

---

## Laminate Analysis – Part I

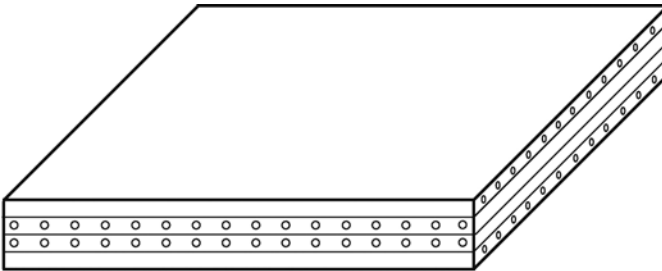
### 7.1 Basic Equations

Fiber-reinforced materials consist usually of multiple layers of material to form a *laminate*. Each layer is thin and may have a different fiber orientation – see Fig. 7.1. Two laminates may have the same number of layers and the same fiber angles but the two laminates may be different because of the arrangement of the layers.

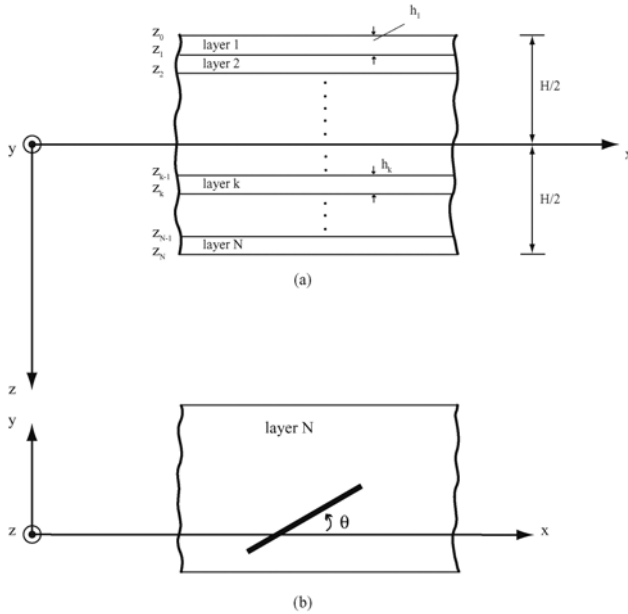
In this chapter, we will evaluate the influence of fiber directions, stacking arrangements and material properties on laminate and structural response. We will study a simplified theory called *classical lamination theory* for this purpose (see [1]).

Figure 7.2 shows a global Cartesian coordinate system and a general laminate consisting of  $N$  layers. The laminate thickness is denoted by  $H$  and the thickness of an individual layer by  $h$ . Not all layers necessarily have the same thickness, so the thickness of the  $k$ th layer is denoted by  $h_k$ .

The origin of the through-thickness coordinate, designated  $z$ , is located at the laminate geometric midplane. The geometric midplane may be within a particular layer or at an interface between layers. We consider the  $+z$  axis to be downward and the laminate extends in the  $z$  direction from  $-H/2$  to  $+H/2$ . We refer to the layer at the most negative location as layer 1, the next layer in as layer 2, the layer at an arbitrary location as layer  $k$ , and the layer at the most positive  $z$  position as layer  $N$ . The locations of the layer interfaces are denoted by a subscripted  $z$ ; the



**Fig. 7.1.** Schematic illustration of a laminate with four layers



**Fig. 7.2.** Schematic illustration showing a cross-section and a plan view

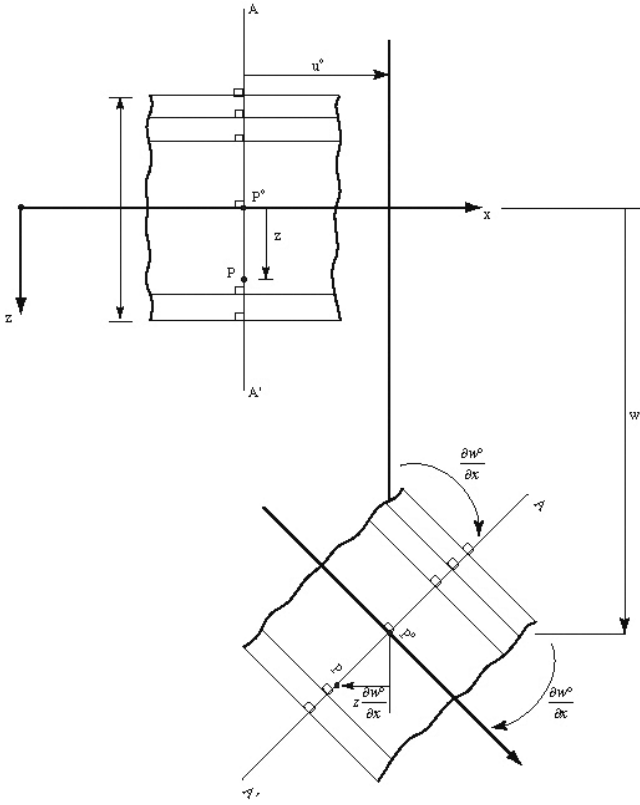
first layer is bounded by locations  $z_0$  and  $z_1$ , the second layer by  $z_1$  and  $z_2$ , the  $k$ th layer by  $z_{k-1}$  and  $z_k$ , and the  $N$ th layer by  $z_{N-1}$  and  $z_N$  [1].

