CHAPTER 3

STRAINS BEYOND THE ELASTIC LIMIT

Summary

For rectangular-sectioned beams strained up to and beyond the elastic limit, i.e. for *plastic bending*, the bending moments (B.M.) which the beam can withstand at each particular stage are:

maximum elastic moment $$M_E = \frac{BD^2}{6}\sigma_y$$

partially plastic moment $$M_{PP} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

fully plastic moment $$M_{FP} = \frac{BD^2}{4}\sigma_y$$

where σ_y is the stress at the elastic limit, or *yield stress*.

$$\text{Shape factor } \lambda = \frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

For **I-section beams**:

$$M_E = \sigma_y \left[\frac{BD^3}{12} - \frac{bd^3}{12}\right]\frac{2}{D}$$

$$M_{FP} = \sigma_y \left[\frac{BD^2}{4} - \frac{bd^2}{4}\right]$$

The position of the neutral axis (N.A.) for fully plastic unsymmetrical sections is given by:

$$\text{area of section above or below N.A.} = \tfrac{1}{2} \times \text{ total area of cross-section}$$

Deflections of partially plastic beams are calculated on the basis of the elastic areas only.

In plastic limit or ultimate collapse load procedures the normal elastic safety factor is replaced by a load factor as follows:

$$\text{load factor} = \frac{\text{collapse load}}{\text{allowable working load}}$$

For **solid shafts**, radius R, strained up to and beyond the elastic limit in shear, i.e. for *plastic torsion*, the torques which can be transmitted at each stage are

maximum elastic torque $$T_E = \frac{\pi R^3}{2}\tau_y$$

partially plastic torque $$T_{PP} = \frac{\pi\tau_y}{6}[4R^3 - R_1^3] \quad \text{(yielding to radius } R_1\text{)}$$

fully plastic torque $$T_{FP} = \frac{2\pi R^3}{3}\tau_y$$

where τ_y is the shear stress at the elastic limit, or shear yield stress. Angles of twist of partially plastic shafts are calculated on the basis of the elastic core only.

For **hollow shafts**, inside radius R_1, outside radius R yielded to radius R_2,

$$T_{PP} = \frac{\pi\tau_y}{6R_2}[4R^3R_2 - R_2^4 - 3R_1^4]$$

$$T_{FP} = \frac{2\pi\tau_y}{3}[R^3 - R_1^3]$$

For **eccentric loading** of rectangular sections the fully plastic moment is given by

$$M_{FP} = \frac{BD^2}{4}\sigma_y - \frac{P^2N^2}{4B\sigma_y}$$

where P is the axial load, N the load factor and B the width of the cross-section.

The maximum allowable moment is then given by

$$M = \frac{BD^2}{4N}\sigma_y - \frac{P^2N}{4B\sigma_y}$$

For a **solid rotating disc**, radius R, the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho R^2}$$

where ρ is the density of the disc material.

For **rotating hollow discs** the collapse speed is found from

$$\omega_p^2 = \frac{3\sigma_y}{\rho}\left[\frac{R_2 - R_1}{R_2^3 - R_1^3}\right]$$

Introduction

When the design of components is based upon the elastic theory, e.g. the simple bending or torsion theory, the dimensions of the components are arranged so that the maximum stresses which are likely to occur under service loading conditions do not exceed the allowable working stress for the material in either tension or compression. The allowable working stress is taken to be the yield stress of the material divided by a convenient safety factor (usually based on design codes or past experience) to account for unexpected increase in the level of service loads. If the maximum stress in the component is likely to exceed the allowable working stress, the component is considered unsafe, yet it is evident that complete failure of the component is unlikely to occur even if the yield stress is reached at the outer fibres provided that some portion of the component remains elastic and capable of carrying load, i.e. the strength of a component will normally be much greater than that assumed on the basis of initial yielding at any position. To take advantage of the inherent additional strength, therefore, a different design procedure is used which is often referred to as *plastic limit design*. The revised design procedures are based upon a number of basic assumptions about the material behaviour.

Figure 3.1 shows a typical stress–strain curve for annealed low carbon steel indicating the presence of both upper and lower yield points and strain-hardening characteristics.

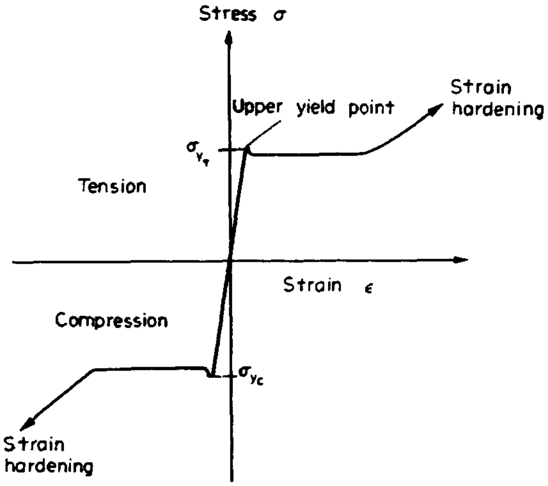

Fig. 3.1. Stress–strain curve for annealed low-carbon steel indicating upper and lower yield points and strain-hardening characteristics.

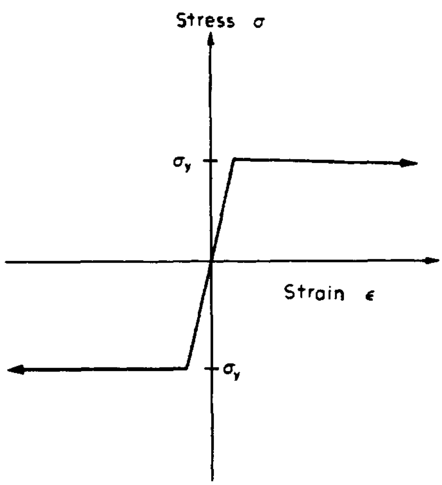

Fig. 3.2. Assumed stress–curve for plastic theory – no strain-hardening, equal yield points, $\sigma_{y_t} = \sigma_{y_c} = \sigma_y$.

Figure 3.2 shows the assumed material behaviour which:

(a) ignores the presence of upper and lower yields and suggests only a single yield point;
(b) takes the yield stress in tension and compression to be equal;

(c) assumes that yielding takes place at constant strain thereby ignoring any strain-hardening characteristics. Thus, once the material has yielded, stress is assumed to remain constant throughout any further deformation.

It is further assumed, despite assumption (c), that transverse sections of beams in bending remain plane throughout the loading process, i.e. strain is proportional to distance from the neutral axis.

It is now possible on the basis of the above assumptions to determine the moment which must be applied to produce:

(a) the maximum or limiting elastic conditions in the beam material with yielding just initiated at the outer fibres;

(b) yielding to a specified depth;

(c) yielding across the complete section.

The latter situation is then termed a fully plastic state, or *"plastic hinge"*. Depending on the support and loading conditions, one or more plastic hinges may be required before complete collapse of the beam or structure occurs, the load required to produce this situation then being termed the *collapse load*. This will be considered in detail in §3.6.

3.1. Plastic bending of rectangular-sectioned beams

Figure 3.3(a) shows a rectangular beam loaded until the yield stress has just been reached in the outer fibres. The beam is still completely elastic and the bending theory applies, i.e.

$$M = \frac{\sigma I}{y}$$

$$\therefore \quad \text{maximum elastic moment} = \sigma_y \times \frac{BD^3}{12} \times \frac{2}{D}$$

$$M_E = \frac{BD^2}{6}\sigma_y \tag{3.1}$$

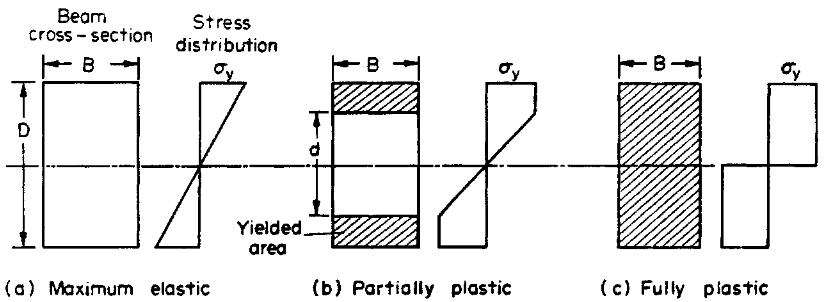

Fig. 3.3. Plastic bending of rectangular-section beam.

If loading is then increased, it is assumed that instead of the stress at the outside increasing still further, more and more of the section reaches the yield stress σ_y. Consider the stage shown in Fig. 3.3(b).

Partially plastic moment,

$$M_{PP} = \text{moment of elastic portion} + \text{total moment of plastic portion}$$

$$\therefore \quad M_{PP} = \frac{Bd^2}{6}\sigma_y + 2\left\{\underset{\text{stress}}{\sigma_y} \times \underset{\text{area}}{B\left[\frac{D}{2} - \frac{d}{2}\right]} \underset{\text{moment arm}}{\left[\frac{1}{2}\left(\frac{D}{2} - \frac{d}{2}\right) + \frac{d}{2}\right]}\right\}$$

$$M_{PP} = \sigma_y\left[\frac{Bd^2}{6} + \frac{B}{4}(D-d)(D+d)\right]$$

$$= \frac{B\sigma_y}{12}[2d^2 + 3(D^2 - d^2)] = \frac{\boldsymbol{B\sigma_y}}{\mathbf{12}}[\mathbf{3}\boldsymbol{D}^2 - \boldsymbol{d}^2] \tag{3.2}$$

When loading has been continued until the stress distribution is as in Fig. 3.3(c) (assumed), the beam with collapse. The moment required to produce this fully plastic state can be obtained from eqn. (3.2), since d is then zero,

i.e.

$$\text{fully plastic moment, } M_{FP} = \frac{B\sigma_y}{12} \times 3D^2 = \frac{\boldsymbol{BD^2}}{\mathbf{4}}\boldsymbol{\sigma_y} \tag{3.3}$$

This is the moment therefore which produces a plastic hinge in a rectangular-section beam.

3.2. Shape factor – symmetrical sections

The shape factor is defined as the ratio of the moments required to produce fully plastic and maximum elastic states:

$$\textbf{shape factor } \boldsymbol{\lambda} = \frac{\mathbf{M_{FP}}}{\mathbf{M_E}} \tag{3.4}$$

It is a factor which gives a measure of the increase in strength or load-carrying capacity which is available beyond the normal elastic design limits for various shapes of section, e.g. for the *rectangular section* above,

$$\text{shape factor } = \frac{BD^2}{4}\sigma_y \Big/ \frac{BD^2}{6}\sigma_y = \mathbf{1.5}$$

Thus rectangular-sectioned beams can carry 50% additional moment to that which is required to produce initial yielding at the edge of the beam section before a fully plastic hinge is formed. (It will be shown later that even greater strength is available beyond this stage depending on the support conditions used.) It must always be remembered, however, that should the stresses exceed the yield at any time during service there will be some associated *permanent set* or deflection when load is removed, and consideration should be given to whether or not this is acceptable. Bearing in mind, however, that normal design office practice involves the use of a safety factor to take account of abnormalities of loading, it should be evident that even at this stage considerable advantages are obtained by application of this factor to the fully plastic condition rather than the limiting elastic case. It is then possible to arrange for all normal loading situations to be associated with elastic stresses in the beam, the additional strength in the partially plastic condition being used as the safety margin to take account of unexpected load increases.

Figure 3.4 shows the way in which moments build up with increasing depth or penetration of yielding and associated radius of curvature as the beam bends.

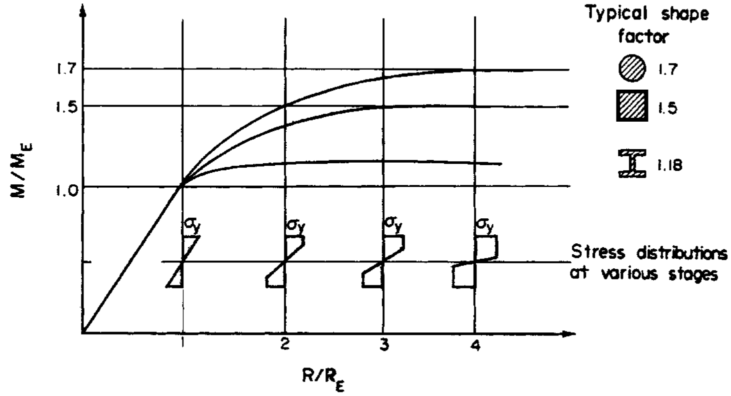

Fig. 3.4. Variation of moment of resistance of beams of various cross-section with depth of plastic penetration and associated radius of curvature.

Here the moment M carried by the beam at any particular stage and its associated radius of curvature R are considered as ratios of the values at the maximum elastic or initial yield condition. It will be noticed that at large curvature ratios, i.e. high plastic penetrations, the values of M/M_E approach the shape factor of the sections indicated, e.g. 1.5 for the rectangular section.

Shape factors of other symmetrical sections such as the I-section beam are found as follows (Fig. 3.5).

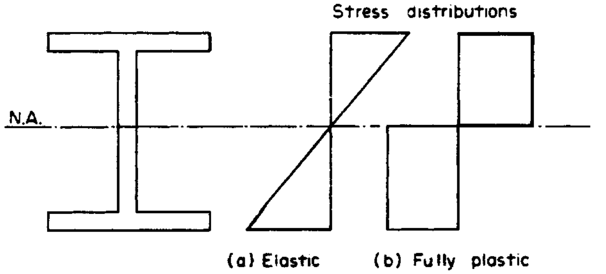

Fig. 3.5. Plastic bending of symmetrical (I-section) beam.

First determine the value of the maximum elastic moment M_E by applying the simple bending theory

$$\frac{M}{I} = \frac{\sigma}{y}$$

with y the maximum distance from the N.A. (the axis of symmetry passing through the centroid) to an outside fibre and $\sigma = \sigma_y$, the yield stress.

Then, in the fully plastic condition, the stress will be uniform across the section at σ_y and the section can be divided into any convenient number of rectangles of area A and centroid distance h from the neutral axis.

Then
$$M_{FP} = \sum(\sigma_y A)h \tag{3.5}$$

The shape factor M_{FP}/M_E can then be determined.

3.3. Application to I-section beams

When the B.M. applied to an I-section beam is just sufficient to initiate yielding in the extreme fibres, the stress distribution is as shown in Fig. 3.5(a) and the value of the moment is obtained from the simple bending theory by subtraction of values for convenient rectangles.

i.e.
$$M_E = \frac{\sigma I}{y}$$
$$= \sigma_y \left[\frac{BD^3}{12} - \frac{bd^3}{12}\right]\frac{2}{D}$$

If the moment is then increased to produce full plasticity across the section, i.e. a plastic hinge, the stress distribution is as shown in Fig. 3.5(b) and the value of the moment is obtained by applying eqn. (3.3) to the same convenient rectangles considered above.

$$M_{FP} = \sigma_y \left[\frac{BD^2}{4} - \frac{bd^2}{4}\right]$$

The value of the shape factor can then be obtained as the ratio of the above equations M_{FP}/M_E. A typical value of shape factor for commercial rolled steel joists is 1.18, thus indicating only an 18% increase in "strength" capacity using plastic design procedures compared with the 50% of the simple rectangular section.

3.4. Partially plastic bending of unsymmetrical sections

Consider the T-section beam shown in Fig. 3.6. Whilst stresses remain within the elastic limit the position of the N.A. can be obtained in the usual way by taking moments of area

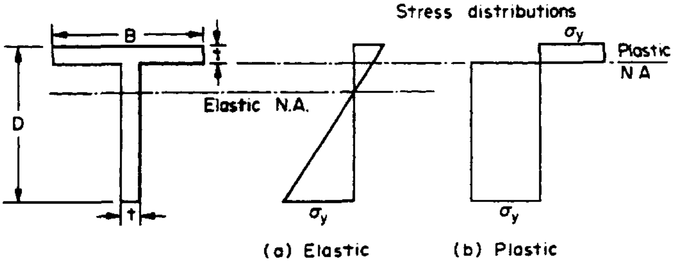

Fig. 3.6. Plastic bending of unsymmetrical (T-section) beam.

about some convenient axis as described in Chapter 4.† A typical position of the elastic N.A. is shown in the figure. Application of the simple blending theory about the N.A. will then yield the value of M_E as described in the previous paragraph.

Whatever the state of the section, be it elastic, partially plastic or fully plastic, equilibrium of forces must always be maintained, i.e. at any section the tensile forces on one side of the N.A. must equal the compressive forces on the other side.

$$\sum \text{stress} \times \text{area above N.A.} = \sum \text{stress} \times \text{area below N.A.}$$

In the *fully plastic* condition, therefore, when the stress is equal throughout the section, the above equation reduces to

$$\sum \textbf{areas above N.A.} = \sum \textbf{areas below N.A.} \tag{3.6}$$

and in the special case shown in Fig. 3.5 the N.A. will have moved to a position coincident with the lower edge of the flange. Whilst this position is peculiar to the particular geometry chosen for this section it is true to say that for all unsymmetrical sections the N.A. will move from its normal position when the section is completely elastic as plastic penetration proceeds. In the ultimate stage when a plastic hinge has been formed the N.A. will be positioned such that eqn. (3.6) applies, or, often more conveniently,

$$\textbf{area above or below N.A.} = \tfrac{1}{2}\ \textbf{total area} \tag{3.7}$$

In the partially plastic state, as shown in Fig. 3.7, the N.A. position is again determined by applying equilibrium conditions to the forces above and below the N.A. The section is divided into convenient parts, each subjected to a force = average stress × area, as indicated, then

$$F_1 + F_2 = F_3 + F_4 \tag{3.8}$$

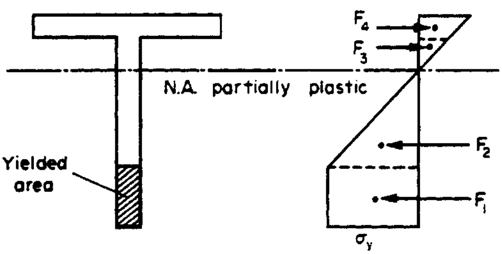

Fig. 3.7. Partially plastic bending of unsymmetrical section beam.

and this is an equation in terms of a single unknown $\bar{y}_p$, which can then be determined, as can the independent values of F_1, F_2, F_3 and F_4.

