

Elastic Constants Based on Micromechanics

3.1 Basic Equations

The purpose of this chapter is to predict the material constants (also called elastic constants) of a composite material by studying the micromechanics of the problem, i.e. by studying how the matrix and fibers interact. These are the same material constants used in Chap. 2 to calculate the compliance and stiffness matrices. Computing the stresses within the matrix, within the fiber, and at the interface of the matrix and fiber is very important for understanding some of the underlying failure mechanisms. In considering the fibers and surrounding matrix, we have the following assumptions [1]:

1. Both the matrix and fibers are linearly elastic.
2. The fibers are infinitely long.
3. The fibers are spaced periodically in square-packed or hexagonal packed arrays.

There are three different approaches that are used to determine the elastic constants for the composite material based on micromechanics. These three approaches are [1]:

1. Using numerical models such as the finite element method.
2. Using models based on the theory of elasticity.
3. Using rule-of-mixtures models based on a strength-of-materials approach.

Consider a unit cell in either a square-packed array (Fig. 3.1) or a hexagonal-packed array (Fig. 3.2) – see [1]. The ratio of the cross-sectional area of the fiber to the total cross-sectional area of the unit cell is called the *fiber volume fraction* and is denoted by V^f . The fiber volume fraction satisfies the relation $0 < V^f < 1$ and is usually 0.5 or greater. Similarly, the *matrix volume fraction* V^m is the ratio of the cross-sectional area of the matrix to the total cross-sectional area of the unit cell. Note that V^m also satisfies

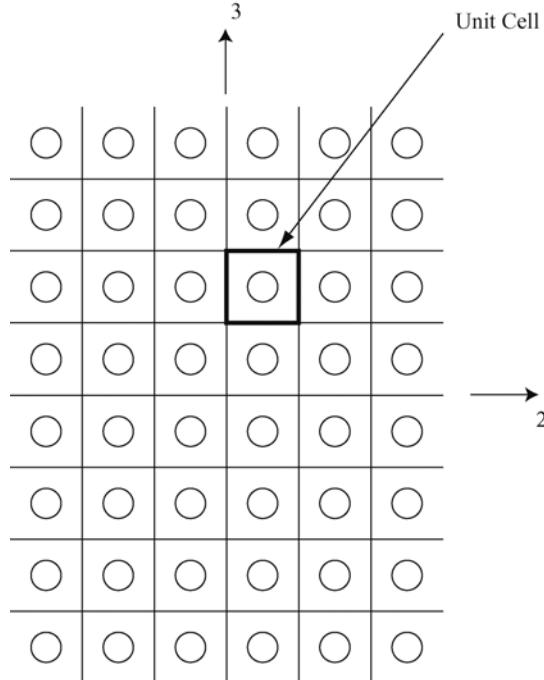


Fig. 3.1. A unit cell in a square-packed array of fiber-reinforced composite material

$0 < V^m < 1$. The following relation can be shown to exist between V^f and V^m :

$$V^f + V^m = 1 \quad (3.1)$$

In the above, we use the notation that a superscript m indicates a matrix quantity while a superscript f indicates a fiber quantity. In addition, the matrix material is assumed to be isotropic so that $E_1^m = E_2^m = E^m$ and $\nu_{12}^m = \nu^m$. However, the fiber material is assumed to be only transversely isotropic such that $E_3^f = E_2^f$, $\nu_{13}^f = \nu_{12}^f$, and $\nu_{23}^f = \nu_{32}^f = \nu^f$.

Using the strength-of-materials approach and the simple rule of mixtures, we have the following relations for the elastic constants of the composite material (see [1]). For Young's modulus in the 1-direction (also called the longitudinal stiffness), we have the following relation:

$$E_1 = E_1^f V^f + E^m V^m \quad (3.2)$$

where E_1^f is Young's modulus of the fiber in the 1-direction while E^m is Young's modulus of the matrix. For Poisson's ratio ν_{12} , we have the following relation:

$$\nu_{12} = \nu_{12}^f V^f + \nu^m V^m \quad (3.3)$$

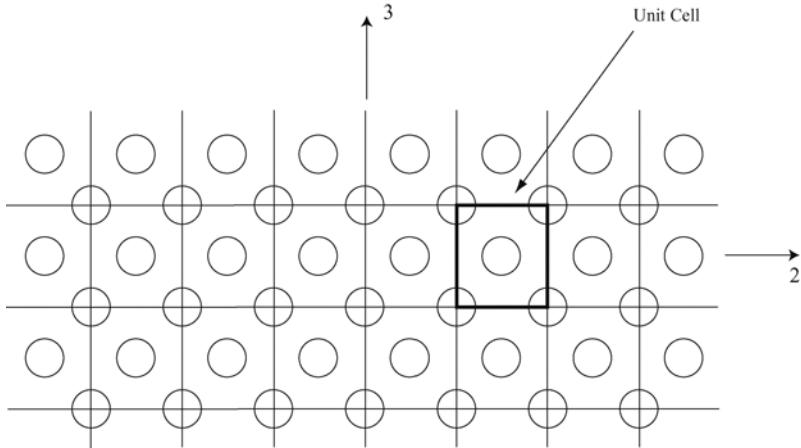


Fig. 3.2. A unit cell in a hexagonal-packed array of fiber-reinforced composite material

