CHAPTER 6

BUILT-IN BEAMS

Summary

The maximum bending moments and maximum deflections for built-in beams with standard loading cases are as follows:

MAXIMUM B.M. AND DEFLECTION FOR BUILT-IN BEAMS

Loading case	Maximum B.M.	Maximum deflection
Central concentrated load W	$\dfrac{WL}{8}$	$\dfrac{WL^3}{192EI}$
Uniformly distributed load w/metre (total load W)	$\dfrac{wL^2}{12} = \dfrac{WL}{12}$	$\dfrac{wL^4}{384EI} = \dfrac{WL^3}{384EI}$
Concentrated load W not at mid-span	$\dfrac{Wab^2}{L^2}$ or $\dfrac{Wa^2b}{L^2}$	$\dfrac{2Wa^3b^2}{3EI(L+2a)^2}$ at $x = \dfrac{2aL}{(L+2a)}$ $\left(\text{where } a < \dfrac{L}{2}\right)$ $= \dfrac{Wa^3b^3}{3EIL^3}$ under load
Distributed load w' varying in intensity between $x = x_1$ and $x = x_2$	$M_A = -\displaystyle\int_{x_1}^{x_2} \frac{w'(L-x)^2}{L^2}dx$ $M_B = -\displaystyle\int_{x_1}^{x_2} \frac{w'(L-x)x^2}{L^2}dx$	

Effect of movement of supports

If one end B of an initially horizontal built-in beam AB moves through a distance δ relative to end A, end moments are set up of value

$$M_A = -M_B = \frac{6EI\delta}{L^2}$$

and the reactions at each support are

$$R_A = -R_B = \frac{12EI\delta}{L^3}$$

Thus, in most practical situations where loaded beams sink at the supports the above values represent *changes* in fixing moment and reaction values, their directions being indicated in Fig. 6.6.

Introduction

When both ends of a beam are rigidly fixed the beam is said to be *built-in*, *encastred* or *encastré*. Such beams are normally treated by a modified form of Mohr's area–moment method or by Macaulay's method.

Built-in beams are assumed to have zero slope at each end, so that the total change of slope along the span is zero. Thus, from Mohr's first theorem,

$$\text{area of } \frac{M}{EI} \text{ diagram across the span} = 0$$

or, if the beam is uniform, EI is constant, and

$$\textbf{area of B.M. diagram} = \mathbf{0} \qquad (6.1)$$

Similarly, if both ends are level the deflection of one end relative to the other is zero. Therefore, from Mohr's second theorem:

$$\text{first moment of area of } \frac{M}{EI} \text{ diagram about one end} = 0$$

and, if EI is constant,

$$\textbf{first moment of area of B.M. diagram about one end} = \mathbf{0} \qquad (6.2)$$

To make use of these equations it is convenient to break down the B.M. diagram for the built-in beam into two parts:

(a) that resulting from the loading, assuming simply supported ends, and known as the *free-moment diagram*;
(b) that resulting from the end moments or fixing moments which must be applied at the ends to keep the slopes zero and termed the *fixing-moment diagram.*

6.1. Built-in beam carrying central concentrated load

Consider the centrally loaded built-in beam of Fig. 6.1. A_a is the area of the free-moment diagram and A_b that of the fixing-moment diagram.

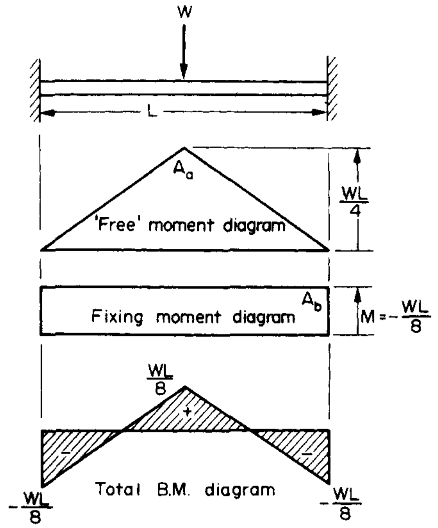

Fig. 6.1.

