CHAPTER 12

SPRINGS

Summary

Close-coiled springs

(a) Under axial load W

Maximum shear stress set up in the material of the spring

$$= \tau_{max} = \frac{2WR}{\pi r^3} = \frac{8WD}{\pi d^3}$$

Total deflection of the spring for n turns

$$= \delta = \frac{4WR^3 n}{Gr^4} = \frac{8WD^3 n}{Gd^4}$$

where r is the radius of the wire and R the mean radius of the spring coils.

i.e.

$$\text{Spring rate} = \frac{W}{\delta} = \frac{Gd^4}{8nD^3}$$

(b) Under axial torque T

$$\text{Maximum bending stress set up} = \sigma_{max} = \frac{4T}{\pi r^3} = \frac{32T}{\pi d^3}$$

$$\text{Wind-up angle} = \theta = \frac{8TRn}{Er^4} = \frac{64TDn}{Ed^4}$$

$$\therefore \text{Torque per turn} = \frac{T}{\theta/2\pi} = \frac{\pi E d^4}{32Dn}$$

The stress formulae given in (a) and (b) may be modified in practice by the addition of 'Wahl' correction factors.

Open-coiled springs

(a) Under axial load W

$$\text{Deflection } \delta = 2\pi n W R^3 \sec\alpha \left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

$$\text{Angular rotation } \theta = 2\pi n W R^2 \sin\alpha \left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

(*b*) *Under axial torque T*

$$\text{Wind-up angle } \theta = 2\pi nRT \sec\alpha\left[\frac{\sin^2\alpha}{GJ}+\frac{\cos^2\alpha}{EI}\right]$$

where α is the helix angle of the spring.

$$\text{Axial deflection } \delta = 2\pi nTR^2 \sin\alpha\left[\frac{1}{GJ}-\frac{1}{EI}\right]$$

Springs in series

$$\text{Stiffness } S = \frac{S_1 S_2}{(S_1+S_2)}$$

Springs in parallel

$$\text{Stiffness } S = S_1 + S_2$$

Leaf or carriage springs

(*a*) *Semi-elliptic*

Under a central load W:

$$\text{maximum bending stress} = \frac{3WL}{2nbt^2}$$

$$\text{deflection } \delta = \frac{3WL^3}{8Enbt^3}$$

where L is the length of spring, b is the breadth of each plate, t is the thickness of each plate, and n is the number of plates.

$$\text{Proof load } W_p = \frac{8Enbt^3}{3L^3}\delta_p$$

where δ_p is the initial central "deflection".

$$\text{Proof or limiting stress } \sigma_p = \frac{4tE}{L^2}\delta_p$$

(*b*) *Quarter-elliptic*

$$\text{Maximum bending stress} = \frac{6WL}{nbt^2}$$

$$\text{Deflection } \delta = \frac{6WL^3}{Enbt^3}$$

Plane spiral springs

$$\text{Maximum bending stress} = \frac{6Ma}{RBt^2}$$

or, assuming $a = 2R$,

$$\text{maximum bending stress} = \frac{12M}{Bt^2}$$

$$\text{wind-up angle } \theta = \frac{ML}{EI}$$

where M is the applied moment to the spring spindle, R is the radius of spring from spindle to pin, a is the maximum dimension of the spring from the pin, B is the breadth of the material of the spring, t is the thickness of the material of the spring, L is equal to $\frac{1}{2}(\pi n)(a+b)$, and b is the diameter of the spindle.

Introduction

Springs are energy-absorbing units whose function it is to store energy and to release it slowly or rapidly depending on the particular application. In motor vehicle applications the springs act as buffers between the vehicle itself and the external forces applied through the wheels by uneven road conditions. In such cases the shock loads are converted into strain energy of the spring and the resulting effect on the vehicle body is much reduced. In some cases springs are merely used as positioning devices whose function it is to return mechanisms to their original positions after some external force has been removed.

From a design point of view "good" springs store and release energy but do not significantly absorb it. Should they do so then they will be prone to failure.

Throughout this chapter reference will be made to strain energy formulae derived in Chapter 11 and it is suggested that the reader should become familiar with the equations involved.

12.1. Close-coiled helical spring subjected to axial load *W*

(a) Maximum stress

A close-coiled helical spring is, as the name suggests, constructed from wire in the form of a helix, each turn being so close to the adjacent turn that, for the purposes of derivation of formulae, the helix angle is considered to be so small that it may be neglected, i.e. each turn may be considered to lie in a horizontal plane if the central axis of the spring is vertical. Discussion throughout the subsequent section on both close-coiled and open-coiled springs will be limited to those constructed from wire of circular cross-section and of constant coil diameter.

Consider, therefore, one half-turn of a close-coiled helical spring shown in Fig. 12.1. Every cross-section will be subjected to a torque WR tending to twist the section, a bending moment tending to alter the curvature of the coils and a shear force W. Stresses set up owing to the shear force are usually insignificant and with close-coiled springs the bending stresses

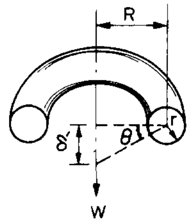

Fig. 12.1. Close-coiled helical spring subjected to axial load W.

are found to be negligible compared with the torsional stresses. Thus the maximum stress in the spring material may be determined to a good approximation using the torsion theory.

$$\tau_{max} = \frac{Tr}{J} = \frac{WRr}{\pi r^4/2}$$

i.e.

$$\textbf{maximum stress} = \frac{2WR}{\pi r^3} = \frac{8WD}{\pi d^3} \qquad (12.1)$$

(*b*) *Deflection*

Again, for one half-turn, if one cross-section twists through an angle θ relative to the other, then from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{WR(\pi R)}{G} \times \frac{2}{\pi r^4} = \frac{2WR^2}{Gr^4}$$

But

$$\delta' = R\theta = \frac{2WR^3}{Gr^4}$$

$\therefore$

$$\textbf{total deflection } \delta = 2n\delta' = \frac{4WR^3n}{Gr^4} = \frac{8WD^3n}{Gd^4} \qquad (12.2)$$

$$\textbf{Spring rate} = \frac{W}{\delta'} = \frac{Gd^4}{8nD^3}$$

12.2. Close-coiled helical spring subjected to axial torque T

(*a*) ***Maximum stress***

In this case the material of the spring is subjected to pure bending which tends to reduce the radius R of the coils (Fig. 12.2). The bending moment is constant throughout the spring and equal to the applied axial torque T. The maximum stress may thus be determined from the bending theory

$$\sigma_{max} = \frac{My}{I} = \frac{Tr}{\pi r^4/4}$$

i.e.
$$\textbf{maximum bending stress} = \frac{4T}{\pi r^3} = \frac{32T}{\pi d^3} \qquad (12.3)$$

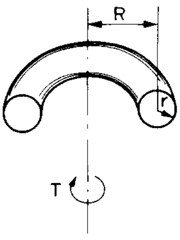

Fig. 12.2. Close-coiled helical spring subjected to axial torque T.

(*b*) *Deflection* (*wind-up angle*)

Under the action of an axial torque the deflection of the spring becomes the "wind-up angle" of the spring, i.e. the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, which, according to Mohr's area–moment theorem (see § 5.7), is the area of the M/EI diagram between the ends.

$$\therefore \qquad \theta = \int_0^L \frac{M\,dL}{EI} = \frac{TL}{EI}$$

where L = total length of the wire = $2\pi Rn$.

$$\therefore \qquad \theta = \frac{T\,2\pi Rn}{E} \times \frac{4}{\pi r^4}$$

i.e.
$$\textbf{wind-up angle } \theta = \frac{8TRn}{Er^4} \qquad (12.4)$$

N.B. The stress formulae derived above are slightly inaccurate in practice, particularly for small D/d ratios, since they ignore the higher stress produced on the inside of the coil due to the high curvature of the wire. "Wahl" correction factors are therefore introduced – see page 307.