Let us examine the deformation of an  $x$ - $z$  cross-section [1]. Figure 7.3 shows in detail the deformation of a cross-section, and in particular the displacements of point  $P$ , a point located at an arbitrary distance  $z$  below point  $P^0$ , a point on the reference surface, with points  $P$  and  $P^0$  being on line  $AA'$ . The superscript 0 will be reserved to denote the kinematics of point  $P^0$  on the reference surface. In particular, the horizontal translation of point  $P^0$  in the  $x$  direction will be denoted by  $u^0$ . The vertical translation will be denoted by  $w^0$ . The rotation of the reference surface about the  $y$  axis at point  $P^0$  is  $\partial w^0/\partial x$ . An important part of the Kirchhoff hypothesis is the assumption that line  $AA'$  remains perpendicular to the reference surface. Because of this, the rotation of line  $AA'$  is the same as the rotation of the reference surface, and thus the rotation of line  $AA'$ , as viewed in the  $x$ - $z$  plane, is  $\partial w^0/\partial x$ . It is assumed that [1]:

$$\frac{\partial w^0}{\partial x} < 1 \tag{7.1}$$

By less than unity is meant that sines and tangents of angles of rotation are replaced by the rotations themselves, and cosines of the angles of rotation are replaced by 1. With this approximation, then, the rotation of point  $P^0$  causes point  $P$  to translate horizontally in the minus  $x$  direction by an amount equal to:

$$z = \frac{\partial w^0}{\partial x} \tag{7.2}$$



**Fig. 7.3.** Schematic illustration showing the kinematics of deformation of a laminate

Therefore, the horizontal translation of a point  $P$  with coordinates  $(x, y, z)$  in the direction of the  $x$ -axis is then given by:

$$u(x, y, z) = u^0(x, y) - z \frac{\partial w^0(x, y)}{\partial x} \tag{7.3}$$

Also, the vertical translation of point  $P$  in the direction of the  $z$ -axis is given by:

$$w(x, y, z) = w^0(x, y) \tag{7.4}$$

The horizontal translation of point  $P$  in the direction of the  $y$ -axis is similar to that in the direction of the  $x$ -axis and is given by:

$$v(x, y, z) = v^0(x, y) - z \frac{\partial w^0(x, y)}{\partial y} \tag{7.5}$$

Therefore, we now have the following relations:

$$u(x, y, z) = u^0(x, y) - z \frac{\partial w^0(x, y)}{\partial x} \tag{7.6a}$$

$$v(x, y, z) = v^0(x, y) - z \frac{\partial w^0(x, y)}{\partial y} \quad (7.6b)$$

$$w(x, y, z) = w^0(x, y) \quad (7.6c)$$

Next, we investigate the strains that result from the displacements according to the Kirchhoff hypothesis. This can be done by using the strain-displacement relations from the theory of elasticity. Using these relations and (7.6a,b,c), we can compute the strains at any point within the laminate, and by using these laminate strains in the stress-strain relations, we can compute the stresses at any point within the laminate.

From the strain-displacement relations and (7.6a), the extensional strain in the  $x$  direction,  $\varepsilon_x$ , is given by:

$$\varepsilon_x(x, y, z) \equiv \frac{\partial u(x, y, z)}{\partial x} = \frac{\partial u^0(x, y)}{\partial x} - z \frac{\partial^2 w^0(x, y)}{\partial x^2} \quad (7.7)$$

Equation (7.7) may be re-written as follows:

$$\varepsilon_x(x, y, z) = \varepsilon_x^0(x, y) + z\kappa_x^0(x, y) \quad (7.8)$$

where the following notation is used:

$$\varepsilon_x^0(x, y) = \frac{\partial u^0(x, y)}{\partial x} \quad (7.9a)$$

$$\kappa_x^0(x, y) = -\frac{\partial^2 w^0(x, y)}{\partial x^2} \quad (7.9b)$$

The quantity  $\varepsilon_x^0$  is referred to as the extensional strain of the reference surface in the  $x$  direction, and  $\kappa_x^0$  is referred to as the curvature of the reference surface in the  $x$  direction. The other five strain components are given by:

$$\varepsilon_y(x, y, z) \equiv \frac{\partial v(x, y, z)}{\partial y} = \varepsilon_y^0(x, y) + z\kappa_y^0(x, y) \quad (7.10a)$$

$$\varepsilon_z(x, y, z) \equiv \frac{\partial w(x, y, z)}{\partial z} = \frac{\partial w^0(x, y)}{\partial z} = 0 \quad (7.10b)$$

$$\begin{aligned} \gamma_{yz}(x, y, z) &\equiv \frac{\partial w(x, y, z)}{\partial y} + \frac{\partial v(x, y, z)}{\partial z} \\ &= \frac{\partial w^0(x, y)}{\partial y} - \frac{\partial w^0(x, y)}{\partial y} = 0 \end{aligned} \quad (7.10c)$$

$$\begin{aligned} \gamma_{xz}(x, y, z) &\equiv \frac{\partial w(x, y, z)}{\partial x} + \frac{\partial u(x, y, z)}{\partial z} \\ &= \frac{\partial w^0(x, y)}{\partial x} - \frac{\partial w^0(x, y)}{\partial x} = 0 \end{aligned} \quad (7.10d)$$

$$\gamma_{xy}(x, y, z) \equiv \frac{\partial v(x, y, z)}{\partial x} + \frac{\partial u(x, y, z)}{\partial y} = \gamma_{xy}^0 + z\kappa_{xy}^0 \quad (7.10e)$$

where the following notation is used:

$$\varepsilon_y^0(x, y) = \frac{\partial v^0(x, y)}{\partial y} \quad (7.11a)$$

$$\kappa_y^0(x, y) = -\frac{\partial^2 w^0(x, y)}{\partial y^2} \quad (7.11b)$$

$$\gamma_{xy}^0(x, y) = \frac{\partial v^0(x, y)}{\partial x} + \frac{\partial u^0(x, y)}{\partial y} \quad (7.11c)$$

$$\kappa_{xy}^0(x, y) = -2\frac{\partial^2 w^0(x, y)}{\partial x \partial y} \quad (7.11d)$$

The quantities  $\varepsilon_y^0$ ,  $\kappa_y^0$ ,  $\gamma_{xy}^0$ , and  $\kappa_{xy}^0$  are referred to as the reference surface extensional strain in the  $y$  direction, the reference surface curvature in the  $y$  direction, the reference surface inplane shear strain, and the reference surface twisting curvature, respectively.