By symmetry the fixing moments are equal at both ends. Now from eqn. (6.1)

$$A_a + A_b = 0$$

$$\therefore \quad \tfrac{1}{2} \times L \times \frac{WL}{4} = -ML$$

$$\therefore \quad \boldsymbol{M = \frac{WL}{8}} \tag{6.3}$$

The B.M. diagram is therefore as shown in Fig. 6.1, the maximum B.M. occurring at both the ends and the centre.

Applying Mohr's second theorem for the deflection at mid-span,

$$\delta = \left[\begin{array}{c}\text{first moment of area of B.M. diagram between centre and}\\ \text{one end about the centre}\end{array}\right] \times \frac{1}{EI}$$

$$= \frac{1}{EI}\left[\frac{1}{2}\left(\frac{1}{2} \times \frac{WL}{4} \times L\right)\left(\frac{1}{3} \times \frac{L}{2}\right) + \left(\frac{ML}{2} \times \frac{L}{4}\right)\right]$$

$$= \frac{1}{EI}\left[\frac{WL^3}{96} + \frac{ML^2}{8}\right] = \frac{1}{EI}\left[\frac{WL^3}{96} - \frac{WL^3}{64}\right]$$

$$= -\boldsymbol{\frac{WL^3}{192EI}} \quad \text{(i.e. downward deflection)} \tag{6.4}$$

6.2. Built-in beam carrying uniformly distributed load across the span

Consider now the uniformly loaded beam of Fig. 6.2.

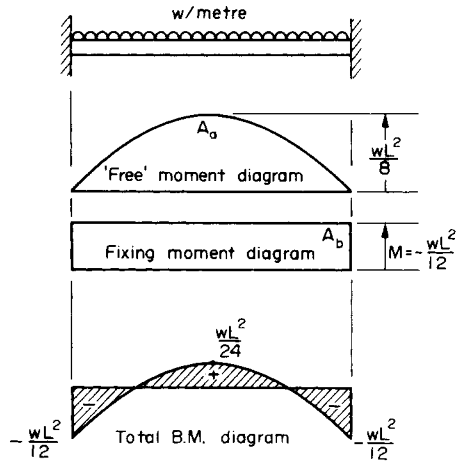

Fig. 6.2.

Again, for zero change of slope along the span,

$$A_a + A_b = 0$$

$$\therefore \qquad \frac{2}{3} \times \frac{wL^2}{8} \times L = -ML$$

$$\therefore \qquad M = \frac{wL^2}{12} \tag{6.5}$$

The deflection at the centre is again given by Mohr's second theorem as the moment of one-half of the B.M. diagram about the centre.

$$\therefore \qquad \delta = \left[\left(\frac{2}{3} \times \frac{wL^2}{8} \times \frac{L}{2}\right)\left(\frac{3}{8} \times \frac{L}{2}\right) + \left(\frac{ML}{2} \times \frac{L}{4}\right)\right]\frac{1}{EI}$$

$$= \frac{1}{EI}\left[\frac{3wL^4}{384} + \frac{ML^2}{8}\right] = \frac{1}{EI}\left[\frac{3wL^4}{384} - \frac{wL^4}{96}\right]$$

$$= -\frac{wL^4}{384EI} \tag{6.6}$$

The negative sign again indicates a downwards deflection.

6.3. Built-in beam carrying concentrated load offset from the centre

Consider the loaded beam of Fig. 6.3.

Since the slope at both ends is zero the change of slope across the span is zero, i.e. the total area between A and B of the B.M. diagram is zero (Mohr's theorem).