The sum of the moments of these forces about the N.A. then yields the value of the partially plastic moment M_{PP}. Example 3.2 describes the procedure in detail.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

3.5. Shape factor – unsymmetrical sections

Whereas with symmetrical sections the position of the N.A. remains constant as the axis of symmetry through the centroid, in the case of unsymmetrical sections additional work is required to take account of the movement of the N.A. position. However, having determined the position of the N.A. in the fully plastic condition using eqn. (3.6) or (3.7), the procedure outlined in §3.2 can then be followed to evaluate shape factors of unsymmetrical sections – see Example 3.2.

3.6. Deflections of partially plastic beams

Deflections of partially plastic beams are normally calculated on the assumption that the yielded areas, having yielded, offer no resistance to bending. Deflections are calculated therefore on the basis of the elastic core only, i.e. by application of simple bending theory and/or the standard deflection equations of Chapter 5† to the elastic material only. Because the second moment of area I of the central core is proportional to the fourth power of d, and I appears in the denominator of deflection formulae, deflections increase rapidly as d approaches zero, i.e. as full plasticity is approached.

If an experiment is carried out to measure the deflection of beams as loading, and hence B.M., is increased, the deflection graph for simply supported end conditions will appear as shown in Fig. 3.8. Whilst the beam is elastic the graph remains linear. The initiation of yielding in the outer fibres of the beam is indicated by a slight change in slope, and when plastic penetration approaches the centre of the section deflections increase rapidly for very small increases in load. For rectangular sections the ratio M_{FP}/M_E will be 1.5 as determined theoretically above.

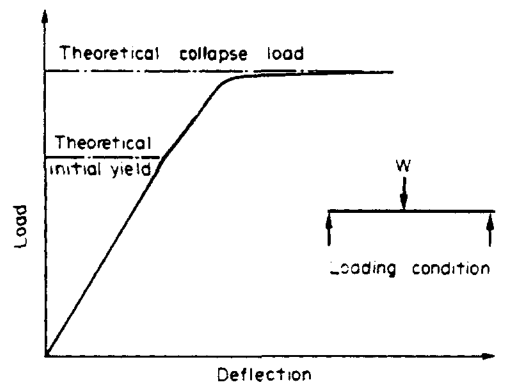

Fig. 3.8. Typical load-deflection curve for plastic bending.

3.7. Length of yielded area in beams

Consider a simply supported beam of rectangular section carrying a central concentrated load W. The B.M. diagram will be as shown in Fig. 3.9 with a maximum value of $WL/4$ at

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

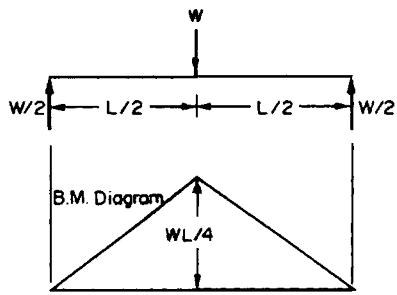

Fig. 3.9.

the centre. If loading is increased, yielding will commence therefore at the central section when $(WL/4) = (BD^2/6)\sigma_y$ and will gradually penetrate from the outside fibres towards the N.A. As this proceeds with further increase in loads, the B.M. at points away from the centre will also increase, and in some other positions near the centre it will also reach the value required to produce the initial yielding, namely $BD^2\sigma_y/6$. Thus, when full plasticity is achieved at the central section with a load W_p, there will be some other positions on either side of the centre, distance x from the supports, where yielding has just commenced at the outer fibres; between these two positions the beam will be in some elastic–plastic state. Now at distance x from the supports:

$$\text{B.M.} = W_p\frac{x}{2} = \frac{2}{3}M_{FP} = \frac{2}{3}\frac{W_pL}{4}$$

$$\therefore \qquad x = \frac{L}{3}$$

The central third of the beam span will be affected therefore by plastic yielding to some depth. At any general section within this part of the beam distance x' from the supports the B.M. will be given by

$$\text{B.M.} = W_p\frac{x'}{2} = \frac{B\sigma_y}{12}[3D^2 - d^2] \qquad (1)$$

Now since $$\frac{BD^2}{4}\sigma_y = W_p\frac{L}{4} \qquad \sigma_y = \frac{W_pL}{BD^2}$$

Therefore substituting in (1),

$$W_p\frac{x'}{2} = \frac{B}{12}[3D^2 - d^2]\frac{W_pL}{BD^2}$$

$$x' = \frac{(3D^2 - d^2)}{6D^2}L$$

$$x' = \frac{L}{2}\left[1 - \frac{d^2}{3D^2}\right]$$

This is the equation of a parabola with

$$x' = L/2 \text{ when } d = 0 \text{ (i.e. fully plastic section)}$$

and $x' = L/3$ when $d = D$ (i.e. section elastic)

The yielded portion of the beam is thus as indicated in Fig. 3.10.

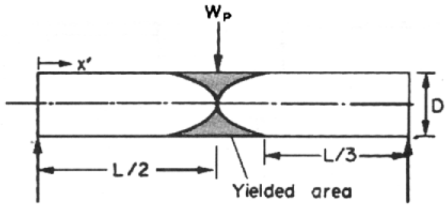

Fig. 3.10. Yielded area in beam carrying central point load.

Other beam support and loading cases may be treated similarly. That for a simply supported beam carrying a uniformly distributed load produces linear boundaries to the yielded areas as shown in Fig. 3.11.

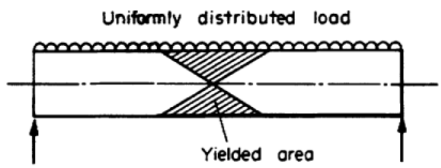

Fig. 3.11. Yielded area in beam carrying uniformly distributed load.

3.8. Collapse loads – plastic limit design

Having determined the moment required to produce a plastic hinge for the shape of beam cross-section used in any design it is then necessary to decide from a knowledge of the support and loading conditions how many such hinges are required before complete collapse of the beam or structure takes place, and to calculate the corresponding load. Here it is necessary to consider a plastic hinge as a pin-joint and to decide how many pin-joints are required to convert the structure into a "mechanism". If there are a number of points of "local" maximum B.M., i.e. peaks in the B.M. diagram, the first plastic hinge will form at the numerical maximum; if further plastic hinges are required these will occur successively at the next highest value of maximum B.M. etc. It is assumed that when a plastic hinge has developed at any cross-section the moment of resistance at that point remains constant until collapse of the whole structure takes place owing to the formation of the required number of further plastic hinges.

Consider, therefore, the following loading cases.

(a) Simply supported beam or cantilever

Whatever the loading system there will only be one point of maximum B.M. and plastic collapse will occur with **one** plastic hinge at this point (Fig. 3.12).

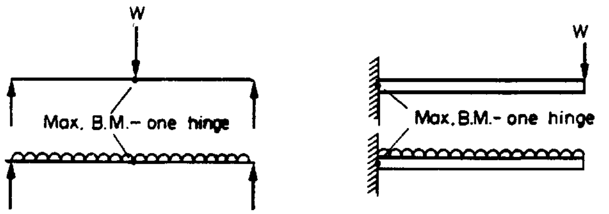

Fig. 3.12.

(b) Propped cantilever

In the case of propped cantilevers, i.e. cantilevers carrying opposing loads, the B.M. diagram is as shown in Fig. 3.13. The maximum B.M. then occurs at the built-in support and a plastic hinge forms first at this position. Due to the support of the prop, however, the beam does not collapse at this stage but requires another plastic hinge before complete failure or collapse occurs. This is formed at the other local position of maximum B.M., i.e. at the prop position, the moments at the support remaining constant until that at the prop also reaches the value required to form a plastic hinge.

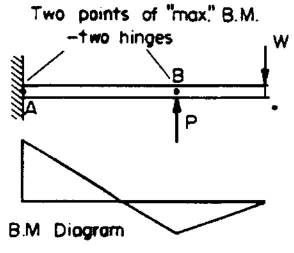

Fig. 3.13.

Collapse therefore occurs when $M_A = M_B = M_{FP}$, and thus **two** plastic hinges are required.

(c) Built-in beam

In this case there are three positions of local maximum **B.M.**, two of them being at the supports, and **three** plastic hinges are required for collapse (Fig. 3.14).

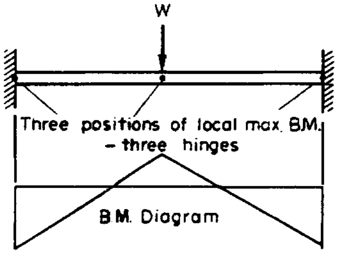

Fig. 3.14.

Other structures may require even more plastic hinges depending on their particular support conditions and degree of redundancy, but these need not be considered here. It should be evident, however, that there is now even more strength or load-carrying capacity available beyond that suggested by the shape factor, i.e. with a knowledge of the yield stress and hence the maximum elastic moment for any particular cross-section, the shape factor determines the increase in moment required to produce a fully plastic section or plastic hinge; depending on the support and loading conditions it may then be possible to increase the moment beyond this value until a sufficient number of plastic hinges have been formed to produce complete collapse. In order to describe the increased strength available using this "plastic limit" or "collapse load" procedure a *load factor* is introduced defined as

$$\textbf{load factor} = \frac{\textbf{collapse load}}{\textbf{allowable working load}} \tag{3.9}$$

This is completely different from, and must not be confused with, the safety factor, which is a factor to be applied to the yield stress in simple *elastic* design procedures.

3.9. Residual stresses after yielding: elastic-perfectly plastic material

Reference to the results of simple tensile or proof tests detailed in §1.7[†] shows that when materials are loaded beyond the yield point the resulting deformation does not disappear completely when load is removed and the material is subjected to permanent deformation or so-called *permanent set* (Fig. 3.15). In bending applications, therefore, when beams may be subjected to moments producing partial plasticity, i.e. part of the beam section remains elastic whilst the outer fibres yield, this permanent set associated with the yielded areas prevents those parts of the material which are elastically stressed from returning to their unstressed state when load is removed. *Residual stress* are therefore produced. In order to determine the magnitude of these residual stresses it is normally assumed that the unloading process, from either partially plastic or fully plastic states, is completely elastic (see Fig. 3.15). The

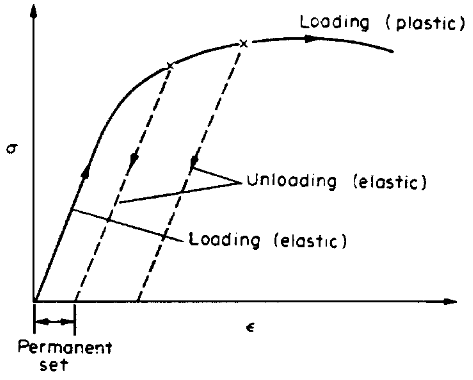

Fig. 3.15. Tensile test stress–strain curve showing elastic unloading process from any load condition.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

unloading stress distribution is therefore linear and it can be subtracted graphically from the stress distribution in the plastic or partially plastic state to obtain the residual stresses.

Consider, therefore, the rectangular beam shown in Fig. 3.16 which has been loaded to its fully plastic condition as represented by the stress distribution rectangles *oabc* and *odef*. The bending stresses which are then superimposed during the unloading process are given by the line *goh* and are opposite to sign. Subtracting the two distributions produces the shaded areas which then indicate the residual stresses which remain after unloading the plastically deformed beam. In order to quantify these areas, and the values of the residual stresses, it should be observed that the loading and unloading moments must be equal, i.e. the moment of the force due to the rectangular distribution *oabc* about the N.A. must equal the moment of the force due to the triangular distribution *oag*.

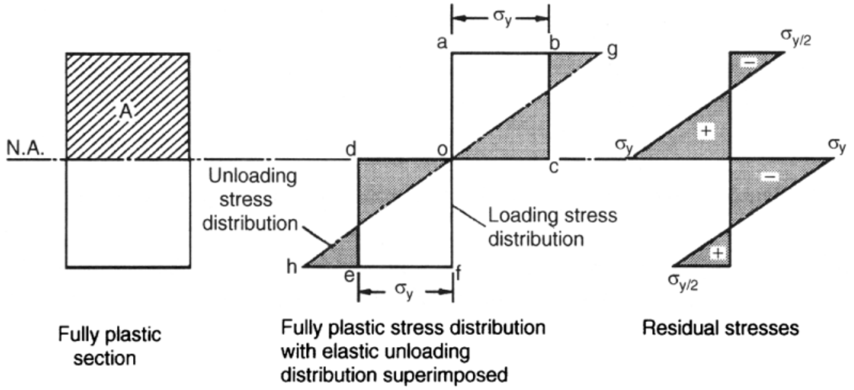

Fig. 3.16. Residual stresses produced after unloading a rectangular-section beam from a fully plastic state.

Now, moment due to *oabc*

$$= \text{stress} \times \text{area} \times \text{moment arm}$$

$$= ab \times A \times oa/2$$

and moment due to *oag*

$$= \text{average stress} \times \text{area} \times \text{moment arm}$$

$$= ag/2 \times A \times 2oa/3$$

Equating these values of moment yields

$$ag = \tfrac{3}{2}ab$$

Now $\quad ab = \text{yield stress } \sigma_y \quad \therefore ag = 1\tfrac{1}{2}\sigma_y$

Thus the residual stresses at the outside surfaces of the beam $= \frac{1}{2}\sigma_y$. The maximum residual stresses occur at the N.A. and are equal to the yield stress. The complete residual stress distribution is shown in Fig. 3.16.

In loading cases where only partial plastic bending has occurred in the beam prior to unloading the stress distributions obtained, using a similar procedure to that outlined above, are shown in Fig. 3.17. Again, the unloading process is assumed elastic and the line *goh* in

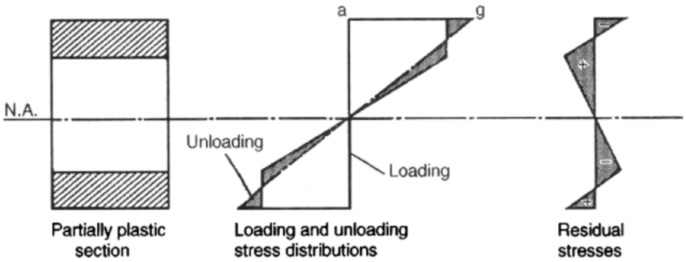

Fig. 3.17. Residual stress produced after unloading a rectangular-section beam from a partially plastic state.

this case is positioned such that the moments of the loading and unloading stress distributions are once more equal, i.e. the stress at the outside fibre ag is determined by considering the plastic moment M_{pp} applied to the beam assuming it to be elastic; thus

$$ag = \sigma = \frac{M_y}{I} = \frac{M_{PP}}{I}\frac{D}{2}$$

Whereas in the previous case the maximum residual stress occurs at the centre of the beam, in this case it may occur either at the outside or at the inner boundary of the yielded portion depending on the depth of plastic penetration. There is no residual stress at the centre of the beam.

Because of the permanent set mentioned above and the resulting stresses, beams which have been unloaded from plastic or partially plastic states will be deformed from their original shape. The straightening moment which is required at any section to return the beam to its original position is that which is required to remove the residual stresses from the elastic core (see Example 3.3).

The residual or permanent radius of curvature R after load is removed can be found from

$$\frac{1}{R} = \frac{1}{R_E} - \frac{1}{R_P} \tag{3.10}$$

where R_P is the radius of curvature in the plastic condition and R_E is the elastic spring-back, calculated by applying the simple bending theory to the complete section with a moment of M_{PP} or M_{FP} as the case may be.

3.10. Torsion of shafts beyond the elastic limit – plastic torsion

The method of treatment of shafts subjected to torques sufficient to initiate yielding of the material is similar to that used for plastic bending of beams (§3.1), i.e. it is usual to assume a stress–strain curve for the shaft material of the form shown in Fig. 3.2, the stress being proportional to strain up to the elastic limit and constant thereafter. It is also assumed that plane cross-sections remain plane and that any radial line across the section remains straight.

Consider, therefore, the cross-section of the shaft shown in Fig. 3.18(a) with its associated shear stress distribution. Whilst the shaft remains elastic the latter remains linear, and as the torque increases the shear stress in the outer fibres will eventually reach the yield stress in shear of the material τ_y. The torque at this point will be the maximum that the shaft can withstand whilst it is completely elastic.

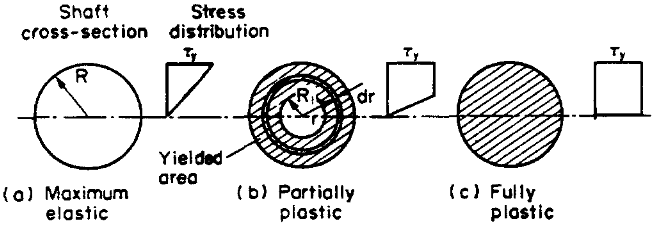

Fig. 3.18. Plastic torsion of a circular shaft.

