12

Introduction to Damage Mechanics of Composite Materials

12.1 Basic Equations

The objective of this final chapter is to introduce the reader to the subject of damage mechanics of composite materials. For further details, the reader is referred to the comprehensive book on this subject written by the authors: *Advances in Damage Mechanics: Metals and Metal Matrix Composites.* This chapter does not contain any MATLAB functions or examples.

In this chapter, only elastic composites are considered. The fibers are assumed to be continuous and perfectly aligned. In addition, a perfect bond is assumed to exist at the matrix-fiber interface. A consistent mathematical formulation is presented in the next sections to derive the equations of damage mechanics for these composite materials using two different approaches: one overall and one local. The elastic stiffness matrix is derived using both these two approaches and is shown to be identical in both cases.

For simplicity, the composite system is assumed to consist of a matrix reinforced with continuous fibers. Both the matrix and fibers are linearly elastic with different material constants. Let $\bar{C}$ denote the configuration of the undamaged composite system and let $\bar{C}^m$ and $\bar{C}^f$ denote the configurations of the undamaged matrix and fibers, respectively. Since the composite system assumes a perfect bond at the matrix-fiber interface, it is clear that $\bar{C}^m \cap \bar{C}^f = \phi$ and $\bar{C}^m \cup \bar{C}^f = \bar{C}$. In the overall approach, the problem reduces to transforming the undamaged configuration $\bar{C}$ into the damaged configuration C. In contrast, two intermediate configurations C^m and C^f are considered in the local approach for the matrix and fibers, respectively. In the latter approach, the problem is reduced to transforming each of the undamaged configurations $\bar{C}^m$ and $\bar{C}^f$ into the damaged configurations C^m and C^f, respectively.

In case of elastic fiber-reinforced composites, the following linear relation is used for each constituent in its respective undamaged configuration:

$$\bar{\sigma}^k = \bar{E}^k : \bar{\varepsilon}^k, \qquad k = m, f \tag{12.1}$$

where $\bar{\sigma}^k$ is the effective constituent stress tensor, $\bar{\varepsilon}^k$ is the effective strain tensor, and $\bar{E}^k$ is the effective constituent elasticity tensor. The operation: denotes the tensor contraction operation over two indices. For the case of an isotropic constituent, $\bar{E}^k$ is given by the following formula:

$$\bar{E}^k = \bar{\lambda}^k I_2 \otimes I_2 + 2\bar{u}^k I_4 \tag{12.2}$$

where $\bar{\lambda}^k$ and $\bar{\mu}^k$ are the effective constituent Lame's constants, I_2 is the second-rank identity tensor, and I_4 is the fourth-rank identity tensor. The operation $\otimes$ is the tensor cross product between two second-rank tensors to produce a fourth-rank tensor.

Within the framework of the micromechanical analysis of composite materials, the effective constituent stress tensor $\bar{\sigma}^k$ is related to the effective composite stress tensor $\bar{\sigma}$ by the following relation:

$$\bar{\sigma}^k = \bar{B}^k : \bar{\sigma} \tag{12.3}$$

The fourth-rank tensor $\bar{B}^k$ is the constituent *stress concentration tensor*. It can be determined using several available homogenization models such as the Voigt and Mori-Tanaka models. The effective constituent strain tensor $\bar{\varepsilon}^k$ is determined in a similar way by the following relation:

$$\bar{\varepsilon}^k = \bar{A}^k : \bar{\varepsilon} \tag{12.4}$$

where $\bar{\varepsilon}$ is the effective composite strain tensor and $\bar{A}^k$ is the fourth-rank constituent strain concentration tensor.

Next, the overall and local approaches to damage in elastic composites are examined in the following two sections.

12.2 Overall Approach

In this approach, damage is incorporated in the composite system as a whole through one damage tensor called the *overall damage tensor*. The two steps needed in this approach are shown schematically in Fig. 12.1 for a two-phase composite system consisting of a matrix and fibers. In the first step, the elastic equations are formulated in an undamaged composite system. This is performed here using the law of mixtures as follows:

$$\bar{\sigma} = \bar{c}^m \bar{\sigma}^m + \bar{c}^f \bar{\sigma}^f \tag{12.5}$$

where $\bar{c}^m$ and $\bar{c}^f$ are the effective matrix and fiber volume fractions, respectively.

