Recent Developments in the Analytical Design of Textile Membranes

Lothar Gründig[1], Dieter Ströbel[2] and Peter Singer[2]

[1] Technical University Berlin
Sekr. H20, Strasse des 17. Juni 135
10623 Berlin, Germany
gruendig@inge3.bv.tu-berlin.de

[2] technet GmbH Stuttgart
Pestalozzistrasse 8, 70563 Stuttgart, Germany
Dieter.Stroebel@technet-gmbh.com
http:/www:technet-gmbh.de

Summary. *The task of determining appropriate forms for stressed membrane surface structures is considered. Following a brief introduction to the field, the primitive form-finding techniques which were traditionally used for practical surface design are described. The general concepts common to all equilibrium modelling systems are presented next, and then a more detailed exposition of the Force Density Method follows. The extension of the Force Density Method to geometrically non-linear elastic analysis is described. A brief overview of the Easy lightweight structure design system is given with particular emphasis paid to the formfinding and statical analysis suite. Finally, some examples are used to illustrate the flexibility and power of Easy's formfinding tools.*
The task of generating planar cutting patterns for stressed membrane surface structures is considered next. Following a brief introduction to the general field of cutting pattern generation, the practical constraints which influence textile surface structures are presented. Several approaches which have been used in the design of practical structures are outlined. These include the physical paper strip modelling technique, together with geodesic string relaxation and flattening approaches. The combined flattening and planar sub-surface regeneration strategy used in the Easy design system is described in detail. Finally, examples are given to illustrate the capabilities of Easys cutting pattern generation tools.

1 Introduction

Contrary to the design of conventional structures a form finding procedure is needed with respect to textile membrane surfaces because of the direct relationship between the geometrical form and the force distribution. A membrane surface is always in the state of equilibrium of acting forces, and is not defined under unstressed conditions. In general there are two possibilities to perform the formfinding procedures: the

E. Oñate and B. Kröplin (eds.), Textile Composites and Inflatable Structures, 29–45.

physical formfinding procedure and the analytical one. The physical modelling of lightweight structures is characterized by stretchening a soft rubber type material between the chosen boundary positions in order to generate a physically feasible geometry. It has limitations with respect to an accurate description due the small scale of the model. The computational model allows for a proper description by discretizing the surface by a large number of points: a scale problem does not exist any more. Therefore the computational modelling of lightweight structures becomes more and more important; without this technology advanced lightweight structures cannot be built.

2 Analytical Formfinding

The analytical formfinding theories are based on *Finite Element Methods* in general: the surfaces are divided into a number of small finite elements like link elements or triangular elements for example. In such a way all possible geometries can be calculated. There are two theories established in practice: The linear Force Density Approach which uses links as finite elements and the nonlinear *Dynamic Relaxation Method* based on finite triangles.

The Force Density Method

The *Force Density Method* was first published in [1] and extended in [2-3, 9]. It is a mathematical approach for solving the equations of equilibrium for any type of cable network, without requiring any initial coordinates of the structure. This is achieved through the exploitation of a mathematical trick. The essential ideas are as follows. Pin-jointed network structures assume the state of equilibrium when internal forces s and external forces p are balanced.

In the case of node i in Fig. 1

$$\begin{aligned}
s_a \cos(a,x) + s_b \cos(b,x) + s_c \cos(c,x) + s_d \cos(d,x) &= p_x \\
s_a \cos(a,y) + s_b \cos(b,y) + s_c \cos(c,y) + s_d \cos(d,y) &= p_y \\
s_a \cos(a,z) + s_b \cos(b,z) + s_c \cos(c,z) + s_d \cos(d,z) &= p_z
\end{aligned}$$

where s_a, s_b, s_c and s_d are the bar forces and f.i. $\cos(a,x)$ is the normalised projection length of the cable a on the x-axis. These normalised projection lengths can also be expressed in the form $(x_m - x_i)/a$. Substituting the above cos values with these coordinate difference expressions results in

$$\begin{aligned}
\frac{s_a}{a}(x_m - x_i) + \frac{s_b}{b}(x_j - x_i) + \frac{s_c}{c}(x_k - x_i) + \frac{s_d}{d}(x_l - x_i) &= p_x \\
\frac{s_a}{a}(y_m - y_i) + \frac{s_b}{b}(y_j - y_i) + \frac{s_c}{c}(y_k - y_i) + \frac{s_d}{d}(y_l - y_i) &= p_y \\
\frac{s_a}{a}(z_m - z_i) + \frac{s_b}{b}(z_j - z_i) + \frac{s_c}{c}(z_k - z_i) + \frac{s_d}{d}(z_l - z_i) &= p_z
\end{aligned}$$

In these equations, the lengths a, b, c and d are nonlinear functions of the coordinates. In addition, the forces may be dependent on the mesh widths or on areas of partial surfaces if the network is a representation of a membrane. If we now apply

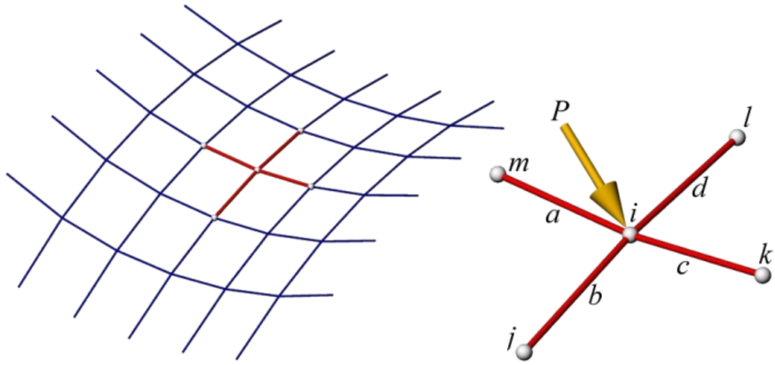

Fig. 1. Part of a cable network

the trick of fixing the force density ratio $s_a/a = q_a$ for every link, linear equations result.