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{r}$$

Therefore maximum elastic torque

$$T_E = \frac{\tau_y J}{R} = \frac{\tau_y}{R}\frac{\pi R^4}{2}$$

$$= \frac{\pi R^3}{2}\tau_y \qquad (3.11)$$

If the torque is now increased further it is assumed that, instead of the stress in the outer fibre increasing beyond τ_y, more and more of the material will yield and take up the stress τ_y, giving the stress distribution shown in Fig. 3.18(b). Consider the case where the material has yielded to a radius R_1, then:
Partially plastic torque

$$T_{PP} = \text{torque owing to elastic core} + \text{torque owing to plastic portion}$$

The first part is obtained directly from eqn. (3.11) with R_1 replacing R,

i.e. $$\frac{\pi R_1^3}{2}\tau_y$$

For the second part consider an element of radius r and thickness dr, carrying a stress τ_y, (see Fig. 3.18(b)),

$$\text{force on element} = \tau_y \times 2\pi r\, dr$$

$$\text{contribution to torque} = \text{force} \times \text{radius}$$

$$= (\tau_y \times 2\pi r\, dr)r$$

$$= 2\pi r^2\, dr\tau_y$$

$\therefore$ $$\text{total contribution} = \int_{R_1}^{R} \tau_y 2\pi r^2\, dr$$

$$= 2\pi\tau_y \left[\frac{r^3}{3}\right]_{R_1}^{R}$$

$$= \frac{2\pi\tau_y}{3}[R^3 - R_1^3]$$

Therefore, partially plastic torque

$$T_{PP} = \frac{\pi R_1^3}{2}\tau_y + \frac{2\pi}{3}\tau_y[R^3 - R_1^3]$$

$$= \frac{\pi\tau_y}{6}[4R^3 - R_1^3] \tag{3.12}$$

In Fig. 3.18(c) the torque has now been increased until the whole cross-section has yielded, i.e. become plastic. The torque required to reach this situation is then easily determined from eqn. (3.12) since $R_1 = 0$.

$\therefore$ fully plastic torque

$$T_{FP} = \frac{\pi\tau_y}{6} \times 4R^3$$

$$= \frac{2\pi}{3}R^3\tau_y \tag{3.13}$$

There is thus a considerable torque capacity beyond that required to produce initial yield, the ratio of fully plastic to maximum elastic torques being

$$\frac{T_{FP}}{T_E} = \frac{2\pi R^3}{3}\tau_y \times \frac{2}{\pi R^3 \tau_y}$$

$$= \frac{4}{3}$$

The fully plastic torque for a solid shaft is therefore 33% greater than the maximum elastic torque. As in the case of beams this can be taken account of in design procedures to increase the allowable torque which can be carried by the shaft or it may be treated as an additional safety factor. In any event it must be remembered that should stresses in the shaft at any time exceed the yield point for the material, then some permanent deformation will occur.

3.11. Angles of twist of shafts strained beyond the elastic limit

Angles of twist of shafts in the partially plastic condition are calculated on the basis of the elastic core only, thus assuming that once the outer regions have yielded they no longer offer any resistance to torque. This is in agreement with the basic assumption listed earlier that radial lines remain straight throughout plastic torsion, i.e. $\theta_{PP} = \theta_E$ for the core.

For the elastic core, therefore,

$$\frac{\tau_y}{R_1} = \frac{G\theta}{L}$$

i.e.

$$\theta_{PP} = \frac{\tau_y L}{R_1 G} \tag{3.14}$$

3.12. Plastic torsion of hollow tubes

Consider the hollow tube of Fig. 3.19 with internal radius R_1 and external radius R subjected to a torque sufficient to produce yielding to a radius R_2. The torque carried by the

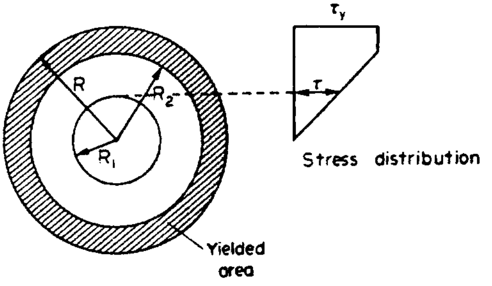

Fig. 3.19. Plastic torsion of a hollow shaft.

equivalent partially plastic solid shaft, i.e. ignoring the central hole, is given by eqn. (3.12) with R_2 replacing R_1 as

$$\frac{\pi\tau_y}{6}[4R^3 - R_2^3]$$

The torque carried by the hollow tube can then be determined by subtracting from the above the torque which would be carried by a solid shaft of diameter equal to the central hole and subjected to a shear stress at its outside fibre equal to τ.
i.e. from eqn. (3.11) torque on imaginary shaft

$$= \frac{\pi R_1^3}{2}\tau$$

but by proportions of the stress distribution diagram

$$\tau = \frac{R_1}{R_2}\tau_y$$

Therefore torque on imaginary shaft equal in diameter to the hollow core

$$= \frac{\pi R_1^4}{2R_2}\tau_y$$

Therefore, partially plastic torque for the hollow tube

$$T_{PP} = \frac{\pi\tau_y}{6}[4R^3 - R_2^3] - \frac{\pi}{2}\frac{R_1^4}{R_2}\tau_y$$

$$= \frac{\pi\tau_y}{6R_2}[4R^3R_2 - R_2^4 - 3R_1^4] \qquad (3.15)$$

The fully plastic torque is then obtained when $R_2 = R_1$,

i.e. $$T_{FP} = \frac{\pi\tau_y}{6R_1}[4R^3R_1 - 4R_1^4] = \frac{2\pi\tau_y}{3}[R^3 - R_1^3] \qquad (3.16)$$

This equation could also have been obtained by adaptation of eqn. (3.13), subtracting a fully plastic core of diameter equal to the central hole.

As an aid in visualising the stresses and torque capacities of members loaded to the fully plastic condition an analogy known as the *sand-heap analogy* has been introduced. Whilst full details have been given by Nadai† it is sufficient for the purpose of this text to note that

† A. Nadai, *Theory of Flow and Fracture of Solids*, Vol. 1, 2nd edn., McGraw-Hill, New York, 1950.

if dry sand is poured on to a raised flat surface having the same shape as the cross-section of the member under consideration, the sand heap will assume a constant slope, e.g. a cone on a circular disc and a pyramid on a square base. The volume of the sand heap, and hence its weight, is then found to be directly proportional to the fully plastic torque which would be carried by that particular shape of cross-section. Thus by calibration, i.e. with a knowledge of the fully plastic torque for a circular shaft, direct comparison of the weight of appropriate sand heaps yields an immediate indication of the fully plastic torque of some other more complicated section.

3.13. Plastic torsion of case-hardened shafts

Consider now the case-hardened shaft shown in Fig. 3.20. Whilst it is often assumed in such cases that the shear-modulus is the same for the material of the case and core, this is certainly not the case for the yield stresses; indeed, there is often a considerable difference, the value for the case being generally much larger than that for the core. Thus, when the shaft is subjected to a torque sufficient to initiate yielding at the outside fibres, the normal triangular elastic stress distribution required to maintain straight radii must be modified, since this would imply that some of the core material is stressed beyond its yield stress. Since the basic assumption used throughout this treatment is that stress remains constant at the yield stress for any increase in strain, it follows that the stress distribution must be as indicated in Fig. 3.20. The shaft thus contains at this stage a plastic region sandwiched between two elastic layers. Torques for each portion must be calculated separately, therefore, and combined to yield the partially plastic torque for the case-hardened shaft. (Example 3.5.)

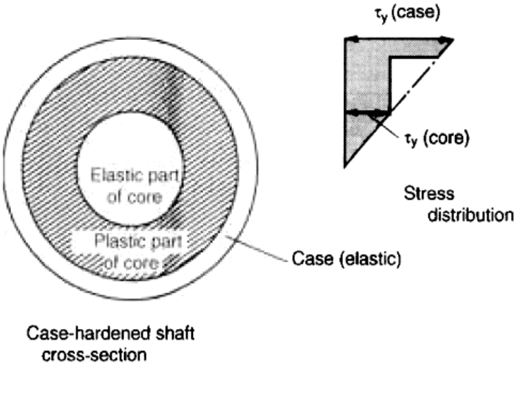

Fig. 3.20. Plastic torsion of a case-hardened shaft.

3.14. Residual stresses after yield in torsion

If shafts are stressed at any time beyond their elastic limit to a partially plastic state as described previously, a permanent deformation will remain when torque is removed. Associated with this plastic deformation will be a system of residual stresses which will affect the strength of the shaft in subsequent loading cycles. The magnitudes of the residual stresses are determined using the method described in detail for beams strained beyond the elastic limit on page 73, i.e. the removal of torque is assumed to be a completely elastic process so that the associated stress distribution is linear. The residual stresses are thus obtained by subtracting the elastic unloading stress distribution from that of the partially plastic loading condition. Now, from eqn. (3.12), partially plastic torque $= T_{PP}$.

Therefore elastic torque to be applied during unloading $= T_{PP}$.

The stress τ' at the outer fibre of the shaft which would be achieved by this torque, assuming elastic material, is given by the torsion theory

$$\frac{T}{J} = \frac{\tau}{R}, \quad \text{i.e. } \tau' = \frac{T_{PP}R}{J}$$

Thus, for a solid shaft the residual stress distribution is obtained as shown in Fig. 3.21.

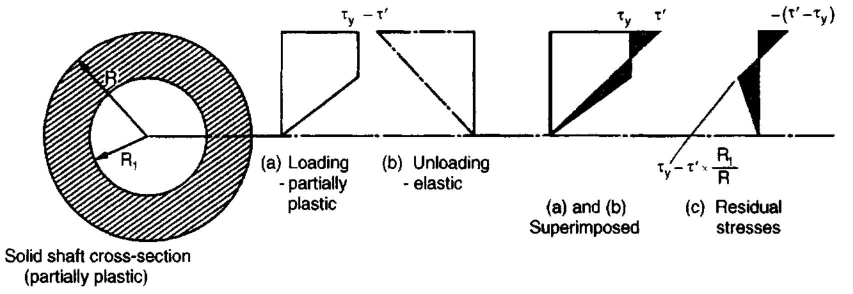

Fig. 3.21. Residual stresses produced in a solid shaft after unloading from a partially plastic state.

Similarly, for hollow shafts, the residual stress distribution will be as shown in Fig. 3.22.

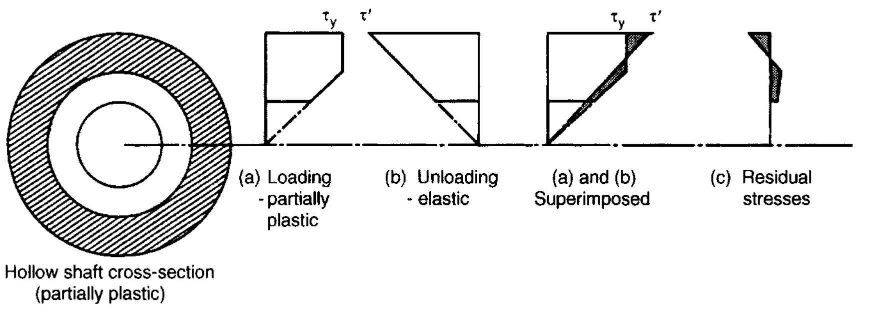

Fig. 3.22. Residual stresses produced in a hollow shaft after unloading from a partially plastic state.

3.15. Plastic bending and torsion of strain-hardening materials

(a) Inelastic bending

Whilst the material in this case no longer follows Hookes' law it is necessary to assume that cross-sections of the beam remain plane during bending so that strains remain proportional to distance from the neutral axis.

Consider, therefore, the rectangular section beam shown in Fig. 3.23(b) with its neutral axis positioned at a distance h_1 from the lower surface and h_2 from the upper surface. Bearing in mind the assumption made in the preceding paragraph we can now locate the neutral axis position by the usual equilibrium conditions.

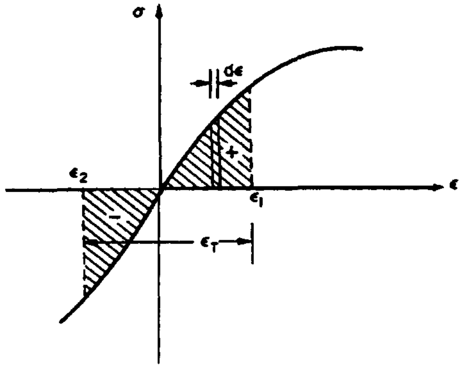

Fig. 3.23(a). Stress–strain curve for a beam in bending constructed from a strain–hardening material.

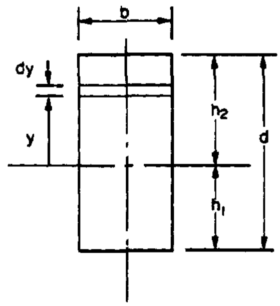

Fig. 3.23(b).

i.e. Since the sum of forces normal to any cross-section must always be zero then:

$$\int \sigma \, dA = \int_{-h_1}^{h_2} \sigma \cdot b \, dy = 0$$

But, from eqn (4.1)[†]

$$y = R\frac{\sigma}{E} = R\varepsilon \qquad \therefore dy = R\,d\varepsilon$$

$\therefore$
$$\int_{-h_1}^{h_2} \sigma b R \, d\varepsilon = 0$$

or
$$\int_{\varepsilon_1}^{\varepsilon_2} \sigma b R \, d\varepsilon = 0.$$

where ε_1 and ε_2 are the strains in the top and bottom surfaces of the beam, respectively. They are also indicated on Fig. 3.23(a).

Since b and R are constant then the position of the neutral axis must be such that:

$$\int_{\varepsilon_1}^{\varepsilon_2} \sigma \, d\varepsilon = 0 \qquad (3.17)$$

i.e. the total area under the $\sigma-\varepsilon$ curve between ε_1 and ε_2 must be zero. This is achieved by marking the length ε_T on the horizontal axis of Fig. 3.23(a) in such a way as to make the positive and negative areas of the diagram equal. This identifies the appropriate values for ε_1 and ε_2 with:

$$\varepsilon_T = |\varepsilon_1 + \varepsilon_2| = \frac{h_1}{R} + \frac{h_2}{R} = \frac{1}{R}(h_1 + h_2)$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

i.e. $$\varepsilon_T = \frac{d}{R} \tag{3.18}$$

Because strains have been assumed linear with distance from the neutral axis the position of the N.A. is then obtained by simple proportions:

$$\frac{h_1}{h_2} = \frac{\varepsilon_1}{\varepsilon_2} \tag{3.19}$$

The value of the applied bending moment M is then given by the sum of the moments of forces above and below the neutral axis.

i.e. $$M = \int \sigma \, dA \cdot y = \int_{-h_1}^{h_2} \sigma \cdot b \, dy \cdot y$$

and, since $dy = R \cdot d\varepsilon$ and $y = R\varepsilon$.

$$M = \int_{\varepsilon_1}^{\varepsilon_2} \sigma b \cdot R^2 \varepsilon \, d\varepsilon = R^2 b \int_{\varepsilon_1}^{\varepsilon_2} \sigma\varepsilon \, d\varepsilon.$$

Substituting, from eqn. (3.18), $R = d/\varepsilon_T$:

$$M = \frac{bd^2}{\varepsilon_T^2} \int_{\varepsilon_1}^{\varepsilon_2} \sigma\varepsilon \, d\varepsilon. \tag{3.20}$$

The integral part of this expression is the first moment of area of the shaded parts of Fig. 3.23(a) about the vertical axis and evaluation of this integral allows the determination of M for any assumed value of ε_T.

An alternative form of the expression is obtained by multiplying the top and bottom of the expression by $12R$ using $R = d/\varepsilon_T$ for the numerator,

i.e. $$M = 12 \frac{(d/\varepsilon_T)}{12R} \left[\frac{bd^2}{\varepsilon_T^2} \int_{\varepsilon_1}^{\varepsilon_2} \sigma\varepsilon \, d\varepsilon \right] = \frac{1}{R} \cdot \frac{bd^3}{12} \cdot \frac{12}{\varepsilon_T^3} \int_{\varepsilon_1}^{\varepsilon_2} \sigma\varepsilon \, d\varepsilon.$$

which can be reduced to a form similar to the standard bending eqn. (4.3)† $M = EI/R$

i.e. $$M = \frac{E_r I}{R} \tag{3.21}$$

with E_r known as the *reduced modulus* and given by:

$$E_r = \frac{12}{\varepsilon_T^3} \int_{\varepsilon_1}^{\varepsilon_2} \sigma\varepsilon \, d\varepsilon. \tag{3.22}$$

The appropriate value of the reduced modulus E_r for any particular curvature is best obtained from a curve of E_r against ε_T. This is constructed rather laboriously by determining the relevant values of ε_1 and ε_2 for a set of assumed ε_T values using the condition of equal positive and negative areas for each ε_T value and then evaluating the integral of eqn. (3.22). Having found E_r, the value of the bending moment for any given curvature R is found from eqn. (3.21).

It is sometimes useful to remember that, because strains are linear with distance from the neutral axis, the distribution of bending stresses across the beam section will take exactly the same form as that of the stress–strain diagram of Fig. 3.23(a) turned through 90° with

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

ε_T replaced by the beam depth d. The position of the neutral axis indicated by eqn. (3.19) is then readily observed.

(b) Inelastic torsion

A similar treatment can be applied to the torsion of shafts constructed from materials which exhibit strain hardening characteristics. Figure 3.24 shows the shear stress–shear strain curve for such a material.

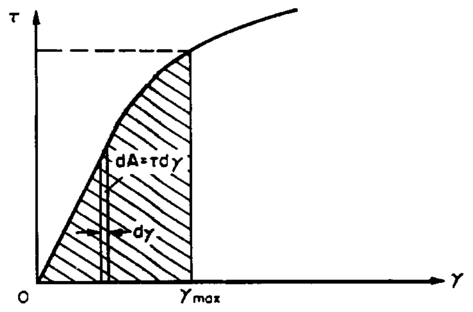

Fig. 3.24. Shear stress–shear strain curve for torsion of materials exhibiting strain-hardening characteristics.

Once again it is necessary to assume that cross-sections of the shaft remain plane and that the radii remain straight under torsion. The shear strain at any radius r is then given by eqn. (8.9)† as:

$$\gamma = \frac{r\theta}{L}$$

For a shaft of radius R the maximum shearing strain is thus

$$\gamma_{\max} = \frac{R\theta}{L}$$

the corresponding shear stress being given by the relevant ordinate of Fig. 3.24.

Now the torque T has been shown in §8.1† to be given by:

$$T = \int_0^R 2\pi r^2 \tau' \, dr$$

where τ' is the shear stress at any general radius r.

