sciences,* Vol. 12, 285–297, 1945. See also in J. R Vinson, *Sandwich Structures of Isotropic and Composite Materials.* Technomic, Lancaster, Pennsylvania, 1999, p. 239.

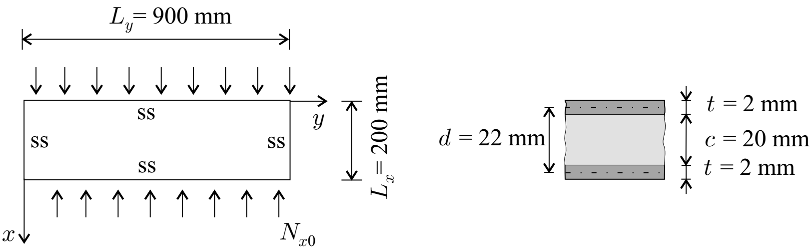

Figure 5.20: The sandwich plate in Example 5.4.

Honeycomb core – isotropic facesheets. Lastly, we consider a sandwich plate with honeycomb core and isotropic facesheets. The half wavelength l_{cr} may be short or long. In this case Eqs. (5.110), (5.111), and (5.109) yield

$$\left(N_\xi^{\mathrm{f}}\right)_{\mathrm{cr,min}} = t\left[\sqrt{\frac{2}{3}\frac{E_{\mathrm{f}}E_{\mathrm{c}}}{c\left(1-\nu_{\mathrm{f}}^2\right)}}\right] \qquad \begin{array}{c}\text{honeycomb core}\\ \text{isotropic facesheets}\\ \text{short or long wave.}\end{array} \tag{5.115}$$

This equation is identical to that given by Heath.[4]

The lowest buckling loads and the half wavelengths corresponding to these buckling loads are summarized in Table 5.4.

These expressions take into account face wrinkling in one direction only. They are accurate when only uniaxial load is applied to an orthotropic plate in one of the orthotropy directions. In case of biaxial loading these expressions may overestimate the buckling load. For most practical composite sandwich plates the error in the buckling loads given by these expressions is less than 30 percent.

5.4 Example. *A* 0.2*-m-long and* 0.9*-m-wide rectangular sandwich plate is made of a* 0.02*-m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is* $[\pm 45_2^{\mathrm{f}}/0_{12}/\pm 45_2^{\mathrm{f}}]$*, and the thickness of each facesheet is* 0.002 *m. The 0-degree plies are parallel to the short edge of the plate. The plate is simply supported along all four edges (Fig. 5.20). The plate is subjected to unidirectional in-plane loads* N_{x0}*. Estimate the load at which the facesheet wrinkles. The core is isotropic (*$E_{\mathrm{c}} = 2 \times 10^6$ *kN/m*^{2}*,* $\nu_{\mathrm{c}} = 0.3$*).*

Solution. Let us first assume that the facesheet wrinkles parallel to the y-axis with long waves ($\alpha = 0$, Fig. 5.16). In this direction Ψ_ξ is (Eq. 5.100)

$$\Psi_\xi = D_{11}^{\mathrm{f}} = D_{11} = 45.30\ \mathrm{N\cdot m}. \tag{5.116}$$

[4] W. G. Heath, Sandwich Construction, Part 2: The Optimum Design of Flat Sandwich Panels. *Aircraft Engineering*, Vol. 32, 230–235, 1960. See also in J. R Vinson, *Sandwich Structures of Isotropic and Composite Materials.* Technomic, Lancaster, Pennsylvania, 1999, p. 239.

The value of $D_{11}^{\rm f}$ is given in Table 3.7 (page 84). The lowest buckling load and the corresponding half wavelength are (Table 5.4, page 193)

$$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 2\sqrt{\Psi_\xi \frac{E_{\rm c}}{c/2}} = 19\,036\ \frac{\rm kN}{\rm m} \tag{5.117}$$

$$l_{\rm cr} = \pi \sqrt[4]{\frac{\Psi_\xi c/2}{E_{\rm c}}} = 0.0069\ {\rm m} = 6.9\ {\rm mm}. \tag{5.118}$$

The assumption that the wave is long is valid when $l_{\rm cr}$ is large compared with the core thickness. Here, $l_{\rm cr}$ is only about a third of the core thickness and, therefore, the long-wave approximation is invalid.

