5

Global Coordinate System

5.1 Basic Equations

In this chapter, we will refer the response of each layer (lamina) of material to the same global system. We accomplish this by transforming the stress-strain relations for the lamina 1-2-3 coordinate system into the *global coordinate system*. This transformation will be done for the state of plane stress using the standard transformation relations for stresses and strains given in introductory courses in mechanics of materials [1].

Consider an isolated infinitesimal element in the principal material coordinate system (1-2-3 system) that will be transformed into the x-y-z global coordinate system as shown in Fig. 5.1. The fibers are oriented at angle θ with respect to the $+x$ axis of the global system. The fibers are parallel to the x-y plane and the 3 and z axes coincide. The orientation angle θ will be considered positive when the fibers rotate counterclockwise from the $+x$ axis toward the $+y$ axis.

The stresses on the small volume of element are now identified with respect to the x-y-z system. The six components of stress are now σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, and τ_{xy}, while the six components of strain are ε_x, ε_y, ε_z, γ_{yz}, γ_{xz}, and γ_{xy} (see Fig. 5.2).

Note that in a plane stress state, it follows that the out-of-plane stress components in the x-y-z global coordinate system are zero, i.e. $\sigma_z = \tau_{yz} = \tau_{xz} = 0$ (see Problem 5.1).

The stress transformation relation is given as follows for the case of plane stress:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{5.1}$$

where $m = \cos\theta$ and $n = \sin\theta$. The above relation is written in compact form as follows:

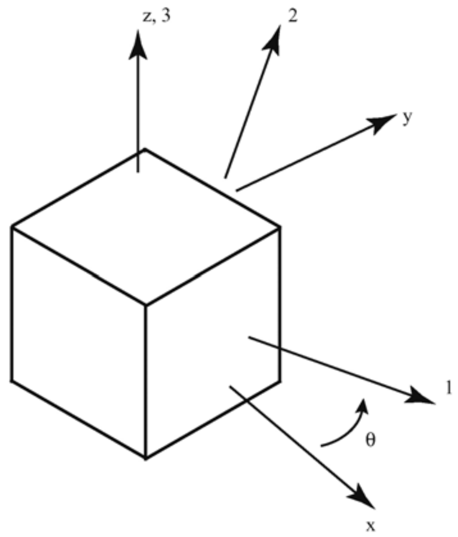

Fig. 5.1. A infinitesimal fiber-reinforced composite element showing the local and global coordinate systems

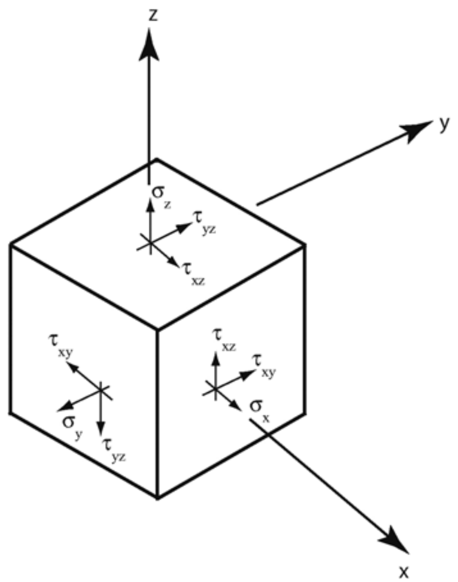

Fig. 5.2. An infinitesimal fiber-reinforced composite element showing the stress components in the global coordinate system

$$\left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\} = [T] \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} \tag{5.2}$$

where $[T]$ is the transformation matrix given as follows:

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \tag{5.3}$$

The inverse of the matrix $[T]$ is $[T]^{-1}$ given as follows (see Problem 5.3):

$$[T]^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix} \tag{5.4}$$

where $[T]^{-1}$ is used in the following equation:

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = [T]^{-1} \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\} \tag{5.5}$$

Similar transformation relations hold for the strains as follows:

$$\left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{array} \right\} = [T] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{array} \right\} \tag{5.6}$$

$$\left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{array} \right\} = [T]^{-1} \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{array} \right\} \tag{5.7}$$

Note that the strain transformation (5.6) and (5.7) include a factor of 1/2 with the engineering shear strain. Therefore (4.5) and (4.6) of Chap. 4 are modified now to include this factor as follows:

$$\left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{array} \right\} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & \frac{1}{2}S_{66} \end{bmatrix} \left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\} \tag{5.8}$$

$$\left\{ \begin{array}{c} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{array} \right\} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{array} \right\} \tag{5.9}$$