These read

$$\begin{aligned}
q_a(x_m - x_i) + q_b(x_j - x_i) + q_c(x_k - x_i) + q_d(x_l - x_i) &= p_x \\
q_a(y_m - y_i) + q_b(y_j - y_i) + q_c(y_k - y_i) + q_d(y_l - y_i) &= p_y \\
q_a(z_m - z_i) + q_b(z_j - z_i) + q_c(z_k - z_i) + q_d(z_l - z_i) &= p_z
\end{aligned}$$

The force density values q have to be choosen in advance depending on the desired prestress. The procedure results in practical networks which are reflecting the architectural shapes and beeing harmonically stressed. The system of equations assembled is extremely sparse and can be efficiently solved using the Method of Conjugate Gradients as described in [3].

3 Analytical Formfinding with Technet's Easy Software

The 3 main steps of the Analytical Formfinding of Textile Membrane with the technet's EASY Software are described as follows:

1. Definition of all design parameters, of all boundary conditions as: the coordinates of the fixed points, the warp -and weft direction, the mesh-size and mesh-mode (rectangular or radial meshes), the prestress in warp- and weft direction, the boundary cable specifications (sag or force can be chosen).
2. The linear Analytical Formfinding with Force Densities is performed: the results are: the surface in equilibrium of forces, described by all coordinates of points on the surface, the stress in warp- and weft direction, the boundary cable-forces, the reaction forces of the fixed points. The stresses in warp and weft-direction and the boundary forces may differ in a small range with respect to the desired one from Step 1.

3. Evaluation and visualization tools in order to judge the result of the formfinding. The stresses and forces can be visualized, layer reactions can be shown, contour-lines can be calculated and visualized, cut-lines through the structure can be made.

4 Force Density Statical Analysis

The *Force Density Method* can be extended efficiently to perform the elastic analysis of geometrically non-linear structures. The theoretical background is described in detail in [3] where it was also compared to the *Method of Finite Elements*. It was shown that the *Finite Element Method*'s formulae can be derived directly from the *Force Density Method*'s approach. In addition, the Force Density Method may be seen in a more general way. According to [3] it has been proven to be numerically more stable for the calculation of structures subject to large deflections, where sub-areas often become slack. The nonlinear force density method shows powerful damping characteristics.

Prior to any statical analysis, the form-found structure has to be materialized. Applying Hooke's law the bar force sa is given by:

$$s_a = EA\frac{a - a_0}{a_0}$$

where A is the area of influence fore bar a, E is the modulus of elasticity, and a_0 is the unstressed length of bar a. Substituting s_a by q_a according to $q_a = s_a/a$ results in

$$a_0 = \frac{EAa}{q_a a + EA}$$

Because of a being a function of the coordinates of the bar end points, the materialized unstressed length is a function of the force density q_a, the stressed length a and the element stiffness EA.

In order to perform a statical structural analysis subject to external load, the unstressed lengths have to be kept fixed. This can be achieved mathematically by enforcing the equations of materialization together with the equations of equilibrium. This system of equations is no longer linear. The unknown variables of the enlarged system of equations are now the coordinates x, y, z and the force density values q. Eliminating q from the equations of equilibrium, by applying the formula above to each bar element, leads to a formulation of equations which are identically to those resulting from the *Finite Element Method*. Directly solving the enlarged system has been shown to be highly numerically stable, as initial coordinates for all nodes are available, and positive values or zero values for q can be enforced through the application of powerful damping techniques.

The usual relationship between stress and strain for the orthotropic membrane material is given by:

$$\begin{bmatrix} \sigma_{uu} \\ \sigma_{vv} \end{bmatrix} = \begin{bmatrix} e_{1111} & 0 \\ 0 & e_{2222} \end{bmatrix} \begin{bmatrix} \varepsilon_{uu} \\ \varepsilon_{vv} \end{bmatrix}$$

The warp-direction u and the weft-direction v are independent from each other; this means: the stress in warp-direction σ_{uu} f.i. is only caused by the modulus of elasticity e_{1111} and the strain ε_{uu} in this direction. Because of this independency cable net theories can be used also for Textile membranes.

In [4] the *Force Density Method* has been applied very favorably to triangular surface elements. This triangle elements allow the statical analysis taking into consideration a more precise material behavior in case of Textile membranes. Actually the both material directions u and v are depending from each other; a strain ε_{uu} leads not only to a stress in u-direction but also to a stress σ_{vv} in v-direction caused by the modulus of elasticity e_{1122}. The fact that shear-stress depends on a shear-stiffness e_{1212} seems not to be important for membranes because of its smallness

$$\begin{bmatrix} \sigma_{uu} \\ \sigma_{vv} \\ \sigma_{uv} \end{bmatrix} = \begin{bmatrix} e_{1111} & e_{1122} & 0 \\ e_{2211} & e_{2222} & 0 \\ 0 & 0 & e_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{uu} \\ \varepsilon_{vv} \\ \varepsilon_{uv} \end{bmatrix}$$

Using these constitutive equations *Finite Element Methods* should be applied. We are using in this case the finite triangle elements.

5 Further Extensions of the Force Density Approach

The force density approach can be favorably exploited for further applications.

According to [3] the following system of equations of equilibrium is valid:

$$\mathbf{C}^t\mathbf{Q}\mathbf{C}\mathbf{x} = \mathbf{p}$$

$\mathbf{C}$ is the matrix describing the topology of the system, $\mathbf{Q}$ is the diagonal matrix storing the force density values, $\mathbf{x}$ contains the coordinates of the nodes of the figure of equilibrium and $\mathbf{p}$ the external forces acting on the structure. For linear formfinding $\mathbf{C}$, $\mathbf{Q}$ and $\mathbf{p}$ are given, $\mathbf{x}$ is the result of the above equation.