The second important assumption of classical lamination theory is that each point within the volume of a laminate is in a state of plane stress. Therefore, we can compute the stresses if we know the strains and curvatures of the reference surface. Accordingly, using the strains that result from the Kirchhoff hypothesis, (7.8) and (7.10a, e), we find that the stress-strain relations for a laminate become:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 + z\kappa_x^0 \\ \varepsilon_y^0 + z\kappa_y^0 \\ \gamma_{xy}^0 + z\kappa_{xy}^0 \end{Bmatrix} \quad (7.12)$$

Finally the force and moment resultants in the laminate can be computed using the stresses as follows:

$$N_x = \int_{-H/2}^{H/2} \sigma_x dz \quad (7.13a)$$

$$N_y = \int_{-H/2}^{H/2} \sigma_y dz \quad (7.13b)$$

$$N_{xy} = \int_{-H/2}^{H/2} \tau_{xy} dz \quad (7.13c)$$

$$M_x = \int_{-H/2}^{H/2} \sigma_x z dz \quad (7.13d)$$

$$M_y = \int_{-H/2}^{H/2} \sigma_y z dz \quad (7.13e)$$

$$M_{xy} = \int_{-H/2}^{H/2} \tau_{xy} z dz \quad (7.13f)$$

## 7.2 MATLAB Functions Used

The only MATLAB function used in this chapter to calculate the strains is:

*Strains*(*eps\_xo*, *eps\_yo*, *gam\_xyo*, *kap\_xo*, *kap\_yo*, *kap\_xyo*, *z*) – This function calculates the three strains  $\varepsilon_x$ ,  $\varepsilon_y$ , and  $\gamma_{xy}$  at any point  $P$  on the normal line given the three strains  $\varepsilon_x^0$ ,  $\varepsilon_y^0$ ,  $\gamma_{xy}^0$  and the three curvatures  $\kappa_x^0$ ,  $\kappa_y^0$ ,  $\kappa_{xy}^0$  at point  $P^0$ , and the distance  $z$  between  $P$  and  $P^0$ . There are seven input arguments to this function. The function returns the  $3 \times 1$  strain vector.

The following is a listing of the MATLAB source code for this function:

---

```
function y = Strains(eps_xo,eps_yo,gam_xyo,kap_xo,kap_yo,kap_xyo,z)
%Strains This function returns the strain vector at any point P
% along the normal line at distance z from point P0 which
% lies on the reference surface. There are seven input
% arguments for this function - namely the three strains
% and three curvatures at point P0 and the distance z.
% The size of the strain vector is 3 x 1.
epsilon_x = eps_xo + z * kap_xo;
epsilon_y = eps_yo + z * kap_yo;
gamma_xy = gam_xyo + z * kap_xyo;
y = [epsilon_x ; epsilon_y ; gamma_xy];
```

---

## MATLAB Example 7.1

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.500 mm and is stacked as a  $[0/90]_S$  laminate. The four layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\varepsilon_x^0 &= 400 \times 10^{-6} \\ \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_x^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

### Solution

This example is solved using MATLAB. First the strains are calculated at the five interface locations using the MATLAB function *Strains* as follows:

```
>> epsilon1 = Strains(400e-6,0,0,0,0,0,-0.250e-3)
```

```
epsilon1 =
```

```
1.0e-003 *  
0.4000  
0  
0
```

```
>> epsilon2 = Strains(400e-6,0,0,0,0,0,-0.125e-3)
```

```
epsilon2 =
```

```
1.0e-003 *  
0.4000  
0  
0
```

```
>> epsilon3 = Strains(400e-6,0,0,0,0,0,0)
```

```
epsilon3 =
```

```
1.0e-003 *  
0.4000  
0  
0
```

```
>> epsilon4 = Strains(400e-6,0,0,0,0,0,0.125e-3)
```

```
epsilon4 =
```

```
1.0e-003 *  
0.4000  
0  
0
```

```
>> epsilon5 = Strains(400e-6,0,0,0,0,0,0.250e-3)
```

```
epsilon5 =
```

```
1.0e-003 *  
0.4000  
0  
0
```

Next, the reduced stiffness  $[Q]$  in GPa is calculated for this material using the MATLAB function *ReducedStiffness* as follows:

```
>> Q = ReducedStiffness(155.0, 12.10, 0.248, 4.40)
```

Q =

155.7478	3.0153	0
3.0153	12.1584	0
0	0	4.4000

The transformed reduced stiffnesses  $[\bar{Q}]$  in GPa for the four layers are now calculated using the MATLAB function *Qbar* as follows:

```
>> Qbar1 = Qbar(Q,0)
```

Qbar1 =

155.7478	3.0153	0
3.0153	12.1584	0
0	0	4.4000

```
>> Qbar2 = Qbar(Q,90)
```

Qbar2 =

12.1584	3.0153	-0.0000
3.0153	155.7478	0.0000
-0.0000	0.0000	4.4000

```
>> Qbar3 = Qbar(Q,90)
```

Qbar3 =

12.1584	3.0153	-0.0000
3.0153	155.7478	0.0000
-0.0000	0.0000	4.4000

```
>> Qbar4 = Qbar(Q,0)
```

Qbar4 =

155.7478	3.0153	0
3.0153	12.1584	0
0	0	4.4000

Next, the stresses in each layer are calculated in MPa. Note that the stress vector is calculated twice for each layer – once at the top of the layer and once at the bottom of the layer.



```
>> sigma1a = Qbar1*epsilon1*1e3
```

```
sigma1a =
```

```
62.2991  
1.2061  
0
```

```
>> sigma1b = Qbar1*epsilon2*1e3
```

```
sigma1b =
```

```
62.2991  
1.2061  
0
```

```
>> sigma2a = Qbar2*epsilon2*1e3
```

```
sigma2a =
```

```
4.8634  
1.2061  
-0.0000
```

```
>> sigma2b = Qbar2*epsilon3*1e3
```

```
sigma2b =
```

```
4.8634  
1.2061  
-0.0000
```

```
>> sigma3a = Qbar3*epsilon3*1e3
```

```
sigma3a =
```

```
4.8634  
1.2061  
-0.0000
```

```
>> sigma3b = Qbar3*epsilon4*1e3
```

```
sigma3b =
```

```
4.8634  
1.2061  
-0.0000
```

```
>> sigma4a = Qbar4*epsilon4*1e3
```

```
sigma4a =
```

```
62.2991
 1.2061
      0
```

```
>> sigma4b = Qbar4*epsilon5*1e3
```

```
sigma4b =
```

```
62.2991
 1.2061
      0
```

