10

Failure Theories of a Lamina

10.1 Basic Equations

In this chapter we present various failure theories for one single layer of the composite laminate, usually called a lamina. We use the following notation throughout this chapter for the various strengths or ultimate stresses:

σ_1^T : tensile strength in longitudinal direction.
σ_1^C : compressive strength in longitudinal direction.
σ_2^T : tensile strength in transverse direction.
σ_2^C : compressive strength in transverse direction.
τ_{12}^F : shear strength in the 1-2 plane.

where the strength means the ultimate stress or failure stress, the longitudinal direction is the fiber direction (1-direction), and the transverse direction is the 2-direction (perpendicular to the fiber).

We also use the following notation for the ultimate strains:

ε_1^T : ultimate tensile strain in the longitudinal direction.
ε_1^C : ultimate compressive strain in the longitudinal direction.
ε_2^T : ultimate tensile strain in the transverse direction.
ε_2^C : ultimate compressive strain in the transverse direction.
γ_{12}^F : ultimate shear strain in the 1-2 plane.

It is assumed that the lamina behaves in a linear elastic manner. For the longitudinal uniaxial loading of the lamina (see Fig. 10.1), we have the following elastic relations:

$$\sigma_1^T = E_1 \varepsilon_1^T \tag{10.1}$$
$$\sigma_1^C = E_1 \varepsilon_1^C \tag{10.2}$$

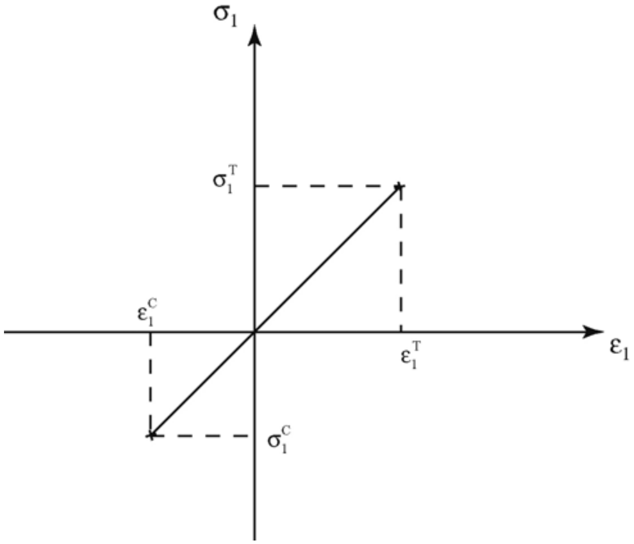

Fig. 10.1. Stress-strain curve for the longitudinal uniaxial loading of a lamina

where E_1 is Young's modulus of the lamina in the longitudinal (fiber) direction.

For the transverse uniaxial loading of the lamina (see Fig. 10.2), we have the following elastic relations:

$$\sigma_2^T = E_2 \varepsilon_2^T \tag{10.3}$$
$$\sigma_2^C = E_2 \varepsilon_2^C \tag{10.4}$$

where E_2 is Young's modulus of the lamina in the transverse direction. For the shear loading of the lamina (see Fig. 10.3), we have the following elastic relation:

$$\tau_{12}^F = G_{12} \gamma_{12}^F \tag{10.5}$$

where G_{12} is the shear modulus of the lamina.

10.1.1 Maximum Stress Failure Theory

In the *maximum stress failure theory*, failure of the lamina is assumed to occur whenever any normal or shear stress component equals or exceeds the corresponding strength. This theory is written mathematically as follows:

$$\sigma_1^C < \sigma_1 < \sigma_1^T \tag{10.6}$$
$$\sigma_2^C < \sigma_2 < \sigma_2^T \tag{10.7}$$
$$|\tau_{12}| < \tau_{12}^F \tag{10.8}$$