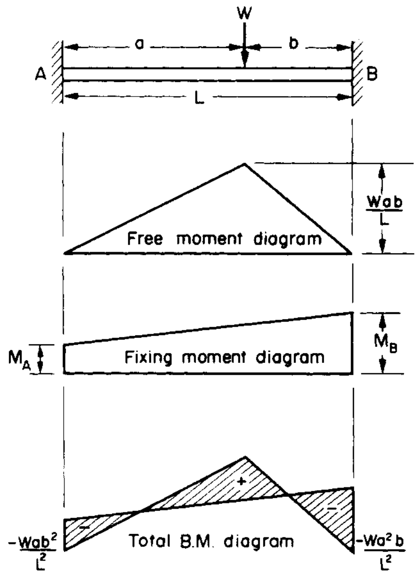

Fig. 6.3.

$$\therefore \qquad \left(\frac{1}{2} \times \frac{Wab}{L} \times L\right) + \tfrac{1}{2}(M_A + M_B)L = 0$$

$$M_A + M_B = -\frac{Wab}{L} \qquad (1)$$

Also the deflection of A relative to B is zero; therefore the moment of the B.M. diagram between A and B about A is zero.

$$\therefore \quad \left[\frac{1}{2} \times \frac{Wab}{L} \times a\right]\frac{2a}{3} + \left[\frac{1}{2} \times \frac{Wab}{L} \times b\right]\left(a + \frac{b}{3}\right) + \left(\tfrac{1}{2}M_A L \times \frac{L}{3}\right) + \left(\tfrac{1}{2}M_B L \times \frac{2L}{3}\right) = 0$$

$$\frac{L^2}{6}(M_A + 2M_B) + \frac{Wa^3b}{3L} + \frac{Wab^2}{3L}\left(a + \frac{b}{3}\right) = 0$$

$$M_A + 2M_B = -\frac{Wab}{L^3}[2a^2 + 3ab + b^2] \qquad (2)$$

Subtracting (1),

$$M_B = -\frac{Wab}{L^3}[2a^2 + 3ab + b^2 - L^2]$$

but $L = a + b$,

$$\therefore \qquad M_B = -\frac{Wab}{L^3}[2a^2 + 3ab + b^2 - a^2 - 2ab - b^2]$$

$$= -\frac{Wab}{L^3}[a^2 + ab] = -\frac{Wa^2bL}{L^3}$$

$$= -\frac{\boldsymbol{Wa^2b}}{\boldsymbol{L^2}} \qquad (6.7)$$

Substituting in (1),

$$M_A = -\frac{Wab}{L} + \frac{Wa^2b}{L^2}$$

$$= -\frac{Wab(a+b)}{L^2} + \frac{Wa^2b}{L^2}$$

$$= -\frac{Wab^2}{L^2} \tag{6.8}$$

6.4. Built-in beam carrying a non-uniform distributed load

Let w' be the distributed load varying in intensity along the beam as shown in Fig. 6.4. On a short length dx at a distance x from A there is a load of $w'dx$. Contribution of this load to M_A

$$= -\frac{Wab^2}{L^2} \quad (\text{where } W = w'dx)$$

$$= -\frac{w'dx \times x(L-x)^2}{L^2}$$

$$\therefore \qquad \text{total } M_A = -\int_0^L \frac{w'x(L-x)^2 dx}{L^2} \tag{6.9}$$

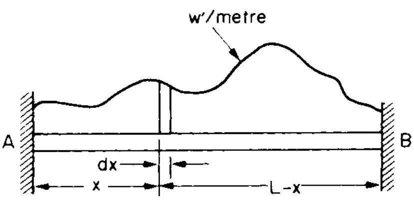

Fig. 6.4. Built-in (*encastré*) beam carrying non-uniform distributed load.