12.3. Open-coiled helical spring subjected to axial load W

(*a*) *Deflection*

In an open-coiled spring the coils are no longer so close together that the effect of the helix angle α can be neglected and the spring is subjected to comparable bending and twisting effects. The axial load W can now be considered as a direct load W acting on the spring at the mean radius R, together with a couple WR about AB (Fig. 12.3). This couple has a component about AX of $WR \cos \alpha$ tending to twist the section, and a component about AY

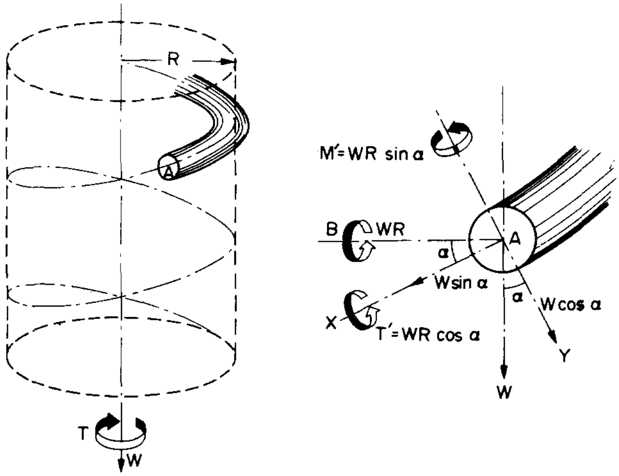

Fig. 12.3. Open-coiled helical spring.

of $WR \sin\alpha$ tending to reduce the curvature of the coils, i.e. a bending effect. Once again the shearing effect of W across the spring section is neglected as being very small in comparison with the other effects.

Thus $$T' = WR\cos\alpha \quad \text{and} \quad M' = WR\sin\alpha$$

Now, the total strain energy, neglecting shear,

$$U = \frac{T^2 L}{2GJ} + \frac{M^2 L}{2EI} \quad \text{(see §§ 11.3 and 11.4)}$$

$$= \frac{L(WR\cos\alpha)^2}{2GJ} + \frac{L(WR\sin\alpha)^2}{2EI}$$

$$= \frac{LW^2R^2}{2}\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right] \tag{12.5}$$

and this must equal the total work done $\frac{1}{2}W\delta$.

$$\therefore \quad \tfrac{1}{2}W\delta = \frac{LW^2R^2}{2}\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

From the helix form of Fig. 12.4

$$2\pi Rn = L\cos\alpha$$

$$\therefore \quad L = 2\pi Rn\sec\alpha$$

$$\therefore \quad \textbf{deflection } \boldsymbol{\delta = 2\pi n\, WR^3 \sec\alpha\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]} \tag{12.6}$$

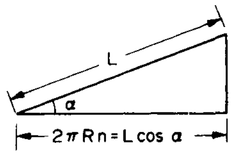

Fig. 12.4.

Since the stiffness of a spring S is normally defined as the value of W required to produce unit deflection,

$$\text{stiffness } S = \frac{W}{\delta}$$

$$\therefore \qquad \frac{1}{S} = \frac{\delta}{W} = 2\pi n R^3 \sec\alpha \left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right] \qquad (12.7)$$

Alternatively, the deflection in the direction of W is given by Castigliano's theorem (see § 11.11) as

$$\delta = \frac{\partial U}{\partial W} = \frac{\partial}{\partial W}\left[\frac{LW^2R^2}{2}\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right)\right]$$

$$= LWR^2\left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

and with $L = 2\pi Rn \sec\alpha$

$$\delta = 2\pi n WR^3 \sec\alpha \left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right] \qquad (12.8)$$

This is the same equation as obtained previously and illustrates the flexibility and ease of application of Castigliano's energy theorem.

(*b*) *Maximum stress*

The material of the spring is subjected to combined bending and torsion, the maximum stresses in each mode of loading being determined from the appropriate theory.

From the bending theory

$$\sigma = \frac{My}{I} \quad \text{with} \quad M = WR\sin\alpha$$

and from the torsion theory

$$\tau = \frac{Tr}{J} \quad \text{with} \quad T = WR\cos\alpha$$

The principal stresses at any point can then be obtained analytically or graphically using the procedures described in § 13.4.

(*c*) *Angular rotation*

Consider an imaginary axial torque T applied to the spring, together with W producing an angular rotation θ of one end of the spring relative to the other.

The combined twisting moment on the spring cross-section is then

$$\bar{T} = WR\cos\alpha + T\sin\alpha$$

and the combined bending moment

$$\bar{M} = T\cos\alpha - WR\sin\alpha$$

The total strain energy of the system is then

$$U = \frac{\bar{T}^2 L}{2GJ} + \frac{\bar{M}^2 L}{2EI}$$

$$= \frac{(WR\cos\alpha + T\sin\alpha)^2 L}{2GJ} + \frac{(T\cos\alpha - WR\sin\alpha)^2 L}{2EI}$$

Now from Castigliano's theorem the angle of twist in the direction of the axial torque T is given by $\theta = \dfrac{\partial U}{\partial T}$ and since $T = 0$ all terms including T may be ignored.

$$\therefore \qquad \theta = \frac{2WR\cos\alpha\sin\alpha\, L}{2GJ} + \frac{(-2WR\sin\alpha\cos\alpha)\, L}{2EI}$$

$$= WRL\cos\alpha\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]$$

i.e.

$$\boldsymbol{\theta = 2\pi n WR^2 \sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]} \qquad (12.9)$$

12.4. Open-coiled helical spring subjected to axial torque T

(a) *Wind-up angle*

When an axial torque T is applied to an open-coiled helical spring it has components as shown in Fig. 12.5, i.e. a torsional component $T\sin\alpha$ about AX and a flexural (bending) component $T\cos\alpha$ about AY, the latter tending to increase the curvature of the coils.

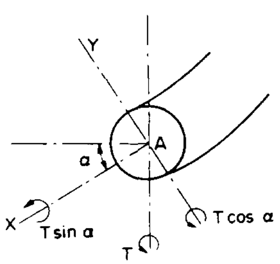

Fig. 12.5. Open-coiled helical spring subjected to axial torque T.

As for the close-coiled spring the total strain energy is given by

$$\text{strain energy } U = \frac{T^2L}{2GJ} + \frac{M^2L}{2EI}$$

$$= \frac{L}{2}\left[\frac{(T\sin\alpha)^2}{GJ} + \frac{(T\cos\alpha)^2}{EI}\right]$$

$$= \frac{T^2L}{2}\left[\frac{\sin^2\alpha}{GJ} + \frac{\cos^2\alpha}{EI}\right] \tag{12.10}$$

and this is equal to the work done by T, namely, $\frac{1}{2}T\theta$, where θ is the angle turned through by one end relative to the other, i.e. the wind-up angle of the spring.

$$\therefore \qquad \tfrac{1}{2}T\theta = \tfrac{1}{2}T^2L\left[\frac{\sin^2\alpha}{GJ} + \frac{\cos^2\alpha}{EI}\right]$$

and, with $L = 2\pi Rn \sec\alpha$ as before,

$$\textbf{wind-up angle } \boldsymbol{\theta = 2\pi nRT\sec\alpha\left[\frac{\sin^2\alpha}{GJ} + \frac{\cos^2\alpha}{EI}\right]} \tag{12.11}$$

(*b*) *Maximum stress*

The maximum stress in the spring material will be found by the procedure outlined in § 12.3(b) with a bending moment of $T\cos\alpha$ and a torque of $T\sin\alpha$ applied to the section.