In some applications it might be of interest to know, how close a given geometrical surface will represent a figure of equilibrium. In this case $\mathbf{C}$, $\mathbf{x}$ and $\mathbf{p}$ are given and $\mathbf{q}$ is searched for. As there might be no exact solution to the task described above the best approximating solution is achieved allowing for minimal corrections to the external forces. The system now reads:

$$\mathbf{C}^t\mathbf{U}\mathbf{q} = \mathbf{p} + \mathbf{v}$$

$\mathbf{U}$ now represents a diagonal matrix of coordinate differences ($\mathbf{C}\ \mathbf{x}$), $\mathbf{q}$ the force density values, and $\mathbf{v}$ the residuals of the systems to be minimal.

Solving the system of equations, applying the method of least squares, results in best approximating force density values for any given surface under external loads or subject to internal prestress, if some force density values are chosen as fixed in the structure.

As shown in [3] the system can be extended even further, by choosing $\mathbf{q}$ and $\mathbf{x}$ as observables and enforcing the equations of equilibrium, according to the method of least squares condition equations. In this case an architectural design can be best approximated computationally, enforcing the necessary conditions. This extension proves to be a powerful optimization strategy.

6 Statical Analysis with Technet's Easy Software

The statical Analysis of lightweight structures under external loads can be performed after three introducing steps:

1. To define stiffness values to all finite membrane and cable elements.
2. To calculate the unstressed link lengths by using the assigned stiffness valueses and the prestress of the membrane and the forces in the cables of the formfinding result.
3. To check if the result of the statical analysis with the loads of the formfinding procedure is identical with the formfinding result.

After these three steps, the statical analysis without beam elements under external loads can be achieved very easily.

1. To calculate the external load vectors as for example snow, wind or normal loads.
2. To perform the nonlinear statical analysis: the approximate values, which are needed in this nonlinear process, are given by the formfinding result.
3. Evaluation- and visualization tools in order to judge the result of the statical analysis. The stresses and forces can be visualized and compared with the maximum possible values. Stresses, forces and layer reactions can be shown, contour-lines can be calculated and visualized, cut-lines through the structure can be made, deflection of the nodes can be calculated.

If beam elements are included, the statical analysis under external loads has to be done as follows: all data for the beam-elements as cross-section areas, moments of inertia, local coordinate systems, joints, etc. have to be defined firstly. In order to set all these values in a convenient way the user is supported by a Beam-Editor in EasyBeam.

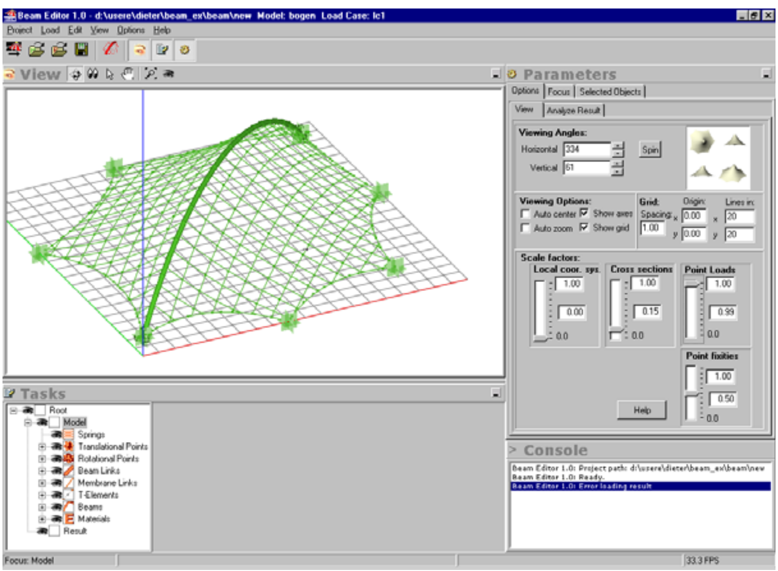

Fig. 2. The Easy beam editor

Then -see above- the steps 1-3 follow. The Beam Editor is also used for checking the results as internal forces and moments, layer-reactions, flexibility-ellipsoids, etc.

7 The Complete Easy Lightweight Structure Design System

The Easy system is composed of a number of program suites. These are represented schematically in Fig. 3.

EasyForm Formfinding of lightweight structures
EasySan Nonlinear Statical Load Analysis (without Beam elements)
EasyCut Cutting pattern generation
EasyBeam Nonlinear hybride Membrane structures including Beam elements
EasyVol Formfinding and Load Analysis of pneumatic constructions