where ν_{12}^f and ν^m are Poisson's ratios for the fiber and matrix, respectively. For Young's modulus in the 2-direction (also called the transverse stiffness), we have the following relation:

$$\frac{1}{E_2} = \frac{V^f}{E_2^f} + \frac{V^m}{E^m} \quad (3.4)$$

where E_2^f is Young's modulus of the fiber in the 2-direction while E^m is Young's modulus of the matrix. For the shear modulus G_{12} , we have the following relation:

$$\frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{V^m}{G^m} \quad (3.5)$$

where G_{12}^f and G^m are the shear moduli of the fiber and matrix, respectively.

For the coefficients of thermal expansion α_1 and α_2 (see Problem 2.9), we have the following relations:

$$\alpha_1 = \frac{\alpha_1^f E_1^f V^f + \alpha^m E^m V^m}{E_1^f V^f + E^m V^m} \quad (3.6)$$

$$\begin{aligned} \alpha_2 = & \left[\alpha_2^f - \left(\frac{E^m}{E_1} \right) \nu_1^f (\alpha^m - \alpha_1^f) V^m \right] V^f \\ & + \left[\alpha^m + \left(\frac{E_1^f}{E_1} \right) \nu^m (\alpha^m - \alpha_1^f) V^f \right] V^m \end{aligned} \quad (3.7)$$

where α_1^f and α_2^f are the coefficients of thermal expansion for the fiber in the 1- and 2-directions, respectively, and α^m is the coefficient of thermal expansion

for the matrix. However, we can use a simple rule-of-mixtures relation for α_2 as follows:

$$\alpha_2 = \alpha_2^f V^f + \alpha^m V^m \quad (3.8)$$

A similar simple rule-of-mixtures relation for α_1 cannot be used simply because the matrix and fiber must expand or contract the same amount in the 1-direction when the temperature is changed.

While the simple rule-of-mixtures models used above give accurate results for E_1 and ν_{12} , the results obtained for E_2 and G_{12} do not agree well with finite element analysis and elasticity theory results. Therefore, we need to modify the simple rule-of-mixtures models shown above. For E_2 , we have the following modified rule-of-mixtures formula:

$$\frac{1}{E_2} = \frac{\frac{V^f}{E_2^f} + \frac{\eta V^m}{E^m}}{V^f + \eta V^m} \quad (3.9)$$

where η is the stress-partitioning factor (related to the stress σ_2). This factor satisfies the relation $0 < \eta < 1$ and is usually taken between 0.4 and 0.6. Another alternative rule-of-mixtures formula for E_2 is given by:

$$\frac{1}{E_2} = \frac{\eta^f V^f}{E_2^f} + \frac{\eta^m V^m}{E^m} \quad (3.10)$$

where the factors η^f and η^m are given by:

$$\eta^f = \frac{E_1^f V^f + \left[\left(1 - \nu_{12}^f \nu_{21}^f\right) E^m + \nu^m \nu_{21}^f E_1^f \right] V^m}{E_1^f V^f + E^m V^m} \quad (3.11)$$

$$\eta^m = \frac{\left[\left(1 - \nu^{m^2}\right) E_1^f - \left(1 - \nu^m \nu_{12}^f\right) E^m \right] V^f + E^m V^m}{E_1^f V^f + E^m V^m} \quad (3.12)$$

The above alternative model for E_2 gives accurate results and is used whenever the modified rule-of-mixtures model of (3.9) cannot be applied, i.e. when the factor η is not known.

The modified rule-of-mixtures model for G_{12} is given by the following formula:

$$\frac{1}{G_{12}} = \frac{\frac{V^f}{G_{12}^f} + \frac{\eta' V^m}{G^m}}{V^f + \eta' V^m} \quad (3.13)$$

where η' is the shear stress-partitioning factor. Note that η' satisfies the relation $0 < \eta' < 1$ but using $\eta' = 0.6$ gives results that correlate with the elasticity solution.