Substitute (5.6) and (5.2) into (5.8) and rearrange the terms to obtain (also multiply the third row through by a factor of 2):

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{5.10}$$

where the *transformed reduced compliance matrix* $[\bar{S}]$ is given by:

$$[\bar{S}] = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} = [T]^{-1} \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} [T] \tag{5.11}$$

Equation (5.11) represents the complex relations that describe the response of an element of fiber-reinforced composite material in a state of plane stress that is subjected to stresses not aligned with the fibers, nor perpendicular to the fibers. In this case, normal stresses cause shear strains and shear stresses cause extensional strains. This coupling found in fiber-reinforced composite materials is called *shear-extension coupling.*

Similarly, we can derive the *transformed reduced stiffness matrix* $[\bar{Q}]$ by substituting (5.2) and (5.6) into (5.9) and rearranging the terms. We therefore obtain:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{5.12}$$

where $[\bar{Q}]$ is given by:

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} [T] \tag{5.13}$$

Equation (5.13) further supports the shear-extension coupling of fiber-reinforced composite materials. Note that the following relations hold between $[\bar{S}]$ and $[\bar{Q}]$:

$$[\bar{Q}] = [\bar{S}]^{-1} \tag{5.14a}$$

$$[\bar{S}] = [\bar{Q}]^{-1} \tag{5.14b}$$

5.2 MATLAB Functions Used

The four MATLAB functions used in this chapter to calculate the four major matrices are:

T(*theta*) – This function calculates the transformation matrix $[T]$ given the angle "theta". The orientation angle "theta" must be given in degrees. The returned matrix has size 3×3.

Tinv(*theta*) – This function calculates the inverse of the transformation matrix $[T]$ given the angle "theta". The orientation angle "theta" must be given in degrees. The returned matrix has size 3×3.

Sbar(*S,theta*) – This function calculates the transformed reduced compliance matrix $[\bar{S}]$ for the lamina. Its input consists of two arguments representing the reduced compliance matrix $[S]$ and the orientation angle "theta". The returned matrix has size 3×3.

Qbar(*Q,theta*) – This function calculates the transformed reduced stiffness matrix $[\bar{Q}]$ for the lamina. Its input consists of two arguments representing the reduced stiffness matrix $[Q]$ and the orientation angle "theta". The returned matrix has size 3×3.

The following is a listing of the MATLAB source code for each function:

```
function y = T(theta)
%T   This function returns the transformation matrix T
%    given the orientation angle "theta".
%    There is only one argument representing "theta"
%    The size of the matrix is 3 x 3.
%    The angle "theta" must be given in degrees.
m = cos(theta*pi/180);
n = sin(theta*pi/180);
y = [m*m n*n 2*m*n ; n*n m*m -2*m*n ; -m*n m*n m*m-n*n];
```

```
function y = Tinv(theta)
%Tinv   This function returns the inverse of the
%       transformation matrix T
%       given the orientation angle "theta".
%       There is only one argument representing "theta"
%       The size of the matrix is 3 x 3.
%       The angle "theta" must be given in degrees.
m = cos(theta*pi/180);
n = sin(theta*pi/180);
y = [m*m n*n -2*m*n ; n*n m*m 2*m*n ; m*n -m*n m*m-n*n];
```

```
function y = Sbar(S,theta)
%Sbar   This function returns the transformed reduced
%       compliance matrix "Sbar" given the reduced
%       compliance matrix S and the orientation
%       angle "theta".
%       There are two arguments representing S and "theta"
%       The size of the matrix is 3 x 3.
%       The angle "theta" must be given in degrees.
m = cos(theta*pi/180);
```

```
n = sin(theta*pi/180);
T = [m*m n*n 2*m*n ; n*n m*m -2*m*n ; -m*n m*n m*m-n*n];
Tinv = [m*m n*n -2*m*n ; n*n m*m 2*m*n ; m*n -m*n m*m-n*n];
y = Tinv*S*T;
```

```
function y = Qbar(Q,theta)
%Qbar   This function returns the transformed reduced
%       stiffness matrix "Qbar" given the reduced
%       stiffness matrix Q and the orientation
%       angle "theta".
%       There are two arguments representing Q and "theta"
%       The size of the matrix is 3 x 3.
%       The angle "theta" must be given in degrees.
m = cos(theta*pi/180);
n = sin(theta*pi/180);
T = [m*m n*n 2*m*n ; n*n m*m -2*m*n ; -m*n m*n m*m-n*n];
Tinv = [m*m n*n -2*m*n ; n*n m*m 2*m*n ; m*n -m*n m*m-n*n];
y = Tinv*Q*T;
```

Example 5.1

Using (5.11), derive explicit expressions for the elements $\bar{S}_{ij}$ in terms of S_{ij} and θ (use m and n for θ).