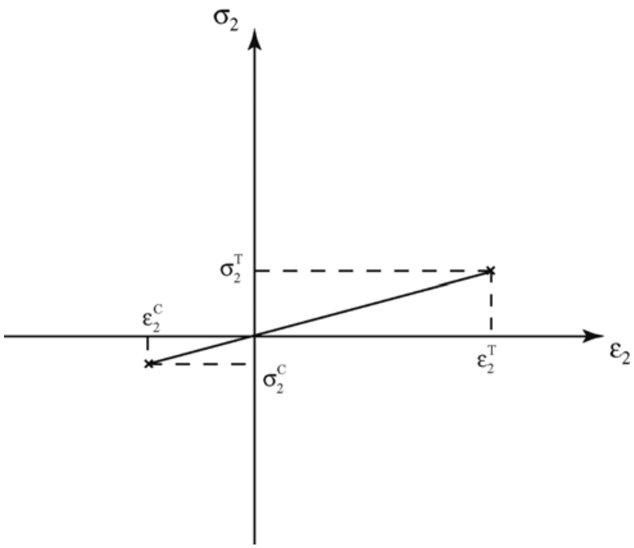

Fig. 10.2. Stress-strain curve for the transverse uniaxial loading of a lamina

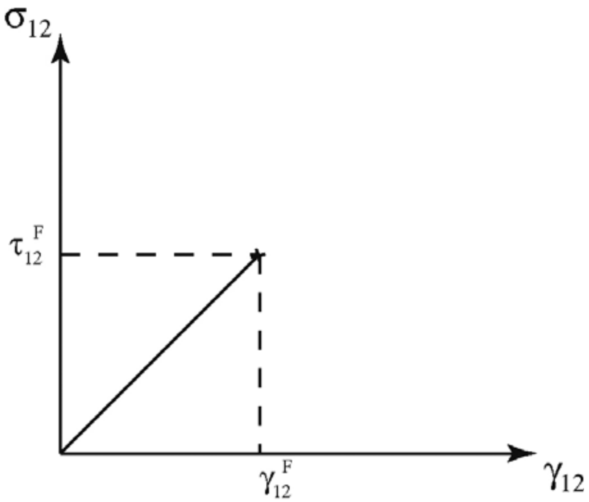

Fig. 10.3. Stress-strain curve for the shear loading of a lamina

where σ_1 and σ_2 are the maximum material normal stresses in the lamina, while τ_{12} is the maximum shear stress in the lamina.

The failure envelope for this theory is clearly illustrated in Fig. 10.4. The advantage of this theory is that it is simple to use but the major disadvantage is that there is no interaction between the stress components.

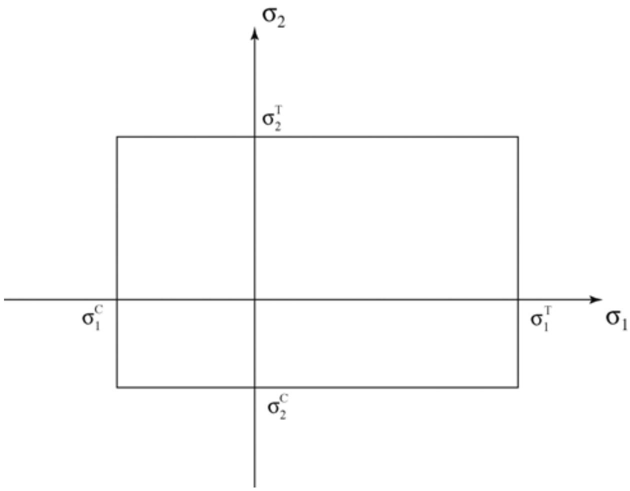

Fig. 10.4. Failure envelope for the maximum stress failure theory

10.1.2 Maximum Strain Failure Theory

In the *maximum strain failure theory*, failure of the lamina is assumed to occur whenever any normal or shear strain component equals or exceeds the corresponding ultimate strain. This theory is written mathematically as follows:

$$\varepsilon_1^C < \varepsilon_1 < \varepsilon_1^T \tag{10.9}$$

$$\varepsilon_2^C < \varepsilon_2 < \varepsilon_2^T \tag{10.10}$$

$$|\gamma_{12}| < \gamma_{12}^F \tag{10.11}$$

where ε_1, ε_2, and γ_{12} are the principal material axis strain components. In this case, we have the following relation between the strains and the stresses in the longitudinal direction:

$$\varepsilon_1 = \frac{\sigma_1^T}{E_1} = \frac{\sigma_1}{E_1} - \nu_{12}\frac{\sigma_2}{E_1} \tag{10.12}$$

Simplifying (10.12), we obtain:

$$\sigma_2 = \frac{\sigma_1 - \sigma_1^T}{\nu_{12}} \tag{10.13}$$

Similarly, we have the following relation between the strains and the stresses in the transverse direction:

$$\varepsilon_2 = \frac{\sigma_2^T}{E_2} = \frac{\sigma_2}{E_2} - \nu_{21}\frac{\sigma_1}{E_2} \tag{10.14}$$

Simplifying (10.14), we obtain:

$$\sigma_2 = \nu_{21}\sigma_1 + \sigma_2^T \tag{10.15}$$

The failure envelope for this theory is clearly shown in Fig. 10.5 (based on (10.13) and (10.15)). The advantage of this theory is that it is simple to use but the major disadvantage is that there is no interaction between the strain components.