Similarly,

$$M_B = -\int_0^L \frac{w'(L-x)x^2}{L^2} dx \tag{6.10}$$

If the distributed load is across only part of the span the limits of integration must be changed to take account of this: i.e. for a distributed load w' applied between $x = x_1$ and $x = x_2$ and varying in intensity,

$$M_A = -\int_{x_1}^{x_2} \frac{w'x(L-x)^2}{L^2} dx \tag{6.11}$$

$$M_B = -\int_{x_1}^{x_2} \frac{w'(L-x)x^2}{L^2} dx \tag{6.12}$$

6.5. Advantages and disadvantages of built-in beams

Provided that perfect end fixing can be achieved, built-in beams carry smaller maximum B.M.s (and hence are subjected to smaller maximum stresses) and have smaller deflections than the corresponding simply supported beams with the same loads applied; in other words built-in beams are stronger and stiffer. Although this would seem to imply that built-in beams should be used whenever possible, in fact this is not the case in practice. The principal reasons are as follows:

(1) The need for high accuracy in aligning the supports and fixing the ends during erection increases the cost.
(2) Small subsidence of either support can set up large stresses.
(3) Changes of temperature can also set up large stresses.
(4) The end fixings are normally sensitive to vibrations and fluctuations in B.M.s, as in applications introducing rolling loads (e.g. bridges, etc.).

These disadvantages can be reduced, however, if hinged joints are used at points on the beam where the B.M. is zero, i.e. at *points of inflexion or contraflexure*. The beam is then effectively a central beam supported on two end cantilevers, and for this reason the construction is sometimes termed the *double-cantilever* construction. The beam is then free to adjust to changes in level of the supports and changes in temperature (Fig. 6.5).

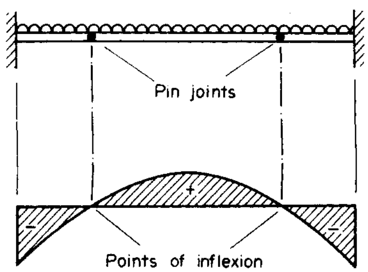

Fig. 6.5. Built-in beam using "double-cantilever" construction.

6.6. Effect of movement of supports

Consider a beam AB initially unloaded with its ends at the same level. If the slope is to remain horizontal at each end when B moves through a distance δ relative to end A, the moments must be as shown in Fig. 6.6. Taking moments about B

$$R_A \times L = M_A + M_B$$

and, by symmetry,
$$M_A = M_B = M$$

$\therefore$
$$R_A = \frac{2M}{L}$$

Similarly,
$$R_B = \frac{2M}{L}$$

in the direction shown.

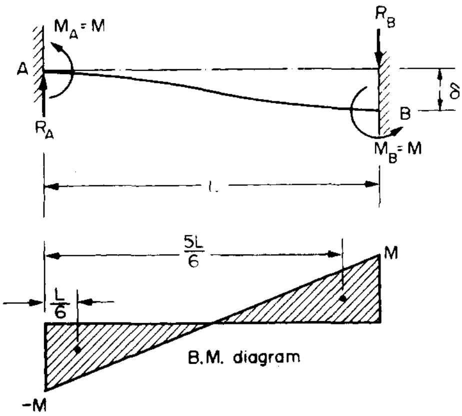

Fig. 6.6. Effect of support movement on B.M.s.

Now from Mohr's second theorem the deflection of A relative to B is equal to the first moment of area of the B.M. diagram about $A \times 1/EI$.

$$\therefore \quad \delta = \left[\left(-\tfrac{1}{2}M \times \frac{L}{2}\right)\frac{L}{6} + \left(\tfrac{1}{2}M \times \frac{L}{2}\right)\frac{5L}{6}\right]\frac{1}{EI}$$

$$= \frac{ML^2}{24EI}(-1+5) = \frac{ML^2}{6EI} \qquad (6.13)$$

$$\therefore \quad M = \frac{6EI\delta}{L^2} \quad \text{and} \quad R_A = R_B = \frac{12EI\delta}{L^3} \qquad (6.14)$$

in the directions shown in Fig. 6.6.

These values will also represent the *changes* in the fixing moments and end reactions for a beam under load when one end sinks relative to the other.