Finally, the elasticity solution gives the following formula for G_{12} :

$$G_{12} = G^m \left[\frac{(G^m + G_{12}^f) - V^f(G^m - G_{12}^f)}{(G^m + G_{12}^f) + V^f(G^m - G_{12}^f)} \right] \quad (3.14)$$

3.2 MATLAB Functions Used

The six MATLAB functions used in this chapter to calculate the elastic material constants are:

E1(Vf, E1f, Em) – This function calculates the longitudinal Young's modulus E_1 for the lamina. Its input consists of three arguments as illustrated in the listing below.

NU12(Vf, NU12f, NUM) – This function calculates Poisson's ratio ν_{12} for the lamina. Its input consists of three arguments as illustrated in the listing below.

E2(Vf, E2f, Em, Eta, NU12f, NU21f, NUM, E1f, p) – This function calculates the transverse Young's modulus E_2 for the lamina. Its input consists of nine arguments as illustrated in the listing below. Use the value zero for any argument not needed in the calculations.

G12(Vf, G12f, Gm, EtaPrime, p) – This function calculates the shear modulus G_{12} for the lamina. Its input consists of five arguments as illustrated in the listing below. Use the value zero for any argument not needed in the calculations.

Alpha1(Vf, E1f, Em, Alpha1f, Alpham) – This function calculates the coefficient of thermal expansion α_1 for the lamina. Its input consists of five arguments as illustrated in the listing below.

Alpha2(Vf, Alpha2f, Alpham, E1, E1f, Em, NU1f, NUM, Alpha1f, p) – This function calculates the coefficient of thermal expansion α_2 for the lamina. Its input consists of ten arguments as illustrated in the listing below. Use the value zero for any argument not needed in the calculations.

The following is a listing of the MATLAB source code for each function:

```
function y = E1(Vf,E1f,Em)
%E1  This function returns Young's modulus in the
%    longitudinal direction. Its input are three values:
%    Vf - fiber volume fraction
%    E1f - longitudinal Young's modulus of the fiber
%    Em - Young's modulus of the matrix
%    This function uses the simple rule-of-mixtures formula
%    of equation (3.2)
Vm = 1 - Vf;
y = Vf*E1f + Vm*Em;
```

```
function y = NU12(Vf,NU12f,NUM)
%NU12  This function returns Poisson's ratio NU12
%    Its input are three values:
%    Vf - fiber volume fraction
%    NU12f - Poisson's ratio NU12 of the fiber
%    NUM - Poisson's ratio of the matrix
```

```

% This function uses the simple rule-of-mixtures
% formula of equation (3.3)
Vm = 1 - Vf;
y = Vf*NU12f + Vm*NUM;

function y = E2(Vf,E2f,Em,Eta,NU12f,NU21f,NUM,E1f,p)
%E2 This function returns Young's modulus in the
% transverse direction. Its input are nine values:
% Vf - fiber volume fraction
% E2f - transverse Young's modulus of the fiber
% Em - Young's modulus of the matrix
% Eta - stress-partitioning factor
% NU12f - Poisson's ratio NU12 of the fiber
% NU21f - Poisson's ratio NU21 of the fiber
% NUM - Poisson's ratio of the matrix
% E1f - longitudinal Young's modulus of the fiber
% p - parameter used to determine which equation to use:
%      p = 1 - use equation (3.4)
%      p = 2 - use equation (3.9)
%      p = 3 - use equation (3.10)
% Use the value zero for any argument not needed
% in the calculations.
Vm = 1 - Vf;
if p == 1
    y = 1/(Vf/E2f + Vm/Em);
elseif p == 2
    y = 1/((Vf/E2f + Eta*Vm/Em)/(Vf + Eta*Vm));
elseif p == 3
    deno = E1f*Vf + Em*Vm;
    etaf = (E1f*Vf + ((1-NU12f*NU21f)*Em + NUM*NU21f*E1f)*Vm) /deno;
    etam = (((1-Num*NUM)*E1f - (1-Num*NU12f)*Em)*Vf + Em*Vm) /deno;
    y = 1/(etaf*Vf/E2f + etam*Vm/Em);
end

function y = G12(Vf,G12f,Gm,EtaPrime,p)
%G12 This function returns the shear modulus G12
% Its input are five values:
% Vf - fiber volume fraction
% G12f - shear modulus G12 of the fiber
% Gm - shear modulus of the matrix
% EtaPrime - shear stress-partitioning factor
% p - parameter used to determine which equation to use:
%      p = 1 - use equation (3.5)
%      p = 2 - use equation (3.13)
%      p = 3 - use equation (3.14)
% Use the value zero for any argument not needed
% in the calculations.
Vm = 1 - Vf;

