11

Introduction to Homogenization of Composite Materials

11.1 Eshelby Method

In this chapter, we present a brief overview of the homogenization of composite materials. Homogenization refers to the process of considering a statistically homogeneous representation of the composite material called a *representative volume element (RVE)*. This homogenized element is considered for purposes of calculating the stresses and strains in the matrix and fibers. We will emphasize mainly the *Eshelby method* in the homogenization process. For more details, the reader is referred to the book *An Introduction to Metal Matrix Composites* by Clyne and Withers.

Since the composite system is composed of two different materials (matrix and fibers) with two different stiffnesses, internal stresses will arise in both the two constituents. Eshelby in the 1950s demonstrated that an analytical solution may be obtained for the special case when the fibers have the shape of an ellipsoid. Furthermore, the stress is assumed to be uniform within the ellipsoid. Eshelby's method is summarized by representing the actual inclusion (i.e fibers) by one made of the matrix material (called the *equivalent homogeneous inclusion*). This equivalent inclusion is assumed to have an appropriate strain (called the *equivalent transformation strain*) such that the stress field is the same as for the actual inclusion. This is the essence of the homogenization process.

The following is a summary of the steps followed in the homogenization procedure according to the Eshelby method (see Fig. 11.1):

1. Consider an initially unstressed elastic homogeneous material (see Fig. 11.1a). Imagine cutting an ellipsoidal region (i.e. inclusion) from this material. Imagine also that the inclusion undergoes a shape change free from the constraining matrix by subjecting it to a transformation strain ε_{ij}^T (see Fig. 11.1b) where the indices i and j take the values 1, 2, and 3.
2. Since the inclusion has now changed in shape, it cannot be replaced directly into the hole in the matrix material. Imagine applying surface tractions to

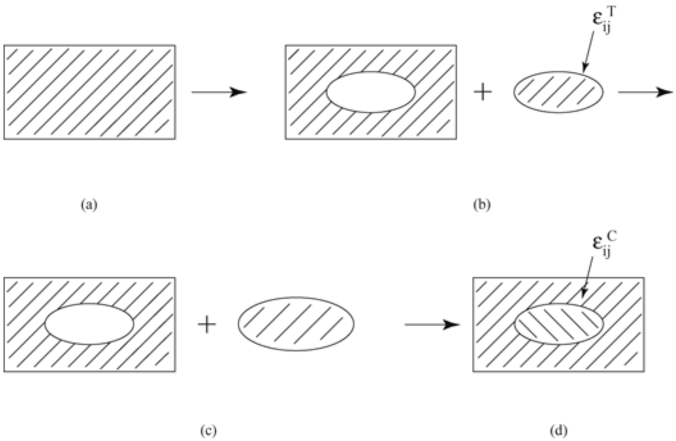

Fig. 11.1. Schematic illustration of homogenization according to the Eshelby method

the inclusion to return it to its original shape, then imagine returning it back to the matrix material (see Fig. 11.1c).

3. Imagine welding the inclusion and matrix material together then removing the surface tractions. The matrix and inclusion will then reach an equilibrium state when the inclusion has a *constraining strain* ε_{ij}^{C} relative to the initial shape before it was removed (see Fig. 11.1d).
4. The stress in the inclusion σ_{ij}^{I} can now be calculated as follows assuming the strain is uniform within the inclusion:

$$\sigma_{ij}^{I} = C_{ijkl}^{M} \left(\varepsilon_{kl}^{C} - \varepsilon_{kl}^{T} \right) \tag{11.1}$$

where C_{ijkl}^{M} are the components of the elasticity tensor of the matrix\material.

5. Eshelby has shown that the constraining strain ε_{ij}^{C} can be calculated in terms of the transformation strain ε_{ij}^{T} using the following equations:

$$\varepsilon_{ij}^{C} = S_{ijkl} \varepsilon_{kl}^{T} \tag{11.2}$$

where S_{ijkl} are the components of the Eshelby tensor **S**. The Eshelby tensor **S** is a fourth-rank tensor determined using Poisson's ratio of the inclusion material and the inclusion's aspect ration.

6. Finally, the stress in the inclusion is determined by substituting (11.2) into (11.1) and simplifying to obtain:

$$\sigma_{ij}^{I} = C_{ijkl}^{M} \left(S_{klmn} - I_{klmn} \right) \varepsilon_{mn}^{T} \tag{11.3}$$

where I_{klmn} are the components of the fourth-rank identity tensor given by:

$$I_{klmn} = \frac{1}{2}\left(\delta_{km}\delta_{\ln} + \delta_{kn}\delta_{lm}\right) \tag{11.4}$$

and δ_{ij} are the components of the Kronecker delta tensor.

Using matrices, (11.3) is re-written as follows:

$$\left\{\sigma^I\right\} = \left[C^M\right]\left([S] - [I]\right)\left\{\varepsilon^T\right\} \tag{11.5}$$

where the braces are used to indicate a vector while the brackets are used to indicate a matrix.

Next, expressions of the Eshelby tensor **S** are presented for the case of long infinite cylindrical fibers. In this case, the values of the Eshelby tensor depend on Poisson's ratio ν of the fibers and are determined as follows:

$$S_{1111} = S_{2222} = \frac{5-\nu}{8(1-\nu)} \tag{11.6a}$$

$$S_{3333} = 0 \tag{11.6b}$$

$$S_{1122} = S_{2211} = \frac{-1+4\nu}{8(1-\nu)} \tag{11.6c}$$

$$S_{1133} = S_{2233} = \frac{\nu}{2(1-\nu)} \tag{11.6d}$$

$$S_{3311} = S_{3322} = 0 \tag{11.6e}$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{3-4\nu}{8(1-\nu)} \tag{11.6f}$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \frac{1}{4} \tag{11.6g}$$

$$S_{3232} = S_{3223} = S_{2332} = S_{2323} = \frac{1}{4} \tag{11.6h}$$

$$S_{ijkl} = 0\,, \qquad \text{otherwise} \tag{11.6i}$$

In addition to Eshelby's method of determining the stresses and strains in the fibers and matrix, there are other methods based on Hill's stress and strain concentration factors.

Problems

Problem 11.1

Derive the equations of the Eshelby method for the case of a misfit strain due to a differential thermal contraction assuming that the matrix and inclusion have different thermal expansion coefficients.

Problem 11.2

Derive the equations of the method for the case of internal stresses in externally loaded composites. Assume the existence of an external load that is responsible for the transfer of load to the inclusion.

Problem 11.3

The formulation in this chapter has been based on what are called dilute composite systems, i.e. a single inclusion is embedded within an infinite matrix. In this case, the inclusion volume fraction is less than a few percent. Consider non-dilute systems where the inclusion volume fraction is much higher with many inclusions. What modifications to the equations of the Eshelby method are needed to formulate the theory for non-dilute systems.