Next, we setup the  $y$ -axis for the three plots:

```
>> y = [0.250 0.125 0.125 0 0 -0.125 -0.125 -0.250]
```

```
y =
```

```
0.2500  0.1250  0.1250  0  0  -0.1250  -0.1250
-0.2500
```

The distribution of the stress  $\sigma_x$  along the depth of the laminate is now plotted as follows (see Fig. 7.4):

```
>> x = [sigma4b(1) sigma4a(1) sigma3b(1) sigma3a(1) sigma2b(1)
        sigma2a(1) sigma1b(1) sigma1a(1)]
```

```
x =
```

```
62.2991  62.2991  4.8634  4.8634  4.8634  4.8634  62.2991
62.2991
```

```
>> plot(x,y)
>> xlabel('\sigma_x (MPa)')
>> ylabel('z (mm)')
```

The distribution of the stress  $\sigma_y$  along the depth of the laminate is now plotted as follows (see Fig. 7.5):

```
>> x = [sigma4b(2) sigma4a(2) sigma3b(2) sigma3a(2) sigma2b(2)
        sigma2a(2) sigma1b(2) sigma1a(2)]
```

```
x =
```

```
1.2061  1.2061  1.2061  1.2061  1.2061  1.2061  1.2061
1.2061
```

```
>> plot(x,y)
>> ylabel('z (mm)')
>> xlabel('\sigma_y (MPa)')
```

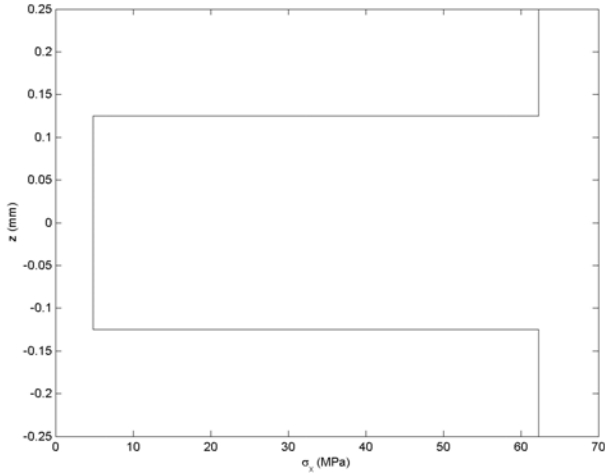


Fig. 7.4. Variation of  $\sigma_x$  versus  $z$  for Example 7.1

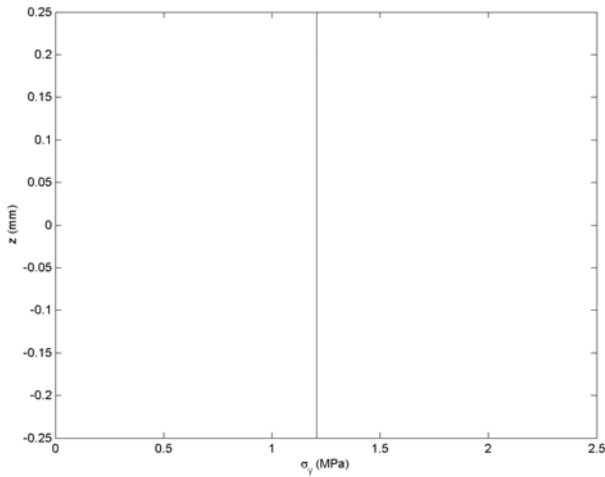


Fig. 7.5. Variation of  $\sigma_y$  versus  $z$  for Example 7.1

The distribution of the stress  $\tau_{xy}$  along the depth of the laminate is now plotted as follows (see Fig. 7.6):

```
>> x = [sigma4b(3) sigma4a(3) sigma3b(3) sigma3a(3) sigma2b(3) sigma2a(3)
sigma1b(3) sigma1a(3)]
```

```
x =
```

```
1.0e-015 *
```

```
0 0 -0.1162 -0.1162 -0.1162 -0.1162 0 0
```

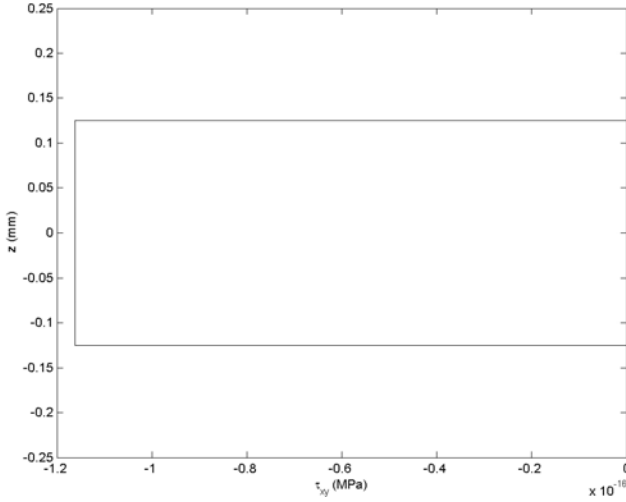


Fig. 7.6. Variation of  $\tau_{xy}$  versus  $z$  for Example 7.1

```
>> plot(x,y)
>> ylabel('z (mm)')
>> xlabel('\tau_{xy} (MPa)')
```

Next, the three force resultants are calculated in MN/m using (7.13a,b,c) as follows:

```
>> Nx = 0.125e-3 * (sigma1a(1) + sigma2a(1) + sigma3a(1) + sigma4a(1))
```

Nx =  
0.0168

```
>> Ny = 0.125e-3 * (sigma1a(2) + sigma2a(2) + sigma3a(2) + sigma4a(2))
```

Ny =  
6.0306e-004

```
>> Nxy = 0.125e-3 * (sigma1a(3) + sigma2a(3) + sigma3a(3) + sigma4a(3))
```