```

```

if p == 1
    y = 1/(Vf/G12f + Vm/Gm);
elseif p == 2
    y = 1/((Vf/G12f + EtaPrime*Vm/Gm)/(Vf + EtaPrime*Vm));
elseif p == 3
    y = Gm*((Gm + G12f) - Vf*(Gm - G12f))/((Gm + G12f) +
        Vf*(Gm - G12f));
end


---


function y = Alpha1(Vf,E1f,Em,Alpha1f,Alpham)
%Alpha1 This function returns the coefficient of thermal
% expansion in the longitudinal direction.
% Its input are five values:
% Vf - fiber volume fraction
% E1f - longitudinal Young's modulus of the fiber
% Em - Young's modulus of the matrix
% Alpha1f - coefficient of thermal expansion in the
%           1-direction for the fiber
% Alpham - coefficient of thermal expansion for the matrix
Vm = 1 - Vf;
y = (Vf*E1f*Alpha1f + Vm*Em*Alpham)/(E1f*Vf + Em*Vm);


---


function y = Alpha2(Vf,Alpha2f,Alpham,E1,E1f,Em,NU1f,NUm,
                    Alpha1f,p)
%Alpha2 This function returns the coefficient of thermal
% expansion in the transverse direction.
% Its input are ten values:
% Vf - fiber volume fraction
% Alpha2f - coefficient of thermal expansion in the
%           2-direction for the fiber
% Alpham - coefficient of thermal expansion for the matrix
% E1 - longitudinal Young's modulus of the lamina
% E1f - longitudianl Young's modulus of the fiber
% Em - Young's modulus of the matrix
% NU1f - Poisson's ratio of the fiber
% NUm - Poisson's ratio of the matrix
% Alpha1f - coefficient of thermal expansion in the
%           1-direction
% p - parameter used to determine which equation to use
%     p = 1 - use equation (3.8)
%     p = 2 - use equation (3.7)
% Use the value zero for any argument not needed in
% the calculation
Vm = 1 - Vf;
if p == 1
    y = Vf*Alpha2f + Vm*Alpham;
elseif p == 2
    y = (Alpha2f - (Em/E1)*NU1f*(Alpham - Alpha1f)*Vm)*Vf +
        (Alpham + (E1f/E1)*NUm*(Alpham - Alpha1f)*Vf)*Vm;
end

```

Example 3.1

Derive the simple rule-of-mixtures formula for the calculation of the longitudinal modulus E_1 given in (3.2).

Solution

Consider a longitudinal cross-section of length L of the fiber and matrix in a lamina as shown in Fig. 3.3. Let A^f and A^m be the cross-sectional areas of the fiber and matrix, respectively. Let also F_1^f and F_1^m be the longitudinal forces in the fiber and matrix, respectively. Then we have the following relations:

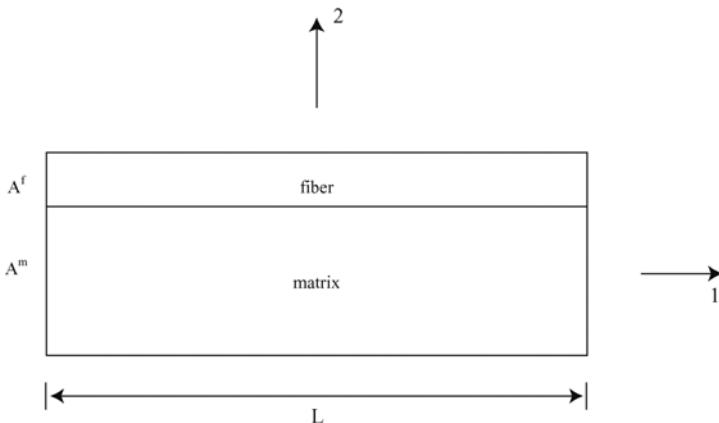


Fig. 3.3. A longitudinal cross-section of fiber-reinforced composite material for Example 3.1

$$F_1^f = \sigma_1^f A^f \quad (3.15a)$$

$$F_1^m = \sigma_1^m A^m \quad (3.15b)$$

where σ_1^f and σ_1^m are the longitudinal normal stresses in the fiber and matrix, respectively. These stresses are given in terms of the longitudinal strains ε_1^f and ε_1^m as follows:

$$\sigma_1^f = E_1^f \varepsilon_1^f \quad (3.16a)$$

$$\sigma_1^m = E^m \varepsilon_1^m \quad (3.16b)$$

where E_1^f is the longitudinal modulus of the fiber and E^m is the modulus of the matrix.

