
Plane Stress

4.1 Basic Equations

In the analysis of fiber-reinforced composite materials, the assumption of *plane stress* is usually used for each layer (lamina). This is mainly because fiber-reinforced materials are utilized in beams, plates, cylinders, and other structural shapes which have at least one characteristic geometric dimension in an order of magnitude less than the other two dimensions. In this case, the stress components σ_3 , τ_{23} , and τ_{13} are set to zero with the assumption that the 1-2 plane of the principal material coordinate system is in the plane of the layer (lamina) – see [1]. Therefore, the stresses σ_1 , σ_2 , and τ_{12} lie in a plane, while the stresses σ_3 , τ_{23} , and τ_{13} are perpendicular to this plane and are zero (see Fig. 4.1).

Using the assumption of plane stress, it is seen that the stress-strain relations of Chap. 2 are greatly simplified. Setting $\sigma_3 = \tau_{23} = \tau_{13} = 0$ in (2.1) leads to the following:

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{pmatrix} \quad (4.1)$$

As a result of the plane stress assumption and using (4.1), we conclude that:

$$\gamma_{23} = 0 \quad (4.2)$$

$$\gamma_{13} = 0 \quad (4.3)$$

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 \neq 0 \quad (4.4)$$

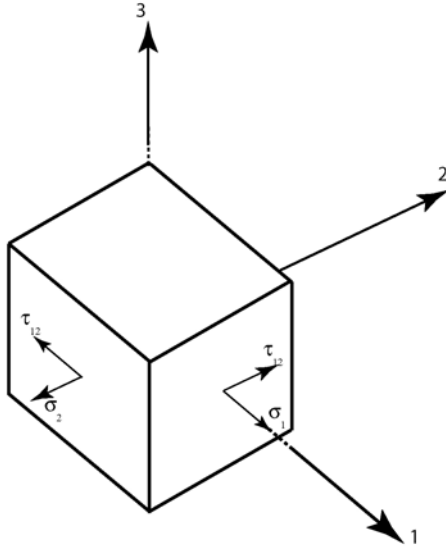


Fig. 4.1. An infinitesimal fiber-reinforced composite element in a state of plane stress

Therefore (4.1) reduces to the following equation:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (4.5)$$

The 3×3 matrix in (4.5) is called the *reduced compliance matrix*. The inverse of the reduced compliance matrix is the *reduced stiffness matrix* given as follows:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (4.6)$$

where the elements Q_{ij} are given as follows:

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} \quad (4.7a)$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} \quad (4.7b)$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} \quad (4.7c)$$

$$Q_{66} = \frac{1}{S_{66}} \quad (4.7d)$$

4.2 MATLAB Functions Used

The four MATLAB functions used in this chapter to calculate the reduced compliance and stiffness matrices are:

ReducedCompliance(E1, E2, NU12, G12) – This function calculates the reduced compliance matrix for the lamina. Its input consists of four arguments representing the four elastic constants E_1 , E_2 , ν_{12} , and G_{12} . See Problem 4.1.

ReducedStiffness(E1, E2, NU12, G12) – This function calculates the reduced stiffness matrix for the lamina. Its input consists of four arguments representing the four elastic constants E_1 , E_2 , ν_{12} , and G_{12} . See Problem 4.2.

ReducedIsotropicCompliance(E, NU) – This function calculates the reduced isotropic compliance matrix for the lamina. Its input consists of two arguments representing the two elastic constants E and ν . See Problem 4.3.

ReducedIsotropicStiffness(E, NU) – This function calculates the reduced isotropic stiffness matrix for the lamina. Its input consists of two arguments representing the two elastic constants E and ν . See Problem 4.4.