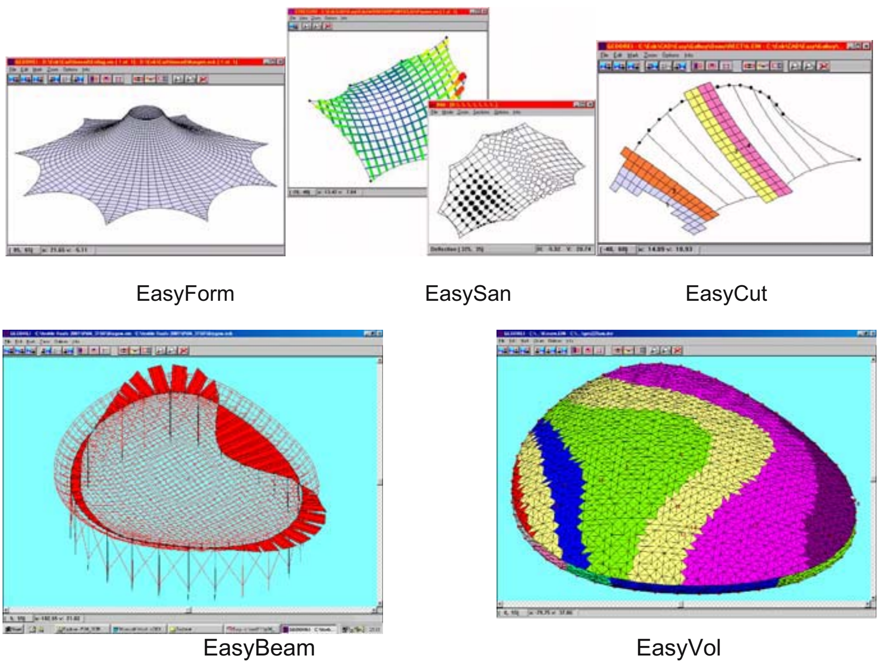

Fig. 3. The Easy program suites

EasyForm comprises the programs used for data generation together with force density form-finding. When the EasySan programs are additionally installed statical structural analysis of non-linear structures becomes possible. The EasyCut programs enable the generation of high quality planar cutting patterns from EasyForm output.

In most situations the incorporation of geometrically non-linear bending elements to lightweight structure models is not economically appropriate. Rather, it is more convenient to treat the beam supports as fully fixed points. The resulting reaction forces on these points are then exported to conventional rigid frame design packages as applied loads. Such a decoupled analysis is appropriate if the resulting deflections are low. However for a sensitive structure decoupling may not be adequate, due the strong interaction of forces causing geometry changes of the membrane surface and the beam elements. In this case the EasyBeam add-on module permits the incorporation of geometrically non-linear frame elements [5].

EasyForm and EasySan together can deal with all standard pneumatic structural configurations which have defined internal pressure prestress. In situations with closed volumes, such as high pressure air beams, this assumption is not valid. It becomes necessary to use more sophisticated algorithms which constrain the cell volumes to prescribed values, and vary the internal pressure accordingly.

Fig. 4. Formfinding and statical analysis under inner pressure and buoyancy

8 Cutting Pattern Generation of Textile Structures

The theory, being used to project a 2D surface in 3 dimensional space to a 2D surface in a plane is very old; it is part of the mathematical field named map projection theory. For example the Mercator Projection dates back to the 17th century.

The surfaces, which are used in practical membrane structure design are in general not developable without distortions. In addition there does not exist a geometrical shape without prestress. The map projection theories -used for the flattening of textile membranes- minimize the distortions with respect to lengths, angles and areas respectively.

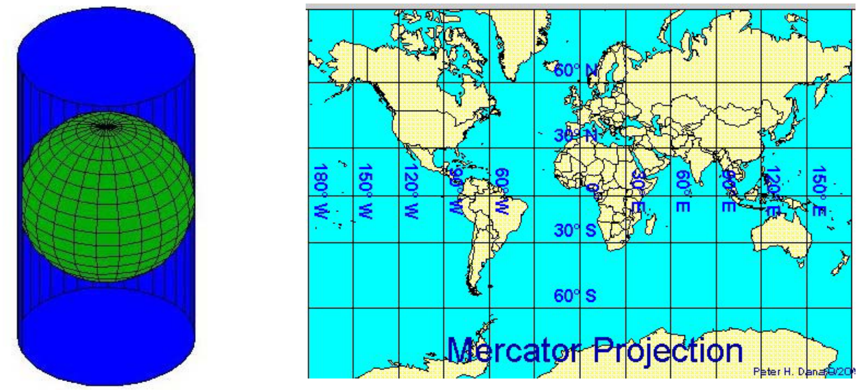

Fig. 5. Mercator projection

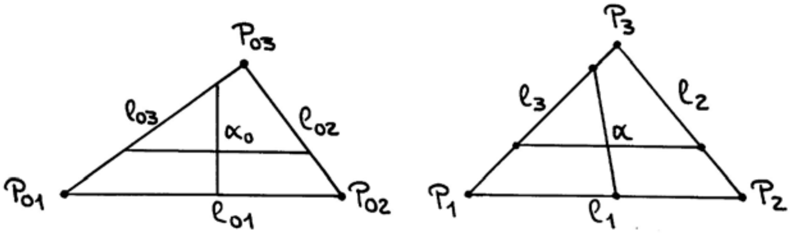

Fig. 6. Triangles non deformed (3D) and deformed (2D)

The theory to be applied optimizes the total distortion energy by means of the adjustment theory.

The surface, which has to be flattened is described using finite triangles. The distortion between the non deformed and deformed situation can be calculated and has to be minimized for all triangles.

The paper strip method is exactly described in [10]. Practical examples are described in [6-8].

Fig. 7 illustrates the paper strip method. A paper is pressed on the surface of the physical model in such a way, that the seam line and the border of the paper are touching each other as good as (in general the paper strip will touch the surface only in one common line, with an increasing distance from this line the difference between paper strip and surface becomes larger.) In the next step a needle is used to perforate the paper strip in a certain number of equidistant points so that the neighboring seam or the boundary line is reached on the shortest way. In doing so the direction of the needle has to be perpendicular to the surface. The connection of the needle holes by straight lines on the flat paper strip leads to the patterns.