(*c*) *Axial deflection*

Assuming an imaginary axial load W applied to the spring the total strain energy is given by eqn. (11.5) as

$$U = \frac{(WR\cos\alpha + T\sin\alpha)^2L}{2GJ} + \frac{(T\cos\alpha - WR\sin\alpha)^2L}{2EI}$$

Now from Castigliano's theorem the deflection in the direction of W is given by

$$\delta = \frac{\partial U}{\partial W}$$

$$= TRL\cos\alpha\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right] \quad \text{when } W = 0$$

$$\therefore \qquad \textbf{deflection } \boldsymbol{\delta = 2\pi nTR^2\sin\alpha\left[\frac{1}{GJ} - \frac{1}{EI}\right]} \tag{12.12}$$

12.5. Springs in series

If two springs of different stiffness are joined end-on and carry a common load W, they are said to be *connected in series* and the combined stiffness and deflection are given by the following equations.

$$\text{Deflection} = \frac{W}{S} = \delta_1 + \delta_2 = \frac{W}{S_1} + \frac{W}{S_2}$$

$$= W\left[\frac{1}{S_1} + \frac{1}{S_2}\right] \qquad (12.13)$$

$\therefore$
$$\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2}$$

and
$$\textbf{stiffness } S = \frac{S_1 S_2}{S_1 + S_2} \qquad (12.14)$$

12.6. Springs in parallel

If two springs are joined in such a way that they have a common deflection δ they are said to be *connected in parallel.* In this case the load carried is shared between the two springs and

$$\text{total load } W = W_1 + W_2 \qquad (1)$$

Now
$$\delta = \frac{W}{S} = \frac{W_1}{S_1} = \frac{W_2}{S_2} \qquad (12.15)$$

so that
$$W_1 = \frac{S_1 W}{S} \quad \text{and} \quad W_2 = \frac{S_2 W}{S}$$

Substituting in eqn. (1)

$$W = \frac{S_1 W}{S} + \frac{S_2 W}{S}$$

$$= \frac{W}{S}\left[S_1 + S_2\right]$$

i.e.
$$\textbf{combined stiffness } S = S_1 + S_2 \qquad (12.16)$$

12.7. Limitations of the simple theory

Whilst the simple torsion theory can be applied successfully to bars with small curvature without significant error the theory becomes progressively more inappropriate as the curvatures increase and become high as in most helical springs. The stress and deflection equations derived in the preceding sections, are, therefore, slightly inaccurate in practice, particularly for small D/d ratios. For accurate assessment of stresses and deflections account should be taken of the influence of curvature and slope by applying factors due to Wahl† and Ancker and Goodier‡. These are discussed in Roark and Young§ where the more accurate

† A. M. Wahl, *Mechanical Springs*, 2nd edn. (McGraw-Hill, New York 1963).
‡ C. J. Ancker (Jr) and J. N. Goodier, "Pitch and curvature correction for helical springs", *ASME J. Appl. Mech.*, 25(4), Dec. 1958.
§ R. J. Roark and W. C. Young, *Formulas for Stress and Strain*, 5th edn. (McGraw-Hill, Kogakusha, 1965).

expressions for circular, square and rectangular section springs are introduced. For the purposes of this text it is considered sufficient to indicate the use of these factors on circular section wire.

For example, Ancker and Goodier write the stress and deflection equations for circular section springs subjected to an axial load W in the following form (which can be related directly to eqns. (12.1) and (12.2)).

Maximum stress $$\tau_{max} = K_1 \left(\frac{2WR}{\pi r^3}\right) = K_1 \left(\frac{8WD}{\pi d^3}\right)$$

and deflection $$\delta = K_2 \left(\frac{4WR^3 n}{Gr^4}\right) = K_2 \left(\frac{8WD^3 n}{Gd^4}\right)$$

where $$K_1 = \left[1 + \frac{5}{8}\left(\frac{d}{R}\right) + \frac{7}{32}\left(\frac{d}{R}\right)^2\right]$$

and $$K_2 = \left[1 - \frac{3}{64}\left(\frac{d}{R}\right)^2 + \frac{(3+\nu)}{2(1+\nu)}(\tan\alpha)^2\right]$$

where α is the pitch angle of the spring.

In an exactly similar way Wahl also proposes the introduction of correction factors which are related to the so-called spring index $C = D/d$.

Thus, for central load W:

maximum stress $$\tau_{max} = K\left[\frac{8WD}{\pi d^3}\right]$$

with $$K = \frac{(4C-1)}{(4C-4)} + \frac{0.615}{C}$$

The British Standard for spring design, BS1726, quotes a simpler equation for K, namely:

$$K = \left[\frac{C+0.2}{C-1}\right]$$

The Standard also makes the point that the influence of the correction factors is often small in comparison with the uncertainty regarding what should be selected as the true number of working coils (depending on the method of support, etc).

Values of K for different ratios of spring index are given in Fig. 12.6 on page 308.

12.8. Extension springs – initial tension.

The preceding laws and formulae derived for compression springs apply equally to extension springs except that the latter are affected by initial tension. When springs are closely wound a force is required to hold the coils together and this can seldom be controlled to a greater accuracy than $\pm$ 10%. This does not increase the ultimate load capacity but must be included in the stress calculation. As an approximate guide, the initial tension obtained in hand-coiled commercial-quality springs is taken to be equivalent to the rate of the spring, although this can be far exceeded if special coiling methods are used.

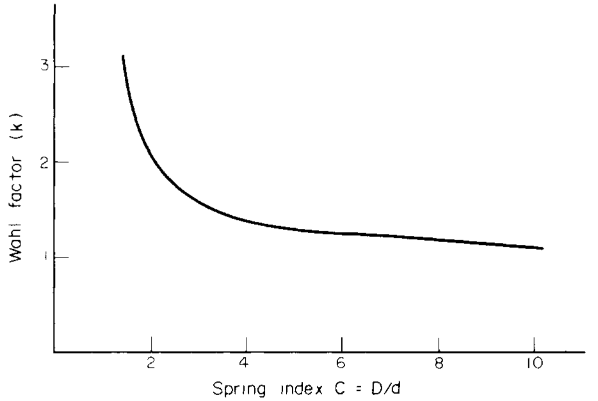

Fig. 12.6. Wahl correction factors for maximum shear stress.

12.9. Allowable stresses

As a rough approximation, the torsional elastic limit of commercial wire materials is taken to be 40 % of the tensile strength. This is applied equally to ferrous and non-ferrous materials such as phosphor bronze and brass.

Typical values of allowable stress for hard-drawn spring steel piano wire based on the above assumption are given in Table 12.1.† These represent the corrected stress and generally should not be exceeded unless exceptionally high grade materials are used.

TABLE 12.1. Allowable stresses for hard-drawn steel spring wire

Wire size *SWG*	*Allowable stress* (MN/m^2)	
	Compression/Extension	*Torsion*
44–39	1134	1409
48–35	1079	1340
34–31	1031	1272
30–28	983	1203
27–24	928	1169
23–18	859	1066
17–13	770	963
12–10	688	859
9–7	619	756
6–5	550	688
4–3	516	619

Care must be exercised in the application of the quoted values bearing in mind the presence of any irregularities in the form or clamping method and the duty the spring is to perform. For example the quoted values may be far too high for springs to operate at high frequency, particularly in the presence of stress raisers, when fatigue failure would soon result. Under

† Spring Design, *Engineering Materials And Design*, Feb. 1980.

such conditions a high-grade annealed spring steel suitably heat-treated should be considered.

A useful comparison of the above theories together with further ones due to Rover, Honegger, Göhner and Bergsträsser is given in the monograph† *Helical Springs*, which then goes on to consider the effect of pitch angle, failure considerations, vibration frequency and spring surge (speed of propagation of wave along the axis of a spring).

12.10. Leaf or carriage spring: semi-elliptic

The principle of using a beam in bending as a spring has been known for many years and widely used in motor-vehicle applications. If the beam is arranged as a simple cantilever, as in Fig. 12.7a, it is called a *quarter-elliptic* spring, and if as a simply supported beam with central load, as in Fig. 12.7b, it is termed a *half* or *semi-elliptic* spring. The latter will be discussed first.