Nxy =  
-2.9043e-020

Next, the three moment resultants are calculated in MN.m/m using (7.13d,e,f) as follows:

```
>> Mx = sigma1a(1)*((-0.125e-3)^2 - (-0.250e-3)^2)/2 + sigma2a(1)*(0 -
(-0.125e-3)^2)/2 + sigma3a(1)*((0.125e-3)^2 - 0)/2 + sigma4a(1)*
((0.250e-3)^2 - (0.125e-3)^2)/2
```

```
Mx =
```

```
0
```

```
>> My = sigma1a(2)*((-0.125e-3)^2 - (-0.250e-3)^2)/2 + sigma2a(2)*(0 -
(-0.125e-3)^2)/2 + sigma3a(2)*((0.125e-3)^2 - 0)/2 + sigma4a(2)*
((0.250e-3)^2 - (0.125e-3)^2)/2
```

```
My =
```

```
3.3087e-024
```

```
>> Mxy = sigma1a(3)*((-0.125e-3)^2 - (-0.250e-3)^2)/2 + sigma2a(3)*(0 -
(-0.125e-3)^2)/2 + sigma3a(3)*((0.125e-3)^2 - 0)/2 + sigma4a(3)*
((0.250e-3)^2 - (0.125e-3)^2)/2
```

```
Mxy =
```

```
0
```

Next, the transformation matrix is calculated for each one of the four layers using the MATLAB function  $T$  as follows:

```
>> T1 = T(0)
```

```
T1 =
```

```
1    0    0
0    1    0
0    0    1
```

```
>> T2 = T(90)
```

```
T2 =
```

```
0.0000    1.0000    0.0000
1.0000    0.0000   -0.0000
-0.0000    0.0000   -1.0000
```

```
>> T3 = T(90)
```

```
T3 =
```

```
0.0000    1.0000    0.0000
1.0000    0.0000   -0.0000
-0.0000    0.0000   -1.0000
```

```
>> T4 = T(0)
```

```
T4 =
```

```

1    0    0
0    1    0
0    0    1
```

The strain vector is now calculated in each layer with respect to the principal material system as follows. Note that the strain vector is calculated twice for each layer – once at the top of the layer and once at the bottom of the layer. Notice also that in this case there is no need to correct the strain vector for the factor of  $\frac{1}{2}$  since the shear strain is zero in this example.

```
>> eps1a = T1*epsilon1
```

```
eps1a =
```

```

1.0e-003 *
0.4000
0
0
```

```
>> eps1b = T1*epsilon2
```

```
eps1b =
```

```

1.0e-003 *
0.4000
0
0
```

```
>> eps2a = T2*epsilon2
```

```
eps2a =
```

```

1.0e-003 *
0.0000
0.4000
-0.0000
```

```
>> eps2b = T2*epsilon3
```

```
eps2b =  
  
    1.0e-003 *  
  
    0.0000  
    0.4000  
   -0.0000  
  
>> eps3a = T3*epsilon3  
  
eps3a =  
  
    1.0e-003 *  
  
    0.0000  
    0.4000  
   -0.0000  
  
>> eps3b = T3*epsilon4  
  
eps3b =  
  
    1.0e-003 *  
  
    0.0000  
    0.4000  
   -0.0000  
  
>> eps4a = T4*epsilon4  
  
eps4a =  
  
    1.0e-003 *  
  
    0.4000  
         0  
         0  
  
>> eps4b = T4*epsilon5  
  
eps4b =  
  
    1.0e-003 *  
  
    0.4000  
         0  
         0
```

Finally, the stress vector is calculated in MPa for each layer with respect to the principal material systems as follows:

```
>> sig1 = T1*sigma1a
```

```
sig1 =
```

```
62.2991
 1.2061
 0
```

```
>> sig2 = T2*sigma2a
```

```
sig2 =
```

```
1.2061
4.8634
-0.0000
```

```
>> sig3 = T3*sigma3a
```

```
sig3 =
```

```
1.2061
4.8634
-0.0000
```

```
>> sig4 = T4*sigma4a
```

```
sig4 =
```

```
62.2991
 1.2061
 0
```

## MATLAB Example 7.2

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.900 mm and is stacked as a  $[\pm 30/0]_S$  laminate. The six layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\kappa_x^0 &= 2.5 \text{ m}^{-1} \\ \varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:



- (a) the three components of strain at the interface locations.
- (b) the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- (c) the force and moment resultants in the laminate.
- (d) the three components of strain at the interface locations with respect to the principal material system.
- (e) the three components of stress in each layer with respect to the principal material system.

### Solution

This example is solved using MATLAB. First, the strains are calculated at the seven interface locations using the MATLAB function *Strains* as follows:

```
>> epsilon1 = Strains(0,0,0,2.5,0,0,-0.450e-3)
```

```
epsilon1 =
```

```
   -0.0011
         0
         0
```

```
>> epsilon2 = Strains(0,0,0,2.5,0,0,-0.300e-3)
```

```
epsilon2 =
```

```
   1.0e-003 *
   -0.7500
         0
         0
```

```
>> epsilon3 = Strains(0,0,0,2.5,0,0,-0.150e-3)
```

```
epsilon3 =
```

```
   1.0e-003 *
   -0.3750
         0
         0
```

```
>> epsilon4 = Strains(0,0,0,2.5,0,0,0)
```

```
epsilon4 =
```

```
   0
   0
   0
```

```
>> epsilon5 = Strains(0,0,0,2.5,0,0,0.150e-3)
```

```
epsilon5 =

    1.0e-003 *

    0.3750
         0
         0
```

```
>> epsilon6 = Strains(0,0,0,2.5,0,0,0.300e-3)
```

```
epsilon6 =

    1.0e-003 *

    0.7500
         0
         0
```

```
>> epsilon7 = Strains(0,0,0,2.5,0,0,0.450e-3)
```

```
epsilon7 =

    0.0011
         0
         0
```

Next, the reduced stiffness  $[Q]$  in GPa is calculated for this material using the MATLAB function *ReducedStiffness* as follows:

```
>> Q = ReducedStiffness(155.0, 12.10, 0.248, 4.40)
```

```
Q =

    155.7478    3.0153         0
         3.0153    12.1584         0
         0         0    4.4000
```