Examples

Example 6.1

An encastre beam has a span of 3 m and carries the loading system shown in Fig. 6.7. Draw the B.M. diagram for the beam and hence determine the maximum bending stress set up. The beam can be assumed to be uniform, with $I = 42 \times 10^{-6}$ m^4 and with an overall depth of 200 mm.

Solution

Using the *principle of superposition* the loading system can be reduced to the three cases for which the B.M. diagrams have been drawn, together with the fixing moment diagram, in Fig. 6.7.

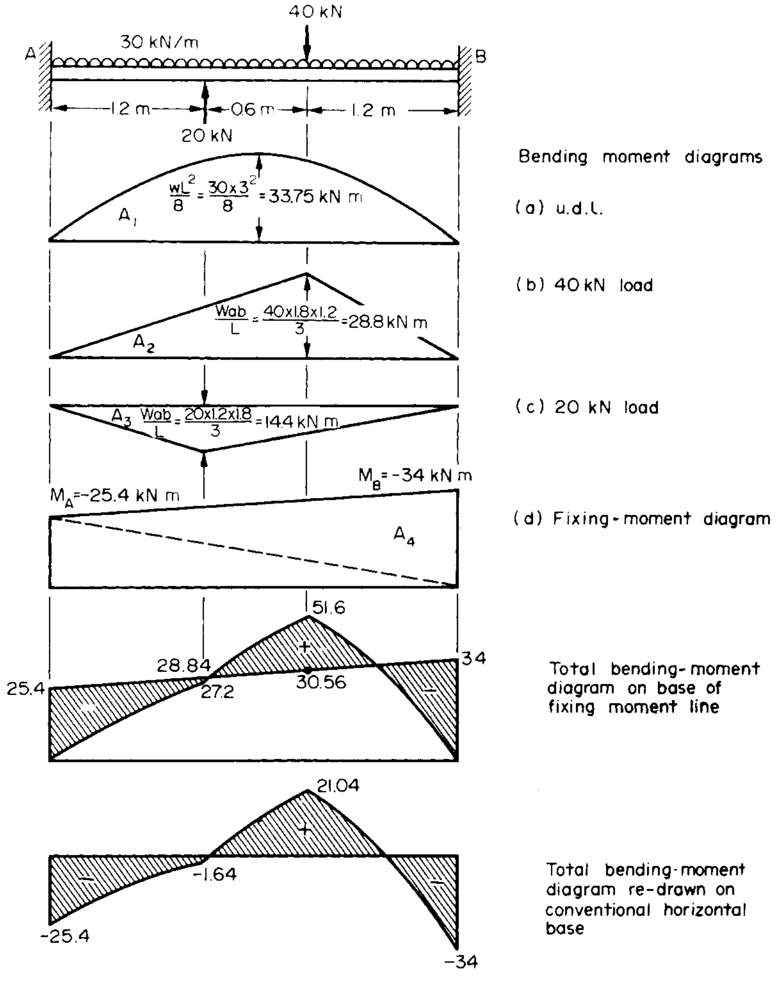

Fig. 6.7. Illustration of the application of the "principle of superposition" to Mohr's area–moment method of solution.

Now from eqn. (6.1)

$$A_1 + A_2 + A_4 = A_3$$

$$(\tfrac{2}{3} \times 33.75 \times 10^3 \times 3) + (\tfrac{1}{2} \times 28.8 \times 10^3 \times 3) + [\tfrac{1}{2}(M_A + M_B)3] = (\tfrac{1}{2} \times 14.4 \times 10^3 \times 3)$$

$$67.5 \times 10^3 + 43.2 \times 10^3 + 1.5(M_A + M_B) = 21.6 \times 10^3$$

$$M_A + M_B = -59.4 \times 10^3 \qquad (1)$$

Also, from eqn. (6.2), taking moments of area about A,

$$A_1\bar{x}_1 + A_2\bar{x}_2 + A_4\bar{x}_4 = A_3\bar{x}_3$$

and, dividing areas A_2 and A_4 into the convenient triangles shown,

$$(67.5\times10^3\times1.5)+(\tfrac{1}{2}\times28.8\times10^3\times1.8)\frac{2\times1.8}{3}+(\tfrac{1}{2}\times28.8\times10^3\times1.2)(1.8+\tfrac{1}{3}\times1.2)$$

