9

Effective Elastic Constants of a Laminate

9.1 Basic Equations

In this chapter, we introduce the concept of *effective elastic constants* for the laminate. These constants are the effective extensional modulus in the x direction $\bar{E}_x$, the effective extensional modulus in the y direction $\bar{E}_y$, the effective Poisson's ratios $\bar{\nu}_{xy}$ and $\bar{\nu}_{yx}$, and the effective shear modulus in the x-y plane $\bar{G}_{xy}$.

The effective elastic constants are usually defined when considering the inplane loading of symmetric balanced laminates. In the following equations, we consider only symmetric balanced or symmetric cross-ply laminates. We therefore define the following three average laminate stresses [1]:

$$\bar{\sigma}_x = \frac{1}{H}\int_{-H/2}^{H/2} \sigma_x dz \tag{9.1}$$

$$\bar{\sigma}_y = \frac{1}{H}\int_{-H/2}^{H/2} \sigma_y dz \tag{9.2}$$

$$\bar{\tau}_{xy} = \frac{1}{H}\int_{-H/2}^{H/2} \tau_{xy} dz \tag{9.3}$$

where H is the thickness of the laminate. Comparing (9.1), (9.2), and (9.3) with (7.13), we obtain the following relations between the average stresses and the force resultants:

$$\bar{\sigma}_x = \frac{1}{H} N_x \tag{9.4}$$

$$\bar{\sigma}_y = \frac{1}{H} N_y \tag{9.5}$$

$$\bar{\tau}_{xy} = \frac{1}{H} N_{xy} \tag{9.6}$$

Solving (9.4), (9.5), and (9.6) for N_x, N_y, and N_{xy}, and substituting the results into (8.11) and (8.12) for symmetric balanced laminates, we obtain:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} a_{11}H & a_{12}H & 0 \\ a_{12}H & a_{22}H & 0 \\ 0 & 0 & a_{66}H \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_x \\ \bar{\sigma}_y \\ \bar{\tau}_{xy} \end{Bmatrix} \tag{9.7}$$

The above 3×3 matrix is defined as the *laminate compliance matrix* for symmetric balanced laminates. Therefore, by analogy with (4.5), we obtain the following effective elastic constants for the laminate:

$$\bar{E}_x = \frac{1}{a_{11}H} \tag{9.8a}$$

$$\bar{E}_y = \frac{1}{a_{22}H} \tag{9.8b}$$

$$\bar{G}_{xy} = \frac{1}{a_{66}H} \tag{9.8c}$$

$$\bar{\nu}_{xy} = -\frac{a_{12}}{a_{11}} \tag{9.8d}$$

$$\bar{\nu}_{yx} = -\frac{a_{12}}{a_{22}} \tag{9.8e}$$

It is clear from the above equations that $\bar{\nu}_{xy}$ and $\bar{\nu}_{yx}$ are not independent and are related by the following reciprocity relation:

$$\frac{\bar{\nu}_{xy}}{\bar{E}_x} = \frac{\bar{\nu}_{yx}}{\bar{E}_y} \tag{9.9}$$

Finally, we note that the expressions of the effective elastic constants of (9.8) can be re-written in terms of the components A_{ij} of the matrix $[A]$ as shown in Example 9.1.

9.2 MATLAB Functions Used

The five MATLAB function used in this chapter to calculate the average laminate elastic constants are:

Ebarx(A, H) – This function calculates the average laminate modulus in the x-direction $\bar{E}_x$. There are two input arguments to this function – they are the thickness of the laminate H and the 3×3 stiffness matrix $[A]$ for balanced symmetric laminates. The function returns a scalar quantity which the desired modulus.

Ebary(A, H) – This function calculates the average laminate modulus in the y-direction $\bar{E}_y$. There are two input arguments to this function – they are the thickness of the laminate H and the 3×3 stiffness matrix $[A]$ for balanced symmetric laminates. The function returns a scalar quantity which the desired modulus.

NUbarxy(A, H) – This function calculates the average laminate Poisson's ratio $\bar{\nu}_{xy}$. There are two input arguments to this function – they are the thickness of the laminate H and the 3×3 stiffness matrix $[A]$ for balanced symmetric laminates. The function returns a scalar quantity which the desired Poisson's ratio.