Let us now assume that the wave is short. With this assumption we have (Table 5.4, page 193)

$$\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 1.5\sqrt[3]{\frac{2\Psi_\xi a^2}{\pi^2}} = 34\,116\ \frac{\rm kN}{\rm m} \tag{5.119}$$

$$l_{\rm cr} = \sqrt[3]{2\frac{\pi^4 \Psi_\xi}{a}} = 0.0063\ {\rm m} = 6.3\ {\rm mm}, \tag{5.120}$$

where (Eq. 5.103)

$$a = \frac{2\pi E_{\rm c}}{(3-\nu_{\rm c})(1+\nu_{\rm c})} = 35.80 \times 10^9\, \frac{\rm N}{\rm m^2}. \tag{5.121}$$

The wave may be assumed to be short when $l_{\rm cr}$ is significantly smaller than the core thickness c. Here, the ratio $l_{\rm cr}/c$ is about one-third. Thus, the short wave approximation is unreasonable.

In this problem, the wave cannot be treated as either long or short. As a conservative estimate we take the lower of the two buckling loads given by the long- and short-wave approximations. Thus, the lowest buckling load is (see Eq. 5.98)

$$N_{x\rm cr} = 2\left(N_\xi^{\rm f}\right)_{\rm cr,min} = 2 \times 19\,036 = 38\,072\ \frac{\rm kN}{\rm m}. \tag{5.122}$$

5.4 Free Vibration of Rectangular Sandwich Plates

In this section we obtain the natural frequencies f and, hence, the periods of vibration ($T = 1/f$) and the circular frequencies ($\omega = 2\pi f$) of sandwich plates. In the analyses that follows we assume that the plate is freely vibrating and is undamped.

5.4.1 Long Plates

We consider a long rectangular sandwich plate whose length is large compared with its width ($L_y \gg L_x$). The edges may be built-in, simply supported, or free, as illustrated in Figure 5.21.

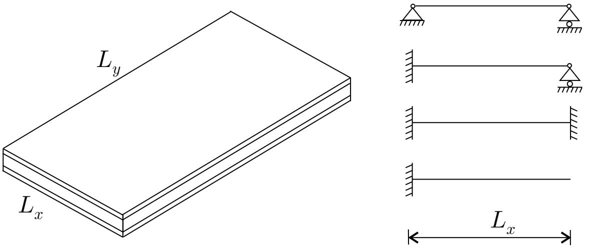

Figure 5.21: The different types of supports along the long edges of long sandwich plates undergoing free undamped vibration.

The deflected surface of the sandwich plate may be assumed to be cylindrical at a considerable distance from the short edges (Fig. 4.4). The equilibrium equations are Eqs. (4.191)–(4.193),

$$\frac{dV_x}{dx} + \rho\,(2\pi f)^2\, w^{\mathrm{o}} = 0 \tag{5.123}$$

$$\frac{dM_x}{dx} - V_x = 0, \tag{5.124}$$

where w^{o} is the deflection and ρ is the mass per unit area of the sandwich plate.

We now consider a sandwich plate that is symmetrical with respect to the midplane. The bending moment and the transverse shear force acting on the sandwich plate are (Eq. 5.44)

$$M_x = -D_{11}\frac{\partial \chi_{xz}}{\partial x} \qquad V_x = \widetilde{S}_{11}\gamma_{xz}. \tag{5.125}$$

Equations (5.123), (5.124), and (5.125), together with Eq. (5.2), give
sandwich plate, symmetrical layup:

$$-D_{11}\frac{d^3\chi_{xz}}{dx^3} + \rho\,(2\pi f)^2\, w^{\mathrm{o}} = 0 \tag{5.126}$$

$$D_{11}\frac{d^2\chi_{xz}}{dx^2} + \widetilde{S}_{11}\left(\frac{dw^{\mathrm{o}}}{dx} - \chi_{xz}\right) = 0. \tag{5.127}$$

For a vibrating sandwich beam the corresponding equations are (see Eqs. 7.178–7.180, $\omega = 2\pi f$)
isotropic sandwich beam:

$$-\widehat{EI}\frac{d^3\chi}{dx^3} + \rho'\,(2\pi f)^2\, w = 0 \tag{5.128}$$

$$\widehat{EI}\frac{d^2\chi}{dx^2} + \widehat{S}\left(\frac{dw}{dx} - \chi\right) = 0, \tag{5.129}$$

where $\widehat{EI}$ and $\widehat{S}$ are the bending and shear stiffnesses of the sandwich beam and ρ' is the mass per unit length.