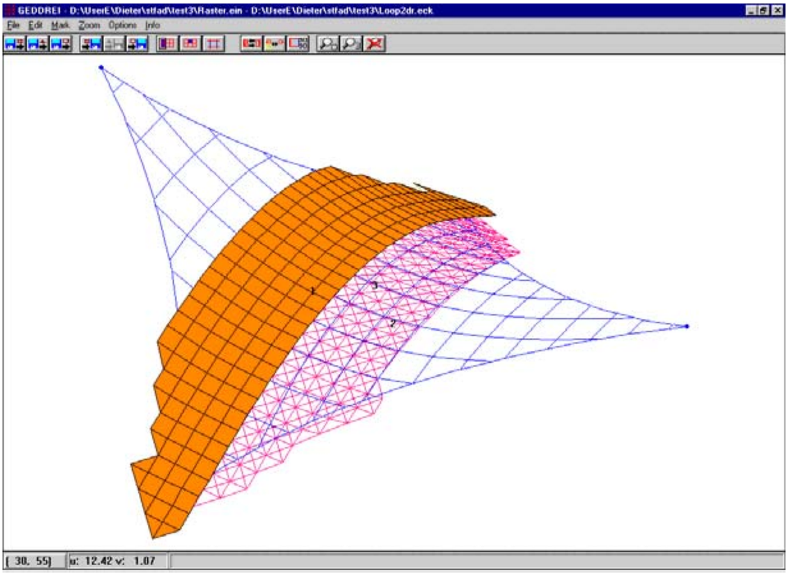

Fig. 7. Paper strip method

9 Cutting Pattern Generation with Technet's Easy Software

The Cutting pattern generation can be performed in the following steps:

1. Geodesic lines are created as seam lines.
2. Cutting procedures are used to cut the surface into different sub-surfaces according to these geodesic lines.
3. Ways of flattening are achieved: map projection, paper strip method.
4. Spline algorithms are applied to create equidistant points on the planar circumference.
5. Boundary adjustment is performed in order to produce identical seam lengths.
6. Compensation values are defined to compensate the strips.
7. Job-drawings are produced.

10 Flexibility Ellipsoids for the Evaluation of Mechanical Structures

The geodetic network adjustment determines the geometrical position of points connected by link observables. In general the adjustment theory was invented by C.F. GAUSS by mimizing the residuals of the observations. The process is well known as Least-Squares-Method.

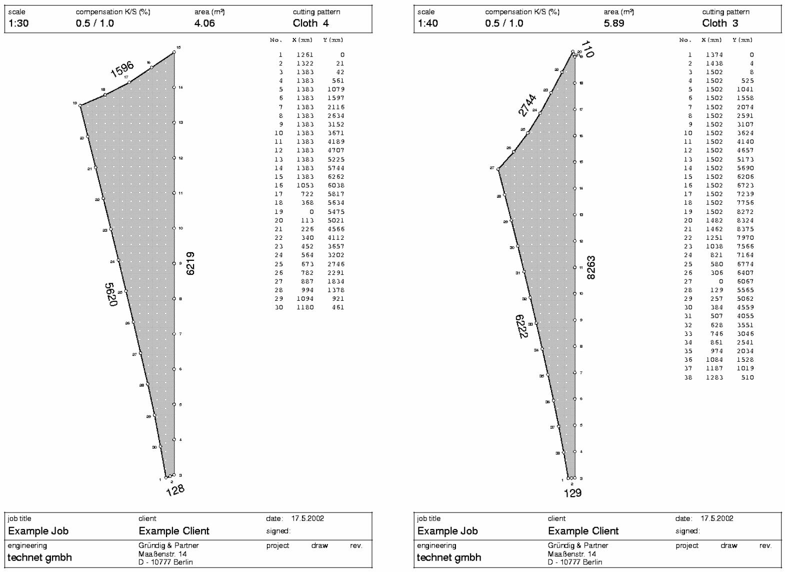

Fig. 8. Cutting patterns

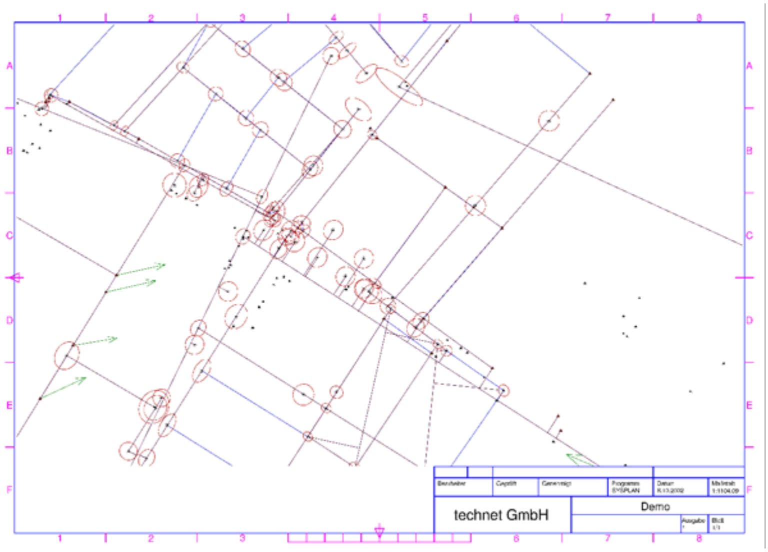

Fig. 9. Error-ellipses in a geodetic network

Analogical relationships between this adjustment theory and the calculation of mechanical structures exist due to the fact that in mechanical structures minimal-principles are valid too; the total energy is minimal.[5] The error-ellipsoids of the geodetic network adjustment have been an important tool for the evaluation of the point quality for more than 100 years, they show very descriptively the accuracy of the points [14].