NUbaryx(A, H) – This function calculates the average laminate Poisson's ratio $\bar{\nu}_{yx}$. There are two input arguments to this function – they are the thickness of the laminate H and the 3×3 stiffness matrix $[A]$ for balanced symmetric laminates. The function returns a scalar quantity which the desired Poisson's ratio.

Gbarxy(A, H) – This function calculates the average laminate shear modulus $\bar{G}_{xy}$. There are two input arguments to this function – they are the thickness of the laminate H and the 3×3 stiffness matrix $[A]$ for balanced symmetric laminates. The function returns a scalar quantity which the desired shear modulus.

The following is a listing of the MATLAB source code for these functions:

```
function y = Ebarx(A,H)
%Ebarx   This function returns the average laminate modulus
%        in the x-direction. Its input are two arguments:
%        A  - 3 x 3 stiffness matrix for balanced symmetric
%             laminates.
%        H - thickness of laminate
a = inv(A);
y = 1/(H*a(1,1));
```

```
function y = Ebary(A,H)
%Ebary   This function returns the average laminate modulus
%        in the y-direction. Its input are two arguments:
%        A  - 3 x 3 stiffness matrix for balanced symmetric
%             laminates.
%        H - thickness of laminate
a = inv(A);
y = 1/(H*a(2,2));
```

```
function y = NUbarxy(A,H)
%NUbarxy   This function returns the average laminate
%          Poisson's ratio NUxy. Its input are two arguments:
%          A  - 3 x 3 stiffness matrix for balanced symmetric
%               laminates.
%          H - thickness of laminate
a = inv(A);
y = -a(1,2)/a(1,1);
```

```
function y = NUbaryx(A,H)
%NUbaryx   This function returns the average laminate
%          Poisson's ratio NUyx. Its input are two arguments:
%          A  - 3 x 3 stiffness matrix for balanced symmetric
%               laminates.
%          H - thickness of laminate
a = inv(A);
y = -a(1,2)/a(2,2);
```

```
function y = Gbarxy(A,H)
%Gbarxy   This function returns the average laminate shear
%         modulus. Its input are two arguments:
%        A  - 3 x 3 stiffness matrix for balanced symmetric
```

```
%              laminates.
%         H - thickness of laminate
a = inv(A);
y = 1/(H*a(3,3));
```

Example 9.1

Show that the effective elastic constants for the laminate can be written in terms of the components A_{ij} of the $[A]$ matrix as follows:

$$\bar{E}_x = \frac{A_{11}AA_{22} - A_{12}^2}{A_{22}H} \tag{9.10a}$$

$$\bar{E}_y = \frac{A_{11}AA_{22} - A_{12}^2}{A_{11}H} \tag{9.10b}$$

$$\bar{\nu}_{xy} = \frac{A_{12}}{A_{22}} \tag{9.10c}$$

$$\bar{\nu}_{yx} = \frac{A_{12}}{A_{11}} \tag{9.10d}$$

$$\bar{G}_{xy} = \frac{A_{66}}{H} \tag{9.10e}$$

Solution

Starting with (8.11) and (8.12) as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} \tag{9.11}$$

take the inverse of (9.11) to obtain:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \tag{9.12}$$

where

$$a_{11} = \frac{A_{22}}{A_{11}A_{22} - A_{12}^2} \tag{9.13a}$$

$$a_{22} = \frac{A_{11}}{A_{11}A_{22} - A_{12}^2} \tag{9.13b}$$

$$a_{12} = \frac{A_{12}}{A_{11}A_{22} - A_{12}^2} \tag{9.13c}$$

$$a_{66} = \frac{1}{A_{66}} \tag{9.13d}$$

Next, substitute (9.13) into (9.8) to obtain the required expressions as follows:

$$\bar{E}_x = \frac{A_{11}AA_{22} - A_{12}^2}{A_{22}H} \tag{9.14a}$$

$$\bar{E}_y = \frac{A_{11}AA_{22} - A_{12}^2}{A_{11}H} \tag{9.14b}$$

$$\bar{\nu}_{xy} = \frac{A_{12}}{A_{22}} \tag{9.14c}$$

$$\bar{\nu}_{yx} = \frac{A_{12}}{A_{11}} \tag{9.14d}$$

$$\bar{G}_{xy} = \frac{A_{66}}{H} \tag{9.14e}$$

MATLAB Example 9.2

Consider a four-layer $[0/90]_S$ graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.800 mm. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