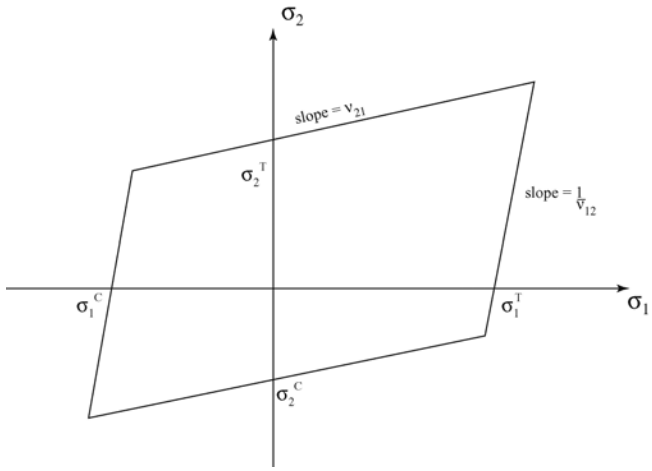

Fig. 10.5. Failure envelope for the maximum strain failure theory

Figure 10.6 shows the two failure envelopes of the maximum stress theory and the maximum strain theory superimposed on the same plot for comparison.

10.1.3 Tsai-Hill Failure Theory

The *Tsai-Hill failure theory* is derived from the von Mises distortional energy yield criterion for isotropic materials but is applied to anisotropic materials with the appropriate modifications. In this theory, failure is assumed to occur whenever the distortional yield energy equals or exceeds a certain value related to the strength of the lamina. In this theory, there is no distinction between the tensile and compressive strengths. Therefore, we use the following notation for the strengths of the lamina:

σ_1^F : strength in longitudinal direction.
σ_2^F : strength in transverse direction.
τ_{12}^F : shear strength in the 1-2 plane.

The Tsai-Hill failure theory is written mathematically for the lamina as follows:

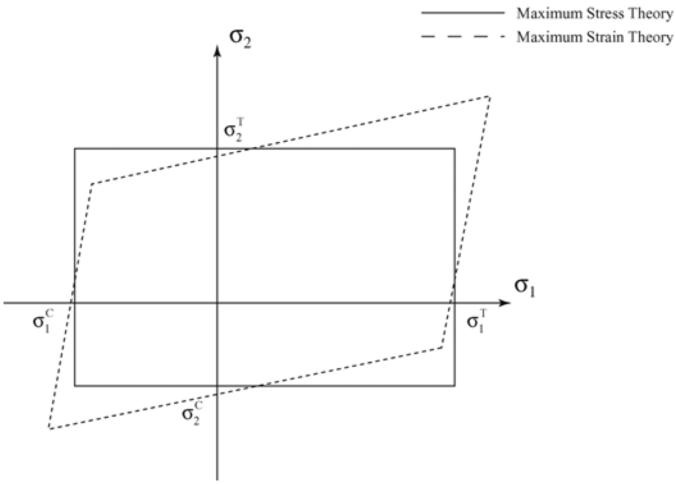

Fig. 10.6. Comparison of the failure envelopes for the maximum stress theory and maximum strain theory

$$\frac{\sigma_1^2}{\left(\sigma_1^F\right)^2} - \frac{\sigma_1\sigma_2}{\left(\sigma_1^F\right)^2} + \frac{\sigma_2^2}{\left(\sigma_2^F\right)^2} + \frac{\tau_{12}^2}{\left(\tau_{12}^F\right)^2} \leq 1 \tag{10.16}$$

The failure envelope for this theory is clearly shown in Fig. 10.7. The advantage of this theory is that there is interaction between the stress components. However, this theory does not distinguish between the tensile and compressive strengths and is not as simple to use as the maximum stress theory or the maximum strain theory.