The error-ellipsoids of the geodetic network adjustment are an essential tool used for the evaluation of mechanical structures too. The relation between the deflection v and the external loads p in the global coordinate system is given using the stiffness matrix $\mathbf{S}$

$$\mathbf{Sv} = \mathbf{p}$$

The inverse relation is

$$\mathbf{S}^{-1}\mathbf{p} = \mathbf{Fp} = \mathbf{v}$$

The inverse stiffness matrix $\mathbf{F}$ is usually called flexibility matrix; the matrix defines the relation between the external loads at a node and the according deflections. In detail, we receive for a node with the coordinates (x, y, z):

$$\begin{bmatrix} \ddots & \vdots & \vdots \\ & \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ & f_{yy} & f_{yz} \\ sym. & & f_{zz} \end{bmatrix} & \vdots \\ sym. & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \\ \vdots \end{bmatrix}$$

For the following deductions the submatrix $\mathbf{F}$ as 3×3 matrix has been introduced for any point P and the 3×1 vectors p and v referring also to this point. $\mathbf{F}$ is now not any more the total flexibility matrix, but only a sub matrix with 9 elements, p are the 3 components of the external load and v the deflections of the point. Let us assume, that all point loads are zero with the exception of the considered point, then the sub matrix $\mathbf{F}$ describes the relation between the point load and the deflections of the point. All other elements in $\mathbf{F}$ are not important in this context, because they are multiplied with zero. Hence, we receive for a free point:

$$\mathbf{Fp} = \mathbf{v}$$

In detail

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ & f_{yy} & f_{yz} \\ sym. & & f_{zz} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

How the deflections v and the external loads p are transformed by an orthogonal transformation $\mathbf{R}$ (orthogonal means: the transposed matrix equals the inverse matrix $\mathbf{R}^t = \mathbf{R}^{-1}$), the deflections are in the transformed system u and the loads q

$$\mathbf{u} = \mathbf{R}^t\mathbf{v} \rightarrow \mathbf{v} = \mathbf{Ru}$$
$$\mathbf{q} = \mathbf{R}^t\mathbf{p} \rightarrow \mathbf{p} = \mathbf{Rq}$$

The relation between the deflections u and the loads q is after some conversions:

$$\mathbf{u} = \mathbf{R}^t\mathbf{FRq} = \mathbf{Dq}$$

Now the question arises, if there is a matrix $\mathbf{R}$ generating a matrix $\mathbf{D}$ having only diagonal elements. The answer is yes, if $\mathbf{F}$ is symmetric with real values. $\mathbf{F}$ is always real symmetric in the described applications and also positive definite if there is a stable equilibrium; therefore all eigenvalues are real and all eigenvectors are orthogonal to each other. The diagonal elements of the matrix $\mathbf{D}$ are the eigenvalues of $\mathbf{F}$, the matrix of rotations $\mathbf{R}$ is generated by the eigenvectors.

This can easily be proofed. We suppose the eigenvalues of 3×3. Matrix $\mathbf{F}$ to be d_1, d_2 and d_3 and the corresponding eigenvectors to be $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. The unit matrix is $\mathbf{E}$. We have at the beginning for the calculations of the eigenvalues:

$$\begin{aligned} \mathbf{F}r_1 = d_1 r_1 &\rightarrow (\mathbf{F} - d_1\mathbf{E})r_1 = 0 \\ \mathbf{F}r_2 = d_2 r_2 &\rightarrow (\mathbf{F} - d_2\mathbf{E})r_2 = 0 \\ \mathbf{F}r_3 = d_3 r_3 &\rightarrow (\mathbf{F} - d_3\mathbf{E})r_3 = 0 \end{aligned}$$

We define the diagonal matrix $\mathbf{D}$ with the diagonal elements d_1, d_2 and d_3 and the rotation matrix $\mathbf{R} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. Doing so, we receive from the upper equations:

$$\mathbf{FR} = \mathbf{RD}$$

The eigenvectors $\mathbf{R}$ are pair wise orthogonal to each other, therefore the orthogonality of $\mathbf{R}$ is obvious, hence $\mathbf{RR}^t = \mathbf{E}$. Multiplication from the left hand side with $\mathbf{R}^t$ leads to

$$\mathbf{R}^t\mathbf{FR} = \mathbf{R}^t\mathbf{RD} = \mathbf{ED} = \mathbf{D}$$

We have seen, that the eigenvalues of the matrix $\mathbf{F}$ are the diagonal elements of the diagonal matrix $\mathbf{D}$. This matrix give the relation in a rotated coordinate system between the deflection u and the external loads q

$$\mathbf{u} = \mathbf{Dq}$$

whereby we receive the new coordinate system by creating the rotation matrix $\mathbf{R}$ using the eigenvectors. The upper equation reads in detail.

Now we are investigating which surface is created by the deflections $u = (u, v, w)$, if the external load

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} q_u \\ q_v \\ q_w \end{bmatrix}$$

vector $q = (q_u, q_v, q_w)$ is showing in any direction having the size 1. We receive:

$$q_u q_u + q_v q_v + q_w q_w = 1$$

In general the matrix $\mathbf{F}$ is positive definite; all eigenvalues are therefore positive. We get immediately with $q_u = u/d_1$, $q_v = v/d_2$ and $q_w = w/d_3$ the equation of an ellipsoid

$$q_u q_u + q_v q_v + q_w q_w = \frac{u^2}{d_1^2} + \frac{v^2}{d_2^2} + \frac{w^2}{d_3^2} = 1$$

as a surface being created by the deflections, if the unit load vector is rotating around a point

$$u_F = fq$$

We see, that the direction of the external load q and the direction of the deflections are only identical in case of a sphere, if $d_1 = d_2 = d_3$. In general we do not have a sphere, therefore we investigate the question, which geometrical figure is created, if the size of the deflection is accorded to the direction of the external load q; the point on the load vector having the size of the total deflection is called root point F; its coordinates are:

$$\begin{bmatrix} u_F \\ v_F \\ w_F \end{bmatrix} = f \begin{bmatrix} q_u \\ q_v \\ q_w \end{bmatrix}$$