The following is a listing of the MATLAB source code for each function:

```
function y = ReducedCompliance(E1,E2,NU12,G12)
%ReducedCompliance This function returns the reduced compliance
% matrix for fiber-reinforced materials.
% There are four arguments representing four
% material constants. The size of the reduced
% compliance matrix is 3 x 3.
y = [1/E1 -NU12/E1 0 ; -NU12/E1 1/E2 0 ; 0 0 1/G12];
```

```
function y = ReducedStiffness(E1,E2,NU12,G12)
%ReducedStiffness This function returns the reduced stiffness
% matrix for fiber-reinforced materials.
% There are four arguments representing four
% material constants. The size of the reduced
% stiffness matrix is 3 x 3.
NU21 = NU12*E2/E1;
y = [E1/(1-NU12*NU21) NU12*E2/(1-NU12*NU21) 0 ; NU12*E2/(1-NU12*NU21)
E2/(1-NU12*NU21) 0 ; 0 0 G12];
```

```
function y = ReducedIsotropicCompliance(E,NU)
%ReducedIsotropicCompliance This function returns the
% reduced isotropic compliance
% matrix for fiber-reinforced materials.
% There are two arguments representing
% two material constants. The size of
% the reduced compliance matrix is 3 x 3.
y = [1/E -NU/E 0 ; -NU/E 1/E 0 ; 0 0 2*(1+NU)/E];
```

```
function y = ReducedIsotropicStiffness(E,NU)
%ReducedIsotropicStiffness This function returns the
% reduced isotropic stiffness
% matrix for fiber-reinforced materials.
% There are two arguments representing
% two material constants. The size of
% the reduced stiffness matrix is 3 x 3.
y = [E/(1-NU*NU) NU*E/(1-NU*NU) 0 ; NU*E/(1-NU*NU) E/(1-NU*NU) 0 ; 0
      0 E/2/(1+NU)];
```

Example 4.1

Derive the following expressions for the elements Q_{ij} of the 3×3 reduced stiffness matrix where C_{ij} are the elements of the 6×6 stiffness matrix of (2.3).

$$Q_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} \quad (4.8a)$$

$$Q_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}} \quad (4.8b)$$

$$Q_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} \quad (4.8c)$$

$$Q_{66} = C_{66} \quad (4.8d)$$

Solution

For the case of plane stress, set $\sigma_3 = \tau_{23} = \tau_{13} = 0$ in (2.3) to obtain (while using the symmetric form of the $[C]$ matrix).

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \quad (4.9)$$

We can therefore write the following three equations based on the first, second, and sixth rows of (4.9):

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 \quad (4.10a)$$

$$\sigma_2 = C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 \quad (4.10b)$$

$$\tau_{12} = C_{66}\gamma_{12} \quad (4.10c)$$

In addition, we can write the following relation based on the third row of (4.9):

$$0 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 \quad (4.11)$$

Solving (4.11) for ε_3 to obtain:

$$\varepsilon_3 = -\frac{C_{13}}{C_{33}}\varepsilon_1 - \frac{C_{23}}{C_{33}}\varepsilon_2 \quad (4.12)$$

Substitute (4.12) into (4.10a,b) and simplify to obtain the following relations:

$$\sigma_1 = \left(C_{11} - \frac{C_{13}^2}{C_{33}} \right) \varepsilon_1 + \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}} \right) \varepsilon_2 \quad (4.13a)$$

$$\sigma_2 = \left(C_{12} - \frac{C_{13}C_{23}}{C_{33}} \right) \varepsilon_1 + \left(C_{22} - \frac{C_{23}^2}{C_{33}} \right) \varepsilon_2 \quad (4.13b)$$

$$\tau_{12} = C_{66}\gamma_{12} \quad (4.13c)$$

Rewriting (4.13a,b,c) in matrix form we obtain (see (4.6)):

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (4.14)$$

where the elements Q_{ij} are given by (see (4.8a,b,c,d).):

$$Q_{11} = C_{11} - \frac{C_{13}^2}{C_{33}} \quad (4.15a)$$

$$Q_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}} \quad (4.15b)$$

$$Q_{22} = C_{22} - \frac{C_{23}^2}{C_{33}} \quad (4.15c)$$

$$Q_{66} = C_{66} \quad (4.15d)$$

MATLAB Example 4.2

Consider a layer of graphite-reinforced composite material 200 mm long, 100 mm wide, and 0.200 mm thick. The layer is subjected to an inplane tensile force of 4 kN in the fiber direction which is perpendicular to the 100-mm width. Assume the layer to be in a state of plane stress and use the elastic constants given in Example 2.2. Use MATLAB to determine the transverse strain ε_3 .