The transformed reduced stiffnesses  $[\bar{Q}]$  in GPa for the six layers are now calculated using the MATLAB function *Qbar* as follows:

```
>> Qbar1 = Qbar(Q,30)
```

```
Qbar1 =

    91.1488    31.7170    95.3179
    31.7170    19.3541    29.0342
    47.6589    14.5171    61.8034
```

```

>> Qbar2 = Qbar(Q,-30)

Qbar2 =

    91.1488    31.7170   -95.3179
    31.7170    19.3541   -29.0342
   -47.6589   -14.5171    61.8034

>> Qbar3 = Qbar(Q,0)

Qbar3 =

   155.7478     3.0153         0
     3.0153    12.1584         0
         0         0     4.4000

>> Qbar4 = Qbar(Q,0)

Qbar4 =

   155.7478     3.0153         0
     3.0153    12.1584         0
         0         0     4.4000

>> Qbar5 = Qbar(Q,-30)

Qbar5 =

    91.1488    31.7170   -95.3179
    31.7170    19.3541   -29.0342
   -47.6589   -14.5171    61.8034

>> Qbar6 = Qbar(Q,30)

Qbar6 =

    91.1488    31.7170    95.3179
    31.7170    19.3541    29.0342
    47.6589    14.5171    61.8034

```

Next, the stresses in each layer are calculated in MPa. Note that the stress vector is calculated twice for each layer – once at the top of the layer and once at the bottom of the layer.

```

>> sigma1a = Qbar1*epsilon1*1e3

sigma1a =

   -102.5424
   -35.6816
   -53.6163

```

```
>> sigma1b = Qbar1*epsilon2*1e3
```

```
sigma1b =
```

```
-68.3616  
-23.7877  
-35.7442
```

```
>> sigma2a = Qbar2*epsilon2*1e3
```

```
sigma2a =
```

```
-68.3616  
-23.7877  
35.7442
```

```
>> sigma2b = Qbar2*epsilon3*1e3
```

```
sigma2b =
```

```
-34.1808  
-11.8939  
17.8721
```

```
>> sigma3a = Qbar3*epsilon3*1e3
```

```
sigma3a =
```

```
-58.4054  
-1.1307  
0
```

```
>> sigma3b = Qbar3*epsilon4*1e3
```

```
sigma3b =
```

```
0  
0  
0
```

```
>> sigma4a = Qbar4*epsilon4*1e3
```

```
sigma4a =
```

```
0  
0  
0
```

```
>> sigma4b = Qbar4*epsilon5*1e3
```

```
sigma4b =
```

```
58.4054
 1.1307
      0
```

```
>> sigma5a = Qbar5*epsilon5*1e3
```

```
sigma5a =
```

```
34.1808
11.8939
-17.8721
```

```
>> sigma5b = Qbar5*epsilon6*1e3
```

```
sigma5b =
```

```
68.3616
23.7877
-35.7442
```

```
>> sigma6a = Qbar6*epsilon6*1e3
```

```
sigma6a =
```

```
68.3616
23.7877
35.7442
```

```
>> sigma6b = Qbar6*epsilon7*1e3
```

```
sigma6b =
```

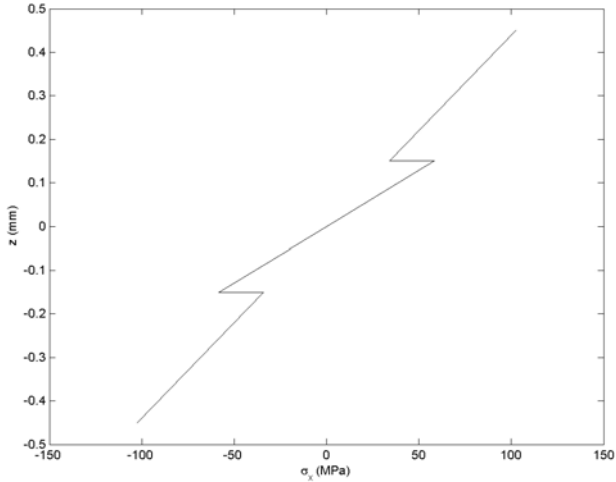
```
102.5424
35.6816
53.6163
```

Next, we setup the  $y$ -axis for the three plots:

```
>> y = [0.450 0.300 0.300 0.150 0.150 0 0 -0.150 -0.150 -0.300 -0.300
      -0.450]
```

```
y =
```

```
0.4500    0.3000    0.3000    0.1500    0.1500    0    0
-0.1500   -0.1500   -0.3000   -0.3000   -0.4500
```



**Fig. 7.7.** Variation of  $\sigma_x$  versus  $z$  for Example 7.2

The distribution of the stress  $\sigma_x$  along the depth of the laminate is now plotted as follows (see Fig. 7.7):

```
>> x = [sigma6b(1) sigma6a(1) sigma5b(1) sigma5a(1) sigma4b(1)
        sigma4a(1) sigma3b(1) sigma3a(1) sigma2b(1) sigma2a(1)
        sigma1b(1) sigma1a(1)]
```

x =

102.5424	68.3616	68.3616	34.1808	58.4054	0	0
-58.4054	-34.1808	-68.3616	-68.3616	-102.5424		

```
>> plot(x,y)
>> xlabel('\sigma_x (MPa)')
>> ylabel('z (mm)')
```