Now, since $\gamma = \frac{r\theta}{L}$ then $d\gamma = \frac{\theta}{L} \cdot dr$

and, substituting for r and dr, we have:

$$T = \int_0^{\gamma_{\max}} 2\pi \left(\frac{\gamma L}{\theta}\right)^2 \tau' \frac{L}{\theta} \cdot d\gamma$$

$$= \frac{2\pi L^3}{\theta^3} \int_0^{\gamma_{\max}} \tau' \gamma^2 \, d\gamma \qquad (3.23)$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

The integral part of the expression is the second moment of area of the shaded portion of Fig. 3.24 about the vertical axis. Thus, determination of this quantity for a given y_{max} value yields the corresponding value of the applied torque T.

As for the case of inelastic bending, the form of the shear stress–strain curve, Fig. 3.24, is identical to the shear stress distribution across the shaft section with the γ axis replaced by radius r.

3.16. Residual stresses – strain-hardening materials

The procedure for determination of residual stresses arising after unloading from given stress states is identical to that described in §3.9 and §3.14.

For example, it has been shown previously that the stress distribution across a beam section in inelastic bending will be similar to that shown in Fig. 3.23(a) with the beam depth corresponding to the strain axis. Application of the elastic unloading stress distribution as described in §3.9 will then yield the residual stress distribution shown in Fig. 3.25. The same procedure should be adopted for residual stresses in torsion situations, reference being made to §3.14.

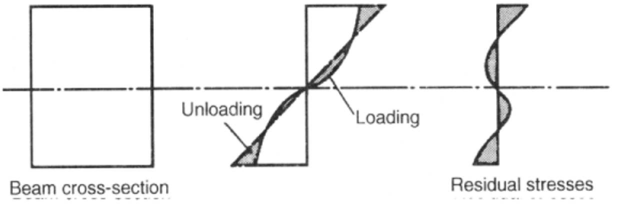

Fig. 3.25. Residual stresses produced in a beam constructed from a strain-hardening material.

3.17. Influence of residual stresses on bending and torsional strengths

The influence of residual stresses on the future loading of members has been summarised by Juvinall† into the following rule:

An overload causing yielding produces residual stresses which are favourable to future overloads in the same direction and unfavourable to future overloads in the opposite direction.

This suggests that the residual stresses represent a favourable stress distribution which has to be overcome by any further load system before any adverse stress can be introduced into the member of structure. This principle is taken advantage of by spring manufacturers, for example, who intentionally yield springs in the direction of anticipated service loads as part of the manufacturing process. A detailed discussion of residual stress can be found in the *Handbook of Experimental Stress Analysis* of Hetényi.‡

† R. C. Juvinall, *Engineering Considerations of Stress, Strain and Strength*, McGraw-Hill, 1967.

‡ M. Hetényi, *Handbook of Experimental Stress Analysis*, John Wiley, 1966.

3.18. Plastic yielding in the eccentric loading of rectangular sections

When a column or beam is subjected to an axial load and a B.M., as in the application of eccentric loads, the elastic stress distribution is as shown in Fig. 3.25(a), the N.A. being displaced from the centroidal axis of the section. As the load increases the yield stress will be reached on one side of the section first as shown in Fig. 3.26(b) and, as in the case of the partially plastic bending of unsymmetrical sections in §3.4, the N.A. will move as plastic penetration proceeds. In the limiting case, when plasticity has spread across the complete section, the N.A. will be situated at a distance h from the centroidal axis (the axis through the centroid of the section) (Fig. 3.26(c)). The precise position of the N.A. is related to the excess of the total tensile force over the total compressive force, i.e. to the area shown shaded in Fig. 3.26(c). In simple bending, for example, there is no resultant force across the section and the shaded area reduces to zero. Thus, the magnitude of the axial load for full plasticity as given by the shaded area

$$= P_{FP} = 2h \times B \times \sigma_y$$

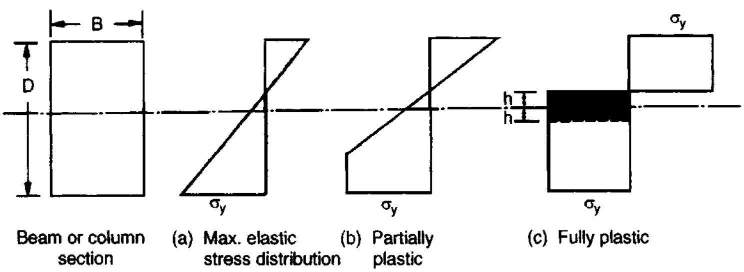

Fig. 3.26. Plastic yielding of eccentrically loaded rectangular section.

where B is the width of the section,

i.e.
$$h = \frac{P_{FP}}{2B\sigma_y} \tag{3.24}$$

The fully plastic load is sometimes written in terms of a load factor N defined as

$$\text{load factor } N = \frac{\text{fully plastic load}}{\text{axial load}} = \frac{P_{FP}}{P}$$

then
$$h = \frac{PN}{2B\sigma_y} \tag{3.25}$$

The fully plastic moment on the section is given by the difference in the moments produced by the stress distributions of Fig. 3.27,

i.e.
$$M_{FP} = \frac{BD^2}{4}\sigma_y - 2(Bh\sigma_y)\frac{h}{2}$$

$$M_{FP} = \frac{BD^2}{4}\sigma_y - Bh^2\sigma_y$$

$$= \frac{BD^2}{4}\sigma_y - \frac{P^2N^2}{4B\sigma_y} \tag{3.26}$$

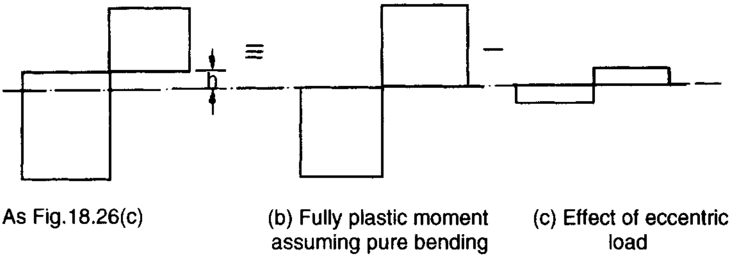

Fig. 3.27.

The fully plastic moment required in eccentric load conditions is therefore reduced from that in the simple bending case by an amount depending on the values of the load, yield stress, section shape and load factor.

The maximum allowable working moment for a single plastic hinge in eccentric loading situations with a load factor N is therefore given by

$$M = \frac{M_{FP}}{N} = \frac{BD^2}{4N}\sigma_y - \frac{P^2N}{4B\sigma_y} \quad (3.27)$$

3.19. Plastic yielding and residual stresses under axial loading with stress concentrations

If a bar with uniform cross-section is loaded beyond its yield point in pure tension (or compression), the bar will experience permanent deformation when load is removed but no residual stresses will be created since all of the material in the cross-section will have yielded simultaneously and all will return to the same unloaded condition. If, however, stress concentrations such as notches, keyways, holes, etc., are present in the bar, these will result in local stress increases or *stress concentrations*, and the material will yield at these positions before the rest of the cross-section. If the local stress concentration factor is K then the maximum stress in the section with an axial load P is given by

$$\sigma_{\max} = K(P/A)$$

where P/A is the mean stress across the section assuming no stress concentration is present.

When the load has been increased to a value P_y, just sufficient to initiate yielding at the root of the notch or other stress concentration, the stress distribution will be as shown in Fig. 3.28(a). Since equilibrium considerations require the mean stress across the section to equal P_y/A it follows that the stress at the centre of the section must be less than P_y/A.

If the load is now increased to P_2, yielding will continue at the root of the notch and plastic penetration will proceed towards the centre of the section. At some stage the stress distribution will appears as in Fig. 3.28(b) with a mean stress value of P_2/A. If the load is then removed the residual stresses may be obtained using the procedure of §§3.9 and 3.14, i.e. by superimposing an elastic stress distribution of opposite sign but equal moment value (shown dotted in Fig. 3.28(b)). The resulting residual stress distribution would then be similar to that shown in Fig. 3.28(c).

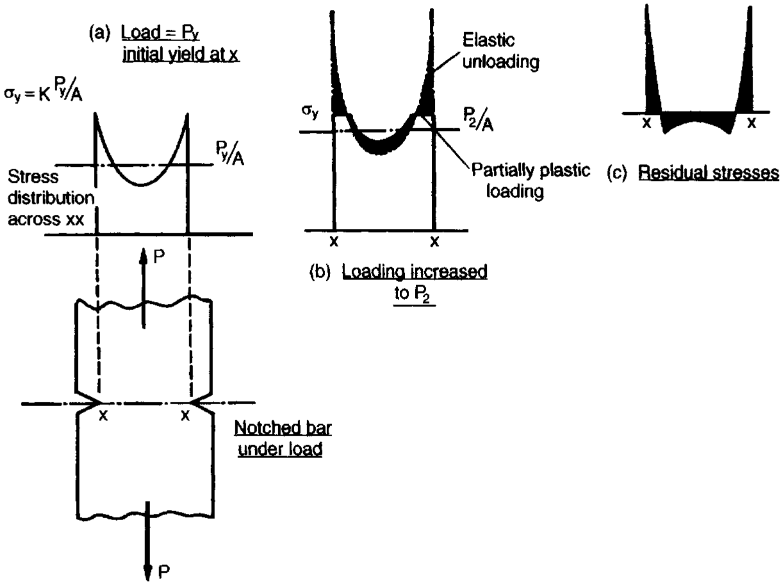

Fig. 3.28. Residual stresses at stress concentrations.

Whilst subsequent application of loads above the value of P_2 will cause further yielding, no yielding will be caused by the application of loads up to the value of P_2 however many times they are applied. With a sufficiently high value of stress concentration factor it is possible to produce a residual stress distribution which exceeds the compressive yield stress at the root of the notch, i.e. the material will be stressed from tensile yield to compressive yield throughout one cycle. Provided that further cycles remain within these limits, the component will not experience additional yielding, and it can be considered safe in, for example, high strain, low-cycle fatigue conditions.

3.20. Plastic yielding of axially symmetric components†

(a) Thick cylinders under internal pressure – collapse pressure

Consider the thick cylinder shown in Fig. 3.29 subjected to an internal pressure P_1 of sufficient magnitude to produce yielding to a radius R_p.

Now for ductile materials, from §10.17,‡ yield is deemed to occur when

$$\sigma_y = \sigma_H - \sigma_r. \qquad (3.28)$$

but from eqn. (10.2),‡, the equilibrium equation,

$$\sigma_H - o_r = r\frac{d\sigma_r}{dr}$$

† J. Heyman, *Proc. I.Mech.E.* **172** (1958). W.R.D. Manning, *High Pressure Engineering*, Bulleid Memorial Lecture, 1963, University of Nottingham.

‡ E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

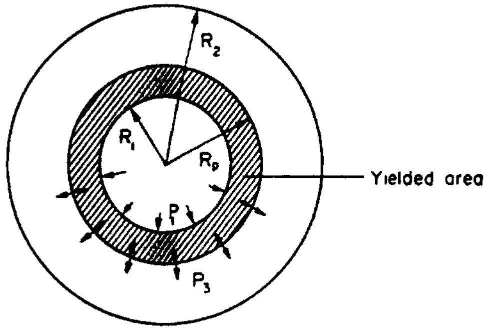

Fig. 3.29.

$\therefore \quad \sigma_r = r\dfrac{d\sigma_r}{dr}$

$\therefore \quad \dfrac{d\sigma_r}{dr} = \dfrac{\sigma_y}{r}$

Integrating: $\sigma_r = \sigma_y \log_e r + \text{ constant}$

Now $\sigma_r = -P_3 \quad \text{at} \quad r = R_p$

$\therefore \quad \text{constant} = -P_3 - \sigma_y \log_e R_p$

$\therefore \quad \sigma_r = \sigma_y \log_e r - P_3 - \sigma_y \log_e R_p$

i.e. $$\boldsymbol{\sigma_r = \sigma_y \log_e \frac{r}{R_p} - P_3} \qquad (3.29)$$

and from eqn. (3.28) $\sigma_H = \sigma_y + \sigma_r$

$\therefore$ $$\boldsymbol{\sigma_H = \sigma_y \left(1 + \log_e \frac{r}{R_p}\right) - P_3} \qquad (3.30)$$

These equations thus yield the hoop and radial stresses *throughout the plastic zone* in terms of the radial pressure at the elastic–plastic interface P_3. The *numerical* value of P_3 may be determined as follows (the sign has been allowed for in the derivation of eqn. (3.29)).

At the stage where plasticity has penetrated partly through the cylinder walls the cylinder may be considered as a compound cylinder with the inner tube plastic and the outer tube elastic, the latter being subjected to an internal pressure P_3. From eqns. (10.5) and (10.6)† the hoop and radial stresses *in the elastic portion* are therefore given by

$$\sigma_r = \frac{P_3 R_p^2}{(R_2^2 - R_p^2)}\left[\frac{R_p^2 - R_2^2}{R_p^2}\right]$$

and $$\sigma_H = \frac{P_3 R_p^2}{(R_2^2 - R_p^2)}\left[\frac{R_p^2 + R_2^2}{R_p^2}\right]$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

i.e. the maximum shear stress is

$$\frac{\sigma_H - \sigma_r}{2} = \frac{P_3 R_p^2}{2(R_2^2 - R_p^2)}\left[\frac{2R_2^2}{R_p^2}\right] = \frac{P_3 R_2^2}{(R_2^2 - R_p^2)}$$

Again, applying the Tresca yield criterion,

$$\frac{\sigma_y}{2} = \frac{P_3 R_2^2}{(R_2^2 - R_p^2)}$$

i.e. the radial pressure at the elastic interface is

$$P_3 = \frac{\sigma_y}{2R_2^2}[R_2^2 - R_p^2] \tag{3.31}$$

Thus from eqns. (3.29) and (3.30) the stresses in the plastic zone are given by

$$\sigma_r = \sigma_y\left[\log_e \frac{r}{R_p} - \frac{1}{2R_2^2}(R_2^2 - R_p^2)\right] \tag{3.32}$$

and

$$\sigma_H = \sigma_y\left[\left(1 + \log_e \frac{r}{R_p}\right) - \frac{1}{2R_2^2}(R_2^2 - R_p^2)\right] \tag{3.33}$$

The pressure required for complete plastic "collapse" of the cylinder is given by eqn. (3.29) when $r = R_1$ and $R_p = R_2$ with $P_3 = P_2 = 0$ (at the outside edge).

For "collapse"

$$\boldsymbol{\sigma_r = -P_1 = \sigma_y \log_e \frac{R_1}{R_2}} \tag{3.34}$$

With a knowledge of this collapse pressure the design pressure can be determined by dividing it by a suitable load factor as described in §3.8.

The *pressure at initial yield* is found from eqn. (3.31) when $R_p = R_1$,

i.e.

$$\textbf{initial yield pressure} = \boldsymbol{\frac{\sigma_y}{2R_2^2}[R_2^2 - R_1^2]} \tag{3.35}$$

Finally, the *internal pressure required to cause yielding to a radius* R_p is given by eqn. (3.32) when $r = R_1$,

i.e.

$$\boldsymbol{\sigma_r = -P_1 = \sigma_y\left[\log_e \frac{R_1}{R_p} - \frac{1}{2R_2^2}(R_2^2 - R_p^2)\right]} \tag{3.36}$$

(b) Thick cylinders under internal pressure ("auto-frettage")

When internal pressure is applied to thick cylinders it has been shown that maximum tensile stresses are set up at the inner surface of the bore. If the internal pressure is increased sufficiently, yielding of the cylinder material will take place at this position and the working safety factor n according to the Tresca theory will be given by

$$\sigma_H - \sigma_r = \sigma_y / n$$

where σ_H and σ_r are the hoop and radial stresses at the bore.

Fortunately, the condition is not too serious at this stage since there remains a considerable bulk of elastic material surrounding the yielded area which contains the resulting strains within reasonable limits. As the pressure is increased further, however, plastic penetration takes place deeper and deeper into the cylinder wall and eventually the whole cylinder will yield. Fatigue life of the cylinder will also be heavily dependent upon the value of the maximum tensile stress at the bore so that any measures which can be taken to reduce the level of this stress will be beneficial to successful operation of the cylinder. Such methods include the use of compound cylinders with force or shrink fits and/or external wire winding; the largest effect is obtained, however, with a process known as *"autofrettage"*.

If the pressure inside the cylinder is increased beyond the initial yield value so that plastic penetration occurs only partly into the cylinder wall then, on release of the pressure, the elastic zone attempts to return to its original dimensions but is prevented from doing so by the permanent deformation or "set" of the yielded material. The result is that residual stresses are introduced, the elastic material being held in a state of residual tension whilst the inside layers are brought into residual compression. On subsequent loading cycles, therefore, the cylinder is able to withstand a higher internal pressure since the compressive residual stress at the bore has to be overcome before this region begins to experience tensile stresses. The autofrettage process has the same effect as shrinking one tube over another without the complications of the shrinking process. With careful selection of cylinder dimensions and autofrettage pressure the resulting residual compressive stresses can significantly reduce or even totally eliminate tensile stresses which would otherwise be achieved at the bore under working conditions. As a result the fatigue life and the safety factor at the bore are considerably enhanced and for this reason gun barrels and other pressure vessels are often pre-stressed in this way prior to service.

Care must be taken in the design process, however, since the autofrettage process introduces a secondary critical stress region at the position of the elastic/plastic interface of the autofrettage pressure loading condition. This will be discussed further below.

The autofrettage pressure required for yielding to any radius R_p is given by the High Pressure Technology Association (HPTA) code of practice† as

$$P_A = \frac{\sigma_y}{2}\left[\frac{K^2 - m^2}{K^2}\right] + \sigma_y \log_e m \qquad (3.37)$$

where $K = R_2/R_1$ and $m = R_p/R_1$, where R_1 the internal radius and R_2 the external radius.

This is simply a modified form of eqn. (3.36) developed in the preceding section.