Solution

Multiply the three matrices in (5.11) as follows:

$$\begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2-n^2 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \quad (5.15)$$

The above multiplication can be performed either manually or using a computer algebra system like MAPLE or MATHEMATICA or the MATLAB Symbolic Math Toolbox. Therefore, we obtain the following expression:

$$\bar{S}_{11} = S_{11}m^4 + (2S_{12} + S_{66})n^2m^2 + S_{22}n^4 \quad (5.16a)$$

$$\bar{S}_{12} = (S_{11} + S_{22} - S_{66})n^2m^2 + S_{12}(n^4 + m^4) \quad (5.16b)$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})nm^3 - (2S_{22} - 2S_{12} - S_{66})n^3m \quad (5.16c)$$

$$\bar{S}_{22} = S_{11}n^4 + (2S_{12} + S_{66})n^2m^2 + S_{22}m^4 \quad (5.16d)$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})n^3m - (2S_{22} - 2S_{12} - S_{66})nm^3 \quad (5.16e)$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})n^2m^2 + S_{66}(n^4 + m^4) \quad (5.16f)$$

MATLAB Example 5.2

Consider a graphite-reinforced polymer composite lamina with the elastic constants as given in Example 2.2. Use MATLAB to plot the values of the six elements $\bar{S}_{ij}$ of the transformed reduced compliance matrix $[\bar{S}]$ as a function of the orientation angle θ in the range $-\pi/2 \leq \theta \leq \pi/2$.

Solution

This example is solved using MATLAB. First, the reduced 3×3 compliance matrix is obtained as follows using the MATLAB function *ReducedCompliance* of Chap. 4.

```
>> S = ReducedCompliance(155.0, 12.10, 0.248, 4.40)

S =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273
```

Next, the transformed reduced compliance matrix $[\bar{S}]$ is calculated at each value of θ between $-90°$ and $90°$ in increments of $10°$ using the MATLAB function *Sbar*.

```
>> S1 = Sbar(S, -90)

S1 =

    0.0826   -0.0016   -0.0000
   -0.0016    0.0065    0.0000
   -0.0000    0.0000    0.2273

>> S2 = Sbar(S, -80)

S2 =

    0.0909   -0.0122   -0.0452
   -0.0122    0.0193    0.0712
   -0.0226    0.0356    0.2061

>> S3 = Sbar(S, -70)

S3 =

    0.1111   -0.0390   -0.0647
   -0.0390    0.0528    0.1137
   -0.0323    0.0568    0.1524
```

```
>> S4 = Sbar(S, -60)

S4 =

    0.1315   -0.0695   -0.0454
   -0.0695    0.0934    0.1114
   -0.0227    0.0557    0.0914

>> S5 = Sbar(S, -50)

S5 =

    0.1390   -0.0894    0.0065
   -0.0894    0.1258    0.0685
    0.0033    0.0342    0.0516

>> S6 = Sbar(S, -40)

S6 =

    0.1258   -0.0894    0.0685
   -0.0894    0.1390    0.0065
    0.0342    0.0033    0.0516

>> S7 = Sbar(S, -30)

S7 =

    0.0934   -0.0695    0.1114
   -0.0695    0.1315   -0.0454
    0.0557   -0.0227    0.0914


>> S8 = Sbar(S, -20)

S8 =

    0.0528   -0.0390    0.1137
   -0.0390    0.1111   -0.0647
    0.0568   -0.0323    0.1524

>> S9 = Sbar(S, -10)

S9 =

    0.0193   -0.0122    0.0712
   -0.0122    0.0909   -0.0452
    0.0356   -0.0226    0.2061
```

```
>> S10 = Sbar(S, 0)

S10 =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273

>> S11 = Sbar(S, 10)

S11 =

    0.0193   -0.0122   -0.0712
   -0.0122    0.0909    0.0452
   -0.0356    0.0226    0.2061

>> S12 = Sbar(S, 20)

S12 =

    0.0528   -0.0390   -0.1137
   -0.0390    0.1111    0.0647
   -0.0568    0.0323    0.1524

>> S13 = Sbar(S, 30)

S13 =

    0.0934   -0.0695   -0.1114
   -0.0695    0.1315    0.0454
   -0.0557    0.0227    0.0914

>> S14 = Sbar(S, 40)

S14 =

    0.1258   -0.0894   -0.0685
   -0.0894    0.1390   -0.0065
   -0.0342   -0.0033    0.0516

>> S15 = Sbar(S, 50)

S15 =

    0.1390   -0.0894   -0.0065
   -0.0894    0.1258   -0.0685
   -0.0033   -0.0342    0.0516

>> S16 = Sbar(S, 60)
```

```
S16 =

    0.1315   -0.0695    0.0454
   -0.0695    0.0934   -0.1114
    0.0227   -0.0557    0.0914

>> S17 = Sbar(S, 70)

S17 =

    0.1111    -0.0390   0.0647
   -0.0390     0.0528  -0.1137
    0.0323    -0.0568   0.1524

>> S18 = Sbar(S, 80)

S18 =

    0.0909    -0.0122   0.0452
   -0.0122     0.0193  -0.0712
    0.0226    -0.0356   0.2061

>> S19 = Sbar(S, 90)

S19 =

    0.0826    -0.0016   0.0000
   -0.0016     0.0065  -0.0000
    0.0000    -0.0000   0.2273
```