Solution

This example is solved using MATLAB. First, the reduced stiffness matrix $[Q]$ for a typical layer using the MATLAB function *ReducedStiffness* as follows:

```
EDU>> Q = ReducedStiffness(155.0, 12.10, 0.248, 4.40)

Q =

  155.7478    3.0153         0
    3.0153   12.1584         0
         0         0    4.4000
```

Next, the transformed reduced stiffness matrix $[\bar{Q}]$ is calculated for each layer using the MATLAB function *Qbar* as follows:

```
EDU>> Qbar1 = Qbar(Q, 0)

Qbar1 =

  155.7478    3.0153         0
    3.0153   12.1584         0
         0         0    4.4000
```

```
EDU>> Qbar2 = Qbar(Q, 90)

Qbar2 =

   12.1584    3.0153   -0.0000
    3.0153  155.7478    0.0000
   -0.0000    0.0000    4.4000

EDU>> Qbar3 = Qbar(Q, 90)

Qbar3 =

   12.1584    3.0153   -0.0000
    3.0153  155.7478    0.0000
   -0.0000    0.0000    4.4000

EDU>> Qbar4 = Qbar(Q, 0)

Qbar4 =

  155.7478    3.0153         0
    3.0153   12.1584         0
         0         0    4.4000
```

Next, the distances $z_k (k = 1, 2, 3, 4, 5)$ are calculated as follows:

```
EDU>> z1 = -0.400

z1 =

   -0.4000

EDU>> z2 = -0.200

z2 =

   -0.2000

EDU>> z3 = 0

z3 =

     0

EDU>> z4 = 0.200

z4 =

    0.2000
```

```
EDU>> z5 = 0.400

z5 =

    0.4000
```

Next, the $[A]$ matrix is calculated using four calls to the MATLAB function *Amatrix* as follows:

```
EDU>> A = zeros(3,3)

A =

     0     0     0
     0     0     0
     0     0     0

EDU>> A = Amatrix(A,Qbar1,z1,z2)

A =

   31.1496    0.6031         0
    0.6031    2.4317         0
         0         0    0.8800

EDU>> A = Amatrix(A,Qbar2,z2,z3)

A =

   33.5812    1.2061   -0.0000
    1.2061   33.5812    0.0000
   -0.0000    0.0000    1.7600

EDU>> A = Amatrix(A,Qbar3,z3,z4)

A =

   36.0129    1.8092   -0.0000
    1.8092   64.7308    0.0000
   -0.0000    0.0000    2.6400

EDU>> A = Amatrix(A,Qbar4,z4,z5)

A =

   67.1625    2.4122   -0.0000
    2.4122   67.1625    0.0000
   -0.0000    0.0000    3.5200
```

Finally, five calls are made to the five MATLAB functions introduced in this chapter to calculate the five effective elastic constants of this laminate.

```
EDU>> H = 0.800
H =

    0.8000

EDU>> Ebarx(A,H)

ans =

   83.8448

EDU>> Ebary(A,H)

ans =

   83.8448

EDU>> NUbarxy(A,H)

ans =

    0.0359

EDU>> NUbaryx(A,H)

ans =

    0.0359

EDU>> Gbarxy(A,H)

ans =

    4.4000
```

MATLAB Example 9.3

Consider a six-layer $[\pm 30/0]_S$ graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.900 mm. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

Solution

This example is solved using MATLAB. First, the reduced stiffness matrix $[Q]$ for a typical layer using the MATLAB function *ReducedStiffness* as follows:

```
EDU>> Q = ReducedStiffness(155.0, 12.10, 0.248, 4.40)

Q =

  155.7478   3.0153         0
    3.0153  12.1584         0
         0        0    4.4000
```