Let F_1 be the total longitudinal force in the lamina where F_1 is given by:

$$F_1 = \sigma_1 A \quad (3.17)$$

where σ_1 is the total longitudinal normal stress in the lamina and A is the total cross-sectional area of the lamina. The total longitudinal normal stress σ_1 is given by:

$$\sigma_1 = E_1 \varepsilon_1 \quad (3.18)$$

However, using force equilibrium, it is clear that we have the following relation between the total longitudinal force and the longitudinal forces in the fiber and matrix:

$$F_1 = F_1^f + F_1^m \quad (3.19)$$

Substituting (3.15a,b) and (3.17) into (3.19), then substituting (3.16a,b) and (3.18) into the resulting equation, we obtain the following relation:

$$E_1 \varepsilon_1 A = E_1^f \varepsilon_1^f A^f + E^m \varepsilon_1^m A^m \quad (3.20)$$

Next, we use the compatibility condition $\varepsilon_1^f = \varepsilon_1^m = \varepsilon_1$ since the matrix, fiber, and lamina all have the same strains. Equation (3.20) is simplified as follows:

$$E_1 A = E_1^f A^f + E^m A^m \quad (3.21)$$

Finally, we divide (3.21) by A and note that $A^f/A = V^f$ and $A^m/A = V^m$ to obtain the required formula for E_1 as follows:

$$E_1 = E_1^f V^f + E^m V^m \quad (3.22)$$

MATLAB Example 3.2

Consider a graphite-reinforced polymer composite lamina with the following material properties for the matrix and fibers [1]:

$$\begin{aligned} E^m &= 4.62 \text{ GPa}, & \nu^m &= 0.360 \\ E_1^f &= 233 \text{ GPa}, & \nu_{12}^f &= 0.200 \\ E_2^f &= 23.1 \text{ GPa}, & \nu_{23}^f &= 0.400 \\ G_{12}^f &= 8.96 \text{ GPa} & G_{23}^f &= 8.27 \text{ GPa} \end{aligned}$$

Use MATLAB and the simple rule-of-mixtures formulas to calculate the values of the four elastic constants E_1 , ν_{12} , E_2 , and G_{12} for the lamina. Use $V^f = 0.6$.

Solution

This example is solved using MATLAB. First, the MATLAB function *E1* is used to calculate the longitudinal modulus E_1 in GPa as follows:

```
>> E1(0.6, 233, 4.62)
```

```
ans =
```

```
141.6480
```

Poisson's ratio ν_{12} is then calculated using the MATLAB function *NU12* as follows:

```
>> NU12(0.6, 0.200, 0.360)
```

```
ans =
```

```
0.2640
```

The transverse modulus E_2 is then calculated in GPa using the MATLAB function *E2* as follows (note that we use the value zero for each parameter not needed in the calculations):

```
>> E2(0.6, 23.1, 4.62, 0, 0, 0, 0, 0, 1)
```

```
ans =
```

```
8.8846
```

The shear modulus for the matrix G^m is calculated in GPa using (2.8) as follows:

```
>> Gm = 4.62/(2*(1 + 0.360))
```

```
Gm =
```

```
1.6985
```

Finally, the shear modulus G_{12} of the lamina is calculated in GPa using the MATLAB function *G12* as follows:

```
>> G12(0.6, 8.96, Gm, 0, 1)
```

```
ans =
```

```
3.3062
```

Note that ν_{23}^f and G_{23}^f are not used in this example.

MATLAB Example 3.3

Consider the graphite-reinforced polymer composite lamina of Example 3.2. Use MATLAB to plot a graph for each one of the four elastic constants (E_1 , ν_{12} , E_2 , G_{12}) versus the fiber volume fraction V^f . Use all values of V^f ranging from 0 to 1 (in increments of 0.1).

Solution

This example is solved using MATLAB. First, the array for the x -axis is set up as follows:

```
>> x = [0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1]

x =
Columns 1 through 10

0 0.1000 0.2000 0.3000 0.4000 0.5000 0.6000 0.7000
0.8000 0.9000

Column 11

1.0000
```

Then, the longitudinal modulus E_1 is calculated in GPa using the MATLAB function $E1$ for all values of V^f between 0 and 1 as follows (in increments of 0.1):

```
>> y(1) = E1(0, 233, 4.62)

y =
4.6200

>> y(2) = E1(0.1, 233, 4.62)

y =
4.6200 27.4580

>> y(3) = E1(0.2, 233, 4.62)

y =
4.6200 27.4580 50.2960

>> y(4) = E1(0.3, 233, 4.62)
```

36 3 Elastic Constants Based on Micromechanics

```
y =
    4.6200  27.4580  50.2960  73.1340

>> y(5) = E1(0.4, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720

>> y(6) = E1(0.5, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100

>> y(7) = E1(0.6, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100  141.6480

>> y(8) = E1(0.7, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100  141.6480
    164.4860

>> y(9) = E1(0.8, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100  141.6480
    164.4860  187.3240

>> y(10) = E1(0.9, 233, 4.62)

y =
    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100  141.6480
    164.4860  187.3240  210.1620

>> y(11) = E1(1, 233, 4.62)

y =
Columns 1 through 10

    4.6200  27.4580  50.2960  73.1340  95.9720  118.8100  141.6480
```

```

164.4860 187.3240 210.1620
Column 11

233.0000