In the effective composite configuration $\bar{C}$, the following linear elastic relation holds:

$$\bar{\sigma} = \bar{E} : \bar{\varepsilon} \tag{12.6}$$

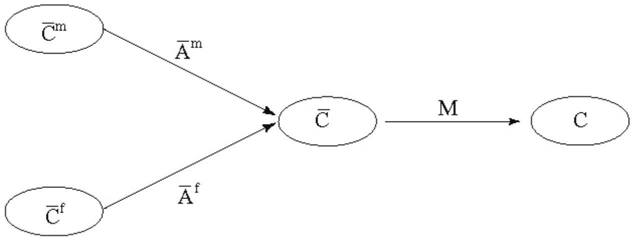

Fig. 12.1. Schematic diagram illustrating the overall approach for composite materials

where $\bar{E}$ is the fourth-rank constant elasticity tensor. Substituting (12.1), (12.4), and (12.6) into (12.5) and simplifying, one obtains the following expression for $\bar{E}$:

$$\bar{E} = \bar{c}^m \bar{E}^m : \bar{A}^m + \bar{c}^f \bar{E}^f : \bar{A}^f \tag{12.7}$$

In the second step of the formulation, damage is induced through the fourth-rank damage effect tensor M as follows:

$$\bar{\sigma} = M : \sigma \tag{12.8}$$

where σ is the composite stress tensor. Equation (12.8) represents the damage transformation equation for the stress tensor. In order to derive a similar relation for the strain tensor, one needs to use the *hypothesis of elastic energy equivalence*. In this hypothesis, the elastic energy of the damage system is equal to the elastic energy of the effective system. Applying this hypothesis to the composite system by equating the two elastic energies, one obtains:

$$\frac{1}{2}\varepsilon : \sigma = \frac{1}{2}\bar{\varepsilon} : \bar{\sigma} \tag{12.9}$$

where ε is the composite strain tensor. Substituting (12.8) into (12.9) and simplifying, one obtains the damage transformation equation for the strain tensor as follows:

$$\bar{\varepsilon} = M^{-T} : \varepsilon \tag{12.10}$$

where the superscript $-T$ denotes the inverse transpose of the tensor.

In order to derive the final relation in the damaged composite system, one substitutes (12.8) and (12.10) into (12.6) to obtain:

$$\sigma = E : \varepsilon \tag{12.11}$$

where the fourth-rank elasticity tensor E is given by:

$$E = M^{-1} : \bar{E} : M^{-T} \tag{12.12a}$$

Substituting for $\bar{E}$ from (12.7) into (12.12a), one obtains:

$$E = M^{-1} : \left(\bar{c}^m \bar{E}^m : \bar{A}^m + \bar{c}^f \bar{E}^f : \bar{A}^f\right) : M^{-T} \tag{12.12b}$$

The above equation represents the elasticity tensor in the damaged composite system according to the overall approach.

12.3 Local Approach

In this approach, damage is introduced in the first step of the formulation using two independent damage tensors for the matrix and fibers. However, more damage tensors may be introduced to account for other types of damage such as debonding and delamination. The two steps involved in this approach are shown schematically in Fig. 12.2. One first introduces the fourth-rank *matrix and fiber damage effect tensors* M^m and M^f, respectively, as follows:

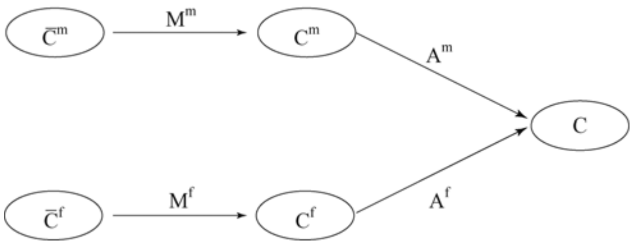

Fig. 12.2. Schematic diagram illustrating the local approach for composite materials

$$\bar{\sigma}^k = M^k : \sigma^k \ , \qquad k = m, f \tag{12.13}$$

The above equation can be interpreted in a similar way to (12.8) except that it applies at the constituent level. It also represents the damage transformation equation for each constituent stress tensor. In order to derive a similar transformation equation for the constituent strain tensor, one applies the hypothesis of elastic energy equivalence to each constituent separately as follows:

$$\frac{1}{2}\varepsilon^k : \sigma^k = \frac{1}{2}\bar{\varepsilon}^k : \bar{\sigma}^k \ , \qquad k = m, f \tag{12.14}$$