$$+(\tfrac{1}{2}M_A\times3\times\tfrac{1}{3}\times3)+(\tfrac{1}{2}M_B\times3\times\tfrac{2}{3}\times3)=(\tfrac{1}{2}\times14.4\times10^3\times1.2)\tfrac{2}{3}\times1.2$$

$$+(\tfrac{1}{2}\times14.4\times10^3\times1.8)\left(1.2+\frac{1.8}{3}\right)$$

$$(101.25+31.1+38.0)10^3+1.5M_A+3M_B=(6.92+23.3)10^3$$

$$1.5\,M_A+3M_B=-140\times10^3$$

$\therefore$
$$M_A+2M_B=-93.4\times10^3 \qquad (2)$$

(2) − (1),
$$M_B=-34\times10^3\,\text{N m}=-34\,\text{kN m}$$

and from (1),
$$M_A=-25.4\times10^3\,\text{N m}=-25.4\,\text{kN m}$$

The fixing moments are therefore negative and not positive as assumed in Fig. 6.7. The total B.M. diagram is then found by combining all the separate loading diagrams and the fixing moment diagram to produce the result shown in Fig. 6.7. It will be seen that the maximum B.M. occurs at the built-in end B and has a value of 34 kN m. This will therefore be the position of the maximum bending stress also, the value being determined from the simple bending theory

$$\sigma_{max}=\frac{My}{I}=\frac{34\times10^3\times100\times10^{-3}}{42\times10^{-6}}$$

$$=81\times10^6=\mathbf{81\ MN/m^2}$$

Example 6.2

A built-in beam, 4 m long, carries combined uniformly distributed and concentrated loads as shown in Fig. 6.8. Determine the end reactions, the fixing moments at the built-in supports and the magnitude of the deflection under the 40 kN load. Take EI = 14 MN m^2.

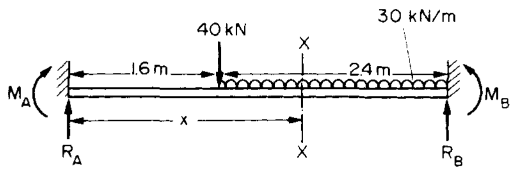

Fig. 6.8.

Solution

Using Macaulay's method (see page 106)

$$M_{xx}=\frac{EI}{10^3}\frac{d^2y}{dx^2}=M_A+R_Ax-40[x-1.6]-30\left[\frac{(x-1.6)^2}{2}\right]$$

Note that the unit of load of kilonewton is conveniently accounted for by dividing EI by 10^3. It can then be assumed in further calculation that R_A is in kN and M_A in kN m.

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = M_A x + R_A\frac{x^2}{2} - \frac{40}{2}[(x-1.6)^2] - \frac{30}{6}[(x-1.6)^3] + A$$

and

$$\frac{EI}{10^3}y = M_A\frac{x^2}{2} + R_A\frac{x^3}{6} - \frac{40}{6}[(x-1.6)^3] - \frac{30}{24}[(x-1.6)^4] + Ax + B$$

Now, when $x = 0$, $y = 0$ $\quad\therefore\quad B = 0$

and when $x = 0$, $\dfrac{dy}{dx} = 0$ $\quad\therefore\quad A = 0$

When $x = 4$, $y = 0$

$$0 = M_A \times \frac{4^2}{2} + R_A \times \frac{4^3}{6} - \frac{40}{6}(2.4)^3 - \frac{30}{24}(2.4)^4$$

$$0 = 8M_A + 10.67\,R_A - 92.16 - 41.47$$

$$133.6 = 8M_A + 10.67\,R_A \qquad (1)$$

When $x = 4$, $\dfrac{dy}{dx} = 0$

$$\therefore \quad 0 = 4M_A + \frac{4^2}{2}R_A - \frac{40}{2}(2.4)^2 - \frac{30}{6}(2.4)^3$$