```

The plot command is then used to plot the graph of E_1 versus V^f as follows. The resulting plot is shown in Fig. 3.4. Notice that the variation is linear.

```

>> plot(x,y)
>> xlabel('V^f');
>> ylabel('E_1 (GPa)');

```

Poisson's ratio ν_{12} is then calculated using the MATLAB function *NU12* for all values of V^f between 0 and 1 as follows (in increments of 0.1):

```
>> z(1) = NU12(0, 0.200, 0.360)
```

```
z =
```

```
0.3600
```

```
>> z(2) = NU12(0.1, 0.200, 0.360)
```

```
z =
```

```
0.3600 0.3440
```

```
>> z(3) = NU12(0.2, 0.200, 0.360)
```

```
z =
```

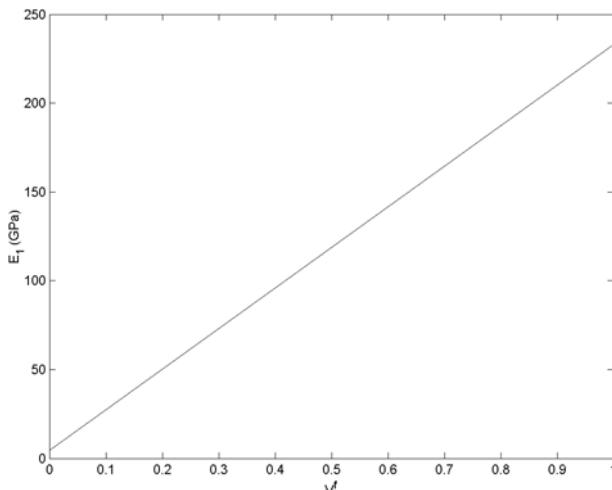


Fig. 3.4. Variation of E_1 versus V^f for Example 3.3

38 3 Elastic Constants Based on Micromechanics

```
0.3600 0.3440 0.3280

>> z(4) = NU12(0.3, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120

>> z(5) = NU12(0.4, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960

>> z(6) = NU12(0.5, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800

>> z(7) = NU12(0.6, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800 0.2640

>> z(8) = NU12(0.7, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800 0.2640 0.2480

>> z(9) = NU12(0.8, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800 0.2640
0.2480 0.2320

>> z(10) = NU12(0.9, 0.200, 0.360)

z =

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800 0.2640 0.2480
0.2320 0.2160

>> z(11) = NU12(1, 0.200, 0.360)
```

```

z =
Columns 1 through 10

0.3600 0.3440 0.3280 0.3120 0.2960 0.2800 0.2640 0.2480
0.2320 0.2160

Column 11

0.2000

```

The plot command is then used to plot the graph of ν_{12} versus V^f as follows. The resulting plot is shown in Fig. 3.5. Notice that the variation is linear.

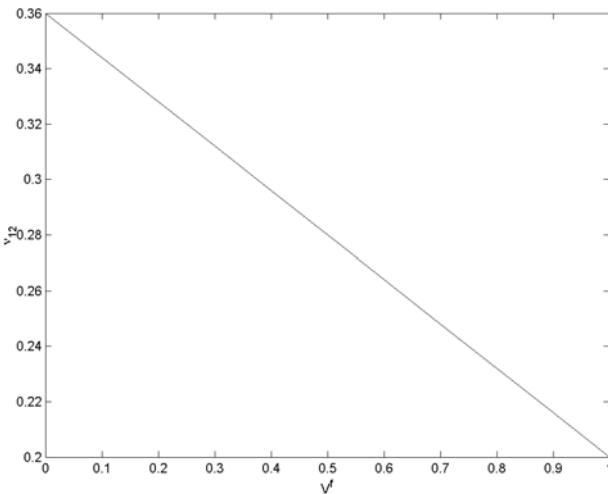


Fig. 3.5. Variation of ν_{12} versus V^f for Example 3.3

```

>> plot(x,z)
>> xlabel('V^f');
>> ylabel('\nu_{12}');

```

The transverse modulus E_2 is then calculated using the MATLAB function $E2$ using all values of V^f between 0 and 1 as follows (in increments of 0.1):

```
>> w(1) = E2(0, 23.1, 4.62, 0, 0, 0, 0, 0, 1)
```

```
w =
4.6200
```

```
>> w(2) = E2(0.1, 23.1, 4.62, 0, 0, 0, 0, 0, 1)
```

