Equilibrium Consistent Anisotropic Stress Fields in Membrane Design

Kai-Uwe Bletzinger[1], Roland Wüchner[2] and Fernass Daoud[3]

[1] Lehrstuhl für Statik
Technische Universität München
80290 München, Germany
kub@bv.tum.de

[2] Lehrstuhl für Statik
Technische Universität München
80290 München, Germany
wuechner@bv.tum.de

[3] Lehrstuhl für Statik
Technische Universität München
80290 München, Germany
daoud@bv.tum.de

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Abstract. *This paper deals with the control of mesh distortions which may appear during the form finding procedure of membrane design. The reason is the unbalance of surface stresses either due to the interaction of edge and surface or incompatibilities along the sewing lines of adjacent membrane patches. An approach is presented which is based on a rational modification of the surface tension field. The criterion is based on the control of the element distortion and derived from differential geometry. Several examples demonstrate the success of the method.*

1 Introduction

Two different lines of research have developed which deal with the generation of structural shapes: the fields of "form finding" and "structural optimization" , respectively. The methods of form finding are usually restricted to tensile structures (cables and membranes) whereas the methods of structural optimization are far more general and can usually be applied to any kind of structure. So far, The differences of the two approaches are not only the level

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of specialization but also their aims. Form finding methods are designed to determine structural shapes from an inverse formulation of equilibrium and are derived from the simulation of physical phenomena of soap films and hanging models. In the case of soap films the structural shape is defined by the equilibrium geometry of a prescribed field of tensile surface stresses. It is well known that the shapes related to isotropic surface stresses are minimal surfaces which have minimal surface area content within given edges. Minimal surfaces have the additional property of zero mean curvature, or, with respect to pneumatically loaded surfaces of constant mean curvature.

Using a variational approach for the solution we realize that only those shape variations are meaningful which result in a variation of the area content. In other words, the variation of the position of any point on the surface must have a component normal to the surface. A variation of the position along the surface will not alter the area content. That means, that if a finite element method is used to solve the problem and the surface is discretized by a mesh of elements and nodes the stiffness with respect to a movement of the nodes tangential to the surface vanishes. This problem is well known since long from shape optimal design also where the design parameters must be chosen such that their modification must have an effect on the structural shape. Shape optimal design is controlled by the modification of the boundary.

There exist several remedies. Two techniques have been accepted as state of the art in shape optimal design: (i) linking the movement of internal nodes to key nodes using mapping techniques from CAGD, the so called design element technique, and, (ii) defining move directions for nodes on the boundary to guarantee relevant shape modifications. The positive side effect is that the number of optimization variables can drastically be reduced by this approach which is very attractive for optimal design. On the other hand, however, the space of possible shapes is also reduced. That is unacceptable if a high variability of shape modification is needed, as e.g. for the shape design of tensile structures or for the problem to find shapes of equilibrium of tension fields at the surface of liquids or related fields. Then methods are needed which are able to stabilize the nodal movement such that all three spatial coordinates of any finite element node may be variables in the shape modification process. For the special case of form finding of tensile structures the updated reference strategy (URS) is designed to find the shape of equilibrium of pre-stressed membranes. It is a general approach which can be applied to any kind of special element formulation (membranes or cables). A stabilization term is used which fades out as the procedure converges to the solution. The method is based on the specific relations of Cauchy and 2nd Piola-Kirchhoff stress tensors which appear to be identical at the converged solution [5], [8], [6]. Other alternative stabilization approaches have the same intention but used other methodologies, one may find many references in [4], [7], [3], [1].

All methods, however, will have problems or even fail if they are applied to physically meaningless situations without solution. E.g. it is not sure if a minimal surface exists for a given edge, or, a practical question from tent design, it is practically impossible to a priori satisfy equilibrium along the common edge of membrane strips which are anisotropically pre-stressed in different directions. The unbalance of stresses can be detected by a cumulative distortion of the FE-mesh during iteration. Methods are needed to control the mesh distortion by adapting the stress distribution. The present approach defines a local criterion to modify the pre-stress such that the element distortion is controlled. Put into the context of the updated reference strategy it appears to converge to homogeneous meshes during the regular time of iteration needed by the form finding procedure. The additional numerical effort is negligible. The results are surfaces of balanced shape representing equilibrium as close as possible at the desired distribution of stresses. The approach will be extended to other situations where mesh control might be necessary, e.g. in general shape optimal design or other surface tension problems [2].