The x-axis is now setup for the plots as follows:

```
>> x = [-90  -80  -70  -60  -50  -40  -30  -20  -10  0  10  20  30 40
       50  60  70  80  90]

x =

   -90  -80  -70  -60  -50  -40  -30  -20  -10  0  10  20  30  40  50
   60  70   80   90
```

The values of $\bar{S}_{11}$ are now calculated for each value of θ between $-90°$ and $90°$ in increments of $10°$.

```
>> y1 = [S1(1,1) S2(1,1) S3(1,1) S4(1,1) S5(1,1) S6(1,1) S7(1,1)
        S8(1,1) S9(1,1) S10(1,1) S11(1,1) S12(1,1) S13(1,1) S14(1,1)
        S15(1,1) 16(1,1) S17(1,1) S18(1,1) S19(1,1)]

y1 =

   Columns 1 through 14
```

```
    0.0826    0.0909    0.1111    0.1315    0.1390    0.1258    0.0934
    0.0528    0.0193    0.0065    0.0193    0.0528    0.0934    0.1258

  Columns 15 through 19

    0.1390    0.1315    0.1111    0.0909    0.0826
```

The plot of the values of $\bar{S}_{11}$ versus θ is now generated using the following commands and is shown in Fig. 5.3. Notice that this compliance is an even function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from $0°$.

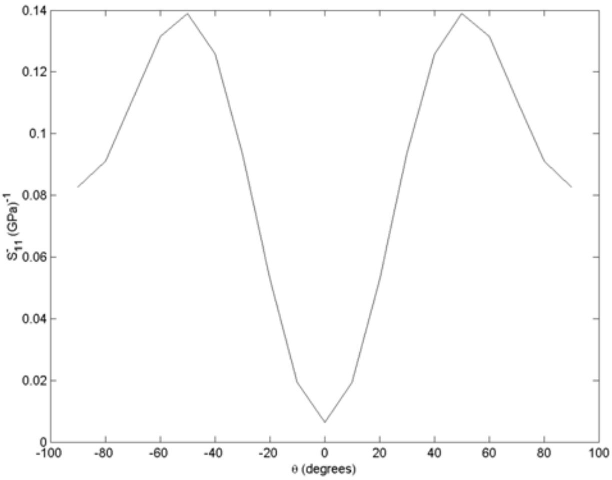

Fig. 5.3. Variation of $\bar{S}_{11}$ versus θ for Example 5.2

```
>> plot(x,y1)
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{11} GPa');
```

The values of $\bar{S}_{12}$ are now calculated for each value of θ between $-90°$ and $90°$ in increments of $10°$.

```
>> y2 = [S1(1,2)  S2(1,2)  S3(1,2)  S4(1,2)  S5(1,2)  S6(1,2) S7(1,2)
        S8(1,2) S9(1,2)  S10(1,2)  S11(1,2)  S12(1,2) S13(1,2)
        S14(1,2)  S15(1,2)  S16(1,2) S17(1,2)  S18(1,2) S19(1,2)]

y2 =

  Columns 1 through 14

   -0.0016   -0.0122   -0.0390   -0.0695   -0.0894   -0.0894   -0.0695
   -0.0390   -0.0122   -0.0016   -0.0122   -0.0390   -0.0695   -0.0894
```

```
  Columns 15 through 19

    -0.0894   -0.0695   -0.0390   -0.0122   -0.0016
```

The plot of the values of $\bar{S}_{12}$ versus θ is now generated using the following commands and is shown in Fig. 5.4. Notice that this compliance is an even function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from 0°.