The maximum allowable autofrettage pressure is then given as that which will produce yielding to the geometric mean radius $R_p = \sqrt{R_1 R_2}$.

Stress distribution under autofrettage pressure loading

From eqns. (3.32) and (3.33) the stresses in the plastic zone at any radius r are given by:

$$\sigma_r = \sigma_y\left[\log_e\left(\frac{r}{R_p}\right) - \frac{1}{2}\left(1 - \frac{R_p^2}{R_2^2}\right)\right] \qquad (3.38)$$

$$\sigma_H = \sigma_y\left[1 + \log_e\left(\frac{r}{R_p}\right) - \frac{1}{2}\left(1 - \frac{R_p^2}{R_2^2}\right)\right] \qquad (3.39)$$

† *High Pressure Safety Code*. High Pressure Technology Association, 1975.

Also in §3.20 (a) it has been shown that stresses at any radius r in the elastic zone are obtained in terms of the radial pressure $P_3 = P_p$ set up at the elastic–plastic interface with:

$$\sigma_r = \frac{P_p R_p^2}{(R_2^2 - R_p^2)}\left[1 - \frac{R_2^2}{r^2}\right] \tag{3.40}$$

$$\sigma_H = \frac{P_p \cdot R_p^2}{(R_2^2 - R_p^2)}\left[1 + \frac{R_2^2}{r^2}\right] \tag{3.41}$$

with

$$P_p = \frac{\sigma_y}{2R_2^2}[R_2^2 - R_p^2] \qquad (3.31)(bis)$$

The above equations yield hoop and radial stress distributions throughout the cylinder wall typically of the form shown in Fig. 3.30.

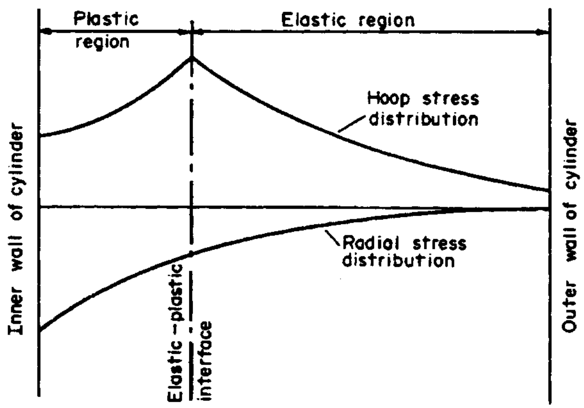

Fig. 3.30. Stress distributions under autofrettage pressure.

Residual stress distributions

Residual stress after unloading can then be obtained using the procedure introduced in §3.9 of elastic unloading, i.e. the autofrettage loading pressure is assumed to be removed (applied in a negative sense) elastically across the whole cylinder, the unloading elastic stress distribution being given by eqns. (10.5) and (10.6)† as:

$$\sigma_r = P_A\left[\frac{1 - \left(\frac{R_2}{r}\right)^2}{K^2 - 1}\right] \tag{3.42}$$

$$\sigma_H = P_A\left[\frac{1 + \left(\frac{R_2}{r}\right)^2}{K^2 - 1}\right] \quad \text{with } K = R_2/R_1 \tag{3.43}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Superposition of these distributions on the previous loading distributions allows the two curves to be subtracted for both hoop and radial stresses and produces residual stresses of the form shown in Figs. 3.31 and 3.32.

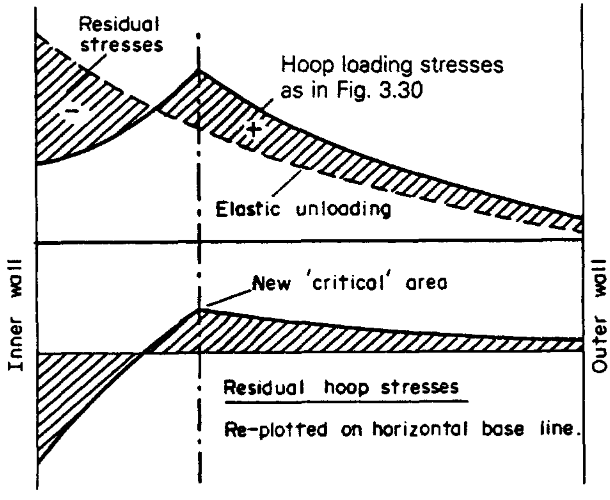

Fig. 3.31. Determination of residual hoop stresses by elastic unloading.

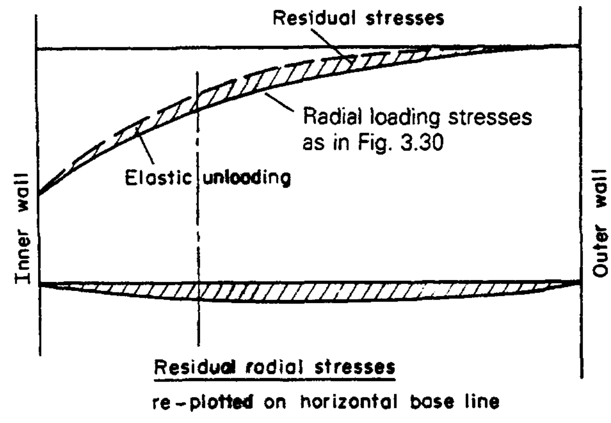

Fig. 3.32. Determination of residual radial stresses by elastic unloading.

Working stress distributions

Finally, if the stress distributions due to an elastic internal working pressure P_w are superimposed on the residual stress state then the final working stress state is produced as in Figs. 3.33 and 3.34.

The elastic working stresses are given by eqns. (3.42) and (3.43) with P_A replaced by P_w. Alternatively a Lamé line solution can be adopted. The final stress distributions show that

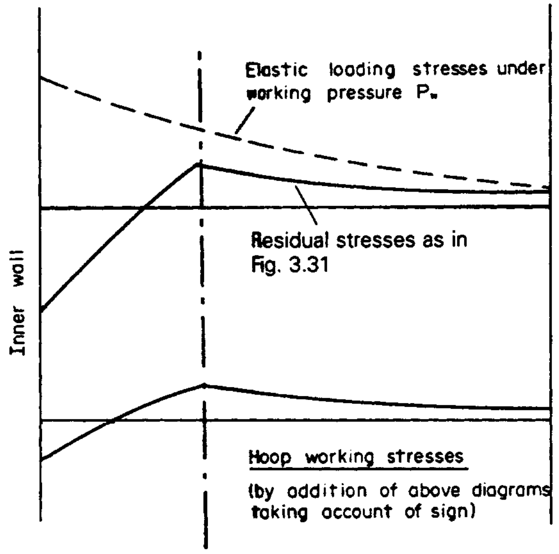

Fig. 3.33. Evaluation of hoop working stresses.

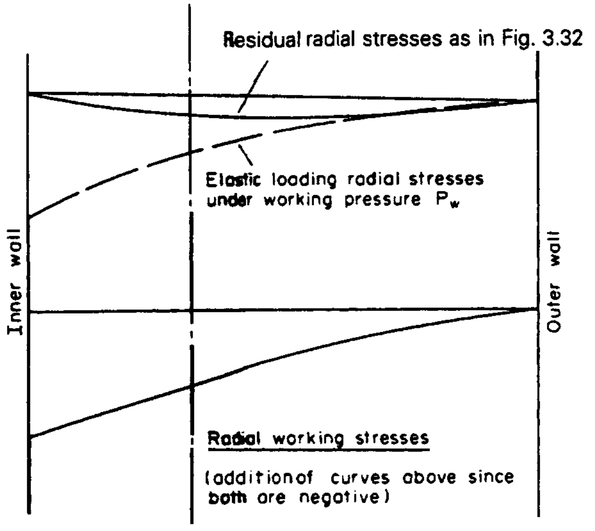

Fig. 3.34. Evaluation of radial working stresses.

the maximum tensile stress, instead of being at the bore as in the plain cylinder, is now at the elastic/plastic interface position. Application of the Tresca maximum shear stress failure criterion:

i.e. $$\sigma_H - \sigma_r = \sigma_y/n$$

also indicates the elastic/plastic interface as now more critical than the internal bore – see Fig. 3.35.

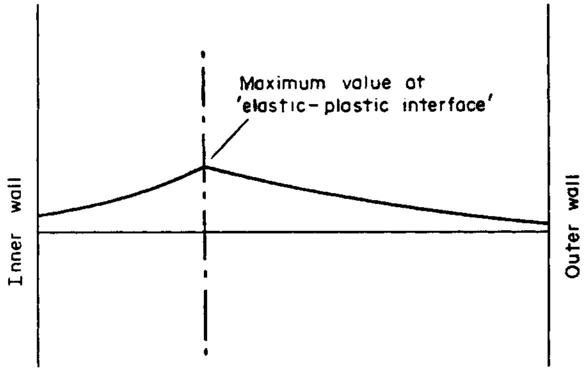

Fig. 3.35. Distribution of maximum shear stress = $\frac{1}{2}(\sigma_\theta - \sigma_r)$ by combination of Figs. 3.33 and 3.34.

Effect of axial stresses and end restraint

Depending on the end conditions which can be assumed for the cylinder during both the autofrettage process and its normal working condition a further complication can arise since the axial stresses σ_z which are produced can affect the application of the Tresca criterion.

Strictly, Tresca requires the use of the greatest difference in the principal stresses which, if σ_z is zero, $= \sigma_H - \sigma_r$. If, however, σ_z has a value it must be used in conjunction with σ_H and σ_r to produce the greatest difference.

The procedure used above to determine residual hoop and radial stresses and subsequent working stresses should therefore be repeated for axial stresses with values in the plastic region being found as suggested by Franklin and Morrison† from:

$$\sigma_z = P_A \frac{(1-2\nu)}{(K^2-1)} + \nu(\sigma_H - \sigma_r) \tag{3.44}$$

and axial stresses under elastic conditions being given by eqn. (10.7)‡ with $P_2 = 0$ and $P_1 = P_A$ or P_W as required.

(c) Rotating discs

It will be shown in Chapter 4 that the centrifugal forces which act on rotating discs produce two-dimensional tensile stress systems. At any given radius the hoop or circumferential stress is always greater than, or equal to, the radial stress, the maximum values occurring at the inside radius. It follows, therefore, that yielding will first occur at the inside surface when the speed of rotation has increased sufficiently to make the circumferential stress equal to the tensile yield stress. With further increase of speed, plastic penetration will gradually proceed towards the centre of the disc and eventually complete plastic collapse will occur.

† G.J. Franklin and J.L.M. Morrison, Autofrettage of cylinders: reduction of pressure/external expansion curves and calculation of residual stresses. *Proc. J. Mech. E.* **174** (35) 1960.

‡ E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Now for a *solid disc* the equilibrium eqn. (4.1) derived on page 120 is, with $\sigma_H = \sigma_y$,

$$\sigma_y - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2\omega^2$$

$$\therefore \qquad \sigma_r + r\frac{d\sigma_r}{dr} = \sigma_y - \rho r^2\omega^2$$

Integrating,

$$r\sigma_r = r\sigma_y - \rho\frac{r^3\omega^2}{3} + A \tag{1}$$

Now since the stress cannot be infinite at the centre where $r = 0$, then A must be zero.

$$\therefore \qquad r\sigma_r = r\sigma_y - \rho\frac{r^3\omega^2}{3}$$

Now at $r = R$, i.e. at the outside of the disc, $\sigma_r = 0$.

$$\therefore \qquad R\sigma_y = \rho\frac{R^3\omega^2}{3}$$

i.e. the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho R^2} \tag{3.45}$$

For a *disc with a central hole*, (1) still applies, but in this case the value of the constant A is determined from the condition

$$\sigma_r = 0 \text{ at } r = R_1 \text{ the inside radius}$$

i.e.

$$A = \rho\frac{R_1^3\omega^2}{3} - R_1\sigma_y$$

Again, $\sigma_r = 0$ at the outside surface where $r = R$.

Substituting in (1),

$$0 = R\sigma_y - \rho\frac{R^3\omega^2}{3} + \rho\frac{R_1^3\omega^2}{3} - R_1\sigma_y$$

$$0 = \sigma_y(R - R_1) - \rho\frac{\omega^2}{3}(R^3 - R_1^3)$$

i.e. the collapse speed ω_p is given by

$$\omega_p^2 = \frac{3\sigma_y}{\rho}\frac{(R - R_1)}{(R^3 - R_1^3)} \tag{3.46}$$

If a rotating disc is stopped after only partial penetration, residual stresses are set up similar to those discussed in the case of thick cylinders under internal pressure (auto-frettage). Their values may be determined in precisely the same manner as that described in earlier sections, namely, by calculating the elastic stress distribution at an appropriate higher speed and subtracting this from the partially plastic stress distribution. Once again, favourable compressive residual stresses are set up on the surface of the central hole which increases the stress range – and hence the speed limit – available on subsequent cycles. This process is sometimes referred to as *overspeeding*.

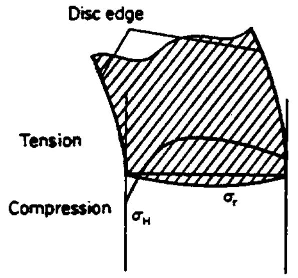

Fig. 3.36. Residual stresses produced after plastic yielding ("overspeeding") of rotating disc with a central hole.

A typical residual stress distribution is shown in Fig. 3.36.

Examples

Example 3.1

(a) A rectangular-section steel beam, 50 mm wide by 20 mm deep, is used as a simply supported beam over a span of 2 m with the 20 mm dimension vertical. Determine the value of the central concentrated load which will produce initiation of yield at the outer fibres of the beam.

(b) If the central load is then increased by 10% find the depth to which yielding will take place at the centre of the beam span.

(c) Over what length of beam will yielding then have taken place?

(d) What are the maximum deflections for each load case?

For steel σ_y in simple tension and compression = 225 MN/m² and $E = 206.8$ GN/m².

Solution

(a) From eqn. (3.1) the B.M. required to initiate yielding is

$$\frac{BD^2}{6}\sigma_y = \frac{50 \times 20^2 \times 10^{-9}}{6} \times 225 \times 10^6 = 750 \text{ N m}$$

But the maximum B.M. on a beam with a central point load is $WL/4$, at the centre.

$$\therefore \qquad \frac{W \times 2}{4} = 750$$

i.e. $$W = \mathbf{1500\ N}$$

The load required to initiate yielding is 1500 N.

(b) If the load is increased by 10% the new load is

$$W' = 1500 + 150 = 1650 \text{ N}$$

The maximum B.M. is therefore increased to

$$M' = \frac{W'L}{4} = \frac{1650 \times 2}{4} = 825 \text{ Nm}$$

and this is sufficient to produce yielding to a depth d, and from eqn. (3.2),

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2] = 825 \text{ Nm}$$

$$\therefore \quad 825 = \frac{50 \times 10^{-3} \times 225 \times 10^6}{12}[3 \times 2^2 - d^2]10^{-4}$$

where d is the depth of the elastic core in centimetres,

$$\therefore \quad 8.8 = 12 - d^2$$

$$d^2 = 3.2 \text{ and } d = 1.79 \text{ cm}$$

$$\therefore \quad \text{depth of yielding} = \tfrac{1}{2}(D - d) = \tfrac{1}{2}(20 - 17.9) = \mathbf{1.05\ mm}$$

(c) With the central load at 1650 N the yielding will have spread from the centre as shown in Fig. 3.37. At the extremity of the yielded region, a distance x from each end of the beam, the section will just have yielded at the extreme surface fibres, i.e. the moment carried at this section will be the maximum elastic moment and given by eqn. (3.1) – see part (a) above.

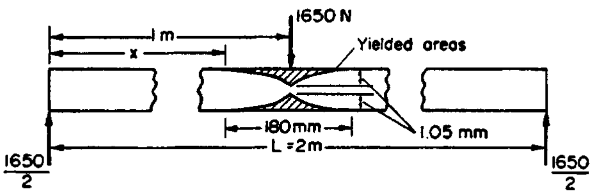

Fig. 3.37.

Now the B.M. at the distance x from the support is

$$\frac{1650x}{2} = \frac{BD^2}{6}\sigma_y = 750$$

$$\therefore \quad x = \frac{2 \times 750}{1650} = 0.91 \text{ m}$$

Therefore length of beam over which yielding has occurred

$$= 2 - 2 \times 0.91 = 0.18 \text{ m} = \mathbf{180\ mm}$$

(d) For $W = 1500$ N the beam is completely elastic and the maximum deflection, at the centre, is given by the standard form of eqn. (5.15)†:

$$\delta = \frac{WL^3}{48EI} = \frac{1500 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 20^3 \times 10^{-12}}$$

$$= 0.0363 \text{ m} = \mathbf{36.3\ mm}$$

With $W = 1650$ N and the beam partially plastic, deflections are calculated on the basis of the elastic core only,

i.e. $$\delta = \frac{W'L^3}{48EI'} = \frac{1650 \times 2^3 \times 12}{48 \times 206.8 \times 10^9 \times 50 \times 17.9^3 \times 10^{-12}}$$

$$= 0.0556 \text{ m} = \mathbf{55.6\ mm}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Example 3.2

(a) Determine the "shape factor" of a T-section beam of dimensions 100 mm × 150 mm × 12 mm as shown in Fig. 3.38.

(b) A cantilever is to be constructed from a beam with the above section and is designed to carry a uniformly distributed load over its complete length of 2 m. Determine the maximum u.d.l. that the cantilever can carry if yielding is permitted over the lower part of the web to a depth of 25 mm. The yield stress of the material of the cantilever is 225 MN/m^2.