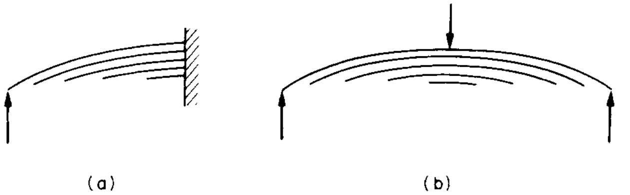

Fig. 12.7. (a) Quarter-elliptic, (b) semi-elliptic, carriage springs.

(a) *Maximum stress*

Consider the semi-elliptic leaf spring shown in Fig. 12.8. With a constant thickness t this design of spring gives a uniform stress throughout and is therefore economical in both material and weight.

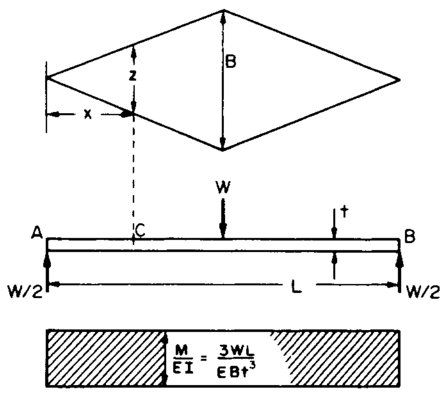

Fig. 12.8. Semi-elliptic leaf spring.

† J. R. Finniecome, *Helical Springs*. Mechanical World Monograph 56 (Emmott & Co., Manchester 1949).

By proportions $$\frac{z}{x} = \frac{B}{L/2} \quad \therefore \; z = \frac{2Bx}{L}$$

$$\text{Bending moment at } C = \frac{Wx}{2} \quad \text{and} \quad I = \frac{zt^3}{12} = \frac{2Bxt^3}{12L}$$

Therefore from the bending theory the stress set up at any section is given by

$$\sigma = \frac{My}{I} = \frac{Wx}{2} \times \frac{t}{2} \times \frac{12L}{2Bxt^3}$$

$$= \frac{3WL}{2Bt^2}$$

i.e. the bending stress in a semi-elliptic leaf spring is independent of x and equal to

$$\frac{\mathbf{3WL}}{\mathbf{2Bt^2}} \tag{12.17}$$

If the spring is constructed from strips and placed one on top of the other as shown in Fig. 12.9, uniform stress conditions are retained, since if the strips are cut along XX and replaced side by side, the equivalent leaf spring is obtained as shown.

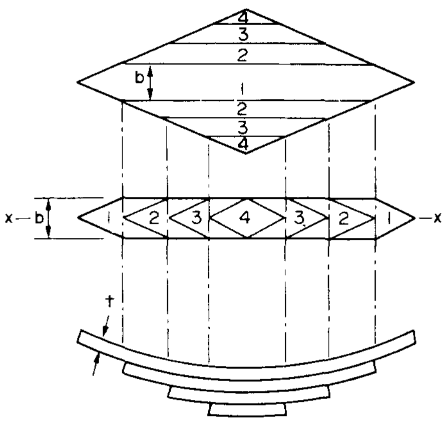

Fig. 12.9. Semi-elliptic carriage spring showing initial pre-forming.

Such a spring is then termed a *carriage spring* with n strips of width b, i.e. $B = nb$. Therefore the bending stress in a semi-elliptic carriage spring is

$$\frac{\mathbf{3WL}}{\mathbf{2nbt^2}} \tag{12.18}$$

The diamond shape of the leaf spring could also be obtained by varying the thickness, but this type of spring is difficult to manufacture and has been found unsatisfactory in practice.

(b) Deflection

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \qquad \therefore \; R = \frac{EI}{M}$$

$$R = E \times \frac{2Bxt^3}{12L} \times \frac{2}{Wx} = \frac{EBt^3}{3WL} \tag{12.19}$$

i.e. for a given spring and given load, R is constant and the spring bends into the arc of a circle.

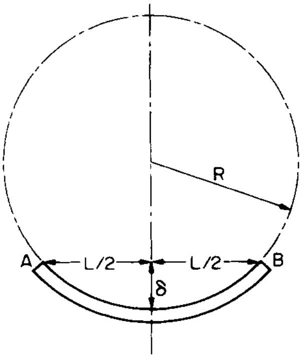

Fig. 12.10.

From the properties of intersecting chords (Fig. 12.10)

$$\delta(2R - \delta) = \frac{L}{2} \times \frac{L}{2}$$

Neglecting δ^2 as the product of small quantities

$$\delta = \frac{L^2}{8R}$$

$$= \frac{L^2}{8} \times \frac{3WL}{EBt^3}$$

i.e. deflection of a semi-elliptic *leaf* spring

$$\boldsymbol{\delta = \frac{3WL^3}{8EBt^3}} \tag{12.20}$$

But $B = nb$, so that the deflection of a semi-elliptic *carriage* spring is given by

$$\boldsymbol{\delta = \frac{3WL^3}{8Enbt^3}} \tag{12.21}$$

(c) Proof load

The proof load of a leaf or carriage spring is the load which is required to straighten the plates from their initial preformed position. From eqn. 12.18 the maximum bending stress for any given load W is

$$\sigma = \frac{3WL}{2nbt^2}$$

Thus if σ_p denotes the stress corresponding to the application of the proof load W_p

$$W_p = \frac{2nbt^2}{3L}\sigma_p \qquad (12.22)$$

Now from eqn. (12.19) and inserting $B = nb$, the load W which would produce bending of a flat carriage spring to some radius R is given by

$$W = \frac{Enbt^3}{3RL}$$

Conversely, therefore, the load which is required to straighten a spring from radius R will be of the same value,

i.e. $$W_p = \frac{Enbt^3}{3RL}$$

Substituting for $$R = \frac{L^2}{8\delta}$$

$\therefore$ $$\textbf{proof load } \boldsymbol{W_p = \frac{8Enbt^3}{3L^3}\delta_p} \qquad (12.23)$$

where δ_p is the initial central "deflection" of the spring.

Equating eqns. (12.22) and (12.23),

$$\frac{2nbt^2}{3L}\sigma_p = \frac{8Enbt^3}{3L^3}\delta_p$$

i.e. $$\textbf{proof stress } \boldsymbol{\sigma_p = \frac{4tE}{L^2}\delta_p} \qquad (12.24)$$

For a given spring material the limiting value of σ_p will be known as will the value of E. The above equation therefore yields the correct relationship between the thickness and initial curvature of the spring plates.

12.11. Leaf or carriage spring: quarter-elliptic

(a) Maximum stress

Consider the *quarter-elliptic* leaf and carriage springs shown in Fig. 12.11. In this case the equations for the semi-elliptic spring of the previous section are modified to

$$z = \frac{Bx}{L} \quad \text{and} \quad \text{B.M. at } C = Wx$$

$\therefore$ $$I = \frac{zt^3}{12} = \frac{Bxt^3}{12L}$$

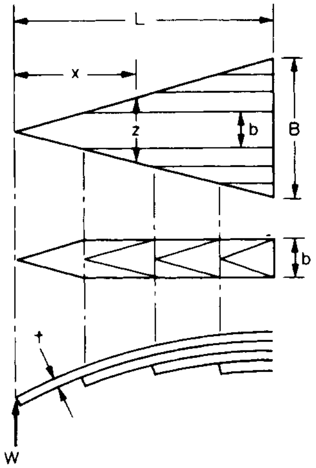

Fig. 12.11. Quarter-elliptic leaf and carriage springs.