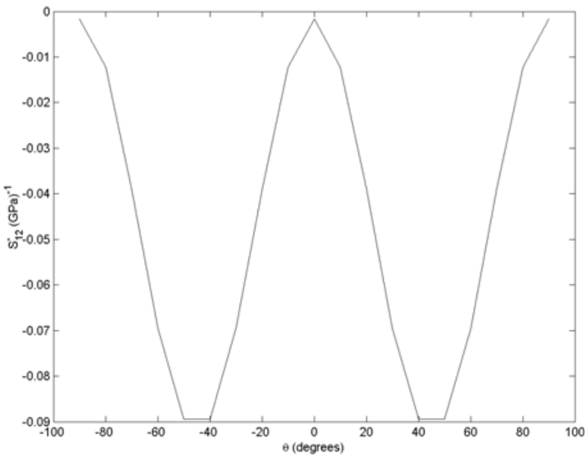

Fig. 5.4. Variation of $\bar{S}_{12}$ versus θ for Example 5.2

```
>> plot(x,y2)
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{12} GPa');
```

The values of $\bar{S}_{16}$ are now calculated for each value of θ between $-90°$ and 90° in increments of 10°.

```
>> y3 = [S1(1,3) S2(1,3) S3(1,3) S4(1,3) S5(1,3) S6(1,3) S7(1,3)
         S8(1,3) S9(1,3) S10(1,3) S11(1,3) S12(1,3) S13(1,3) S14(1,3)
         S15(1,3) S16(1,3) S17(1,3) S18(1,3) S19(1,3)]

y3 =

  Columns 1 through 14

    -0.0000   -0.0452   -0.0647   -0.0454    0.0065    0.0685    0.1114
    0.1137    0.0712          0   -0.0712   -0.1137   -0.1114   -0.0685

 Columns 15 through 19

    -0.0065    0.0454    0.0647    0.0452    0.0000
```

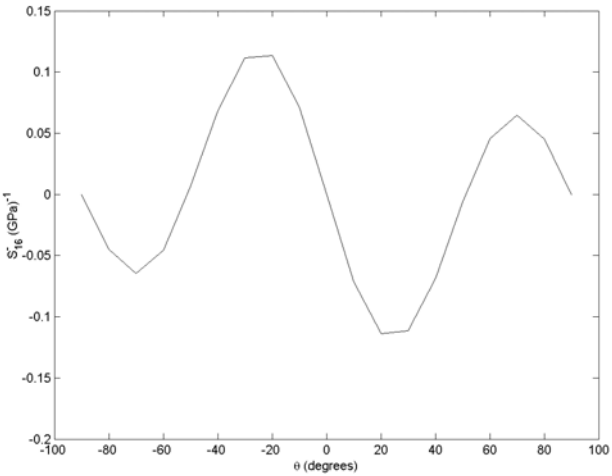

Fig. 5.5. Variation of $\bar{S}_{16}$ versus θ for Example 5.2

The plot of the values of $\bar{S}_{16}$ versus θ is now generated using the following commands and is shown in Fig. 5.5. Notice that this compliance is an odd function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from $0°$.

```
>> plot(x,y3)
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{16} GPa');
```

The values of $\bar{S}_{22}$ are now calculated for each value of θ between $-90°$ and $90°$ in increments of $10°$.

```
>> y4 = [S1(2,2) S2(2,2) S3(2,2) S4(2,2) S5(2,2) S6(2,2) S7(2,2)
        S8(2,2) S9(2,2) S10(2,2) S11(2,2) S12(2,2) S13(2,2) S14(2,2)
        S15(2,2) S16(2,2) S17(2,2) S18(2,2) S19(2,2)]

y4 =

  Columns 1 through 14

    0.0065    0.0193    0.0528    0.0934    0.1258    0.1390 0.1315
    0.1111    0.0909    0.0826    0.0909    0.1111    0.1315    0.1390

  Columns 15 through 19

    0.1258    0.0934    0.0528    0.0193    0.0065
```

The plot of the values of $\bar{S}_{22}$ versus θ is now generated using the following commands and is shown in Fig. 5.6. Notice that this compliance is an even function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from $0°$.

```
>> plot(x,y4)}
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{22} GPa');
```