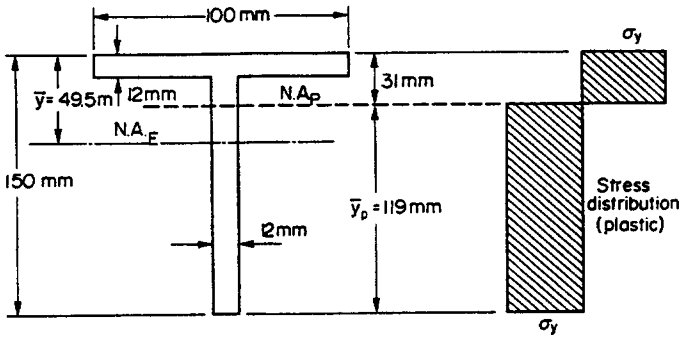

Fig. 3.38.

Solution

(a)
$$\text{Shape factor} = \frac{\text{fully plastic moment}}{\text{maximum elastic moment}}$$

To determine the maximum moment carried by the beam while completely elastic we must first determine the position of the N.A.

Take moments of area about the top edge (see Fig. 3.38):

$$(100 \times 12 \times 6) + (138 \times 12 \times 81) = [(100 \times 12) + (138 \times 12)]\ \bar{y}$$

$$7200 + 134136 = (1200 + 1656)\ \bar{y}$$

$$\therefore \qquad \bar{y} = 49.5 \text{ mm}$$

$$\therefore \qquad I_{NA} = \left[\frac{100 \times 49.5^3}{3} + \frac{12 \times 100.5^3}{3} - \frac{88 \times 37.5^3}{3}\right] 10^{-12} \text{ m}^4$$

$$= \tfrac{1}{3}[121.29 + 121.81 - 46.4]10^{-7}$$

$$= 6.56 \times 10^{-6} \text{ m}^4$$

Now from the simple bending theory the moment required to produce the yield stress at the edge of the section (in this case the lower edge), i.e. the maximum elastic moment, is

$$M_E = \frac{\sigma I}{y_{\max}} = \sigma_y \times \frac{6.56 \times 10^{-6}}{100.5 \times 10^{-3}} = 0.065 \times 10^{-3}\sigma_y$$

When the section becomes fully plastic the N.A. is positioned such that

area below N.A. = half total area

i.e. if the plastic N.A. is a distance $\bar{y}_p$ above the base, then

$$\bar{y}_p \times 12 = \tfrac{1}{2}(1200 + 1656)$$

$$\therefore \qquad \bar{y}_p = 119 \text{ mm}$$

The fully plastic moment is then obtained by considering the moments of forces on convenient rectangular parts of the section, each being subjected to a uniform stress σ_y,

i.e.
$$M_{FP} = \left[\sigma_y(100 \times 12)(31 - 6) + \sigma_y(31 - 12) \times 12 \times \frac{(31-12)}{2} + \sigma_y(119 \times 12)\frac{119}{2}\right] 10^{-9}$$

$$= \sigma_y(30\,000 + 2166 + 84\,966)10^{-9}$$

$$= 0.117 \times 10^{-3}\sigma_y$$

$$\therefore \qquad \text{shape factor} = \frac{M_{FP}}{M_E} = \frac{0.117 \times 10^{-3}}{0.065 \times 10^{-3}} = \mathbf{1.8}$$

(b) For this part of the question the load on the cantilever is such that yielding has progressed to a depth of 25 mm over the lower part of the web. It has been shown in §3.4 that whilst plastic penetration proceeds, the N.A. of the section moves and is always positioned by the rule:

compressive force above N.A. = tensile force below N.A.

Thus if the partially plastic N.A. is positioned a distance y above the extremity of the yielded area as shown in Fig. 3.39, the forces exerted on the various parts of the section may be established (proportions of the stress distribution diagram being used to determine the various values of stress noted in the figure).

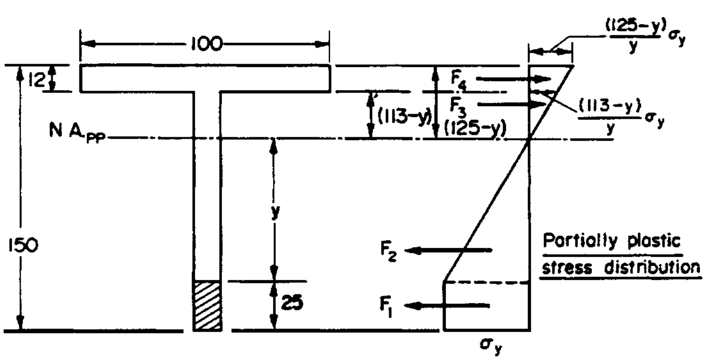

Fig. 3.39.

Force on yielded area $F_1 = \text{stress} \times \text{area}$

$$= 225 \times 10^6(12 \times 25 \times 10^{-6})$$

$$= 67.5 \text{ kN}$$

Force on elastic portion of web below N.A.

$$F_2 = \text{average stress} \times \text{area}$$

$$= \frac{225 \times 10^6}{2}(12 \times y \times 10^{-6})$$

$$= 1.35y \text{ kN}$$

where y is in millimetres.

Force in web above N.A.

$$F_3 = \text{average stress} \times \text{area}$$

$$= \frac{(113 - y)}{2y}(225 \times 10^6)(113 - y)12 \times 10^{-6}$$

$$= 1.35\frac{(113 - y)^2}{y} \text{ kN}$$

Force in flange

$$F_4 = \text{average stress} \times \text{area}$$

$$= \frac{1}{2}\left[\frac{(113 - y)}{y} + \frac{(125 - y)}{y}\right](225 \times 10^6)100 \times 12 \times 10^{-6} \text{ approximately}$$

$$= \frac{(238 - 2y)}{2y} 225 \times 10^6 \times 100 \times 12 \times 10^{-6}$$

$$= 135\frac{(238 - 2y)}{y} \text{ kN}$$

Now for the resultant force across the section to be zero,

$$F_1 + F_2 = F_3 + F_4$$

$$67.5 + 1.35y = \frac{1.35(113 - y)^2}{y} + \frac{135(238 - 2y)}{y}$$

$$\therefore \quad 67.5y + 1.35y^2 = 17.24 \times 10^3 - 305y + 1.35y^2 + 32.13 \times 10^3 - 270y$$

$$642.5y = 49\,370$$

$$y = 76.8 \text{ mm}$$

Substituting back,

$$F_1 = 67.5 \text{ kN} \quad F_2 = 103.7 \text{ kN}$$

$$F_3 = 23 \text{ kN} \quad F_4 = 148.1 \text{ kN}$$

The moment of resistance of the beam can now be obtained by taking the moments of these forces about the N.A. Here, for ease of calculation, it is assumed that F_4 acts at the mid-point of the web. This, in most cases, is sufficiently accurate for practical purposes.

$$\text{Moment of resistance} = \left\{F_1(y+12.5) + F_2\left(\frac{2y}{3}\right) + F_3\left[\frac{2}{3}(113-y)\right]\right.$$

$$\left.+ F_4[(113-y)+6]\right\} 10^{-3} \text{ kNm}$$

$$= (6030 + 5312 + 554 + 6243)10^{-3} \text{ kNm}$$

$$= 18.14 \text{ kNm}$$

Now the maximum B.M. present on a cantilever carrying a u.d.l. is $wL^2/2$ at the support

$$\therefore \qquad \frac{wL^2}{2} = 18.15 \times 10^3$$

The maximum u.d.l. which can be carried by the cantilever is then

$$w = \frac{18.15 \times 10^3 \times 2}{4} = \mathbf{9.1\ kN/m}$$

Example 3.3

(a) A steel beam of rectangular section, 80 mm deep by 30 mm wide, is simply supported over a span of 1.4 m and carries a u.d.l. w. If the yield stress of the material is 240 MN/m^2, determine the value of w when yielding of the beam material has penetrated to a depth of 20 mm from each surface of the beam.

(b) What will be the magnitudes of the residual stresses which remain when load is removed?

(c) What external moment must be applied to the unloaded beam in order to return it to its undeformed (straight) position?

Solution

(a) From eqn. (3.2) the partially plastic moment carried by a rectangular section is given by

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

Thus, for the simply supported beam carrying a u.d.l., the maximum B.M. will be at the centre of the span and given by

$$BM_{\max} = \frac{wL^2}{8} = \frac{B\sigma_y}{12}[3D^2 - d^2]$$

$$\therefore \qquad w = \frac{8 \times 30 \times 10^{-3} \times 240 \times 10^6}{1.4^2 \times 12}[3 \times 80^2 - 40^2]10^{-6}$$

$$= \mathbf{43.1\ kN/m}$$

(b) From the above working

$$M_{pp} = \frac{B\sigma_y}{12}[3D^2 - d^2] = \frac{wL^2}{8}$$

$$= 43.1 \times 10^3 \times \frac{1.4^2}{8} = 10.6 \text{ kNm}$$

During the unloading process a moment of equal value but opposite sense is applied to the beam assuming it to be completely elastic. Thus the equivalent maximum elastic stress σ' introduced at the outside surfaces of the beam by virtue of the unloading is given by the simple bending theory with $M = M_{pp} = 10.6$ kNm,

i.e.
$$\sigma' = \frac{My}{I} = \frac{10.6 \times 10^3 \times 40 \times 10^{-3} \times 12}{30 \times 80^3 \times 10^{-12}}$$

$$= 0.33 \times 10^9 = 330 \text{ MN/m}^2$$

The unloading, elastic stress distribution is then linear from zero at the N.A. to ± 330 MN/m^2 at the outside surfaces, and this may be subtracted from the partially plastic loading stress distribution to yield the residual stresses as shown in Fig. 3.40.

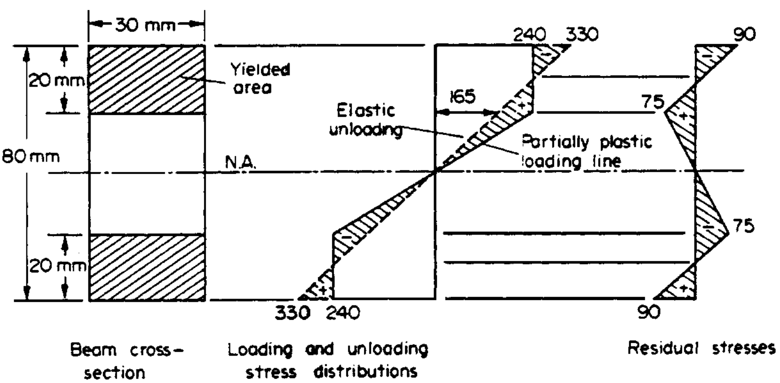

Fig. 3.40.

(c) The residual stress distribution of Fig. 3.40 indicates that the central portion of the beam, which remains elastic throughout the initial loading process, is subjected to a residual stress system when the beam is unloaded from the partially plastic state. The beam will therefore be in a deformed state. In order to remove this deformation an external moment must be applied of sufficient magnitude to return the elastic core to its unstressed state. The required moment must therefore introduce an elastic stress distribution producing stresses of ± 75 MN/m^2 at distances of 20 mm from the N.A. Thus, applying the bending theory,

$$M = \frac{\sigma I}{y} = \frac{75 \times 10^6}{20 \times 10^{-3}} \times \frac{30 \times 80^3 \times 10^{-12}}{12}$$

$$= \mathbf{4.8\ kNm}$$

Alternatively, since a moment of 10.6 kNm produces a stress of 165 MN/m^2 at 20 mm from the N.A., then, by proportion, the required moment is

$$M = 10.6 \times \frac{75}{165} = \mathbf{4.8\ kNm}$$

Example 3.4

A solid circular shaft, of diameter 50 mm and length 300 mm, is subjected to a gradually increasing torque T. The yield stress in shear for the shaft material is 120 MN/m^2 and, up to the yield point, the modulus of rigidity is 80 GN/m^2.

(a) Determine the value of T and the associated angle of twist when the shaft material first yields.

(b) If, after yielding, the stress is assumed to remain constant for any further increase in strain, determine the value of T when the angle of twist is increased to twice that at yield.

Solution

(a) For this part of the question the shaft is elastic and the simple torsion theory applies,

i.e. $$T = \frac{\tau J}{R} = \frac{120 \times 10^6}{25 \times 10^{-3}} \times \frac{\pi(25 \times 10^{-3})^4}{2} = 2950$$

$$= \mathbf{2.95\ kNm}$$

$$\theta = \frac{\tau L}{GR} = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times 25 \times 10^{-3}} = 0.018 \text{ radian}$$

$$= \mathbf{1.03^\circ}$$

If the torque is now increased to double the angle of twist the shaft will yield to some radius R_1. Applying the torsion theory to the elastic core only,

$$\theta = \frac{\tau L}{GR}$$

i.e. $$2 \times 0.018 = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{80 \times 10^9 \times R_1}$$

$\therefore$ $$R_1 = \frac{120 \times 10^6 \times 300 \times 10^{-3}}{2 \times 0.018 \times 80 \times 10^9} = 0.0125 = 12.5 \text{ mm}$$

Therefore partially plastic torque, from eqn. (3.12),

$$= \frac{\pi \tau_y}{6}[4R^3 - R_1^3]$$

$$= \frac{\pi \times 120 \times 10^6}{6}[4 \times 25^3 - 12.5^3]10^{-9}$$

Example 3.5

A 50 mm diameter steel shaft is case-hardened to a depth of 2 mm. Assuming that the inner core remains elastic up to a yield stress in shear of 180 MN/m^2 and that the case can also be assumed to remain elastic up to failure at the shear stress of 320 MN/m^2, calculate:

(a) the torque required to initiate yielding at the outside surface of the case;
(b) the angle of twist per metre length at this stage.

Take $G = 85$ GN/m^2 for both case and core whilst they remain elastic.

Solution

Since the modulus of rigidity G is assumed to be constant throughout the shaft whilst elastic, the angle of twist θ will be constant.

The stress distribution throughout the shaft cross-section at the instant of yielding of the outside surface of the case is then as shown in Fig. 3.41, and it is evident that whilst the failure stress of 320 MN/m^2 has only just been reached at the outside of the case, the yield stress of the core of 180 NM/m^2 has been exceeded beyond a radius r producing a fully plastic annulus and an elastic core.

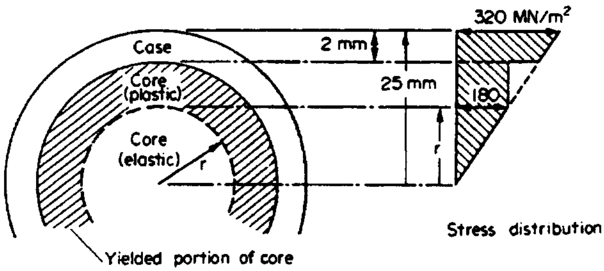

Fig. 3.41.

By proportions, since $G_{\text{case}} = G_{\text{core}}$, then

$$\left(\frac{\tau}{r}\right)_{\text{case}} = \left(\frac{\tau}{r}\right)_{\text{core}}$$

$$\frac{180}{r} = \frac{320}{25}$$

$$\therefore \qquad r = \frac{180}{320} \times 25 = 14.1 \text{ mm}$$

The shaft can now be considered in three parts:

(i) A solid elastic core of 14.1 mm external radius;
(ii) A fully plastic cylindrical region between $r = 14.1$ mm and $r = 23$ mm;
(iii) An elastic outer cylinder of external diameter 50 mm and thickness 2 mm.

$$\text{Torque on elastic core} = \frac{\tau_y J}{R} = \frac{180 \times 10^6}{14.1 \times 10^{-3}} \times \frac{\pi(14.1 \times 10^{-3})^4}{2}$$

$$= 793 \text{ Nm} = 0.793 \text{ kNm}$$

Torque on plastic section $= 2\pi\tau_y \int_{r_1}^{r_2} r^2\,dr$

$$= \frac{2\pi \times 180 \times 10^6}{3}[23^3 - 14.1^3]10^{-9}$$

$$= \frac{2\pi \times 180 \times 10^6 \times 9364 \times 10^{-9}}{3}$$

$= 3.53$ kNm

Torque on elastic outer case $= \frac{\tau_y J}{r} = \frac{320 \times 10^6}{25 \times 10^{-3}}\pi \left[\frac{25^4 - 23^4}{2}\right] 10^{-12}$

$= 2.23$ kNm

Therefore total torque required $= (0.793 + 3.53 + 2.23)10^3$

$=$ **6.55 kNm**

Since the angle of twist is assumed constant across the whole shaft its value may be determined by application of the simple torsion theory to either the case or the elastic core.

For the case: $\frac{\theta}{L} = \frac{\tau}{GR} = \frac{320 \times 10^6}{85 \times 10^9 \times 25 \times 10^{-3}}$

$= 0.15$ rad $=$ **8.6°**

Example 3.6

A hollow circular bar of 100 mm external diameter and 80 mm internal diameter (Fig. 3.42) is subjected to a gradually increasing torque T. Determine the value of T:

(a) when the material of the bar first yields;
(b) when plastic penetration has occurred to a depth of 5 mm;
(c) when the section is fully plastic.

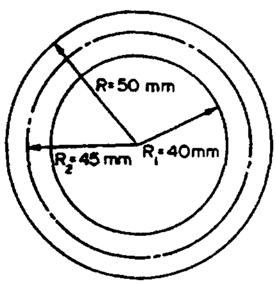

Fig. 3.42.

The yield stress in shear of the shaft material is 120 MN/m^2.

Determine the distribution of the residual stresses present in the shaft when unloaded from conditions (b) and (c).

Solution

(a) Maximum elastic torque from eqn. (3.11)

$$= \frac{\pi\tau_y}{2R}[R^4 - R_1^4] = \frac{\pi \times 120 \times 10^6}{2 \times 50 \times 10^{-3}}(625 - 256)10^{-8}$$

$$= 13900 \text{ Nm} = \mathbf{13.9\ kNm}$$

(b) Partially plastic torque, from eqns. (3.11) and (3.13),

$$= \frac{\pi\tau_y}{2R_2}[R_2^4 - R_1^4] + \frac{2\pi\tau_y}{3}[R^3 - R_2^3]$$

$$= \frac{\pi \times 120 \times 10^6}{2 \times 45 \times 10^{-3}}(4.5^4 - 256)10^{-8} + \frac{2\pi \times 120 \times 10^6}{3}(125 - 91)10^{-6}$$

$$= 6450 + 8550 = 15000 \text{ Nm} = \mathbf{15\ kNm}$$

(c) Fully plastic torque from eqn. (3.16) or eqn. (3.13)

$$= \frac{2\pi\tau_y}{3}[R^3 - R_1^3]$$

$$= \frac{2\pi \times 120 \times 10^6}{3}[125 - 64]10^{-6} = 15330 = \mathbf{15.33\ kNm}$$

In order to determine the residual stresses after unloading, the unloading process is assumed completely elastic.