Now $$\sigma = \frac{My}{I} = \frac{Wxt}{2} \times \frac{12L}{Bxt^3} = \frac{6WL}{Bt^2}$$

Therefore the maximum bending stress for a quarter-elliptic *leaf* spring

$$= \frac{6WL}{Bt^2} \qquad (12.25)$$

and the maximum bending stress for a quarter-elliptic *carriage* spring

$$= \frac{6WL}{nbt^2} \qquad (12.26)$$

(*b*) *Deflection*

With B.M. at $C = Wx$ and replacing $L/2$ by L in the proof of §12.7(b),

$$\delta = \frac{L^2}{2R}$$

and $$R = \frac{EI}{M} = \frac{E}{Wx} \times \frac{Bxt^3}{12L} = \frac{Ebt^3}{12WL}$$

∴ $$\delta = \frac{L^2}{2} \times \frac{12WL}{EBt^3} = \frac{6WL^3}{EBt^3}$$

Therefore deflection of a quarter-elliptic *leaf* spring

$$= \frac{6WL^3}{EBt^3} \qquad (12.27)$$

and deflection of a quarter-elliptic *carriage* spring

$$= \frac{6WL^3}{Enbt^3} \tag{12.28}$$

12.12. Spiral spring

(a) Wind-up angle

Spiral springs are normally constructed from thin rectangular-section strips wound into a spiral in one plane. They are often used in clockwork mechanisms, the winding torque or moment being applied to the central spindle and the other end firmly anchored to a pin at the outside of the spiral. Under the action of this central moment all sections of the spring will be subjected to uniform bending which tends to reduce the radius of curvature at all points.

Consider now the spiral spring shown in Fig. 12.12.

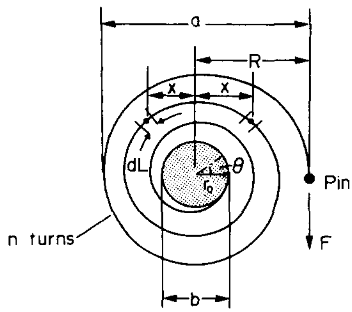

Fig. 12.12. Spiral spring.

Let M = winding moment applied to the spring spindle, R = radius of spring from spindle to pin, a = maximum dimension of the spring from the pin, B = breadth of the material of the spring, t = thickness of the material of the spring, and b = diameter of the spindle.

Assuming the polar equation of the spiral to be that of an Archimedean spiral,

$$r = r_0 + \left(\frac{A}{2\pi}\right)\theta \quad \text{where } A \text{ is some constant}$$

When $\theta = 0, \quad r = r_0 = \dfrac{b}{2}$

and for the nth turn, $\theta = 2n\pi$ and

$$r = \frac{a}{2} = \frac{b}{2} + \left(\frac{A}{2\pi}\right)2n\pi$$

$$\therefore \quad A = \frac{(a-b)}{2n}$$

i.e. the equation to the spiral is

$$r = \frac{b}{2} + \frac{(a-b)}{4\pi n}\,\theta \tag{12.29}$$

When a torque or winding couple M is applied to the spindle a resistive force F will be set up at the pin such that

$$\text{winding couple } M = F \times R$$

Consider now two small elements of material of length dl at distance x to each side of the centre line (Fig. 12.12).

For small deflections, from Mohr's area–moment method the change in slope between two points is

$$\left(\frac{M}{EI}\right)dL \qquad \text{(see §5.7)}$$

For the portion on the left,

$$\text{change in slope} = d\theta_1 = \frac{F(R+x)\,dL}{EI}$$

and similarly for the right-hand portion,

$$\text{change in slope} = d\theta_2 = \frac{F(R-x)\,dL}{EI}$$

The sum of these changes in slope is thus

$$d\theta_1 + d\theta_2 = \frac{F(R+x)\,dL}{EI} + \frac{F(R-x)\,dL}{EI}$$

$$= \frac{2FR\,dL}{EI}$$

If this is integrated along the length of the spring the result obtained will be twice the total change in slope along the spring, i.e. twice the angle of twist.

$$\therefore \qquad \text{angle of twist} = \tfrac{1}{2}\int_0^L \frac{2FR\,dL}{EI} = \frac{FRL}{EI} = \frac{ML}{EI}$$

where M is the applied winding moment and L the total length of the spring.

Now

$$L = \int_0^L dL = \int_0^{2n\pi} r\,d\theta = \int_0^{2n\pi} \frac{b}{2} + \frac{(a-b)}{4\pi n}\,\theta\,d\theta$$

$$= \left[\frac{b\theta}{2} + \frac{(a-b)}{4\pi n}\frac{\theta^2}{2}\right]_0^{2n\pi} = \left[\frac{2nb\pi}{2} + \frac{(a-b)}{4\pi n}\frac{(2n\pi)^2}{2}\right]$$

$$= \pi n\left[b + \frac{(a-b)}{2}\right]$$

$$= \frac{\pi n}{2}\left[a+b\right] \tag{12.30}$$

Therefore the wind-up angle of a spiral spring is

$$\theta = \frac{M}{EI}\left[\frac{\pi n}{2}(a+b)\right] \tag{12.31}$$

(*b*) *Maximum stress*

The maximum bending stress set up in the spring will be at the point of greatest bending moment, since the material of the spring is subjected to pure bending.

$$\text{Maximum bending moment} = F \times a$$

$$\therefore \qquad \text{maximum bending stress} = \frac{My}{I} = \frac{Fa(t/2)}{I}$$

But, for rectangular-section spring material of breadth B and thickness t,

$$I = \frac{Bt^3}{12}$$

$$\therefore \qquad \sigma_{max} = \frac{Fat}{2} \times \frac{12}{Bt^3} = \frac{6Fa}{Bt^2}$$

Now the applied moment $\qquad M = F \times R$

$$\therefore \qquad \textbf{maximum bending stress } \boldsymbol{\sigma_{max}} = \boldsymbol{\frac{6Ma}{RBt^2}} \tag{12.32}$$

or, assuming $a = 2R$,

$$\sigma_{max} = \frac{12M}{Bt^2} \tag{12.33}$$

Examples

Example 12.1

A close-coiled helical spring is required to absorb 2.25×10^3 joules of energy. Determine the diameter of the wire, the mean diameter of the spring and the number of coils necessary if:

(a) the maximum stress is not to exceed $400\,\text{MN/m}^2$;
(b) the maximum compression of the spring is limited to 250 mm;
(c) the mean diameter of the spring can be assumed to be eight times that of the wire.

How would the answers change if appropriate Wahl factors are introduced?
For the spring material $G = 70\,\text{GN/m}^2$.

Solution

The spring is required to absorb 2.25×10^3 joules or 2.25 kN m of energy.

$$\therefore \qquad \text{work done} = \tfrac{1}{2}W\delta = 2.25 \times 10^3$$

But δ is limited to 250 mm.

$$\therefore \quad \tfrac{1}{2}W \times 250 \times 10^{-3} = 2.25 \times 10^{3}$$

$$W = \frac{2.25 \times 10^{3} \times 2}{250 \times 10^{-3}} = 18\,\text{kN}$$

Thus the maximum load which can be carried by the spring is 18 kN.

Now the maximum stress is not to exceed 400 MN/m^2; therefore from eqn. (12.1),

$$\frac{2WR}{\pi r^3} = 400 \times 10^{6}$$

But $R = 8r$

$$\therefore \quad \frac{2 \times 18 \times 10^{3} \times 8r}{\pi r^3} = 400 \times 10^{6}$$

$$r^2 = \frac{2 \times 18 \times 10^{3} \times 8}{\pi \times 400 \times 10^{6}} = 229 \times 10^{-6}$$

$$r = 15.1 \times 10^{-3} = 15.1\,\text{mm}$$

The required diameter of the wire, for practical convenience, is, therefore,

$$2 \times 15 = \mathbf{30\,mm}$$

and, since $R = 8r$, the required mean diameter of the coils is

$$8 \times 30 = \mathbf{240\,mm}$$

Now total deflection

$$\delta = \frac{4WR^3n}{Gr^4} = 250\,\text{mm}$$

$$n = \frac{250 \times 10^{-3} \times 70 \times 10^{9} \times (15 \times 10^{-3})^4}{4 \times 18 \times 10^{3} \times (120 \times 10^{-3})^3}$$

$$= 7.12$$

Again from practical considerations, the number of complete coils necessary = 7. (If 8 coils were chosen the maximum deflection would exceed 250 mm.)