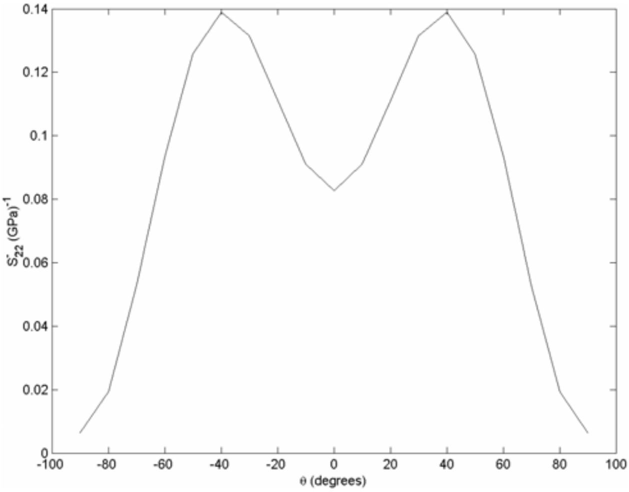

Fig. 5.6. Variation of $\bar{S}_{22}$ versus θ for Example 5.2

The values of $\bar{S}_{26}$ are now calculated for each value of θ between $-90°$ and $90°$ in increments of $10°$

```
>> y5 = [S1(2,3) S2(2,3) S3(2,3) S4(2,3) S5(2,3) S6(2,3) S7(2,3)
        S8(2,3) S9(2,3) S10(2,3) S11(2,3) S12(2,3) S13(2,3) S14(2,3)
        S15(2,3) S16(2,3) S17(2,3) S18(2,3) S19(2,3)]

y5 =

  Columns 1 through 14

    0.0000    0.0712    0.1137    0.1114    0.0685    0.0065   -0.0454
   -0.0647   -0.0452         0    0.0452    0.0647    0.0454   -0.0065

  Columns 15 through 19

   -0.0685   -0.1114   -0.1137   -0.0712   -0.0000
```

The plot of the values of $\bar{S}_{26}$ versus θ is now generated using the following commands and is shown in Fig. 5.7. Notice that this compliance is an odd function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from $0°$.

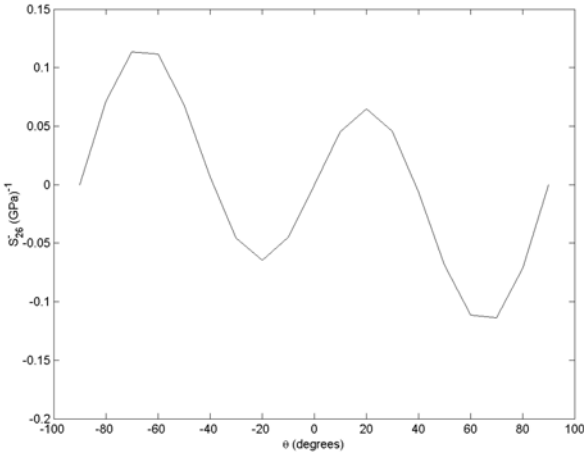

Fig. 5.7. Variation of $\bar{S}_{26}$ versus θ for Example 5.2

```
>> plot(x,y5)
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{26} GPa');}
```

The values of $\bar{S}_{66}$ are now calculated for each value of θ between $-90°$ and $90°$ in increments of $10°$.

```
>> y6 = [S1(3,3) S2(3,3) S3(3,3) S4(3,3) S5(3,3) S6(3,3) S7(3,3)
        S8(3,3) S9(3,3) S10(3,3) S11(3,3) S12(3,3) S13(3,3) S14(3,3)
        S15(3,3) S16(3,3) S17(3,3) S18(3,3) S19(3,3)]

y6 =

  Columns 1 through 14

    0.2273    0.2061    0.1524    0.0914    0.0516    0.0516   0.0914
    0.1524    0.2061    0.2273    0.2061    0.1524    0.0914    0.0516

  Columns 15 through 19

    0.0516    0.0914    0.1524    0.2061    0.2273
```

The plot of the values of $\bar{S}_{66}$ versus θ is now generated using the following commands and is shown in Fig. 5.8. Notice that this compliance is an even function of θ. Notice also the rapid variation of the compliance as θ increases or decreases from $0°$.

```
>> plot(x,y6)
>> xlabel('\theta (degrees)');
>> ylabel('S^{-}_{66} GPa');
```