The preceding set of equations describing the vibration of long sandwich plates (symmetrical layup) and isotropic sandwich beams are identical when D_{11}, $\widetilde{S}_{11}$, and ρ are replaced, respectively, by $\widehat{EI}$, $\widehat{S}$, and ρ'. Therefore, the natural

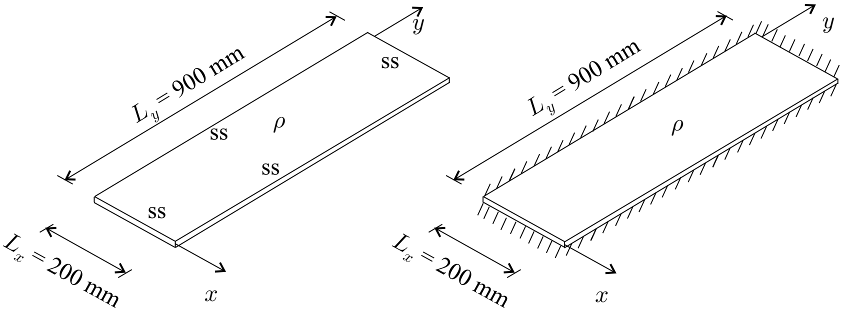

Figure 5.22: The sandwich plates in Example 5.5.

frequencies of a long sandwich plate (symmetrical layup) may be obtained by substituting the values of D_{11}, $\widetilde{S}_{11}$, and ρ for $\widehat{EI}$, $\widehat{S}$, ρ' in the expression given in Section 7.5 for the natural frequencies of the corresponding isotropic sandwich beam.

The natural frequencies of a long unsymmetrical sandwich plate may be obtained by substituting the values of Ψ (Eq.4.52), $\widetilde{S}_{11}$, and ρ for $\widehat{EI}$, $\widehat{S}$, ρ' in the expression for the natural frequencies of the corresponding isotropic sandwich beam.

5.5 Example. *A* 0.9*-m-long and* 0.2*-m-wide rectangular sandwich plate is made of a* 0.02*-m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is* $[\pm 45^{\mathrm{f}}_2/0_{12}/\pm 45^{\mathrm{f}}_2]$*, and the thickness of each facesheet is* 0.002 *m. The 0-degree plies are parallel to the short edge of the plate. The plate is either simply supported or built-in along all four edges (Fig. 5.22). A uniform mass is over the plate such that for the combined mass-plate system* $\rho = 200$ *kg/m*^{2}*. Calculate the circular and the natural frequencies. The core is isotropic (*$E_{\mathrm{c}} = 2 \times 10^6$ *kN/m*^{2}*,* $\nu_{\mathrm{c}} = 0.3$*).*

Solution. The plate may be treated as "long" (Example 5.1, page 180). The circular frequencies of the corresponding beam are (Eq. 7.243, Eq. 6.398, and Table 6.13, page 308)

$$\omega_i = \sqrt{\left(\frac{\rho'}{EI}\frac{L^4}{\mu_{Bi}^4} + \frac{\rho'}{\widehat{S}_{zz}}\frac{L^2}{\mu_{Si}^2}\right)^{-1}}, \tag{5.130}$$

where

$$\mu_{Bi} = \pi,\ 2\pi,\ 3\pi, \ldots \qquad \text{(ss)} \tag{5.131}$$

$$\mu_{Bi} = 4.730,\ 7.853,\ 10.996, \ldots \qquad \text{(built-in)} \tag{5.132}$$

and

$$\mu_{Si} = \pi,\ 2\pi,\ 3\pi, \ldots \qquad \text{(ss or built-in)}. \tag{5.133}$$

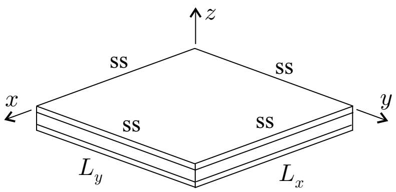

Figure 5.23: Rectangular sandwich plate with simply supported edges.