$$0 = 4M_A + 8R_A - 115.2 - 69.12$$

$$184.32 = 4M_A + 8R_A \qquad (2)$$

Multiply (2) × 2,

$$368.64 = 8M_A + 16R_A \qquad (3)$$

(3) − (1),

$$235.04 = 5.33\,R_A$$

$$R_A = \frac{235.04}{5.33} = \mathbf{44.1\,kN}$$

Now $\quad R_A + R_B = 40 + (2.4 \times 30) = 112\,\text{kN}$

$$\therefore \quad R_B = 112 - 44.1 = \mathbf{67.9\,kN}$$

Substituting in (2),

$$4M_A + 352.8 = 184.32$$

$$\therefore \quad M_A = \tfrac{1}{4}(184.32 - 352.8) = \mathbf{-42.12\,kN\,m}$$

i.e. M_A is in the opposite direction to that assumed in Fig. 6.8.

Taking moments about A,

$$M_B + 4R_B - (40 \times 1.6) - (30 \times 2.4 \times 2.8) - (-42.12) = 0$$

$$\therefore \quad M_B = -(67.9 \times 4) + 64 + 201.6 - 42.12 = \mathbf{-48.12\,kN\,m}$$

i.e. again in the opposite direction to that assumed in Fig. 6.8.

(Alternatively, and more conveniently, this value could have been obtained by substitution into the original Macaulay expression with $x = 4$, which is, in effect, taking moments about B. The need to take additional moments about A is then overcome.)

Substituting into the Macaulay deflection expression,

$$\frac{EI}{10^3}y = -42.1\frac{x^2}{2} + \frac{44.1x^3}{6} - \frac{20}{3}[x-1.6]^3 - \tfrac{5}{4}[x-1.6]^4$$

Thus, under the 40 kN load, where $x = 1.6$ (and neglecting negative Macaulay terms),

$$y = \frac{10^3}{EI}\left[\frac{-(42.1 \times 2.56)}{2} + \frac{(44.1 \times 4.1)}{6} - 0 - 0\right]$$

$$= -\frac{23.75 \times 10^3}{14 \times 10^6} = -1.7 \times 10^{-3}\,\text{m}$$

$$= \mathbf{-1.7\,mm}$$

The negative sign as usual indicates a deflection downwards.

Example 6.3

Determine the fixing moment at the left-hand end of the beam shown in Fig. 6.9 when the load varies linearly from 30 kN/m to 60 kN/m along the span of 4 m.

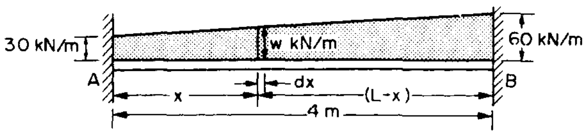

Fig. 6.9.

Solution

From §6.4

$$M_A = -\int_0^L \frac{w'x(L-x)^2}{L^2}\,dx$$

Now $$w' = \left(30 + \frac{30x}{4}\right)10^3 = (30 + 7.5x)10^3\ \text{N/m}$$

$$\therefore \qquad M_A = -\int_0^4 \frac{(30+7.5x)10^3(4-x)^2}{4^2}\,x\,dx$$

$$= -\frac{10^3}{16}\int_0^4 (30+7.5x)(16-8x+x^2)x\,dx$$

$$= -\frac{10^3}{16}\int_0^4 (480x-240x^2+30x^3+120x^2-60x^3+7.5x^4)\,dx$$

$$= -\frac{10^3}{16}\int_0^4 (480x-120x^2-30x^3+7.5x^4)\,dx$$

$$= -\frac{10^3}{16}\left[\frac{480x^2}{2}-\frac{120x^3}{3}-\frac{30x^4}{4}+\frac{7.5x^5}{5}\right]_0^4$$

$$= -\frac{10^3}{16}[240\times 16-40\times 64-30\times 64+2.5\times 1024]$$

$$= -\mathbf{120\times 10^3\,N\,m}$$

The required moment at A is thus 120 kN m in the opposite direction to that shown in Fig. 6.8.