In using (12.14), one assumes that there are no micromechanical or constituent elastic interactions between the matrix and fibers. This assumption is not valid in general. From micromechanical considerations, there should be interactions between the elastic energies in the matrix and fibers. However, such interactions are beyond the scope of this book as the resulting equations will be complicated and the sought relations may consequently be unattainable. It should be clear to the reader that (12.14) is the single most important assumption that is needed to derive the relations of the local approach. It will also be needed later when we show the equivalence of the overall and local approaches. Therefore, the subsequent relations are very special cases when (12.14) is valid.

Substituting (12.13) into (12.14) and simplifying, one obtains the required transformations for the constituent strain tensor as follows:

$$\bar{\varepsilon}^k = M^{k^{-T}} : \varepsilon^k \ , \qquad k = m, f \tag{12.15}$$

The above equation implies a decoupling between the elastic energy in the matrix and fibers. Other methods may be used that include some form of coupling but they will lead to complicated transformation equations that are beyond the scope of this book.

Substituting (12.13) and (12.15) into (12.1) and simplifying, one obtains:

$$\sigma^k = E^k : \varepsilon^k \, , \qquad k = m, f \tag{12.16}$$

where the constituent elasticity tensor E^k is given by:

$$E^k = M^{k^{-1}} : \bar{E}^k : M^{k^{-T}} \, , \qquad k = m, f \tag{12.17}$$

Equation (12.16) represents the elasticity relation for the damaged constituents. The second step of the formulation involves transforming (12.17) into the whole composite system using the law of mixtures as follows:

$$\sigma = c^m \sigma^m + c^f \sigma^f \tag{12.18}$$

where c^m and c^f are the matrix and fiber volume fractions, respectively, in the damaged composite system. Before proceeding with (12.18), one needs to derive a strain constituent equation similar to (12.4). Substituting (12.10) and (12.15) into (12.4) and simplifying, one obtains:

$$\varepsilon^k = A^k : \varepsilon \, , \qquad k = m, f \tag{12.19}$$

where the constituent strain concentration tensor A^k in the damaged state is given by:

$$A^k = M^{k^T} : \bar{A}^k : M^{-T} \, , \qquad k = m, f \tag{12.20}$$

The above equation represents the damage transformation equation for the strain concentration tensor.

Finally, one substitutes (12.11), (12.16), and (12.19) into (12.18) and simplifies to obtain:

$$E = c^m E^m : A^m + c^f E^f : A^f \tag{12.21}$$

Equation (12.21) represents the elasticity tensor in the damaged composite system according to the local approach.

12.4 Final Remarks

In this final section, it is shown that both the overall and local approaches are equivalent elastic composites which are considered here. This proof is performed by showing that both the elasticity tensors given in (12.12b) and (12.21) are exactly the same. In fact, it is shown that (12.21) reduces to (12.12b) after making the appropriate substitution.

First, one needs to find a damage transformation equation for the volume fractions. This is performed by substituting (12.8) and (12.13) into (12.5), simplifying and comparing the result with (12.18). One therefore obtains:

$$c^k I_4 = \bar{c}^k M^{-1} : M^k \ , \qquad k = m, f \tag{12.22}$$

where I_4 is the fourth-rank identity tensor. Substituting (12.17) and (12.20) into (12.21) and simplifying, one obtains:

$$E = \left(c^m M^{m^{-1}} : \bar{E}^m : \bar{A}^m + c^f M^{f^{-1}} : \bar{E}^f : \bar{A}^f \right) : M^{-T} \tag{12.23}$$

Finally, one substitutes (12.22) into (12.23) and simplifies to obtain:

$$E = M^{-1} : \left(\bar{c}^m \bar{E}^m : \bar{A}^m + \bar{c}^f \bar{E}^f : \bar{A}^f \right) : M^{-T} \tag{12.24}$$

It is clear that the above equation is the same as (12.12b). Therefore, both the overall and local approaches yield the same elasticity tensor in the damaged composite system.