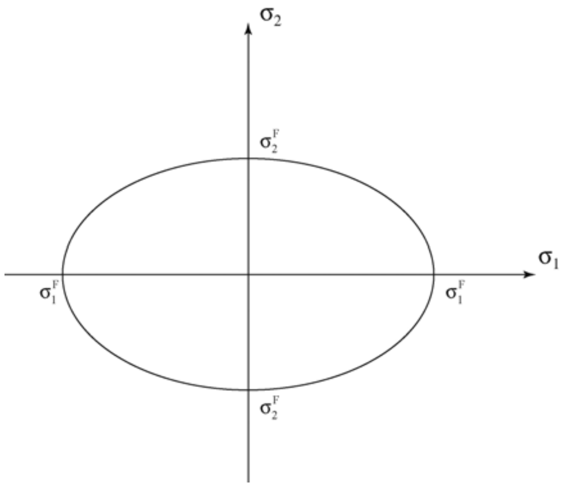

Fig. 10.7. Failure envelope for the Tsai-Hill failure theory

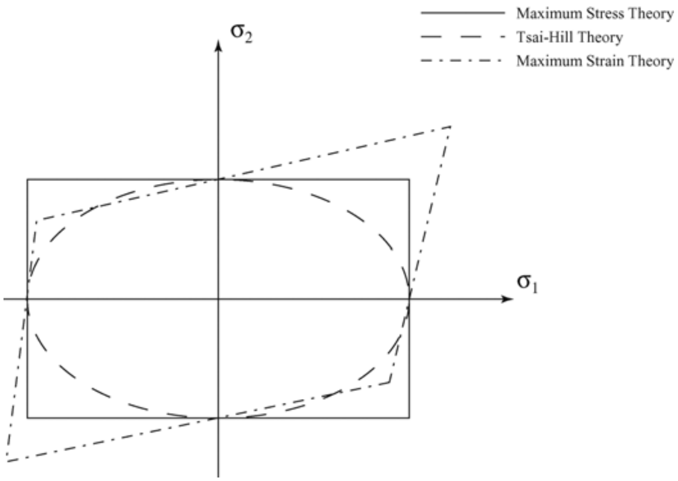

Fig. 10.8. Comparison between the three failure envelopes

Figure 10.8 shows the three failure envelopes of the maximum stress theory, the maximum strain theory, and the Tsai-Hill theory superimposed on the same plot for comparison.

10.1.4 Tsai-Wu Failure Theory

The *Tsai-Wu failure theory* is based on a total strain energy failure theory. In this theory, failure is assumed to occur in the lamina if the following condition is satisfied:

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + F_{12}\sigma_1\sigma_2 \leq 1 \tag{10.17}$$

where the coefficients F_{11}, F_{22}, F_{66}, F_1, F_2, and F_{12} are given by:

$$F_{11} = \frac{1}{\sigma_1^T \sigma_1^C} \tag{10.18}$$

$$F_{22} = \frac{1}{\sigma_2^T \sigma_2^C} \tag{10.19}$$

$$F_1 = \frac{1}{\sigma_1^T} - \frac{1}{\sigma_1^C} \tag{10.20}$$

$$F_1 = \frac{1}{\sigma_2^T} - \frac{1}{\sigma_2^C} \tag{10.21}$$

$$F_{66} = \frac{1}{\left(\tau_{12}^F\right)^2} \tag{10.22}$$

and F_{12} is a coefficient that is determined experimentally. Tsai-Hahn determined F_{12} to be given by the following approximate expression:

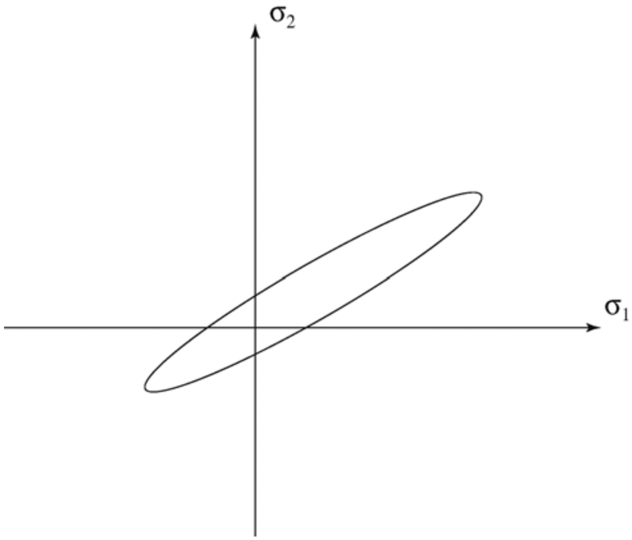

Fig. 10.9. A general failure ellipse for the Tsai-Wu failure theory

$$F_{12} \approx -\frac{1}{2}\sqrt{F_{11}F_{22}} \tag{10.23}$$

The failure envelope for this theory is shown in general in Fig. 10.9. The advantage of this theory is that there is interaction between the stress components and the theory does distinguish between the tensile and compressive strengths. A major disadvantage of this theory is that it is not simple to use.