Next, the transformed reduced stiffness matrix $[\bar{Q}]$ is calculated for each layer using the MATLAB function *Qbar* as follows:

```
EDU>> Qbar1 = Qbar(Q, 30)

Qbar1 =

   91.1488   31.7170   95.3179
   31.7170   19.3541   29.0342
   47.6589   14.5171   61.8034

EDU>> Qbar2 = Qbar(Q, -30)

Qbar2 =

   91.1488   31.7170  -95.3179
   31.7170   19.3541  -29.0342
  -47.6589  -14.5171   61.8034

EDU>> Qbar3 = Qbar(Q, 0)

Qbar3 =

  155.7478    3.0153         0
    3.0153   12.1584         0
         0         0    4.4000

EDU>> Qbar4 = Qbar(Q, 0)

Qbar4 =

  155.7478    3.0153         0
    3.0153   12.1584         0
         0         0    4.4000

EDU>> Qbar5 = Qbar(Q, -30)

Qbar5 =

   91.1488   31.7170  -95.3179
   31.7170   19.3541  -29.0342
  -47.6589  -14.5171   61.8034
```

```
EDU>> Qbar6 = Qbar(Q, 30)

Qbar6 =

   91.1488   31.7170   95.3179
   31.7170   19.3541   29.0342
   47.6589   14.5171   61.8034
```

Next, the distances z_k ($k = 1, 2, 3, 4, 5, 6, 7$) are calculated as follows:

```
EDU>> z1 = -0.450

z1 =

   -0.4500

EDU>> z2 = -0.300

z2 =

   -0.3000

EDU>> z3 = -0.150

z3 =

   -0.1500

EDU>> z4 = 0

z4 =

     0

EDU>> z5 = 0.150

z5 =

    0.1500

EDU>> z6 = 0.300

z6 =

    0.3000

EDU>> z7 = 0.450

z7 =
    0.4500
```

Next, the $[A]$ matrix is calculated using six calls to the MATLAB function *Amatrix* as follows:

```
EDU>> A = zeros(3,3)

A =

     0     0     0
     0     0     0
     0     0     0

EDU>> A = Amatrix(A,Qbar1,z1,z2)

A =

   13.6723    4.7575   14.2977
    4.7575    2.9031    4.3551
    7.1488    2.1776    9.2705

EDU>> A = Amatrix(A,Qbar2,z2,z3)

A =

   27.3446    9.5151    0.0000
    9.5151    5.8062    0.0000
    0.0000    0.0000   18.5410

EDU>> A = Amatrix(A,Qbar3,z3,z4)

A =

   50.7068    9.9674    0.0000
    9.9674    7.6300    0.0000
    0.0000    0.0000   19.2010

EDU>> A = Amatrix(A,Qbar4,z4,z5)

A =

   74.0690   10.4197    0.0000
   10.4197    9.4537    0.0000
    0.0000    0.0000   19.8610

EDU>> A = Amatrix(A,Qbar5,z5,z6)

A =
   87.7413   15.1772  -14.2977
   15.1772   12.3568   -4.3551
   -7.1488   -2.1776   29.1315
```

```
EDU>> A = Amatrix(A,Qbar6,z6,z7)

A =

  101.4136   19.9348    0.0000
   19.9348   15.2599    0.0000
    0.0000    0.0000   38.4020
```

Finally, five calls are made to the five MATLAB functions introduced in this chapter to calculate the five effective elastic constants of this laminate.

```
EDU>> H = 0.900

H =

    0.9000

EDU>> Ebarx(A,H)

ans =

   83.7466

EDU>> Ebary(A,H)

ans =

   12.6015

EDU>> NUbarxy(A,H)

ans =

    1.3063

EDU>> NUbaryx(A,H)

ans =

    0.1966

EDU>> Gbarxy(A,H)

ans =

   42.6689
```

Problems

Problem 9.1

Show that the effective shear modulus $\bar{G}_{xy}$ is *not* related to the effective extensional modulus $\bar{E}_x$ and the effective Poisson's ratio $\bar{\nu}_{xy}$ by the relation:

$$\bar{G}_{xy} = \frac{\bar{E}_x}{2(1 + \bar{\nu}_{xy})} \tag{9.15}$$

MATLAB Problem 9.2

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.600 mm and is stacked as a $[0/90]_S$ laminate. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.3

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.600 mm and is stacked as a $[0/90]_S$ laminate. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.4

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.900 mm and is stacked as a $[\pm 30/0]_S$ laminate. The six layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.5

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.800 mm and is stacked as a $[+30/0]_S$ laminate. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.6

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.800 mm and is stacked as a $[+30/0]_S$ laminate. The four layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.7

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.600 mm and is stacked as a $[+45/0/-30]_T$ laminate. The three layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.

MATLAB Problem 9.8

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.600 mm and is stacked as a $[+45/0/-30]_T$ laminate. The three layers are of equal thickness. Use MATLAB to determine the five effective elastic constants for this laminate.