The circular frequencies of the plate are obtained by replacing EI, $\widehat{S}$, ρ' by D_{11}, $\widetilde{S}_{11}$, ρ (see page 198) as follows:

$$\omega = \sqrt{\left(\frac{\rho}{D_{11}}\frac{L_x^4}{\mu_{Bi}^4} + \frac{\rho}{\widetilde{S}_{11}}\frac{L_x^2}{\mu_{Si}^2}\right)^{-1}}. \tag{5.134}$$

With the values of $D_{11} = 52.16$ kN $\cdot$ m and $\widetilde{S}_{11} = 18\,615\frac{\text{kN}}{\text{m}}$, (see Eqs. 5.53 and 5.54) and with $L_x = 0.2$ m, the first three modes of the circular frequencies of the plate are

$$\omega_1 = 3\,064 \qquad \omega_2 = 8\,214 \qquad \omega_3 = 13\,344 \text{ 1/s} \qquad \text{(ss)} \tag{5.135}$$

$$\omega_1 = 4\,233 \qquad \omega_2 = 8\,945 \qquad \omega_3 = 13\,791 \text{ 1/s} \qquad \text{(built-in).} \tag{5.136}$$

The corresponding natural frequencies are ($f = \omega/2\pi$)

$$f_1 = 488 \qquad f_2 = 1\,307 \qquad f_3 = 2\,124 \text{ Hz} \qquad \text{(ss)} \tag{5.137}$$

$$f_1 = 674 \qquad f_2 = 1\,424 \qquad f_3 = 2\,195 \text{ Hz} \qquad \text{(built-in).} \tag{5.138}$$

5.4.2 Simply Supported Plates – Orthotropic and Symmetrical Layup

A simply supported rectangular sandwich plate with dimensions L_x and L_y is considered (Fig. 5.23). The layup of the plate is orthotropic (page 176) and symmetrical with respect to the plate's midplane. The plate is undergoing free undamped vibration. The deflection of the plate is (Eq. 4.189)

$$w^{\text{o}} = \overline{w}^{\text{o}} \sin(2\pi f t). \tag{5.139}$$

Analogously, we express the rotations of the normals as

$$\chi_{xz} = \overline{\chi}_{xz} \sin(2\pi f t) \tag{5.140}$$

$$\chi_{yz} = \overline{\chi}_{yz} \sin(2\pi f t), \tag{5.141}$$

where $\overline{w}^{\text{o}}$, $\overline{\chi}_{xz}$, and $\overline{\chi}_{yz}$ are as yet unknown functions of x and y. These functions must be chosen such that $\overline{w}^{\text{o}}$, $\overline{\chi}_{xz}$, and $\overline{\chi}_{yz}$ satisfy the boundary conditions given in Section 5.2.2 (Eqs. 5.66–5.70). To determine these functions we introduce Eqs. (5.139)–(5.141) into the expression for the strain energy given by Eq. (5.64). This results in

$$U = \overline{U} \sin^2(2\pi f t), \tag{5.142}$$

where $\overline{U}$ is defined as

$$\overline{U} = \frac{1}{2}\int_0^{L_x}\int_0^{L_y}\Bigg[\left(\frac{\partial \overline{\chi}_{xz}}{\partial x}\right)^2 D_{11} + \left(\frac{\partial \overline{\chi}_{yz}}{\partial y}\right)^2 D_{22} + 2\frac{\partial \overline{\chi}_{xz}}{\partial x}\frac{\partial \overline{\chi}_{yz}}{\partial y} D_{12}$$
$$+ \left(\frac{\partial \overline{\chi}_{xz}}{\partial y} + \frac{\partial \overline{\chi}_{yz}}{\partial x}\right)^2 D_{66} + \left(\frac{\partial \overline{w}^{\text{o}}}{\partial x} - \overline{\chi}_{xz}\right)^2 \widetilde{S}_{11} + \left(\frac{\partial \overline{w}^{\text{o}}}{\partial y} - \overline{\chi}_{yz}\right)^2 \widetilde{S}_{22}\Bigg] dxdy. \tag{5.143}$$