Equation (12.24) can be generalized to an elastic composite system with n constituents as follows:

$$E = M^{-1} : \left(\sum_{k=1}^{n} \bar{c}^k \bar{E}^k : \bar{A}^k \right) : M^{-T} \tag{12.25}$$

The two formulations of the overall and local approaches can be used to obtain the above equation for a composite system with n constituents. The derivation of (12.25) is similar to the derivation of (12.24) – therefore it is not presented here and is left to the problems.

In the remaining part of this section, some additional relations are presented to relate the overall damage effect tensor with the constituent damage effect tensors. Substituting (12.3) into (12.5) and simplifying, one obtains the constraint equation for the stress concentration tensors. The constraint equation is generalized as follows:

$$\sum_{k=1}^{n} \bar{c}^k \bar{B}^k = I_4 \tag{12.26}$$

where I_4 is the fourth-rank identity tensor. To find a relation between the stress concentration tensors in the effective and damaged states, one substitutes (12.8) and (12.13) into (12.3) and simplifies to obtain:

$$\sigma^k = B^k : \sigma \ , \qquad k = 1, 2, 3, \ldots, n \tag{12.27}$$

where B^k is the fourth-rank stress concentration tensor in the damaged configuration and is given by:

$$B^k = M^{k^{-1}} : \bar{B}^k : M \ , \qquad k = 1, 2, 3, \ldots, n \tag{12.28}$$

Substituting (12.27) into (12.18) and simplifying, the resulting constraint is generalized as follows:

$$\sum_{k=1}^{n} c^k B^k = I_4 \tag{12.29}$$

Finally, substituting (12.28) into (12.29) and simplifying, one obtains:

$$M = \left(\sum_{k=1}^{n} c^k M^{k^{-1}} : \bar{B}^k \right)^{-1} \tag{12.30}$$

Equation (12.30) represents the required relation between the overall and local (constituent) damage effect tensors.

Problems

Problem 12.1

Consider a composite system that consists of n constituents. In this case, the overall approach is schematically illustrated in Fig. 12.3. In this case, derive (12.25) in detail.

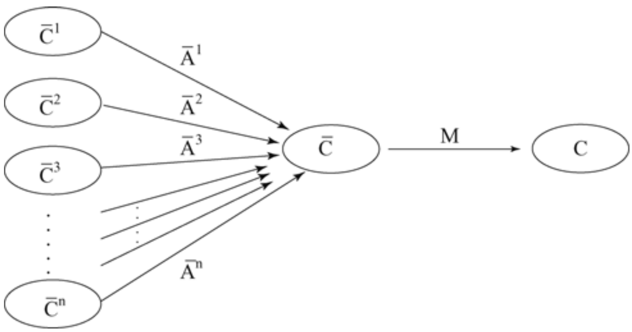

Fig. 12.3. Schematic diagram illustrating the overall approach for composite materials for Problem 12.1

Problem 12.2

Consider a composite system that consists of n constituents. In this case, the local approach is schematically illustrated in Fig. 12.4. In this case, derive (12.25) in detail.

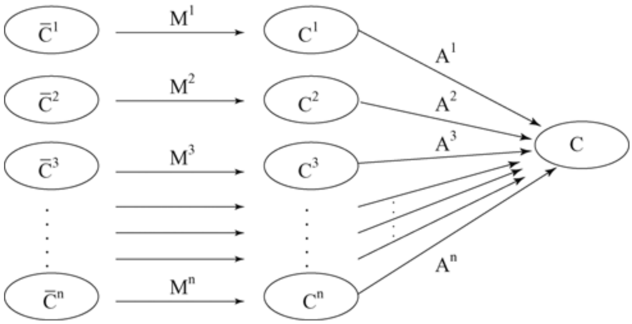

Fig. 12.4. Schematic diagram illustrating the local approach for composite materials for Problem 12.2

Problem 12.3

Derive (12.20) in detail.

Problem 12.4

Derive (12.28) in detail.

Problem 12.5

The stress and strain concentration tensors are usually determined using one of the following four models:

1. The Voigt model.
2. The Reuss model.
3. The Mori-Tanaka model.
4. The Eshelby Tensor.

Make a literature search on the above four models and describe each model briefly writing its basic equations.