Thus, unloading from condition (b) is equivalent to applying a moment of 15 kNm of opposite sense to the loading moment on a complete elastic bar. The effective stress introduced at the outer surface by this process is thus given by the simple torsion theory

$$\frac{T}{J} = \frac{\tau}{R}$$

i.e.

$$\tau = \frac{TR}{J} = \frac{15 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi \times (50^4 - 40^4)10^{-12}}$$

$$= \frac{15 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi(5^4 - 4^4)10^{-8}}$$

$$= 129 \text{ MN/m}^2$$

The unloading stress distribution is then linear, from zero at the centre of the bar to 129 MN/m^2 at the outside. This can be subtracted from the partially plastic loading stress distribution as shown in Fig. 3.43 to produce the residual stress distribution shown.

Similarly, unloading from the fully plastic state is equivalent to applying an elastic torque of 15.33 kNm of opposite sense. By proportion, from the above calculation,

$$\text{equivalent stress at outside of shaft on unloading} = \frac{15.33}{15} \times 129 = 132 \text{ MN/m}^2$$

Subtracting the resulting unloading distribution from the fully plastic loading one gives the residual stresses shown in Fig. 3.44.

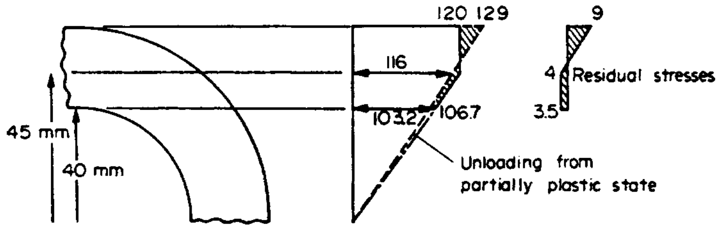

Fig. 3.43.

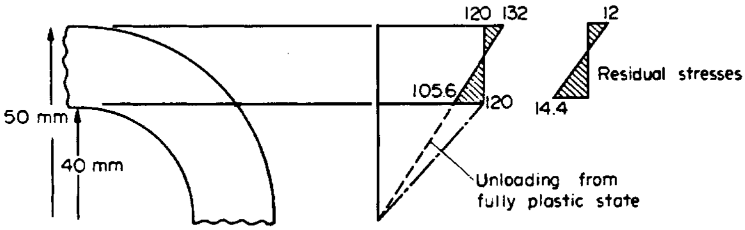

Fig. 3.44.

Example 3.7

(a) A thick cylinder, inside radius 62.5 mm and outside radius 190 mm, forms the pressure vessel of an isostatic compacting press used in the manufacture of ceramic components. Determine, using the Tresca theory of elastic failure, the safety factor on initial yield of the cylinder when an internal working pressure P_W of 240 MN/m^2 is applied.

(b) In view of the relatively low value of the safety factor which is achieved at this working pressure the cylinder is now subjected to an autofrettage pressure P_A of 580 MN/m^2. Determine the residual stresses produced at the bore of the cylinder when the autofrettage pressure is removed and hence determine the new value of the safety factor at the bore when the working pressure $P_W = 240$ MN/m^2 is applied.

The yield stress of the cylinder material is $\sigma_y = 850$ MN/m^2 and axial stresses may be ignored.

Solution

(a) Plain cylinder – working conditions $K = 190/62.5 = 9.24$

From eqn (10.5)†

$$\sigma_{rr} = -P\left[\frac{(R_2/r)^2 - 1}{K^2 - 1}\right] = \frac{-240}{8.24}\left[\frac{0.19^2}{r^2} - 1\right]$$

$$= -240 \text{ MN/m}^2 \text{ at the bore surface } (r = 0.0625 \text{ mm})$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

and from eqn. (10.6)[†]

$$\sigma_{\theta\theta} = P\left[\frac{(R_2/r)^2+1}{K^2-1}\right] = \frac{240}{8.24}\left[\frac{0.19^2}{r^2}+1\right]$$

$$= 298.3 \text{ MN/m}^2 \text{ at the bore surface}$$

Thus, assuming axial stress will be the intermediate stress (σ_2) value, the critical stress conditions for the cylinder at the internal bore are $\sigma_1 = 298.3$ MN/m^2 and $\sigma_3 = -240$ MN/m^2.
$\therefore$ Applying the Tresca theory of failure ($\sigma_1 - \sigma_3 = \sigma_y/n$)

$$\text{Safety factor } n = \frac{850}{298.3-(-240)} = \mathbf{1.58}$$

(b) Autofrettage conditions

From eqn 3.37 the radius R_p of the elastic/plastic interface under autofrettage pressure of 580 MN/m^2 will be given by:

$$P_A = \frac{\sigma_y}{2}\left[\frac{K^2-m^2}{K^2}\right] + \sigma_y \log_e m$$

$$\therefore \qquad 580\times10^6 = \frac{850\times10^6}{2}\left[\frac{3.04^2-m^2}{3.04^2}\right] + 850\times10^6 \log_e m$$

By trial and error:

m	$850 \log_e m$	$\frac{850}{2}\left[\frac{3.04^2-m^2}{3.04^2}\right]$	P_A
1.6	399.5	307.3	706.8
1.4	286.0	334.8	620.8
1.3	223.0	347.3	570.3
1.33	242.4	343.6	585.6
1.325	239.2	344.2	583.4

$\therefore$ to a good approximation $m = 1.325 = R_p/R_1$

$$\therefore R_p = 1.325 \times 62.5 = 82.8 \text{ mm}$$

$\therefore$ From eqns 3.38 and 3.39 stresses in the plastic zone are:

$$\sigma_{rr} = 850\times10^6\left[\log_e\left(\frac{r}{82.8}\right) - \frac{1}{2\times190^2}(190^2-82.8^2)\right]$$

$$= 850\times10^6[\log_e(r/82.8) - 0.405]$$

and $$\sigma_{\theta\theta} = 850\times10^6[\log_e(r/82.8) + 0.595]$$

$\therefore$ At the bore surface where $r = 62.5$ mm the stresses due to autofrettage are:

$$\sigma_{rr} = -580 \text{ MN/m}^2 \text{ and } \sigma_{\theta\theta} = 266.7 \text{ MN/m}^2.$$

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Residual stresses are then obtained by *elastic* unloading of the autofrettage pressure, i.e. by applying $\sigma_{rr} = +580\ \text{MN/m}^2$ at the bore in eqns (10.5) and (10.6)†; i.e. by proportions:

$$\sigma_{rr} = 580\ \text{MN/m}^2 \text{ and } \sigma_{\theta\theta} = -298.3 \times \frac{580}{240} = -721\ \text{MN/m}^2.$$

Giving residual stresses at the bore of:

$$\sigma'_{rr} = 580 - 580 = \mathbf{0}$$

$$\sigma'_{\theta\theta} = 266.7 - 721 = \mathbf{-453\ MN/m^2}$$

Working stresses are then obtained by the addition of elastic loading stresses due to an internal working pressure of 240 MN/m^2

i.e. from part (a) $\sigma_{rr} = -240\ \text{MN/m}^2, \sigma_{\theta\theta} = 298.3\ \text{MN/m}^2$

∴ final working stresses are:

$$\sigma_{rr_w} = 0 - 240 = -240\ \text{MN/m}^2$$

$$\sigma_{\theta\theta_w} = 298.3 - 454.3 = -156\ \text{MN/m}^2.$$

∴ New safety factor according to Tresca theory

$$n = \frac{850}{-156 - (-240)} = \mathbf{10.1}.$$

N.B. It is unlikely that the Tresca theory will give such a high value in practice since the axial working stress (ignored in this calculation) may well become the major principal stress σ_1 in the working condition and increase the magnitude of the denominator to reduce the resulting value of n.

Problems

3.1 (A/B). Determine the shape factors for the beam cross-sections shown in Fig. 3.45, in the case of section (c) treating the section both with and without the dotted area. [1.23, 1.81, 1.92, 1.82.]

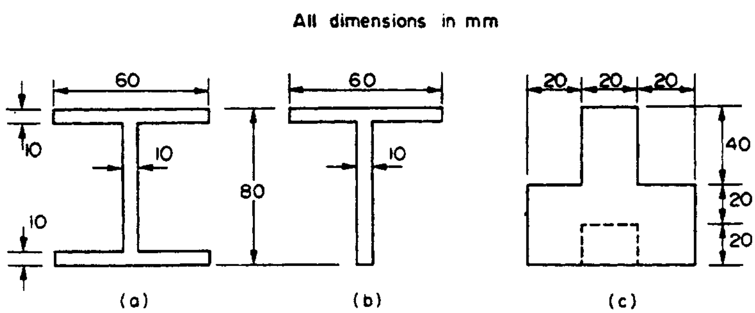

Fig. 3.45.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

3.2 (B). A 50 mm × 20 mm rectangular-section beam is used simply supported over a span of 2 m. If the beam is used with its long edges vertical, determine the value of the central concentrated load which must be applied to produce initial yielding of the beam material.

If this load is then increased by 10% determine the depth to which yielding will take place at the centre of the beam span.

Over what length of beam has yielding taken place?

What are the maximum deflections for each load case? Take $\sigma_y = 225$ MN/m^2 and $E = 206.8$ GN/m^2.

[1.5 kN; 1.05 mm; 180 mm; 36.3, 55.5 mm.]

3.3 (B). A steel bar of rectangular section 72 mm × 30 mm is used as a simply supported beam on a span of 1.2 m and loaded at mid-span. If the yield stress is 280 MN/m^2 and the long edges of the section are vertical, find the loading when yielding first occurs.

Assuming that a further increase in load causes yielding to spread inwards towards the neutral axis, with the stress in the yielded part remaining at 280 MN/m^2, find the load required to cause yielding for a depth of 12 mm at the top and bottom of the section at mid-span, and find the length of beam over which yielding has occurred.

[24.2 kN; 31 kN; 0.264 m.]

3.4 (B). A 300 mm × 125 mm I-beam has flanges 13 mm thick and web 8.5 mm thick. Calculate the shape factor and the moment of resistance in the fully plastic state. Take $\sigma_y = 250$ MN/m^2 and $I_{xx} = 85 \times 10^{-6}$ m^4.

[1.11, 141 kNm.]

3.5 (B). Find the shape factor for a 150 mm × 75 mm channel in pure bending with the plane of bending perpendicular to the web of the channel. The dimensions are shown in Fig. 3.46 and $Z = 21 \times 10^{-6}$ m^3.

[2.2.]

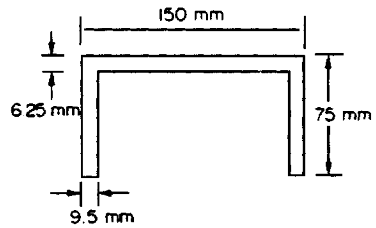

Fig. 3.46.

3.6 (B). A cantilever is to be constructed from a 40 mm × 60 mm T-section beam with a uniform thickness of 5 mm. The cantilever is to carry a u.d.l. over its complete length of 1 m. Determine the maximum u.d.l. that the cantilever can carry if yielding is permitted over the lower part of the web to a depth of 10 mm. $\sigma_y = 225$ MN/m^2.

[2433 N/m.]

3.7 (B). A 305 mm × 127 mm symmetrical I-section has flanges 13 mm thick and a web 5.4 mm thick. Treating the web and flanges as rectangles, calculate the bending moment of resistance of the cross-section (a) at initial yield, (b) for full plasticity of the flanges only, and (c) for full plasticity of the complete cross-section. Yield stress in simple tension and compression = 310 MN/m^2. What is the shape factor of the cross-section?

[167, 175.6, 188.7 kNm; 1.13.]

3.8 (B). A steel bar of rectangular section 80 mm by 40 mm is used as a simply supported beam on a span of 1.4 m and point-loaded at mid-span. If the yield stress of the steel is 300 MN/m^2 in simple tension and compression and the long edges of the section are vertical, find the load when yielding first occurs.

Assuming that a further increase in load causes yielding to spread in towards the neutral axis with the stress in the yielded part remaining constant at 300 MN/m^2, determine the load required to cause yielding for a depth of 10 mm at the top and bottom of the section at mid-span and find the length of beam over which yielding at the top and bottom faces will have occurred. [U.L.] [36.57, 44.6 kN; 0.232 m.]

3.9 (B). A straight bar of steel of rectangular section, 76 mm wide by 25 mm deep, is simply supported at two points 0.61 m apart. It is subjected to a uniform bending moment of 3 kNm over the whole span. Determine the depth of beam over which yielding will occur and make a diagram showing the distribution of bending stress over the full depth of the beam. Yield stress of steel in tension and compression = 280 MN/m^2.

Estimate the deflection at mid-span assuming $E = 200$ GN/m^2 for elastic conditions. [5.73, 44.4 mm.]

3.10 (B). A symmetrical I-section beam of length 6 m is simply supported at points 1.2 m from each end and is to carry a u.d.l. w kN/m run over its entire length. The second moment of area of the cross-section about the neutral axis parallel to the flanges is 6570 cm^4 and the beam cross-section dimensions are: flange width and thickness, 154 mm and 13 mm respectively, web thickness 10 mm, overall depth 254 mm.

(a) Determine the value of w to just cause initial yield, stating the position of the transverse section in the beam length at which it occurs.

(b) By how much must w be increased to ensure full plastic penetration of the flanges only, the web remaining elastic?

Take the yield stress of the beam material in simple tension and compression as 340 MN/m^2.

[B.P.] [195, 20 kN/m.]

3.11 (B). A steel beam of rectangular cross-section, 100 mm wide by 50 mm deep, is bent to the arc of a circle until the material just yields at the outer fibres, top and bottom. Bending takes place about the neutral axis parallel to the 100 mm side. If the yield stress for the steel is 330 MN/m^2 in simple tension and compression, determine the applied bending moment and the radius of curvature of the neutral layer. $E = 207$ GN/m^2.

Find how much the bending moment has to be increased so that the stress distribution is as shown in Fig. 3.47.

[I.Mech.E.] [13.75 kNm; 15.7 m; 16.23 kNm.]

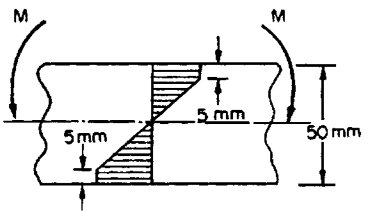

Fig. 3.47.

3.12 (B). A horizontal steel cantilever beam, 2.8 m long and of uniform I-section throughout, has the following cross-sectional dimensions: flanges 150 mm × 25 mm, web 13 mm thick, overall depth 305 mm. It is fixed at one end and free at the other.

(a) Determine the intensity of the u.d.l. which the beam has to carry across its entire length in order to produce fully developed plasticity of the cross-section.

(b) What is the value of the shape factor of the cross-section?

(c) Determine the length of the beam along the top and bottom faces, measured from the fixed end, over which yielding will occur due to the load found in (a).

Yield stress of steel = 330 MN/m^2. [106.2 kN/m; 1.16; 0.2 m.]

3.13 (B). A rectangular steel beam, 60 mm deep by 30 mm wide, is supported on knife-edges 2m apart and loaded with two equal point loads at one-third of the span from each end. Find the load at which yielding just begins, the yield stress of the material in simple tension and compression being 300 MN/m^2.

If the loads are increased to 25% above this value, estimate how far the yielding penetrates towards the neutral axis, assuming that the yield stress remains constant. [U.L.] [8.1 kN; 8.79 mm.]

3.14 (B). A steel bar of rectangular section, 72 mm deep by 30 mm wide, is used as a beam simply supported at each end over a span of 1.2 m and loaded at mid-span with a point load. The yield stress of the material is 280 MN/m^2. Determine the value of the load when yielding first occurs.

Find the load to cause an inward plastic penetration of 12 mm at the top and bottom of the section at mid-span. Also find the length, measured along the top and bottom faces, over which yielding has occurred, and the residual stresses present after unloading. [U.L.] [24.2 kN; 31 kN; 0.26 m, ∓ 79, ± 40.7 MN/m^2.]

3.15 (B). A symmetrical I-section beam, 300 mm deep, has flanges 125 mm wide by 13 mm thick and a web 8.5 mm thick. Determine:

(a) the applied bending moment to cause initial yield;

(b) the applied bending moment to cause full plasticity of the cross-section;

(c) the shape factor of the cross-section.

Take the yield stress = 250 MN/m^2 and assume $I = 85 \times 10^6$ mm^4.

[141×10^6 N mm; 156×10^6 N mm; 1.11.]

3.16 (B). A rectangular steel beam AB, 20 mm wide by 10 mm deep, is placed symmetrically on two knife-edges C and D, 0.5 m apart, and loaded by applying equal loads at the ends A and B. The steel follows a linear stress/strain law ($E = 200$ GN/m^2) up to a yield stress of 300 MN/m^2; at this constant stress considerable plastic deformation occurs. It may be assumed that the properties of the steel are the same in tension and compression.

Calculate the bending moment on the central part of the beam *CD* when yielding commences and the deflection at the centre relative to the supports.

If the loads are increased until yielding penetrates half-way to the neutral axis, calculate the new value of the bending moment and the corresponding deflection. [U.L.] [100 Nm, 9.375 mm; 137.5 Nm, 103 mm.]

3.17 (B). A steel bar of rectangular material, 75 mm × 25 mm, is used as a simply supported beam on a span of 2 m and is loaded at mid-span. The 75 mm dimension is placed vertically and the yield stress for the material is 240 MN/m^2. Find the load when yielding first occurs.