The effect of introducing Wahl correction factors is determined as follows:
From the given data $C = D/d = 8$ $\therefore$ From Fig. 12.6 $K = 1.184$.

$$\text{Now} \quad \tau_{max} = K\left[\frac{8WD}{\pi d^3}\right] = K\left[\frac{2WR}{\pi r^3}\right] = 400 \times 10^{6}$$

$$\therefore \quad 400 \times 10^{6} = \frac{1.184 \times 2 \times 18 \times 10^{3} \times 8r}{\pi r^3}$$

$$\therefore \quad r^2 = \frac{1.184 \times 2 \times 18 \times 10^{3} \times 8}{\pi \times 400 \times 10^{6}} = 271.35 \times 10^{-6}$$

$$\therefore \quad r = 16.47 \times 10^{-3} = 16.47\,\text{mm}$$

i.e. for practical convenience $d = 2 \times 16.5 =$ **33 mm,**
and since $D = 8d$, $D = 8 \times 33 =$ **264 mm.**

Total deflection $$\delta = \frac{4WR^3 n}{Gr^4} = 250\,\text{mm}.$$

$$\therefore \quad n = \frac{250 \times 10^{-3} \times 70 \times 10^9 \times (16.5 \times 10^{-3})^4}{4 \times 18 \times 10^3 \times (132 \times 10^{-3})^3}$$

$$= 7.83.$$

Although this is considerably greater than the value obtained before, the number of complete coils required remains at 7 if maximum deflection is strictly limited to 250 mm.

Example 12.2

A compression spring is required to carry a load of 1.5 kN with a limiting shear stress of 250 MN/m^2. If the spring is to be housed in a cylinder of 70 mm diameter estimate the size of spring wire required. Use appropriate Wahl factors in your solution.

Solution

Maximum shear stress $$\tau_{max} = K\left[\frac{8WD}{\pi d^3}\right]$$

i.e. $$d^3 = \frac{8WDK}{\pi \tau_{max}}$$

$$\therefore \quad d = \sqrt[3]{\frac{8 \times 1.5 \times 10^3\, DK}{\pi \times 250 \times 10^6}}$$

$$= 2.481 \times 10^{-2} \sqrt[3]{DK} \qquad (1)$$

Unfortunately, this cannot readily be solved for d since K is dependent on d, and D the mean diameter is not known except so far as its maximum value is limited to $(70 - d)$ mm.

If, therefore, as a first approximation, D is taken to be 70 mm and K is assumed to be 1, a rough order of magnitude is obtained for d from the above equation (1).

i.e. $$d = 2.481 \times 10^{-2}\,(70 \times 10^{-3} \times 1)^{\frac{1}{3}}$$

$$= 10.22\ \text{mm}.$$

It is now appropriate to apply a graphical solution to the determination of the precise value of d using assumed values of d close to the above rough value, reading the appropriate value of K from Fig. 12.6 and calculating the corresponding d value from eqn. (1).

Assumed d	D $(= 70 - d)$	C $(= D/d)$	K	*Calculated* d (*from eqn.* (1))
10	60	6.0	1.25	10.46
10.5	59.5	5.67	1.27	10.49
11.0	59	5.36	1.29	10.51
11.5	58.5	5.09	1.304	10.522

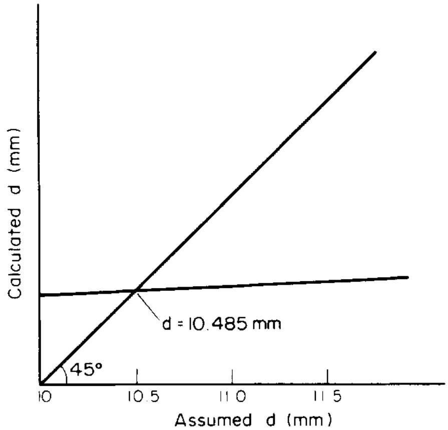

Fig. 12.13.

Plotting the assumed and calculated values gives the nearly horizontal line of Fig. 12.13. The other line is that of the required solution, i.e. it represents all points along which the assumed and calculated d values are the same (i.e. at 45° to the axes). Thus, where this line crosses the previously plotted line is the required value of d, namely 10.485 mm.

The spring wire must therefore have a minimum diameter of **10.485 mm** and a mean diameter of $70 - 10.485 =$ **59.51 mm**.

Example 12.3

A close-coiled helical spring, constructed from wire of 10 mm diameter and with a mean coil diameter of 50 mm, is used to join two shafts which transmit 1 kilowatt of power at 4000 rev/min. If the number of turns of the spring is 10 and the modulus of elasticity of the spring material is 210 GN/m^2 determine:

(a) the relative angle of twist between the two ends of the spring;
(b) the maximum stress set up in the spring material.

Solution

$$\text{Power} = T\omega = 1000\,\text{W}$$

$$T = \frac{1000 \times 60}{4000 \times 2\pi} = 2.39\,\text{N m}$$

Now the wind-up angle of the spring, from eqn. (12.4),

$$= \frac{8TRn}{Er^4}$$

$$\therefore \quad \theta = \frac{8 \times 2.39 \times 25 \times 10^{-3} \times 10}{210 \times 10^9 \times (5 \times 10^{-3})^4}$$

$$= 0.036\,\text{radian} = \mathbf{2.1^\circ}$$

The maximum stress is then given by eqn. (12.3),

$$\sigma_{max} = \frac{4T}{\pi r^3} = \frac{4 \times 2.39}{\pi \times (5 \times 10^{-3})^3}$$

$$= 24.3 \times 10^6 = \mathbf{24.3\,MN/m^2}$$

Example 12.4

Show that the ratio of extension per unit axial load to angular rotation per unit axial torque of a close-coiled helical spring is directly proportional to the square of the mean diameter, and hence that the constant of proportionality is $\frac{1}{4}(1+\nu)$.

If Poisson's ratio $\nu = 0.3$, determine the angular rotation of a close-coiled helical spring of mean diameter 80 mm when subjected to a torque of 3 N m, given that the spring extends 150 mm under an axial load of 250 N.

Solution

From eqns. (12.2) and (12.4)

$$\delta = \frac{4WR^3n}{Gr^4} \quad \text{and} \quad \theta = \frac{8TRn}{Er^4}$$

$$\therefore \quad \frac{\delta}{W} = \frac{4R^3n}{Gr^4} \quad \text{and} \quad \frac{\theta}{T} = \frac{8Rn}{Er^4}$$

$$\therefore \quad \frac{\delta/W}{\theta/T} = \frac{4R^3n}{Gr^4} \times \frac{Er^4}{8Rn} = \frac{R^2E}{2G} = \frac{D^2E}{8G}$$

But $E = 2G(1+\nu)$

$$\therefore \quad \frac{\delta/W}{\theta/T} = \frac{D^2}{8} \times \frac{2G(1+\nu)}{G} = \tfrac{1}{4}(1+\nu)D^2 \qquad (1)$$

Thus the ratio is directly proportional to D^2 and the constant of proportionality is $\frac{1}{4}(1+\nu)$.