Finally, in order to compare the failure envelopes for a composite lamina with the envelopes of isotropic ductile materials, Fig. 10.10 shows the failure envelopes for the usual von Mises and Tresca criteria for isotropic materials.

Problems

Problem 10.1

Determine the maximum value of $\alpha > 0$ if stresses of $\sigma_x = 3\alpha$, $\sigma_y = -2\alpha$, and $\tau_{xy} = 5\alpha$ are applied to a 60°-lamina of graphite/epoxy. Use the maximum stress failure theory. The material properties of this lamina are given as follows:

$$\begin{array}{ll}
V^f = 0.70 & \sigma_1^T = 1500\,\text{MPa} \\
E_1 = 181\,\text{GPa} & \sigma_1^C = 1500\,\text{MPa} \\
E_2 = 10.30\,\text{GPa} & \sigma_2^T = 40\,\text{MPa} \\
\nu_{12} = 0.28 & \sigma_2^C = 246\,\text{MPa} \\
G_{12} = 7.17\,\text{GPa} & \tau_{12}^F = 68\,\text{MPa}
\end{array}$$

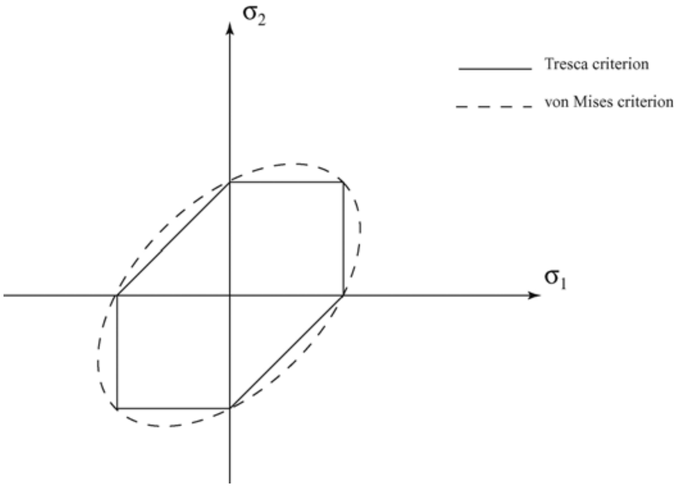

Fig. 10.10. Two failure criteria for ductile homogeneous materials

Problem 10.2

Repeat Problem 10.1 using the maximum strain failure theory instead of the maximum stress failure theory.

Problem 10.3

Repeat Problem 10.1 using the Tsai-Hill failure theory instead of the maximum stress failure theory.

Problem 10.4

Repeat Problem 10.1 using the Tsai-Wu failure theory instead of the maximum stress failure theory.

MATLAB Problem 10.5

Use MATLAB to plot the four failure envelopes using the strengths given in Problem 10.1.

Problem 10.6

Determine the maximum value of $\alpha > 0$ if stresses of $\sigma_x = 3\alpha$, $\sigma_y = -2\alpha$, and $\tau_{xy} = 5\alpha$ are applied to a 30°-lamina of glass/epoxy. Use the maximum stress failure theory. The material properties of this lamina are given as follows:

$$
\begin{aligned}
V^f &= 0.45 & \sigma_1^T &= 1062\,\text{MPa} \\
E_1 &= 38.6\,\text{GPa} & \sigma_1^C &= 610\,\text{MPa} \\
E_2 &= 8.27\,\text{GPa} & \sigma_2^T &= 31\,\text{MPa} \\
\nu_{12} &= 0.26 & \sigma_2^C &= 118\,\text{MPa} \\
G_{12} &= 4.14\,\text{GPa} & \tau_{12}^F &= 72\,\text{MPa}
\end{aligned}
$$

Problem 10.7

Repeat Problem 10.6 using the maximum strain failure theory instead of the maximum stress failure theory.

Problem 10.8

Repeat Problem 10.6 using the Tsai-Hill failure theory instead of the maximum stress failure theory.

Problem 10.9

Repeat Problem 10.6 using the Tsai-Wu failure theory instead of the maximum stress failure theory.

MATLAB Problem 10.10

Use MATLAB to plot the four failure envelopes using the strengths given in Problem 10.6.