Solution

This example is solved using MATLAB. First, the full 6×6 compliance matrix is obtained as follows using the MATLAB function

OrthotropicCompliance of Chap. 2.

```
>> S = OrthotropicCompliance(155.0, 12.10, 12.10, 0.248, 0.458,
    0.248, 4.40, 3.20, 4.40)
```

```
S =
```

```
    0.0065    -0.0016    -0.0016     0     0     0
   -0.0016     0.0826    -0.0379     0     0     0
   -0.0016    -0.0379     0.0826     0     0     0
         0         0         0  0.3125     0     0
         0         0         0     0  0.2273     0
         0         0         0     0     0  0.2273
```

Using the third row of (4.1), we obtain the following expression for the transverse strain ε_3 (see (4.4)):

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 \quad (4.16)$$

Next, the stresses σ_1 and σ_2 are calculated in GPa as follows:

```
>> sigma1 = 4/(100*0.200)
```

```
sigma1 =
```

```
    0.2000
```

```
>> sigma2 = 0
```

```
sigma2 =
```

```
    0
```

Finally, the transverse strain ε_3 is calculated using (4.16) as follows:

```
>> epsilon3 = S(1,3)*sigma1 + S(2,3)*sigma2
```

```
epsilon3 =
```

```
   -3.2000e-004
```

Thus, we obtain the transverse strain $\varepsilon_3 = -3.2 \times 10^{-4}$.

MATLAB Example 4.3

Consider the graphite-reinforced composite material of Example 2.2.

- Use MATLAB to determine the reduced compliance and stiffness matrices.
- Use MATLAB to check that the two matrices obtained in (a) are indeed inverses of each other by multiplying them together to get the identity matrix.

Solution

This example is solved using MATLAB. First, the reduced compliance matrix is obtained as follows using the MATLAB function *ReducedCompliance*.

```
>> S = ReducedCompliance(155.0, 12.10, 0.248, 4.40)
```

S =

```
    0.0065    -0.0016     0
   -0.0016     0.0826     0
         0         0    0.2273
```

Next, the reduced stiffness matrix is obtained as follows using the MATLAB function *ReducedStiffness*:

```
>> Q = ReducedStiffness(155.0, 12.10, 0.248, 4.40)
```

Q =

```
   155.7478    3.0153     0
    3.0153   12.1584     0
         0         0    4.4000
```

Finally, the two matrices are multiplied with each other to get the identity matrix in order to show that they are indeed inverses of each other.

```
>> S*Q
```

ans =

```
    1.0000     0     0
   -0.0000    1.0000     0
         0         0    1.0000
```

Problems**Problem 4.1**

Write the reduced compliance matrix for a fiber-reinforced composite material in terms of the four elastic constants E_1 , E_2 , ν_{12} , and G_{12} .

Problem 4.2

Write the reduced stiffness matrix for a fiber-reinforced composite material in terms of the four elastic constants E_1 , E_2 , ν_{12} , and G_{12} .

Problem 4.3

Write the reduced compliance matrix for an isotropic fiber-reinforced composite material in terms of the two elastic constants E and ν .

Problem 4.4

Write the reduced stiffness matrix for an isotropic fiber-reinforced composite material in terms of the two elastic constants E and ν .

MATLAB Problem 4.5

Consider the glass-reinforced polymer composite material of Problem 2.7.

- (a) Use MATLAB to determine the reduced compliance and stiffness matrices.
- (b) Use MATLAB to check that the two matrices obtained in (a) are indeed inverses of each other by multiplying them together to get the identity matrix.

MATLAB Problem 4.6

Consider the layer of composite material of Example 4.2. Suppose that the layer is subjected to an inplane compressive force of 2.5 kN in the 2-direction instead of the 4 kN force in the 1-direction. Use MATLAB to calculate the transverse strain ε_3 in this case.

MATLAB Problem 4.7

Consider the isotropic material aluminum with $E = 72.4$ GPa and $\nu = 0.3$.

- (a) Use MATLAB to determine the reduced compliance and stiffness matrices.
- (b) Use MATLAB to check that the two matrices obtained in (a) are indeed inverses of each other by multiplying them together to get the identity matrix.

MATLAB Problem 4.8

Suppose in Example 4.2 that the fibers are perpendicular to the 200-mm direction. Use MATLAB to calculate the transverse strain ε_3 in this case.

MATLAB Problem 4.9

Write two MATLAB functions called *ReducedStiffness2* and *ReducedIsotropicStiffness2* where the reduced stiffness matrix in each case is determined by taking the inverse of the reduced compliance matrix.

Problem 4.10

Consider a layer of fiber-reinforced composite material that is subjected to both temperature and moisture variations. Write the 3×3 reduced stress-strain equations that correspond to (4.5) and (4.6). See Problems 2.9 and 2.10 of Chap. 2.