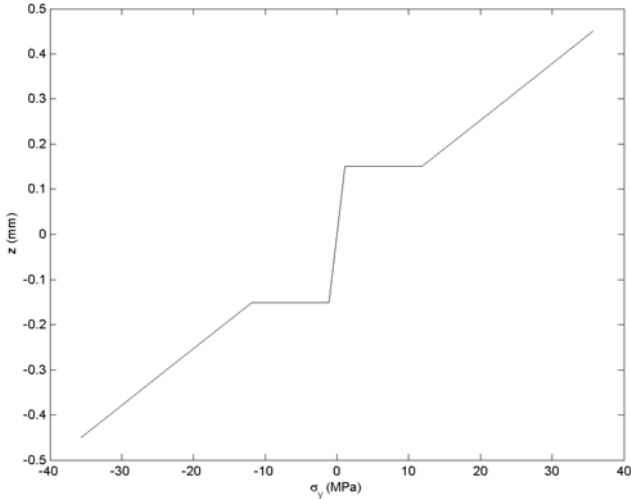
The distribution of the stress  $\sigma_y$  along the depth of the laminate is now plotted as follows (see Fig. 7.8):

```
>> x = [sigma6b(2) sigma6a(2) sigma5b(2) sigma5a(2) sigma4b(2)
        sigma4a(2) sigma3b(2) sigma3a(2) sigma2b(2) sigma2a(2)
        sigma1b(2) sigma1a(2)]
```

x =

35.6816	23.7877	23.7877	11.8939	1.1307	0	0
-1.1307	-11.8939	-23.7877	-23.7877	-35.6816		

```
>> plot(x,y)
>> ylabel('z (mm)')
>> xlabel('\sigma_y (MPa)')
```



**Fig. 7.8.** Variation of  $\sigma_y$  versus  $z$  for Example 7.2

The distribution of the stress  $\tau_{xy}$  along the depth of the laminate is now plotted as follows (see Fig. 7.9):

```
>> x = [sigma6b(3) sigma6a(3) sigma5b(3) sigma5a(3) sigma4b(3)
        sigma4a(3) sigma3b(3) sigma3a(3) sigma2b(3) sigma2a(3)
        sigma1b(3) sigma1a(3)]
```

x =

```
53.6163  35.7442  -35.7442  -17.8721  0  0  0
0  17.8721  35.7442  -35.7442  -53.6163
```

```
>> plot(x,y)
>> ylabel('z (mm)')
>> xlabel('\tau_{xy} (MPa)')
```

Next, the three force resultants are calculated in MN/m using (7.13a,b,c) as follows:

```
>> Nx = 0.150 * (sigma1a(1) + sigma2a(1) + sigma3a(1) + sigma4a(1) +
                sigma5a(1) + sigma6a(1))
```

Nx =

```
-19.0150
```

```
>> Ny = 0.150 * (sigma1a(2) + sigma2a(2) + sigma3a(2) + sigma4a(2) +
                sigma5a(2) + sigma6a(2))
```

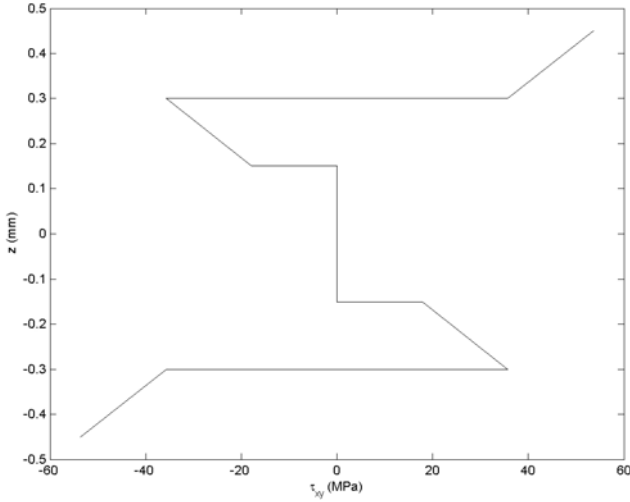


Fig. 7.9. Variation of  $\tau_{xy}$  versus  $z$  for Example 7.2

Ny =

-3.7378

```
>> Nxy = 0.150 * (sigma1a(3) + sigma2a(3) + sigma3a(3) + sigma4a(3) +
    sigma5a(3) + sigma6a(3))
```

Nxy =

0

Next, the three moment resultants are calculated in MN.m/m using (7.13d, e, f) as follows:

```
>> Mx = sigma1a(1) * ((-0.300)^2 - (-0.450)^2)/2 + sigma2a(1) *
    (-0.150)^2 - (-0.300)^2)/2 + sigma3a(1) * (0 - (-0.150)^2)/2 +
    sigma4a(1) * ((0.150)^2 - 0)/2 + sigma5a(1) * ((0.300)^2 -
    (0.150)^2)/2 + sigma6a(1) * ((0.450)^2 - (0.300)^2)/2
```

Mx =

13.7312

```
>> My = sigma1a(2) * ((-0.300)^2 - (-0.450)^2)/2 + sigma2a(2) *
    ((-0.150)^2 - (-0.300)^2)/2 + sigma3a(2) * (0 - (-0.150)^2)/2 +
    sigma4a(2) * ((0.150)^2 - 0)/2 + sigma5a(2) * ((0.300)^2 -
    (0.150)^2)/2 + sigma6a(2) * ((0.450)^2 - (0.300)^2)/2
```



My =

4.5621

```
>> Mxy = sigma1a(3) * ((-0.300)^2 - (-0.450)^2)/2 + sigma2a(3) *
      ((-0.150)^2 - (-0.300)^2)/2 + sigma3a(3) * (0 - (-0.150)^2)/2
      + sigma4a(3) * ((0.150)^2 - 0)/2 + sigma5a(3) * ((0.300)^2 -
      (0.150)^2)/2 + sigma6a(3) * ((0.450)^2 - (0.300)^2)/2
```

Mxy =

3.2170

Next, the transformation matrix is calculated for each one of the six layers using the MATLAB function  $T$  as follows:

```
>> T1 = T(30)
```

T1 =

0.7500	0.2500	0.8660
0.2500	0.7500	-0.8660
-0.4330	0.4330	0.5000

```
>> T2 = T(-30)
```

T2 =

0.7500	0.2500	-0.8660
0.2500	0.7500	0.8660
0.4330	-0.4330	0.5000

```
>> T3 = T(0)
```

T3 =

1	0	0
0	1	0
0	0	1

```
>> T4 = T(0)
```

T4 =

1	0	0
0	1	0
0	0	1

```
>> T5 = T(-30)
```

```
T5 =
```

```
    0.7500    0.2500   -0.8660
    0.2500    0.7500    0.8660
    0.4330   -0.4330    0.5000
```

```
>> T6 = T(30)
```

```
T6 =
```

```
    0.7500    0.2500    0.8660
    0.2500    0.7500   -0.8660
   -0.4330    0.4330    0.5000
```