40 3 Elastic Constants Based on Micromechanics

```
w =
4.6200 5.0217

>> w(3) = E2(0.2, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000

>> w(4) = E2(0.3, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789

>> w(5) = E2(0.4, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941

>> w(6) = E2(0.5, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000

>> w(7) = E2(0.6, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000 8.8846

>> w(8) = E2(0.7, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000 8.8846 10.5000

>> w(9) = E2(0.8, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000 8.8846 10.5000
12.8333

>> w(10) = E2(0.9, 23.1, 4.62, 0, 0, 0, 0, 0, 1)

w =
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000 8.8846 10.5000
12.8333 16.5000

>> w(11) = E2(1, 23.1, 4.62, 0, 0, 0, 0, 0, 1)
```

```
w =
Columns 1 through 10
4.6200 5.0217 5.5000 6.0789 6.7941 7.7000 8.8846 10.5000
12.8333 16.5000

Column 11
23.1000
```

The plot command is then used to plot the graph of E_2 versus V^f as follows. The resulting plot is shown in Fig. 3.6. Notice that the variation is nonlinear.

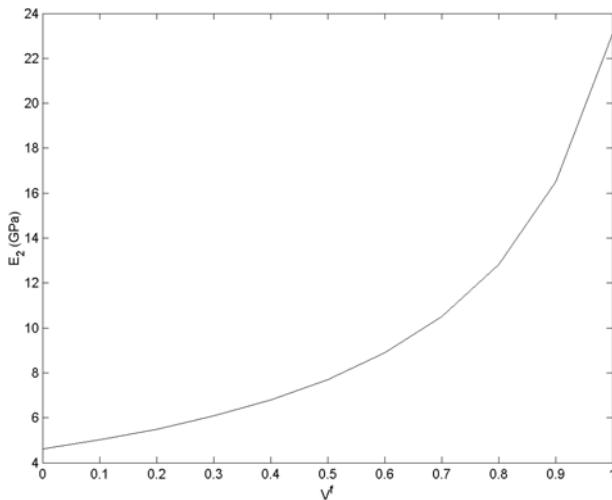


Fig. 3.6. Variation of E_2 versus V^f for Example 3.3

```
>> plot(x,w)
>> xlabel('V^f');
>> ylabel('E_2 (GPa)');
```

Finally, the shear modulus G_{12} is then calculated using the MATLAB function $G12$ using all values of V^f between 0 and 1 as follows (in increments of 0.1). Note that we first calculate G^m using (2.8).

```
>> Gm = 4.62/(2*(1 + 0.360))

Gm =
1.6985

>> u(1) = G12(0, 8.96, Gm, 0, 1)
```

42 3 Elastic Constants Based on Micromechanics

```
u =
    1.6985

>> u(2) = G12(0.1, 8.96, Gm, 0, 1)

u =
    1.6985  1.8483

>> u(3) = G12(0.2, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271

>> u(4) = G12(0.3, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441

>> u(5) = G12(0.4, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441 2.5133

>> u(6) = G12(0.5, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441 2.5133 2.8557

>> u(7) = G12(0.6, 8.96, Gm, 0, 1)
u =
    1.6985 1.8483 2.0271 2.2441 2.5133 2.8557 3.3062

>> u(8) = G12(0.7, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441 2.5133 2.8557 3.3062 3.9254

>> u(9) = G12(0.8, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441 2.5133 2.8557 3.3062 3.9254
    4.8301

>> u(10) = G12(0.9, 8.96, Gm, 0, 1)

u =
    1.6985 1.8483 2.0271 2.2441 2.5133 2.8557 3.3062 3.9254
    4.8301 6.2766
```

```
>> u(11) = G12(1, 8.96, Gm, 0, 1)

u =

Columns 1 through 10

1.6985 1.8483 2.0271 2.2441 2.5133 2.8557 3.3062 3.9254
4.8301 6.2766

Column 11

8.9600
```

The plot command is then used to plot the graph of G_{12} versus V^f as follows. The resulting plot is shown in Fig. 3.7. Notice that the variation is nonlinear.

```
>> plot(x,u)
>> xlabel('V^f');
>> ylabel('G_{12} (GPa)');
```

Problems

Problem 3.1

Derive (3.1) in detail.

Problem 3.2

Derive the simple rule-of-mixtures formula for the calculation of Poisson's ratio ν_{12} given in (3.3).