The load is further increased until the bending moment is 20% greater than that which would cause initial yield. Assuming that the increased load causes yielding to spread inwards towards the neutral axis, with the stress in the yielded part remaining at 240 MN/m^2, find the depth at the top and bottom of the section at mid-span to which the yielding will extend. Over what length of the beam has yielding occurred?

[B.P.] [11.25 kN; 8.45 mm; 0.33 m.]

3.18 (B). The cross-section of a beam is a channel, symmetrical about a vertical centre line. The overall width of the section is 150 mm and the overall depth 100 mm. The thickness of both the horizontal web and each of the vertical flanges is 12 mm. By comparing the behaviour in both the elastic and plastic range determine the shape factor of the section. Work from first principles in both cases. [1.806.]

3.19 (B). The T-section beam shown in Fig. 3.48 is subjected to increased load so that yielding spreads to within 50 mm of the lower edge of the flange. Determine the bending moment required to produce this condition. $\sigma_y = 240$ MN m^2. [44 kN m.]

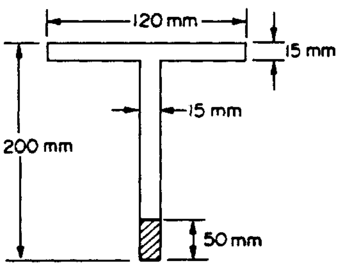

Fig. 3.48.

3.20 (B). A steel beam of I-section with overall depth 300 mm, flange width 125 mm and length 5 m, is simply supported at each end and carries a uniformly distributed load of 114 kN/m over the full span. Steel reinforcing plates 12 mm thick are welded symmetrically to the outside of the flanges producing a section of overall depth 324 mm. If the plate material is assumed to behave in an elastic-ideally plastic manner, determine the plate width necessary such that yielding has just spread through each reinforcing plate at mid-span under the given load.

Determine also the positions along the reinforcing plates at which the outer surfaces have just reached the yield point. At these sections what is the horizontal shearing stress at the interfaces of the reinforcing plates and the flanges?

Take the yield stress $\sigma_y = 300$ MN/m^2 and the second moment of area of the basic I-section to be 80×10^{-6} m^4.

[C.E.I.] [175 mm; 1.926 m; 0.94 MN/m^2.]

3.21 (B). A horizontal cantilever is propped at the free end to the same level as the fixed end. It is required to carry a vertical concentrated load *W* at any position between the supports. Using the normal assumption of plastic limit design, determine the least favourable position of the load. (Note that the calculation of bending moments under elastic conditions is not required.)

Hence calculate the maximum permissible value of *W* which may be carried by a rectangular-section cantilever with depth *d* equal to twice the width over a span *L*. Assume a load factor of *n* and a yield stress for the beam material σ_y. [0.586 *L* from built-in end; $d^3\sigma_y$ / 1.371 *Ln*.]

3.22 (B). (a) Sketch the idealised stress–strain diagram which is used to establish a quantitative relationship between stress and strain in the plastic range of a ductile material. Include the effect of strain-hardening.

(b) Neglecting strain–hardening, sketch the idealised stress–strain diagram and state, in words, the significance of any alteration you make in the diagram shown for part (a) when calculations are made, say, for pure bending beyond the yield point.

(c) A steel beam of rectangular cross-section, 200 mm wide × 100 mm deep, is bent to the arc of a circle, bending taking place about the neutral axis parallel to the 200 mm side.

Determine the bending moment to be applied such that the stress distribution is as shown in (i) Fig. 3.49(a) and (ii) Fig. 3.49(b).

Take the yield stress of steel in tension and compression as 250 MN/m^2. [B.P.] [98.3, 125 kN m.]

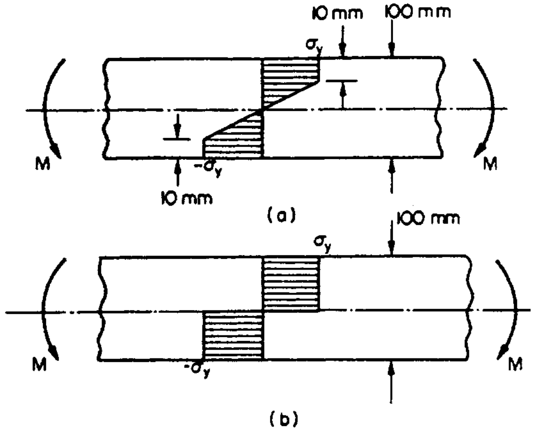

Fig. 3.49.

3.23 (B). (a) A rectangular section beam is 80 mm wide, 120 mm deep and is simply supported at each end over a span of 4 m. Determine the maximum uniformly distributed load that the beam can carry:

(i) if yielding of the beam material is permitted to a depth of 40 mm;
(ii) before complete collapse occurs.

(b) What residual stresses would be present in the beam after unloading from condition (a) (i)?

(c) What external moment must be applied to the beam to hold the deformed bar in a straight position after unloading from condition (a) (i)?

The yield stress of the material of the beam = 280 MN/m^2.

[B.P.] [38.8, 40.3 kN/m; ± 123, ± 146 MN/m^2; 84.3 kN m.]

3.24 (C). A rectangular beam 80 mm wide and 20 mm deep is constructed from a material with a yield stress in tension of 270 MN/m^2 and a yield stress in compression of 300 MN/m^2. If the beam is now subjected to a pure bending moment find the value required to produce:

(a) initial yield;
(b) initial yield on the compression edge;
(c) a fully plastic section. [1.44; 1.59, 2.27 kN m.]

3.25 (C). Determine the load factor of a propped cantilever carrying a concentrated load W at the centre.

Allowable working stress = 150 MN/m^2, yield stress = 270 MN/m^2. The cantilever is of I-section with dimensions 300 mm × 80 mm × 8 mm. [2.48.]

3.26 (C). A 300 mm × 100 mm beam is carried over a span of 7 m the ends being rigidly built in. Find the maximum point load which can be carried at 3 m from one end and the maximum working stress set up.

Take a load factor of 1.8 and σ_y = 240 MN/m^2.

$I = 85 \times 10^{-6}$ m^4 and the shape factor = 1.135. [100 kN; 172 MN/m^2.]

3.27 (C). A 300 mm × 125 mm I-beam is carried over a span of 20 m the ends being rigidly built in. Find the maximum point load which can be carried at 8 m from one end and the maximum working stress set up. Take a load factor of 1.8 and σ_y = 250 MN/m^2; $Z = 56.6 \times 10^{-5}$ m^3 and shape factor $\lambda = 1.11$.

[36 kN; 183 MN/m^2.]

3.28 (C). Determine the maximum intensity of loading that can be sustained by a simply supported beam, 75 mm wide × 100 mm deep, assuming perfect elastic–plastic behaviour with a yield stress in tension and compression of 135 MN/m^2. The beam span is 2 m.

What will be the distribution of residual stresses in the beam after unloading?

[50.6 kN/m; 67, 135, −67 MN/m^2.]

3.29 (C). A short column of 0.05 m square cross-section is subjected to a compressive load of 0.5 MN parallel to but eccentric from the central axis. The column is made from elastic – perfectly plastic material which has a yield stress in tension or compression of 300 MN/m^2. Determine the value of the eccentricity which will result in the section becoming just fully plastic. Also calculate the residual stress at the outer surfaces after elastic unloading from the fully plastic state. [10.4 mm; 250, 150 MN/m^2.]

3.30 (C). A rectangular beam 75 mm wide and 200 mm deep is constructed from a material with a yield stress in tension of 270 MN/m^2 and a yield stress in compression of 300 MN/m^2. If the beam is now subjected to a pure bending moment, determine the value of the moment required to produce (a) initial yield, (b) initial yield on the compression edge, (c) a fully plastic section. [135, 149.2, 213.2 kN m.]

3.31 (C). Figure 3.50 shows the cross-section of a welded steel structure which forms the shell of a gimbal frame used to support the ship-to-shore transport platform of a dock installation. The section is symmetrical about the vertical centre-line with a uniform thickness of 25 mm throughout.

As a preliminary design study what would you assess as the maximum bending moment which the section can withstand in order to prevent:

(a) initial yielding at any point in the structure if the yield stress for the material is 240 MN/m^2,
(b) complete collapse of the structure?

What would be the effect of adverse weather conditions which introduce instantaneous loads approaching, but not exceeding that predicted in (b). Quantify your answers where possible.

State briefly the factors which you would consider important in the selection of a suitable material for such a structure. [309.3 kN m; 423 kN m; local yielding, residual stress max = 279 MN/m^2.]

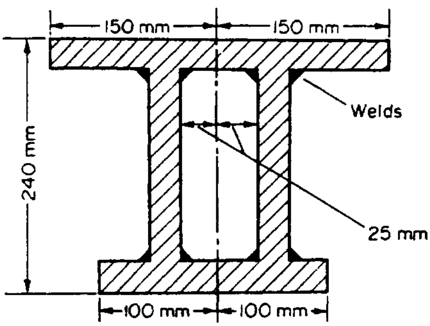

Fig. 3.50.

3.32 (B). A solid shaft 40 mm diameter is made of a steel the yield point of which in shear is 150 MN/m^2. After yielding, the stress remains constant for a very considerable increase in strain. Up to the yield point the modulus of rigidity $G = 80$ GN/m^2. If the length of the shaft is 600 mm calculate:

(a) the angle of twist and the twisting moment when the shaft material first yields;
(b) the twisting moment when the angle of twist is increased to twice that at yield. [3.32°; 1888, 2435 Nm.]

3.33 (B). A solid steel shaft, 76 mm diameter and 1.53 m long, is subjected to pure torsion. Calculate the applied torque necessary to cause initial yielding if the material has a yield stress in pure tension of 310 MN/m^2. Adopt the Tresca criterion of elastic failure.

(b) If the torque is increased to 10% above that at first yield, determine the radial depth of plastic penetration. Also calculate the angle of twist of the shaft at this increased torque. Up to the yield point in shear, $G = 83$ GN/m^2.

(c) Calculate the torque to be applied to cause the cross-section to become fully plastic.

[13.36 kNm; 4.26 mm, 0.085 rad; 17.8 kNm.]

3.34 (B). A hollow steel shaft having outside and inside diameters of 32 mm and 18 mm respectively is subjected to a gradually increasing axial torque. The yield stress in shear is reached at the surface when the torque is 1 kNm, the angle of twist per metre length then being 7.3°. Find the magnitude of the yield shear stress.

If the torque is increased to 1.1 kN m, calculate (a) the depth to which yielding will have penetrated, and (b) the angle of twist per metre length.

State any assumptions made and prove any special formulae used.

[U.L.] [172.7 MN/m^2; 1.8 mm; 8.22°.]

3.35 (B). A hollow shaft, 50 mm diameter and 25 mm bore, is made of steel with a yield stress in shear of 150 MN/m^2 and a modulus of rigidity of 83 GN/m^2. Calculate the torque and the angle of twist when the material first yields, if the shaft has a length of 2 m.

On the assumption that the yield stress, after initial yield, then remains constant for a considerable increase in strain, calculate the depth of penetration of plastic yield for an increase in torque of 10% above that at initial yield. Determine also the angle of twist of the shaft at the increased torque.

[U.L.] [3.45 kN m; 8.29°; 2.3 mm, 9.15°.]

3.36 (C). A steel shaft of length 1.25 m has internal and external diameters of 25 mm and 50 mm respectively. The shear stress at yield of the steel is 125 MN/m^2. The shear modulus of the steel is 80 GN/m^2. Determine the torque and overall twist when (a) yield first occurs, (b) the material has yielded outside a circle of diameter 40 mm, and (c) the whole section has just yielded. What will be the residual stresses after unloading from (b) and (c)?

[2.88, 3.33, 3.58 kN m; 0.0781, 0.0975, 0.1562 rad, (a) 19.7, −9.2, −5.5 MN/m^2, (b) 30.6, −46.75 MN/m^2.]

3.37 (B). A shaft having a diameter of 90 mm is turned down to 87 mm for part of its length. If a torque is applied to the shaft of sufficient magnitude just to produce yielding at the surface of the shaft in the unturned part, determine the depth of yielding which would occur in the turned part. Find also the angle of twist per unit length in the turned part to that in the unturned part of the shaft. [U.L.] [5.3 mm; 1.18.]

3.38 (B). A steel shaft, 90 mm diameter, is solid for a certain distance from one end but hollow for the remainder of its length with an inside diameter of 38 mm. If a pure torque is transmitted from one end of the shaft to the other of such a magnitude that yielding just occurs at the surface of the solid part of the shaft, find the depth of yielding in the hollow part of the shaft and the ratio of the angles of twist per unit length for the two parts of the shaft. [U.L.] [1.5 mm; 1.0345:1.]

3.39 (B). A steel shaft of solid circular cross-section is subjected to a gradually increasing torque. The diameter of the shaft is 76 mm and it is 1.22 m long. Determine for initial yield conditions in the outside surface of the shaft (a) the angle of twist of one end relative to the other, (b) the applied torque, and (c) the total resilience stored.

Assume a yield in shear of 155 MN/m^2 and a shear modulus of 85 GN/m^2. If the torque is increased to a value 10% greater than that at initial yield, estimate (d) the depth of penetration of plastic yielding and (e) the new angle of twist. [B.P.] [3.35°; 13.4 kN m; 391 J; 4.3 mm; 3.8°.]

3.40 (B). A solid steel shaft, 50 mm diameter and 1.22 m long, is transmitting power at 10 rev/s.

(a) Determine the power to be transmitted at this speed to cause yielding of the outer fibres of the shaft if the yield stress in shear is 170 MN/m^2.

(b) Determine the increase in power required to cause plastic penetration to a radial depth of 6.5 mm, the speed of rotation remaining at 10 rev/s. What would be the angle of twist of the shaft in this case? G for the steel is 82 GN/m^2. [B.P.] [262 kW, 52 kW, 7.83°.]

3.41 (B). A marine propulsion shaft of length 6 m and external diameter 300 mm is initially constructed from solid steel bar with a shear stress at yield of 150 MN/m^2.

In order to increase its power/weight ratio the shaft is machined to convert it into a hollow shaft with internal diameter 260 mm, the outer diameter remaining unchanged.

Compare the torques which may be transmitted by the shaft in both its initial and machined states:

(a) when yielding first occurs,

(b) when the complete cross-section has yielded.

If, in service, the hollow shaft is subjected to an unexpected overload during which condition (b) is achieved, what will be the distribution of the residual stresses remaining in the shaft after torque has been removed?

[795 kN m, 346 kN m, 1060 kN m, 370 kN m; −10.2, +11.2 MN/m^2.]

3.42 (C). A solid circular shaft 100 mm diameter is in an elastic–plastic condition under the action of a pure torque of 24 kN m. If the shaft is of steel with a yield stress in shear of 120 MN/m^2 determine the depth of the plastic zone in the shaft and the angle of twist over a 3 m length. Sketch the residual shear stress distribution on unloading. $G = 85$ GN/m^2. [0.95 mm; 4.95°.]

3.43 (C). A column is constructed from elastic – perfectly plastic material and has a cross-section 60 mm square. It is subjected to a compressive load of 0.8 MN parallel to the central longitudinal axis of the beam but eccentric from it. Determine the value of the eccentricity which will produce a fully plastic section if the yield stress of the column material is 280 MN/m^2.

What will be the values of the residual stresses at the outer surfaces of the column after unloading from this condition? [7 mm; 213, −97 MN/m^2.]

3.44 (C). A beam of rectangular cross-section with depth d is constructed from a material having a stress–strain diagram consisting of two linear portions producing moduli of elasticity E_1 in tension and E_2 in compression.

Assuming that the beam is subjected to a positive bending moment M and that cross-sections remain plane, show that the strain on the outer surfaces of the beam can be written in the form

$$\varepsilon_1 = \frac{d}{R}\left[\frac{\sqrt{E_2}}{\sqrt{E_1}+\sqrt{E_2}}\right]$$

where R is the radius of curvature.

Hence derive an expression for the bending moment M in terms of the elastic moduli, the second moment of area I of the beam section and R the radius of curvature.

$$\left[M = \frac{4E_1E_2I}{R(\sqrt{E_1}+\sqrt{E_2})^2}\right]$$

3.45 (C). Explain what is meant by the term "autofrettage" as applied to thick cylinder design. What benefits are obtained from autofrettage and what precautions should be taken in its application?

(b) A thick cylinder, inside radius 62.5 mm and outside radius 190 mm, forms the pressure vessel of an isostatic compacting press used in the manufacture of sparking plug components. Determine, using the Tresca theory of elastic failure, the safety factor on initial yield of the cylinder when an internal working pressure P_w of 240 MN/m^2 is applied.

(c) In view of the relatively low value of safety factor which is achieved at this working pressure, the cylinder is now subjected to an autofrettage pressure of $P_A = 580$ MN/m^2.

Determine the residual stresses produced at the bore of the cylinder when the autofrettage pressure is removed and hence determine the new value of the safety factor at the bore when the working pressure P_w is applied.

The yield stress of the cylinder material $\sigma_y = 850$ MN/m^2 and axial stresses may be ignored.

3.46 (C). A thick cylinder of outer radius 190 mm and radius ratio $K = 3.04$ is constructed from material with a yield stress of 850 MN/m^2 and tensile strength 1 GN/m^2. In order to prepare it for operation at a working pressure of 248 MN/m^2 it is subjected to an initial autofrettage pressure of 584 MN/m^2.

Ignoring axial stresses, compare the safety factors against initial yielding of the bore of the cylinder obtained with and without the autofrettage process. [1.53, 8.95.]

3.47 (C). What is the maximum autofrettage pressure which should be applied to a thick cylinder of the dimensions given in problem 3.46 in order to achieve yielding to the geometric mean radius?

Determine the maximum hoop and radial residual stresses produced by the application and release of this pressure and plot the distributions of hoop and radial residual stress across the cylinder wall.

[758 MN/m^2; -55.2 MN/m^2; -8.5 MN/m^2.]