Following the steps used in the analysis of free vibration of thin plates (Section 4.4.2), we arrive at the following expression for the natural frequency:

$$(2\pi f)^2 = \frac{\overline{U}}{\frac{1}{2}\rho \int_0^{L_x}\int_0^{L_y} \rho \overline{w}^{\text{o}2} dydx}. \tag{5.144}$$

We adopt the following expressions for $\overline{w}^{\text{o}}$, $\overline{\chi}_{xz}$, $\overline{\chi}_{yz}$:

$$\overline{w}^{\text{o}} = \sum_{i=1}^{I}\sum_{j=1}^{J} w_{ij} \sin\frac{i\pi x}{L_x} \sin\frac{j\pi y}{L_y}$$
$$\overline{\chi}_{xz} = \sum_{i=1}^{I}\sum_{j=1}^{J} (\chi_{xz})_{ij} \cos\frac{i\pi x}{L_x} \sin\frac{j\pi y}{L_y} \tag{5.145}$$
$$\overline{\chi}_{yz} = \sum_{i=1}^{I}\sum_{j=1}^{J} (\chi_{yz})_{ij} \sin\frac{i\pi x}{L_x} \cos\frac{j\pi y}{L_y}.$$

With these expressions the deflections and the rotations given by Eqs. (5.139)–(5.141) satisfy the boundary given by Eqs. (5.66)–(5.70). The unknown coefficients w_{ij}, $(\chi_{xz})_{ij}$, $(\chi_{yz})_{ij}$ are determined from the conditions (Eq. 4.217)

$$\frac{\partial f}{\partial (\chi_{xz})_{ij}} = 0 \qquad \frac{\partial f}{\partial (\chi_{yz})_{ij}} = 0 \qquad \frac{\partial f}{\partial w_{ij}} = 0. \tag{5.146}$$

After algebraic manipulations we obtain

$$\left(\frac{L_x L_y}{4}\begin{bmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55}\end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\right)\begin{Bmatrix} w_{ij} \\ (\gamma_{xz})_{ij} \\ (\gamma_{yz})_{ij}\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0\end{Bmatrix}, \tag{5.147}$$

where $(\gamma_{xz})_{ij}$ and $(\gamma_{yz})_{ij}$ are defined by Eq. (5.74), F_{ij} are given in Table 5.3 (page 184), and λ is defined as

$$\lambda = \frac{1}{4}(2\pi f)^2 \rho L_x L_y. \tag{5.148}$$

In the case of free vibration the deflection is nonzero. For nonzero deflections, Eq. (5.147) is satisfied when the determinant of the matrix in the parentheses is zero. At this condition λ signifies the eigenvalues of Eq. (5.147), and we obtain

$$\lambda_{ij} = \frac{\begin{vmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{vmatrix}}{\begin{vmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{vmatrix}} \frac{L_x L_y}{4}. \tag{5.149}$$

The values of λ_{ij} are calculated for different sets of i and j, $(i, j = 1, 2, \dots)$, of which the lowest value is of interest.

The natural frequencies are calculated from Eq. (5.148):

$$f_{ij} = \frac{1}{\pi}\sqrt{\frac{\lambda_{ij}}{\rho L_x L_y}}. \tag{5.150}$$

When the sandwich plate is isotropic, we obtain λ_{ij} by replacing D_{11}, D_{12}, D_{66} in Table 5.3 (page 184) by D^{iso}, $\nu^{\text{iso}} D^{\text{iso}}$, and $\left(1 - \nu^{\text{iso}}\right) D^{\text{iso}}/2$ (see Table 5.2, page 178) and $\widetilde{S}_{11}$, $\widetilde{S}_{22}$ by $\widetilde{S}$ (see Eq. 5.39). With these substitutions Eq. (5.149) simplifies to

$$\lambda_{ij} = \frac{L_x L_y}{4}\pi^2 \left[\left(\frac{i}{L_x}\right)^2 + \left(\frac{j}{L_y}\right)^2\right] \left(N_{D,ij}^{-1} + \widetilde{S}^{-1}\right)^{-1}, \tag{5.151}$$

where $N_{D,ij}$ is defined as

$$N_{D,ij} = \pi^2 D^{\text{iso}} \left[\left(\frac{i}{L_x}\right)^2 + \left(\frac{j}{L_y}\right)^2\right]. \tag{5.152}$$