2 The Updated Reference Strategy

The basic idea of URS will be briefly shown to understand the following chapters. A detailed description is given in [5]. Suppose one wants to find the equilibrium shape of a given surface tension field $\boldsymbol{\sigma}$. The solution is defined by the stationary condition

$$\delta w = \int_a \boldsymbol{\sigma} : \delta \mathbf{u}_{,\mathbf{x}} \; da = \int_A det\mathbf{F}(\boldsymbol{\sigma} \cdot \mathbf{F}^{-T}) : \delta\mathbf{F} \; dA = 0 \qquad (1)$$

which represents the vanishing virtual work of the Cauchy stresses $\boldsymbol{\sigma}$ and $\delta \mathbf{u}_{,\mathbf{x}}$ is the spatial derivative of the virtual displacements, $\mathbf{F}$ is the deformation gradient, a and A are the surface area of the actual and the reference configuration, respectively. As mentioned in the introduction the direct discretization of (1) w.r.t. all spatial coordinates will give a singular system of equations. The problem is stabilized by blending with the alternative formulation of (1) in terms of 2^{nd} Piola-Kirchhoff stresses $\mathbf{S}$:

$$\delta w = \lambda \int_A det\mathbf{F}(\boldsymbol{\sigma} \cdot \mathbf{F}^{-T}) : \delta\mathbf{F} \; dA + (1-\lambda) \int_A (\mathbf{F} \cdot \mathbf{S}) : \delta\mathbf{F} \; dA = 0 \qquad (2)$$

where the coefficients of $\mathbf{S}$ are assumed to be constant and identical to those of $\boldsymbol{\sigma}$. The continuation parameter λ must be chosen small enough, even zero is possible. As the solution of (2) is used as the reference configuration for a following analysis the procedure converges to the solution of (1). Then $\mathbf{F}$ becomes the identity and both stress tensors are identical. The stabilization fades out.

3 Equilibrium of Surface Stresses

As also mentioned in the introduction it is not always possible to find a shape of equilibrium for each combination of edge geometry and surface stress distribution, isotropic or anisotropic. In these cases the size of some elements will steadily increase during iteration although URS is designed to control the mesh quality. That is because it is not possible to satisfy equilibrium of the given surface stresses at the location of those elements. For example consider a catenoid, Fig. 1. If the height exceeds the critical value the surface will collapse as shown. During iteration that is indicated by the increasing length of the related elements. To avoid the collapse the meridian stresses should be increased.

Now, consider the catenoid with an constant anisotropic stress field with larger meridian stresses, Fig. 2. The surface does not collapse anymore, but still the elements at the top get out of control as indicated by the tremendous increase of size. Again, the physical explanation is that equilibrium cannot be found with a constant distribution of meridian stresses over the height.

Fig. 1. Collapsed catenoid with different end rings.

Fig. 2. Catenoid with anisotropic pre-stress (meridian/ring=3/1).

A similar situation arises in membrane design. Typically those structures are sewed together by several strips, each of them made of anisotropic material and anisotropically pre-stressed. Along the common line there exists also an intrinsic unbalance of equilibrium of stresses which must be handled with during the form finding procedure. All cases demand for automatically adjusted stresses.

4 Element Size Control

The element size is used as indicator to adjust the surface stress field. At each Gauss point of the element discretization an additional constraint is intro-

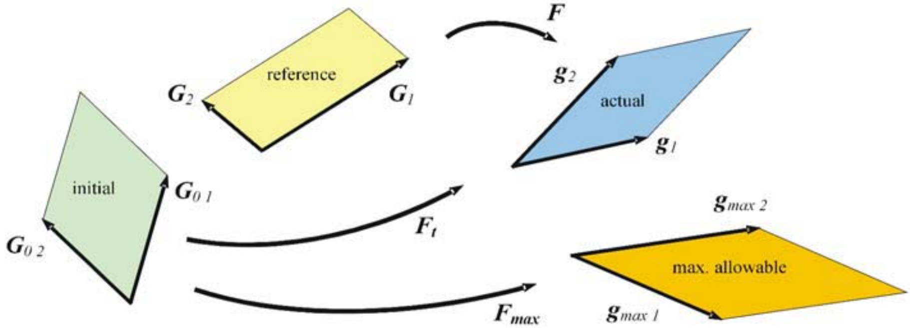


Fig. 3. Configurations during form finding.

duced to check for the length of base vectors against the allowable maximal deformation. The standard optimization approach by the Lagrangian multiplier technique is prohibitive because of the size of the problem. Therefore, a local criterion is defined which can be solved at each Gauss point independently. The approach is analogous to what is done in elastic-plastic analysis. In contrast to that, now, the surface stresses are constant until the critical deformation is reached and will then be adjusted to the further change of deformation.