Problems

6.1 (A/B). A straight beam $ABCD$ is rigidly built-in at A and D and carries point loads of 5 kN at B and C.

$$AB = BC = CD = 1.8\,\text{m}$$

If the second moment of area of the section is $7\times 10^{-6}\,\text{m}^4$ and Young's modulus is $210\,\text{GN/m}^2$, calculate:

(a) the end moments;
(b) the central deflection of the beam. [U.Birm.] [−6 kN m; 4.13 mm.]

6.2 (A/B). A beam of uniform section with rigidly fixed ends which are at the same level has an effective span of 10 m. It carries loads of 30 kN and 50 kN at 3 m and 6 m respectively from the left-hand end. Find the vertical reactions and the fixing moments at each end of the beam. Determine the bending moments at the two points of loading and sketch, approximately to scale, the B.M. diagram for the beam.
[41.12, 38.88 kN; −92, −90.9, 31.26, 64.62 kN m.]

6.3 (A/B). A beam of uniform section and of 7 m span is "fixed" horizontally at the same level at each end. It carries a concentrated load of 100 kN at 4 m from the left-hand end. Neglecting the weight of the beam and working from first principles, find the position and magnitude of the maximum deflection if $E = 210\,\text{GN/m}^2$ and $I = 190\times 10^{-6}\,\text{m}^4$. [3.73 from l.h. end; 4.28 mm.]

6.4 (A/B). A uniform beam, built-in at each end, is divided into four equal parts and has equal point loads, each W, placed at the centre of each portion. Find the deflection at the centre of this beam and prove that it equals the deflection at the centre of the same beam when carrying an equal total load uniformly distributed along the entire length.

[U.C.L.I.] $\left[\dfrac{WL^3}{96EI}.\right]$

6.5 (A/B). A horizontal beam of I-section, rigidly built-in at the ends and 7 m long, carries a total uniformly distributed load of 90 kN as well as a concentrated central load of 30 kN. If the bending stress is limited to 90 MN/m^2 and the deflection must not exceed 2.5 mm, find the depth of section required. Prove the deflection formulae if used, or work from first principles. $E = 210\,\text{GN/m}^2$. [U.L.C.I.] [583 mm.]

6.6 (A/B). A beam of uniform section is built-in at each end so as to have a clear span of 7 m. It carries a uniformly distributed load of 20 kN/m on the left-hand half of the beam, together with a 120 kN load at 5 m from the left-hand end. Find the reactions and the fixing moments at the ends and draw a B.M. diagram for the beam, inserting the principal values. [U.L.] [-105.4, -148 kN; 80.7, 109.3 kN m.]

6.7 (A/B). A steel beam of 10 m span is built-in at both ends and carries two point loads, each of 90 kN, at points 2.6 m from the ends of the beam. The middle 4.8 m has a section for which the second moment of area is $300 \times 10^{-6}\,\text{m}^4$ and the 2.6 m lengths at either end have a section for which the second moment of area is $400 \times 10^{-6}\,\text{m}^4$. Find the fixing moments at the ends and calculate the deflection at mid-span. Take $E = 210\,\text{GN/m}^2$ and neglect the weight of the beam. [U.L.] [$M_A = M_B = 173.2$ kN m; 8.1 mm.]

6.8 (B.) A loaded horizontal beam has its ends securely built-in; the clear span is 8 m and $I = 90 \times 10^{-6}\,\text{m}^4$. As a result of subsidence one end moves vertically through 12 mm. Determine the changes in the fixing moments and reactions. For the beam material $E = 210\,\text{GN/m}^2$. [21.26 kN m; 5.32 kN.]