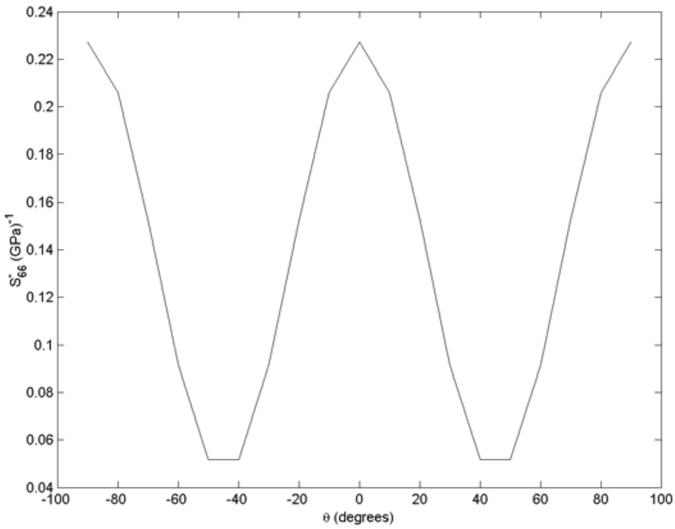

Fig. 5.8. Variation of $\bar{S}_{66}$ versus θ for Example 5.2

MATLAB Example 5.3

Consider a plane element of size 40 mm × 40 mm made of graphite-reinforced polymer composite material whose elastic constants are given in Example 2.2. The element is subjected to a tensile stress $\sigma_x = 200$ MPa in the x-direction. Use MATLAB to calculate the strains and the deformed dimensions of the element in the following two cases:

(a) the fibers are aligned along the x-axis.
(b) the fibers are inclined to the x-axis with an orientation angle $\theta = 30°$.

Solution

This example is solved using MATLAB. First, the reduced compliance matrix is obtained as follows using the MATLAB function *ReducedCompliance* of Chap. 4.

```
>> S = ReducedCompliance(155.0, 12.10, 0.248, 4.40)

S =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273
```

Next, the transformed reduced compliance matrix is calculated for part (a) with $\theta = 0°$ using the MATLAB function *Sbar*.

```
>> S1 = Sbar(S,0)
```

```
S1 =

    0.0065   -0.0016         0
   -0.0016    0.0826         0
         0         0    0.2273
```

Next, the stress vector in the global coordinate system is setup in GPa as follows:

```
>> sigma = [200e-3 ; 0 ; 0]

sigma =

    0.2000
         0
         0
```

The strain vector is now calculated in the global coordinate system using (5.10):

```
>> epsilon = S1*sigma

epsilon =

    0.0013
   -0.0003
         0
```

The change in the length in both the x- and y-direction is calculated next in mm as follows:

```
>> deltax = 40*epsilon(1)

deltax =

    0.0516

>> deltay = 40*epsilon(2)

deltay =

   -0.0128
```

The change in the right angle (in radians) of the element is then calculated using the shear strain obtained from the strain vector above. It is noticed that in this case, this change is zero indicating that the right angle remains a right angle after deformation. This is mainly due to the fibers being aligned along the x-direction.

```
>> gammaxy = epsilon(3)

gammaxy =

     0
```

The deformed dimensions are next calculated as follows:

```
>> dx = 40 + deltax

dx =

   40.0516

>> dy = 40 + deltay

dy =

   39.9872
```

Next, the transformed reduced compliance matrix is calculated for part (b) with $\theta = 30^\circ$ using the MATLAB function *Sbar*.

```
>> S2 = Sbar(S, 30)

S2 =

    0.0934   -0.0695   -0.1114
   -0.0695    0.1315    0.0454
   -0.0557    0.0227    0.0914
```

The strain vector is now calculated in the global coordinate system using (5.10):

```
>> epsilon = S2*sigma

epsilon =

    0.0187
   -0.0139
   -0.0111
```

The change in the length in both the x- and y-direction is calculated next in mm as follows:

```
>> deltax = 40*epsilon(1)

deltax =

    0.7474

>> deltay = 40*epsilon(2)

deltay =

   -0.5562
```

The deformed dimensions are next calculated as follows:

```
>> dx = 40 + deltax
```

```
dx =

   40.7474

>> dy = 40 + deltay

dy =

   39.4438
```

The change in the right angle (in radians) of the element is then calculated using the shear strain obtained from the strain vector above. It is noticed that in this case, there is a negative shear strain indicating that the right angle increases to become more than 90° after deformation. This is mainly due to the fibers being inclined at an angle to the x-direction.

```
>> gammaxy = epsilon(3)

gammaxy =

   -0.0111
```

Problems

Problem 5.1

Show mathematically why the three stresses σ_z, τ_{yz}, and τ_{xz} (with respect to the global coordinate system) vanish in the case of plane stress.

Problem 5.2

Derive (5.1) in detail.

Problem 5.3

Derive the expression for $[T]^{-1}$ given in (5.4). Use (5.3) in your derivation.

Problem 5.4

Show the validity of (5.14a,b).