Consider the following configurations and the related definition of base vectors: the initial configuration, $\mathbf{G_0}$, where the form finding was started, the reference configuration, $\mathbf{G}$, defined by the update procedure of URS, and the actual configuration, $\mathbf{g}$, as the state of equilibrium of the current time step. An additional configuration, $\mathbf{g}_{max}$ is defined which states the maximum allowable element size. There are several deformation gradients $\mathbf{F}$ defined which map differential entities between the configurations, Fig. 3.

The deformation gradient $\mathbf{F}_t^{(k)}$in time step k which describes the total deformation is multiplicatively created by $\mathbf{F}_t^{(k)} = \mathbf{F} \cdot \mathbf{F}_t^{(k-1)}$ where $\mathbf{F} = \mathbf{g}_i \otimes \mathbf{G}^{i(k)}$ and $\mathbf{F}_t^{(k)} = \mathbf{g}_i^{(k)} \otimes \mathbf{G}_0^i$.

If the actual deformation exceeds the allowable limit, i.e. $\| \mathbf{F}_t^{(k)} \| > \| \mathbf{F}_{max} \|$, a modified surface stress tensor $\boldsymbol{\sigma}_{mod}$ is generated by the following rule of nested pull back and push forward operations:

1. apply the surface Cauchy stress $\boldsymbol{\sigma}$ to the "max. allowable" configuration
2. determine the related 2^{nd} Piola-Kirchhoff stress $\mathbf{S}_0$ w.r.t. to the initial configuration by pull back using $\mathbf{F}_{max}$
3. push $\mathbf{S}_0$ forward to the actual configuration applying $\mathbf{F}_t$

The resulting operation is:

$$\boldsymbol{\sigma}_{mod} = \frac{det\,\mathbf{F}_{max}}{det\,\mathbf{F}_t} \mathbf{F}_t \cdot \mathbf{F}_{max}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}_{max}^{-T} \cdot \mathbf{F}_t^T \tag{3}$$

and, considering the 2^{nd} Piola-Kirchhoff stress tensor $\mathbf{S}$ (used in the stabilizing term of URS):

$$\boldsymbol{\sigma}_{mod} = \frac{det\,\mathbf{F}_{max}}{det\,\mathbf{F}\;det\,\mathbf{F}_t}\mathbf{F}_t \cdot \mathbf{F}_{max}^{-1} \cdot \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T \cdot \mathbf{F}_{max}^{-T} \cdot \mathbf{F}_t^T \tag{4}$$

The "max. allowable" configuration is defined by multiples of the base vectors of the actual configuration:

$$\mathbf{g}_{max\ i} = \beta_{(i)}\mathbf{g}_i \tag{5}$$

Then, we can determine the related modified deformation gradient $\mathbf{F}_{mod}$

$$\mathbf{F}_{mod} = \mathbf{F}_t \cdot \mathbf{F}_{max}^{-1} \cdot \mathbf{F} = \frac{1}{\beta_{(i)}}\mathbf{g}_i \otimes \mathbf{G}^i \tag{6}$$

The change of element size is determined by tracing the length change of base vectors during the deformation process. Introducing the ratio $\alpha_i = \parallel \mathbf{g}_{(i)} \parallel / \parallel \mathbf{G}_{(i)} \parallel$ (no summation) the total change of geometry in time step k is defined as: $\alpha_{ti}^{(k)} = \alpha_{(i)}\alpha_{t(i)}^{(k-1)}$. If the total change of geometry is larger than the allowable maximum $\alpha_{max\ i}$ then the constraint is active. The factor β_i which is used to define the "max. allowable" configuration can now be determined as:

$$\text{if} \quad \alpha_{ti}^{(k)} > \alpha_{max\ (i)} \quad \text{then :} \quad \alpha_{max\ (i)} = \alpha_{t(i)}^{(k)}\ \beta_i$$
$$\therefore \quad \beta_i = \frac{\alpha_{max(i)}}{\alpha_{t(i)}^{(k)}} \tag{7}$$

Considering both surface base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$ the determinant of $\mathbf{F}_{mod}$ is now

$$det\,\mathbf{F}_{mod} = (\beta_1\beta_2)^{(-1)}\ det\,\mathbf{F} \tag{8}$$

If constraints are active the state of stresses must change with time and the modified Cauchy stress tensor $\boldsymbol{\sigma}_{mod}^{(k+1)}$ for the next time step $k+1$ which is acting in the "max. allowed" configuration follows as:

$$\boldsymbol{\sigma}_{mod}^{(k+1)} = \frac{\beta_1\beta_2}{det\,\mathbf{F}}\mathbf{F}_{mod} \cdot \mathbf{S}^{(k)} \cdot \mathbf{F}_{mod}^T \tag{9}$$

In the context of URS the components of $\boldsymbol{\sigma}_{mod}^{(k+1)}$ will also be used as the components of $S^{(k+1)}$ in the next time step. They are:

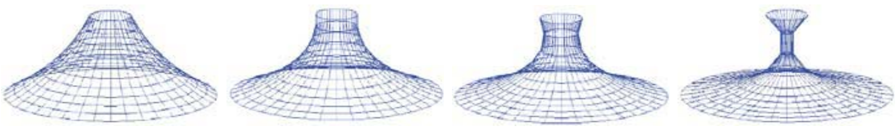
$$\begin{aligned}
\sigma_{mod}^{11(k+1)} &= \frac{\beta_2}{\beta_1\ det\,\mathbf{F}}S^{11(k)} \\
\sigma_{mod}^{22(k+1)} &= \frac{\beta_1}{\beta_2\ det\,\mathbf{F}}S^{22(k)} \\
\sigma_{mod}^{12(k+1)} &= \frac{1}{det\,\mathbf{F}}S^{12(k)}
\end{aligned} \tag{10}$$

The modification rule (10) simply says that the surface stress of an element which became too long must be increased to shorten it and vice versa.

5 Examples

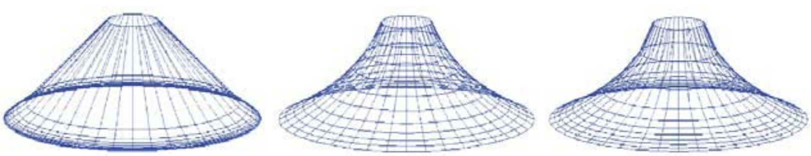
5.1 Catenoid

This example shows the effect of different values of α_{max} on the final shape of a catenoid which was initially isotropically pre-stressed. The choice of α_{max} controls the maximal deformation of the elements. A value of 1.0 means almost no element deformation. Compare with Fig. 1 how collapse could be omitted.



(a) $\alpha_{max} = 1.0$ (b) $\alpha_{max} = 1.2$ (c) $\alpha_{max} = 1.5$ (d) $\alpha_{max} = 2.0$

Fig. 4. Stabilized catenoid of initially isotropic surface stress.



(a) no constraint (b) $\alpha_{max} = 1.2$ (c) $\alpha_{max} = 2.0$

Fig. 5. Initially anisotropically pre-stressed catenoid (meridian/ring=1.2/1).

5.2 Tent Hüfingen

This tent is composed by 5 membrane patches which are anisotropically pre-stressed towards the center of the tent. A top view shows the consequences of unbalanced stresses at the common edges of adjacent patches where the mesh is distorted, Fig. 6, left. The mesh quality is maintained using a value of $\alpha_{max} = 1.0$, Fig. 6, right. Fig. 7 shows the generated shapes using different values for the mesh control factor.

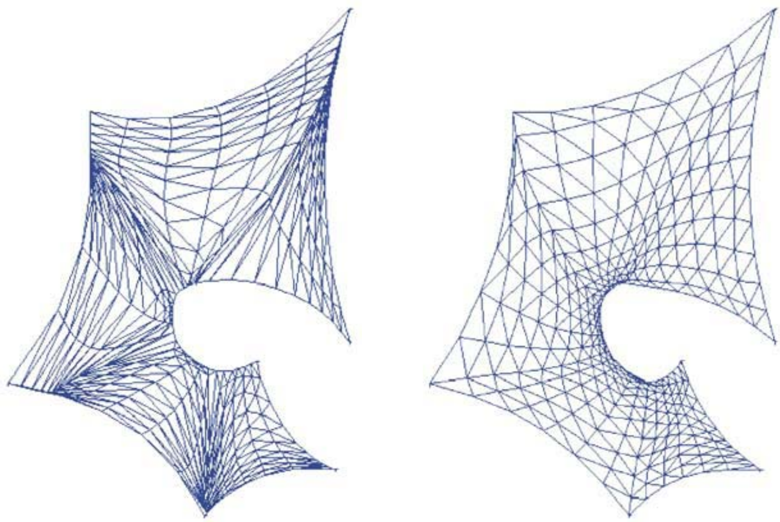


Fig. 6. Top view after form finding, without or with mesh control (left, right).

(a) no mesh control (b) $\alpha_{max} = 1.0$ (c) $\alpha_{max} = 1.5$

Fig. 7. Effects of varied mesh control

6 Conclusion

Motivated by the specific problems of form finding an approach was developed to control the mesh quality of the finite element discretization. The idea of the method is to adjust the value of the applied surface stresses by a simple scaling of the stress components which is derived from geometric considerations. The method is very effective without additional effort. Several examples demonstrate the success. Further developments are directed towards the generalization of the approach for general problems of shape optimal design where similar situations occur.

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