The strain vector is now calculated in each layer with respect to the principal material system. Note that the strain vector is calculated twice for each layer – once at the top of the layer and once at the bottom of the layer.

```
>> eps1a = T1*epsilon1
```

```
eps1a =
```

```
    1.0e-003 *
   -0.8438
   -0.2812
    0.4871
```

```
>> eps1b = T1*epsilon2
```

```
eps1b =
```

```
    1.0e-003 *
   -0.5625
   -0.1875
    0.3248
```

```
>> eps2a = T2*epsilon2
```

```
eps2a =
```

```
    1.0e-003 *
   -0.5625
   -0.1875
   -0.3248
```

```
>> eps2b = T2*epsilon3
```

```
eps2b =  
  
1.0e-003 *  
  
-0.2813  
-0.0937  
-0.1624
```

```
>> eps3a = T3*epsilon3
```

```
eps3a =  
  
1.0e-003 *  
  
-0.3750  
0  
0
```

```
>> eps3b = T3*epsilon4
```

```
eps3b =  
  
0  
0  
0
```

```
>> eps4a = T4*epsilon4
```

```
eps4a =  
  
0  
0  
0
```

```
>> eps4b = T4*epsilon5
```

```
eps4b =  
  
1.0e-003 *  
  
0.3750  
0  
0
```

```
>> eps5a = T5*epsilon5
```

```
eps5a =
```

```
1.0e-003 *
    0.2813
    0.0937
    0.1624
```

```
>> eps5b = T5*epsilon6
```

```
eps5b =
```

```
1.0e-003 *
    0.5625
    0.1875
    0.3248
```

```
>> eps6a = T6*epsilon6
```

```
eps6a =
```

```
1.0e-003 *
    0.5625
    0.1875
   -0.3248
```

```
>> eps6b = T6*epsilon7
```

```
eps6b =
```

```
1.0e-003 *
    0.8438
    0.2812
   -0.4871
```

Next, we correct the shear strain component for the factor of  $1/2$  that appears in the equations.

```
>> eps1a(3) = eps1a(3)*2
```

```
eps1a =
```

```
1.0e-003 *
   -0.8438
   -0.2812
    0.9743
```

```
>> eps2a(3) = eps2a(3)*2
```

```
eps2a =  
  
1.0e-003 *  
  
-0.5625  
-0.1875  
-0.6495
```

```
>> eps3a(3) = eps3a(3)*2
```

```
eps3a =  
  
1.0e-003 *  
  
-0.3750  
0  
0
```

```
>> eps4a(3) = eps4a(3)*2
```

```
eps4a =  
  
0  
0  
0
```

```
>> eps5a(3) = eps5a(3)*2
```

```
eps5a =  
  
1.0e-003 *  
  
0.2813  
0.0937  
0.3248
```

```
>> eps6a(3) = eps6a(3)*2
```

```
eps6a =  
  
1.0e-003 *  
  
0.5625  
0.1875  
-0.6495
```

Finally, the stress vector is calculated in MPa for each layer with respect to the principal material system as follows:

```
>> sig1 = T1*sigma1a
```

```
sig1 =
```

```
-132.2602  
-5.9637  
2.1434
```

```
>> sig2 = T2*sigma2a
```

```
sig2 =
```

```
-88.1735  
-3.9758  
-1.4289
```

```
>> sig3 = T3*sigma3a
```

```
sig3 =
```

```
-58.4054  
-1.1307  
0
```

```
>> sig4 = T4*sigma4a
```

```
sig4 =
```

```
0  
0  
0
```

```
>> sig5 = T5*sigma5a
```

```
sig5 =
```

```
44.0867  
1.9879  
0.7145
```

```
>> sig6 = T6*sigma6a
```

```
sig6 =
```

```
88.1735  
3.9758  
-1.4289
```

## Problems

### MATLAB Problem 7.1

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.600 mm and is stacked as a  $[0/90]_S$  laminate. The four layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\varepsilon_x^0 &= 500 \times 10^{-6} \\ \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_x^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

### MATLAB Problem 7.2

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.600 mm and is stacked as a  $[0/90]_S$  laminate. The four layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\kappa_x^0 &= 2.5 \text{ m}^{-1} \\ \varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

**MATLAB Problem 7.3**

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.600 mm and is stacked as a  $[0/90]_S$  laminate. The four layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\kappa_x^0 &= 2.5 \text{ m}^{-1} \\ \varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

**MATLAB Problem 7.4**

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.900 mm and is stacked as a  $[\pm 30/0]_S$  laminate. The six layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\kappa_x^0 &= 2.5 \text{ m}^{-1} \\ \varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

**MATLAB Problem 7.5**

Consider a graphite-reinforced polymer composite laminate with the elastic constants as given in Example 2.2. The laminate has total thickness of 0.900 mm and is stacked as a  $[\pm 30/0]_S$  laminate. The six layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:



$$\begin{aligned}\varepsilon_x^0 &= 1000 \times 10^{-6} \\ \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_x^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.

### MATLAB Problem 7.6

Consider a glass-reinforced polymer composite laminate with the elastic constants as given in Problem 2.7. The laminate has total thickness of 0.900 mm and is stacked as a  $[\pm 30/0]_S$  laminate. The six layers are of equal thickness. It is deformed so that at a point  $(x, y)$  on the reference surface, we have the following strains and curvatures:

$$\begin{aligned}\varepsilon_x^0 &= 1000 \times 10^{-6} \\ \varepsilon_y^0 = \gamma_{xy}^0 = \kappa_x^0 = \kappa_y^0 = \kappa_{xy}^0 &= 0\end{aligned}$$

Use MATLAB to determine the following:

- the three components of strain at the interface locations.
- the three components of stress in each layer. Plot the stress distribution along the depth of the laminate for each component.
- the force and moment resultants in the laminate.
- the three components of strain at the interface locations with respect to the principal material system.
- the three components of stress in each layer with respect to the principal material system.