5.6 Example. *A* 0.9*-m-long and* 0.2*-m-wide rectangular sandwich plate is made of a* 0.02*-m-thick core covered on both sides by graphite epoxy facesheets. The material properties are given in Table 3.6 (page 81). The layup of each facesheet is* $[\pm 45^{\text{f}}_2/0_{12}/\pm 45^{\text{f}}_2]$*, and the thickness of each facesheet is* 0.002 *m. The 0-degree*

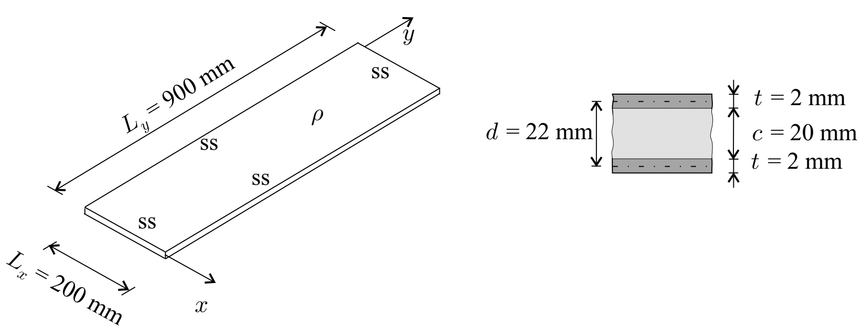

Figure 5.24: The sandwich plate in Example 5.6.

plies are parallel to the short edge of the plate. The plate is simply supported along all four edges (Fig. 5.24). A uniform mass is over the plate such that for the combined mass-plate system $\rho = 200$ kg/m^2. Calculate the circular and the natural frequencies. The core is isotropic ($E_c = 2 \times 10^6$ kN/m^2, $\nu_c = 0.3$).

Solution. The eigenvalues λ_{ij} are (Eq. 5.149)

$$\lambda_{ij} = \frac{\begin{vmatrix} F_{33} & F_{34} & F_{35} \\ F_{34} & F_{44} & F_{45} \\ F_{35} & F_{45} & F_{55} \end{vmatrix}}{\begin{vmatrix} F_{44} & F_{45} \\ F_{45} & F_{55} \end{vmatrix}} \frac{L_x L_y}{4}. \tag{5.153}$$

The parameters F_{ij} are given by Eq. (5.95). The elements of the stiffness matrices are (Eqs. 5.53 and 5.54)

$$D_{11} = 52.16 \text{ kN}\cdot\text{m} \qquad D_{22} = 11.71 \text{ kN}\cdot\text{m} \qquad D_{12} = 7.96 \text{ kN}\cdot\text{m}$$

$$D_{66} = 8.76 \text{ kN}\cdot\text{m} \qquad \widetilde{S}_{11} = 18\,615 \frac{\text{kN}}{\text{m}} \qquad \widetilde{S}_{22} = 18\,615 \frac{\text{kN}}{\text{m}},$$

and $L_x = 0.2$ m, $L_y = 0.9$ m. With these values, Eqs. (5.95) and (5.153) give the following values of $(\lambda_{cr})_{ij} \times 10^{-9}$:

$i \backslash j$	1	2	3
1	0.0884	0.1007	0.1226
2	0.6162	0.6436	0.6904
3	1.6183	1.6656	1.7458 .

(5.154)

The natural frequencies are (Eq. 5.150)

$$f_{ij} = \frac{1}{\pi}\sqrt{\frac{\lambda_{ij}}{\rho L_x L_y}}. \tag{5.155}$$

This yields the following values of f_{ij} (Hz):

$i \backslash j$	1	2	3
1	499	532	587
2	1 317	1 346	1 394
3	2 134	2 165	2 217 .

(5.156)

By treating the plate as long, in Example 5.5 (page 198) we obtained the following values for the first three natural frequencies: 488; 1 307; and 2 124 Hz (Eq. 5.137). These natural frequencies are within 3 percent of the values given above.