Problem 5.5

Using (5.13), derive explicit expressions for the elements $\bar{Q}_{ij}$ in terms of Q_{ij} and θ (use m and n for θ).

MATLAB Problem 5.6

Write a new MATLAB function called *Tinv2* which calculates the inverse of the transformation matrix $[T]$ by calculating first $[T]$ then inverting it using the MATLAB function `inv`. Use the same argument "theta" that was used in the MATLAB function *Tinv*.

MATLAB Problem 5.7

(a) Write a new MATLAB function called *Sbar2* to calculate the transformed reduced compliance matrix $[\bar{S}]$. Use the two arguments S and T instead of S and "theta" as was used in the MATLAB function *Sbar*.
(b) Write a new MATLAB function called *Qbar2* to calculate the transformed reduced stiffness matrix $[\bar{Q}]$. Use the two arguments Q and T instead of Q and "theta" as was used in the MATLAB function *Qbar*.

MATLAB Problem 5.8

Consider a glass-reinforced polymer composite lamina with the elastic constants as given in Problem 2.7. Use MATLAB to plot the values of the six elements $\bar{S}_{ij}$ of the transformed reduced compliance matrix $[\bar{S}]$ as a function of the orientation angle θ in the range $-\pi/2 \leq \theta \leq \pi/2$.

MATLAB Problem 5.9

Consider a graphite-reinforced polymer composite lamina with the elastic constants as given in Example 2.2. Use MATLAB to plot the values of the six elements $\bar{Q}_{ij}$ of the transformed reduced stiffness matrix $[\bar{Q}]$ as a function of the orientation angle θ in the range $-\pi/2 \leq \theta \leq \pi/2$.

MATLAB Problem 5.10

Consider a glass-reinforced polymer composite lamina with the elastic constants as given in Problem 2.7. Use MATLAB to plot the values of the six elements $\bar{Q}_{ij}$ of the transformed reduced stiffness matrix $[\bar{Q}]$ as a function of the orientation angle θ in the range $-\pi/2 \leq \theta \leq \pi/2$.

Problem 5.11

(a) Show that the transformed reduced compliance matrix $[\bar{S}]$ becomes equal to the reduced compliance matrix $[S]$ when $\theta = 0°$.
(b) Show that the transformed reduced stiffness matrix $[\bar{Q}]$ becomes equal to the reduced stiffness matrix $[Q]$ when $\theta = 0°$.

Problem 5.12

Show that $[\bar{S}] = [S]$ for isotropic materials. In particular, show the following relation:

$$[\bar{S}] = [S] = \begin{bmatrix} \dfrac{1}{E} & -\dfrac{\nu}{E} & 0 \\ -\dfrac{\nu}{E} & \dfrac{1}{E} & 0 \\ 0 & 0 & \dfrac{2(1+\nu)}{E} \end{bmatrix} \tag{5.17}$$

Problem 5.13

Show that $[\bar{Q}] = [Q]$ for isotropic materials. In particular, show the following relation:

$$[\bar{Q}] = [Q] = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 \\ 0 & 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix} \tag{5.18}$$

MATLAB Problem 5.14

Consider a plane element of size 50 mm × 50 mm made of glass-reinforced polymer composite material whose elastic constants are given in Problem 2.7. The element is subjected to a tensile stress $\sigma_x = 100\,\text{MPa}$ in the x-direction. Use MATLAB to calculate the strains and the deformed dimensions of the element in the following three cases:

(a) the fibers are aligned along the x-axis.
(b) the fibers are inclined to the x-axis with an orientation angle $\theta = 45°$.
(c) the fibers are inclined to the x-axis with an orientation angle $\theta = -45°$.

Problem 5.15

Consider the case of free thermal and moisture strains. Show that in this case (5.10) and (5.12) take the following modified forms:

$$\begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T - \beta_x \Delta M \\ \varepsilon_y - \alpha_y \Delta T - \beta_y \Delta M \\ \gamma_{xy} - \alpha_{xy} \Delta T - \beta_{xy} \Delta M \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{5.19}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T - \beta_x \Delta M \\ \varepsilon_y - \alpha_y \Delta T - \beta_y \Delta M \\ \gamma_{xy} - \alpha_{xy} \Delta T - \beta_{xy} \Delta M \end{Bmatrix} \tag{5.20}$$

where ΔT and ΔM are the changes in temperature and moisture, respectively, α_x, α_y and α_{xy} are the coefficients of thermal expansion with respect to the global coordinate system, and β_x, β_y, and β_{xy} are the coefficients of moisture deformation with respect to the global coordinate system.