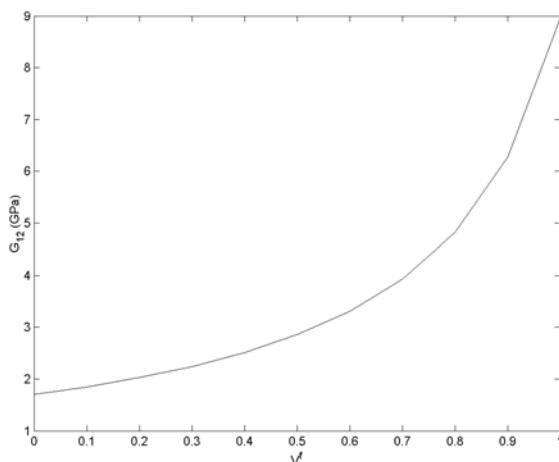


Fig. 3.7. Variation of G_{12} versus V^f for Example 3.3

Problem 3.3

Derive the simple rule-of-mixtures formula for the calculation of the transverse modulus E_2 given in (3.4).

MATLAB Problem 3.4

In the calculation of the transverse modulus E_2 using the simple rule-of-mixtures formula of (3.4), the results can be improved by replacing E^m by $E^{m'}$ where $E^{m'}$ is given by:

$$E^{m'} = \frac{E^m}{1 - \nu^m} \quad (3.23)$$

where ν^m is Poisson's ratio of the matrix. Modify the MATLAB function E_2 with the addition of this formula as a fourth case to be calculated when the parameter p is set to the value 4.

MATLAB Problem 3.5

Consider a carbon/epoxy composite lamina with the following matrix and fiber material properties [2]:

$$E_2^f = 14.8 \text{ GPa}, \quad E^m = 3.45 \text{ GPa}, \quad \nu^m = 0.36$$

Use MATLAB to calculate the transverse modulus E_2 using the following three methods (use $V^f = 0.65$):

- (a) the simple rule-of-mixtures formula of (3.4).
- (b) the modified rule-of-mixtures formula of (3.9) with $\eta = 0.5$.
- (c) the alternative rule-of-mixtures formula of (3.10). For this case, use $E_1^f = 85.6 \text{ GPa}$, $\nu_{12}^f = \nu_{21}^f = 0.3$.

MATLAB Problem 3.6

Consider the glass/epoxy composite lamina of Problem 3.5. Use MATLAB to plot a graph of the transverse modulus E_2 versus the fiber volume fraction V^f for each one of the following cases. Use all values of V^f ranging from 0 to 1 (in increments of 0.1).

- (a) the simple rule-of-mixtures formula of (3.4).
- (b) the modified rule-of-mixtures formula of (3.9) with $\eta = 0.4$.
- (c) the modified rule-of-mixtures formula of (3.9) with $\eta = 0.5$.
- (d) the modified rule-of-mixtures formula of (3.9) with $\eta = 0.6$.
- (e) the alternative rule-of-mixtures formula of (3.10) with the values given in part (c) of Problem 3.5.

Make sure that all five graphs appear on the same plot.

MATLAB Problem 3.7

Consider a carbon/epoxy composite lamina with the following matrix and fiber material properties [2]:

$$G_{12}^f = 28.3 \text{ GPa}, \quad G^m = 1.27 \text{ GPa}$$

Use MATLAB to calculate the shear modulus G_{12} using the following three methods (use $V^f = 0.55$):

- (a) the simple rule-of-mixtures formula of (3.5).
- (b) the modified rule-of-mixtures formula of (3.13) with $\eta' = 0.6$.
- (c) the elasticity formula of (3.14).

MATLAB Problem 3.8

Consider the glass/epoxy composite lamina of Problem 3.7. Use MATLAB to plot a graph of the shear modulus G_{12} versus the fiber volume fraction V^f for each one of the following cases. Use all values of V^f ranging from 0 to 1 (in increments of 0.1).

- (a) the simple rule-of-mixtures formula of (3.5).
- (b) the modified rule-of-mixtures formula of (3.13) with $\eta' = 0.6$.
- (c) the elasticity formula of (3.14).

Make sure that all three graphs appear on the same plot.

MATLAB Problem 3.9

Consider the graphite-reinforced polymer composite lamina of Example 3.2. Let the coefficients of thermal expansion for the matrix and fibers be given as follows [1]:

$$\begin{aligned}\alpha^m &= 41.4 \times 10^{-6}/\text{K} \\ \alpha_1^f &= -0.540 \times 10^{-6}/\text{K} \\ \alpha_2^f &= 10.10 \times 10^{-6}/\text{K}\end{aligned}$$

Use MATLAB to calculate α_1 and α_2 for the lamina. When calculating α_2 , use the two formulas given (3.7) and (3.8).

Problem 3.10

Consider a fiber-reinforced composite lamina assuming the existence of an interface region. Let E^f , E^m , and E^i be Young's moduli for the matrix, fiber, and interface material, respectively. Also, let V^f , V^m , and V^i be the volume fractions of the fiber, matrix, and interface satisfying the relation $V^f + V^m + V^i = 1$. Determine an expression for the longitudinal modulus E_1 of the lamina using a simple rule-of-mixtures formula.