From eqn. (1)
$$\frac{T\delta}{W\theta} = \tfrac{1}{4}(1+\nu)D^2$$

$$\therefore \quad \frac{3 \times 150 \times 10^{-3}}{250 \times \theta} = \tfrac{1}{4}(1+0.3)(80 \times 10^{-3})^2$$

$$\theta = \frac{3 \times 150 \times 10^{-3} \times 4}{250 \times 1.3 \times 6400 \times 10^{-6}}$$

$$= 0.865 \text{ radian} = \mathbf{49.6^\circ}$$

The required angle of rotation is 49.6°.

Example 12.5

(a) Determine the load required to produce an extension of 8 mm on an open coiled helical spring of 10 coils of mean diameter 76 mm, with a helix angle of 20° and manufactured from wire of 6 mm diameter. What will then be the bending and shear stresses in the surface of the wire? For the material of the spring, $E = 210\,\text{GN/m}^2$ and $G = 70\,\text{GN/m}^2$.

(b) What would be the angular twist at the free end of the above spring when subjected to an axial torque of 1.5 N m?

Solution

(a) From eqn. (12.6) the extension of an open-coiled helical spring is given by

$$\delta = 2\pi n W R^3 \sec\alpha \left[\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right]$$

Now $$I = \frac{\pi d^4}{64} = \frac{\pi \times (6\times 10^{-3})^4}{64} = 63.63 \times 10^{-12}\,\text{m}^4$$

and $$J = \frac{\pi d^4}{32} = 127.26 \times 10^{-12}\,\text{m}^4$$

$$\therefore \quad 8\times 10^{-3} = 2\pi \times 10 \times W \times (38\times 10^{-3})^3 \sec 20^\circ \left[\frac{\cos^2 20^\circ}{70\times 10^9 \times 127.26\times 10^{-12}} + \frac{\sin^2 20^\circ}{210\times 10^9 \times 63.63\times 10^{-12}}\right]$$

$$= \frac{20\pi W \times 38^3 \times 10^{-9}}{0.9397}\left[\frac{(0.9397)^2}{8.91} + \frac{(0.342)^2}{13.36}\right]$$

$$= \frac{20\pi W \times 38^3 \times 10^{-9}}{0.9397}[0.1079]$$

$$\therefore \quad W = \frac{8\times 10^{-3} \times 0.9397}{20\pi \times 38^3 \times 10^{-9} \times 0.1079}$$

$$= \mathbf{20\,N}$$

The bending moment acting on the spring is

$$WR\sin\alpha = 20 \times 38 \times 10^{-3} \times 0.342$$

$$= 0.26\,\text{N m}$$

$$\therefore \quad \text{bending stress} = \frac{My}{I} = \frac{0.26 \times 3 \times 10^{-3}}{63.63 \times 10^{-12}} = \mathbf{12.3\,MN/m^2}$$

Similarly, the torque on the spring material is

$$WR\cos\alpha = 20 \times 38 \times 10^{-3} \times 0.9397$$

$$= 0.714\,\text{N m}$$

$$\therefore \quad \text{shear stress} = \frac{Tr}{J} = \frac{0.714 \times 3 \times 10^{-3}}{127.26 \times 10^{-12}}$$

$$= \mathbf{16.8\,MN/m^2}$$

(b) The wind-up angle of the spring under the action of an axial torque is given by eqn. (12.11):

$$\theta = 2\pi nRT\sec\alpha\left[\frac{\sin^2\alpha}{GJ} + \frac{\cos^2\alpha}{EI}\right]$$

$$= \frac{2\pi \times 10 \times 38 \times 10^{-3} \times 1.5}{0.9397}\left[\frac{(0.342)^2}{8.91} + \frac{(0.9397)^2}{13.36}\right]$$

$$= \frac{2\pi \times 10 \times 38 \times 10^{-3} \times 1.5}{0.9397}[0.0792]$$

$$= 0.302\,\text{radian} = \mathbf{17.3^\circ}$$

Example 12.6

Calculate the thickness and number of leaves of a semi-elliptic carriage spring which is required to support a central load of 2 kN on a span of 1 m if the maximum stress is limited to 225 MN/m^2 and the central deflection to 75 mm. The breadth of each leaf can be assumed to be 100 mm.

For the spring material $E = 210\,\text{GN/m}^2$.

Solution

From eqn. (12.18),

$$\text{maximum stress} = \frac{3WL}{2nbt^2} = 225 \times 10^6$$

$$\therefore \quad \frac{3 \times 2000 \times 1}{2 \times n \times 100 \times 10^{-3}t^2} = 225 \times 10^6$$

$$nt^2 = \frac{3 \times 2000}{2 \times 100 \times 10^{-3} \times 225 \times 10^6} = 0.133 \times 10^{-3}$$

And from eqn. (12.21),

$$\text{Deflection } \delta = \frac{3WL^3}{8Enbt^3}$$

$$\therefore \quad 75 \times 10^{-3} = \frac{3 \times 2000 \times 1}{8 \times 210 \times 10^9 \times n \times 100 \times 10^{-3} \times t^3}$$

$$\therefore \quad nt^3 = \frac{3 \times 2000}{75 \times 10^{-3} \times 8 \times 210 \times 10^8}$$

$$= 0.476 \times 10^{-6}$$

$$\therefore \quad \frac{nt^3}{nt^2} = t = \frac{0.476 \times 10^{-6}}{0.133 \times 10^{-3}}$$

$$t = 3.58 \times 10^{-3} = \mathbf{3.58\,mm}$$

and, since $nt^2 = 0.133 \times 10^{-3}$,

$$n = \frac{0.133 \times 10^{-3}}{(3.58 \times 10^{-3})^2}$$

$$= \mathbf{10.38}$$

The nearest whole number of leaves is therefore 10. However, with $n = 10$, the stress limit would be exceeded and this should be compensated for by increasing the thickness t in the ratio $\sqrt{\left(\frac{10.38}{10}\right)} = 1.02$,

i.e. $$t = \mathbf{3.65\,mm}$$

Example 12.7

A flat spiral spring is pinned at the outer end and a winding couple is applied to a spindle attached at the inner end as shown in Fig. 12.11, with $a = 150\,\text{mm}$, $b = 40\,\text{mm}$ and $R = 75\,\text{mm}$. The material of the spring is rectangular in cross-section, 12 mm wide and 2.5 mm thick, and there are 5 turns. Determine:

(a) the angle through which the spindle turns;
(b) the maximum bending stress produced in the spring material when a torque of 1.5 N m is applied to the winding spindle.

For the spring material, $E = 210\,\text{GN/m}^2$.

Solution

(a)

$$\text{The angle of twist} = \frac{ML}{EI}$$

where $$L = \frac{\pi n}{2}(a+b)$$

$$= \frac{\pi \times 5}{2}(150+40)10^{-3}$$

$$= 1492.3 \times 10^{-3}\,\text{m} = 1.492\,\text{m}$$

∴ $$\text{angle of twist} = \frac{1.5 \times 1.492 \times 12}{210 \times 10^9 \times 12 \times 2.5^3 \times 10^{-12}}$$

$$= 0.682\,\text{radian}$$

$$= \mathbf{39.1^\circ}$$

$$\text{Maximum bending moment} = F \times a$$

where $$\text{applied moment} = F \times R = 1.5\,\text{N m}$$

i.e.
$$F = \frac{1.5}{75 \times 10^{-3}} = 20\,\text{N}$$

$\therefore$
$$\text{maximum bending moment} = 20 \times 150 \times 10^{-3} = 3\,\text{N m}$$

$\therefore$
$$\text{maximum bending stress} = \frac{My}{I} = \frac{3 \times (t/2)}{I}$$
$$= \frac{3 \times 1.25 \times 10^{-3} \times 12}{12 \times 2.5^3 \times 10^{-12}}$$
$$= 240 \times 10^6 = \mathbf{240\,MN/m^2}$$

Problems

(Take $E = 210\,\text{GN/m}^2$ and $G = 70\,\text{GN/m}^2$ throughout)

12.1 (A/B). A close-coiled helical spring is to have a stiffness of 90 kN/m and to exert a force of 3 kN; the mean diameter of the coils is to be 75 mm and the maximum stress is not to exceed 240 MN/m^2. Calculate the required number of coils and the diameter of the steel rod from which the spring should be made. [E.I.E.] [8, 13.5 mm.]