The load q leads to the deflections $\mathbf{u} = [\,u \quad v \quad w\,]$. Because of the fact that the load has the size 1, is f the total length of the deflection, hence:

$$f^2 = u_F^2 + v_F^2 + w_F^2 = u^2 + v^2 + w^2$$

Also

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} d_1 q_u \\ d_2 q_v \\ d_3 q_w \end{bmatrix} = \begin{bmatrix} d_1 u_F f^{-1} \\ d_2 v_F f^{-1} \\ d_3 w_F f^{-1} \end{bmatrix}$$

We receive by substituting

$$f^2 = u_F^2 d_1^2 f^{-2} + v_F^2 d_2^2 f^{-2} + w_F^2 d_3^2 f^{-2} = u^2 + v^2 + w^2$$

By a simple conversion we end up with the equation of a so-called Booth-Lemniscate

$$f^4 - u_F^2 d_1^2 - v_F^2 d_2^2 - w_F^2 d_3^2 = 0$$

In general written as follows

$$(u_F^2 + v_F^2 + w_F^2) - u_F^2 d_1^2 - v_F^2 d_2^2 - w_F^2 d_3^2 = 0$$

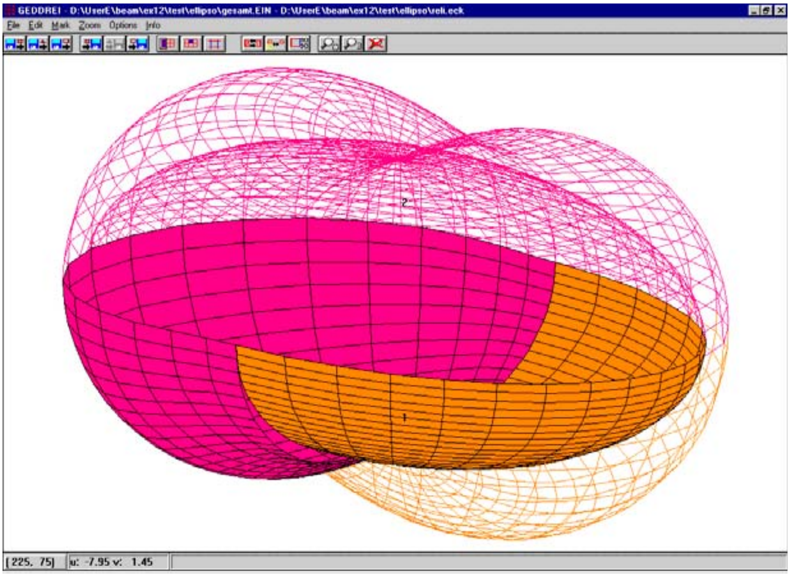

Fig. 10. Flexibility ellipsoide and lemniscate

The lemniscate has 6 common points with the ellipsoid of the deflections, this are the points on the local coordinate axis of the ellipsoid. In this 6 point the direction of the external loads and the deflections of the point are identical.

The example in Fig. 12 clarifies that the points being far from the fixed points are generally more flexible than the closed ones.

In Fig. 13 we see immediately that the points in the upper layer are very flexible in tangential directions. In order to change this behavior diagonals should be introduced [11].

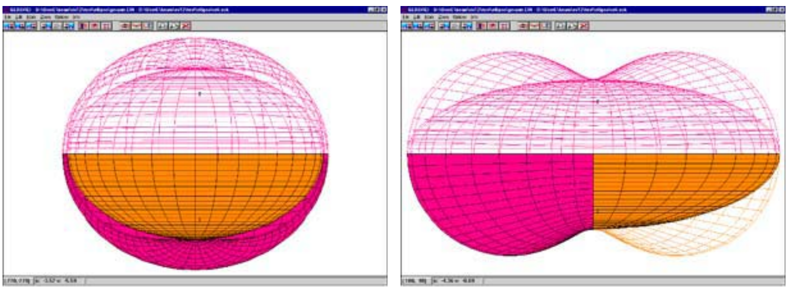

Fig. 11. Side views

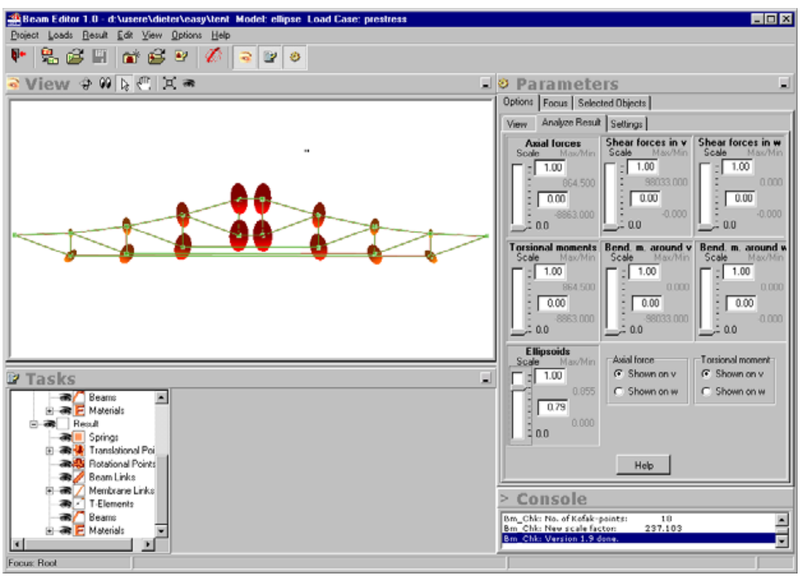

Fig. 12. 2-dimensional structure

Some remarks to the practical realization of the flexibility ellipsoids in computer software.