12.2 (A/B). A close-coiled helical spring is fixed at one end and subjected to axial twist at the other. When the spring is in use the axial torque varies from 0.75 N m to 3 N m, the working angular deflection between these torques being 35°. The spring is to be made from rod of circular section, the maximum permissible stress being 150 MN/m^2. The mean diameter of the coils is eight times the rod diameter. Calculate the mean coil diameter, the number of turns and the wire diameter. [B.P.] [48, 6 mm; 24.]

12.3 (A/B). A close-coiled helical compression spring made from round wire fits over the spindle of a plunger and has to work inside a tube. The spindle diameter is 12 mm and the tube is of 25 mm outside diameter and 0.15 mm thickness. The maximum working length of the spring has to be 120 mm and the minimum length 90 mm. The maximum force exerted by the spring has to be 350 N and the minimum force 240 N. If the shearing stress in the spring is not to exceed 600 MN/m^2 find:
(a) the free length of the spring (i.e. before assembly);
(b) the mean coil diameter;
(c) the wire diameter;
(d) the number of free coils. [185.4, 18.3, 3 mm; 32.]

12.4 (A/B). A close-coiled helical spring of circular wire and mean diameter 100 mm was found to extend 45 mm under an axial load of 50 N. The same spring when firmly fixed at one end was found to rotate through 90° under a torque of 5.7 N m. Calculate the value of Poisson's ratio for the material. [C.U.] [0.3.]

12.5 (B). Show that the total strain energy stored in an open-coiled helical spring by an axial load W applied together with an axial couple T is

$$U = (T\cos\theta - WR\sin\theta)^2 \frac{L}{2EI} + (WR\cos\theta + T\sin\theta)^2 \frac{L}{2GJ}$$

where θ is the helix angle and L the total length of wire in the spring, and the sense of the couple is in a direction tending to wind up the spring. Hence, or otherwise, determine the rotation of one end of a spring of helix angle 20° having 10 turns of mean radius 50 mm when an axial load of 25 N is applied, the other end of the spring being securely fixed. The diameter of the wire is 6 mm. [B.P.] [26°.]

12.6 (B). Deduce an expression for the extension of an open-coiled helical spring carrying an axial load W. Take α as the inclination of the coils, d as the diameter of the wire and R as the mean radius of the coils. Find by what percentage the axial extension is underestimated if the inclination of the coils is neglected for a spring in which $\alpha = 25°$. Assume n and R remain constant. [U.L.] [3.6 %.]

12.7 (B). An open-coiled spring carries an axial vertical load W. Derive expressions for the vertical displacement and angular twist of the free end. Find the mean radius of an open-coiled spring (angle of helix 30°) to give a vertical displacement of 23 mm and an angular rotation of the loaded end of 0.02 radian under an axial load of 40 N. The material available is steel rod of 6 mm diameter. [U.L.] [182 mm.]

12.8 (B). A compound spring comprises two close-coiled helical springs having exactly the same initial length when unloaded. The outer spring has 16 coils of 12 mm diameter bar coiled to a mean diameter of 125 mm and the inner spring has 24 coils with a mean diameter of 75 mm. The working stress in each spring is to be the same. Find (a) the diameter of the steel bar for the inner spring and (b) the stiffness of the compound spring.
[I.Mech.E.] [6.48 mm; 7.33 kN/m.]

12.9 (B). A composite spring has two close-coiled helical springs connected in series; each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the springs if the diameter of wire in the other spring is 2.5 mm and the stiffness of the composite spring is 700 N/m. Estimate the greatest load that can be carried by the composite spring and the corresponding extension for a maximum shearing stress of 180 MN/m^2.
[U.L.] [44.2 N; 63.2 mm.]

12.10 (B). (a) Derive formulae in terms of load, leaf width and thickness, and number of leaves for the maximum deflection and maximum stress induced in a cantilever leaf spring. (b) A cantilever leaf spring is 750 mm long and the leaf width is to be 8 times the leaf thickness. If the bending stress is not to exceed 210 MN/m^2 and the spring is not to deflect more than 50 mm under a load of 5 kN, find the leaf thickness, the least number of leaves required, the deflection and the stress induced in the leaves of the spring.
[11.25 mm, say 12 mm; 9.4, say 10; 47 mm, 197.5 MN/m^2.]

12.11 (B). Make a sketch of a leaf spring showing the shape to which the ends of the plate should be made and give the reasons for doing this. A leaf spring which carries a central load of 9 kN consists of plates each 75 mm wide and 7 mm thick. If the length of the spring is 1 m, determine the least number of plates required if the maximum stress owing to bending is limited to 210 MN/m^2 and the maximum deflection must not exceed 30 mm. Find, for the number of plates obtained, the actual values of the maximum stress and maximum deflection and also the radius to which the plates should be formed if they are to straighten under the given load.
[U.L.] [14; 200 MN/m^2, 29.98 mm; 4.2 m.]

12.12 (B). A semi-elliptic laminated carriage spring is 1 m long and 75 mm wide with leaves 10 mm thick. It has to carry a central load of 6 kN with a deflection of 25 mm. Working from first principles find (a) the number of leaves, (b) the maximum induced stress. [6; 200 MN/m^2.]

12.13 (B). A semi-elliptic leaf spring has a span of 720 mm and is built up of leaves 10 mm thick and 45 mm wide. Find the number of leaves required to carry a load of 5 kN at mid-span if the stress is not to exceed 225 MN/m^2, nor the deflection 12 mm. Calculate also the radius of curvature to which the spring must be initially bent if it must just flatten under the application of the above load. [7; 6.17 m.]

12.14 (B) An open-coiled helical spring has 10 coils of 12 mm diameter steel bar wound with a mean diameter of 150 mm. The helix angle of the coils is 32°. Find the axial extension produced by a load of 250 N. Any formulae used must be established by the application of fundamental principles relating to this type of spring.
[U.L.] [49.7 mm.]

12.15 (B). An open-coiled spring carries an axial load W. Show that the deflection is related to W by

$$\delta = \frac{8\,WnD^3}{Gd^4} \times K$$

where K is a correction factor which allows for the inclination of the coils, n = number of effective coils, D = mean coil diameter, and d = wire diameter.

A close-coiled helical spring is wound from 6 mm diameter steel wire into a coil having a mean diameter of 50 mm. If the spring has 20 effective turns and the maximum shearing stress is limited to 225 MN/m^2, what is the greatest safe deflection obtainable? [U.Birm.] [84.2 mm.]

12.16 (B/C). A flat spiral spring, as shown in Fig. 12.11, has the following dimensions: a = 150 mm, b = 25 mm, R = 80 mm. Determine the maximum value of the moment which can be applied to the spindle if the bending stress in the spring is not to exceed 150 MN/m^2. Through what angle does the spindle turn in producing this stress? The spring is constructed from steel strip 25 mm wide × 1.5 mm thick and has six turns. [0.75 N m, 48°.]

12.17 (B/C). A strip of steel of length 6 m, width 12 mm and thickness 2.5 mm is formed into a flat spiral around a spindle, the other end being attached to a fixed pin. Determine the couple which can be applied to the spindle if the maximum stress in the steel is limited to 300 MN/m^2. What will then be the energy stored in the spring?
[1.875 N m, 3.2 J.]

12.18 (B/C). A flat spiral spring is 12 mm wide, 0.3 mm thick and 2.5 m long. Assuming the maximum stress of 900 MN/m^2 to occur at the point of greatest bending moment, calculate the torque, the work stored and the number of turns to wind up the spring. [U.L.] [0.081, 1.45 J; 5.68.]