Due to the fact, that the inverse stiffness matrix is generally not calculated; conventional methods to solve this matrix are very time consuming and also intensive with respect to storage. In those cases the ellipsoids cannot be calculated. However, we are using in so-called hyper sparse algorithms, which are storing only the non-zero elements [12],[13] and generates the inverse for only those elements needed. Without such a strategy the calculation of those ellipsoids is not possible for large structures.

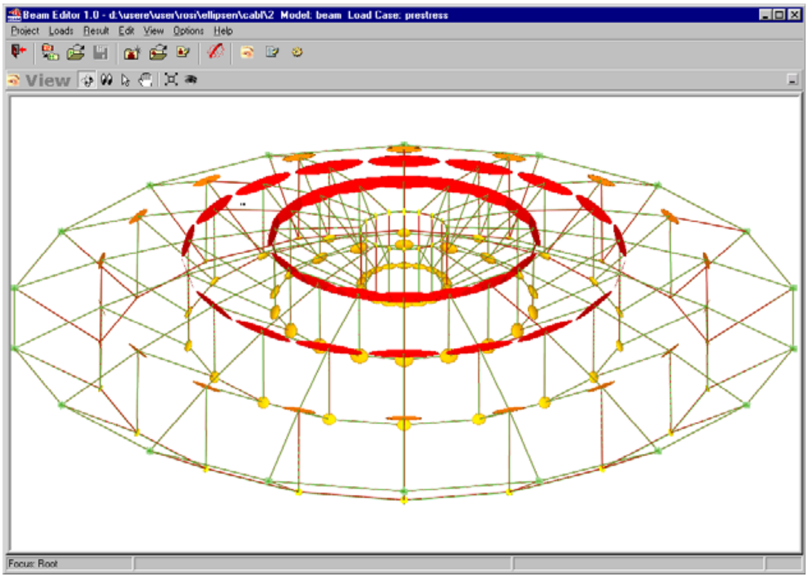

Fig. 13. 3-dimensional space-structure

11 Conclusions

It has been shown that, by using a modular approach for the design of membrane structure surfaces, the resulting system is extremely powerful and flexible. The very large number of structures which have been built using the Easy tools (many thousands) prove the validity of this strategy.

References

1. Linkwitz K and Schek H-J (1971) Einige Bemerkungen zur Berechnung von vorgespannten Seilnetzkonstruktionen. Ingenieur-Archiv 40:145–158.
2. Schek HJ (1974) The force density method for form finding and computation of general net- works. Computer Methods in Applied Mechanics and Engineering 3:115–134.

3. Gründig L (1975) Die Berechnung von vorgespannten Seilnetzen und Hängenetzen unter Berücksichtigung ihrer topologischen und physikalischen Eigenschaften und der Ausgleichungs- rechnung. Dissertationsschrift, DGK Reihe C, Nr. 216, 1976 and SFB 64-Mitteilungen 34/1976.
4. Singer P (1995) Die Berechnung von Minimalflächen, Seifenblasen, Membrane und Pneus aus geodätischer Sicht. Dissertationsschrift, DGK Reihe C, Nr. 448.
5. Ströbel D (1997) Die Anwendung der Ausgleichungsrechnung auf elastomechanische Systeme. DGK, Reihe C, Nr. 478.
6. Gründig L and Moncrieff E (1998) Formfinding, Analysis and Patterning of Regular and Irregular-Mesh Cablenet Structures. in Hough R and Melchers R (Eds.), LSA98: Lightweight Structures in Architecture Engineering and Construction Proceedings IASS 39th Congress, October, Sydney, Australia, IASS/LSAA.
7. Moncrieff E, Gründig L and Ströbel D (1999) The Cutting Pattern Generation of the Pilgrim's Tents for Phase II of the Mina Valley Project. in Astudillo, R. and Madrid, A. J. (Eds.), Proc. IASS 40th Anniversary Congress, September 20-24, 1999, Madrid, Spain, IASS/CEDEX.
8. Gründig L, Moncrieff E, Singer P and Ströbel D (2000) High-performance cutting pattern generation of architectural textile structures. in Papadrakakis E (Ed.), Proc. IASS-IACM 2000 Fourth International Colloquium on Computation of Shell & Spatial Structures, June 4-7, Chania-Crete, Greece.
9. Gründig L, Moncrieff E, Singer P and Ströbel D (2000) A history of the principal developements and applications of the force density method in Germany 1970-1999. in Papadrakakis, E. (Ed.), Proc. IASS-IACM 2000 Fourth International Colloquium on Computation of Shell & Spatial Structures, June 4-7, Chania-Crete, Greece.
10. Moncrieff E and Gründig L (2000) Computation modelling of lightweight structures. 5th international workshop on the design and practical realization of architecture membrane structures, Technical University Berlin, June 22-24, Germany.
11. Wagner R (2202) Tensegrity and Cable Domes. In: The Seventh International Workshop on the Design and Realisation of Architectural Membrane Structures, Technische Hochschule Berlin.
12. Ströbel D (1985) Entwicklung von Hypersparsetechniken zur Gleichungslösung für Probleme in der Ausgleichungsrechnung und Statik. Diplomarbeit, Institut für Anwendungen der Geodäsie im Bauwesen, Technische Universität Stuttgart.
13. Gründig L, Bahndorf J und Ströbel D (1985) Hyper Sparse Techniques Applied to Geodetic Networks. In: 7th International Symposium on Geodetic Computations, Krakau.
14. Linkwitz K (1988) Einige Bemerkungen zur Fehlerellipse und zum Fehlerellipsoid. Vermessung, Photogrammetrie, Kulturtechnik, Schweizerischer Verein fr Vermessungs- und Kulturtechnik (SVVK), S. 345–364, 86. Jahrgang, Heft 7.

http://www.technet-gmbh.com