CHAPTER 12

MISCELLANEOUS TOPICS

12.1. Bending of beams with initial curvature

The bending theory derived and applied in *Mechanics of Materials 1* was concerned with the bending of initially straight beams. Let us now consider the modifications which are required to this theory when the beams are initially curved before bending moments are applied. The problem breaks down into two classes:

(a) initially curved beams where the depth of cross-section can be considered small in relation to the initial radius of curvature, and
(b) those beams where the depth of cross-section and initial radius of curvature are approximately of the same order, i.e. deep beams with high curvature.

In both cases similar assumptions are made to those for straight beams even though some will not be strictly accurate if the initial radius of curvature is small.

(a) Initially curved slender beams

Consider now Fig. 12.1, with Fig. 12.1 (a) showing the initial curvature of the beam before bending, with radius R_1, and Fig. 12.1 (b) the state after the bending moment M has been applied to produce a new radius of curvature R_2. In both figures the radii are measured to the neutral axis.

The strain on any element $A'B'$ a distance y from the neutral axis will be given by:

$$\text{strain on } A'B' = \varepsilon = \frac{A'B' - AB}{AB}$$

$$= \frac{(R_2 + y)\theta_2 - (R_1 + y)\theta_1}{(R_1 + y)\theta_1}$$

$$= \frac{R_2\theta_2 + y\theta_2 - R_1\theta_1 - y\theta_1}{(R_1 + y)\theta_1}$$

Since there is no strain on the neutral axis in either figure $CD = C'D'$ and $R_1\theta_1 = R_2\theta_2$.

$$\therefore \qquad \varepsilon = \frac{y\theta_2 - y\theta_1}{(R_1 + y)\theta_1} = \frac{y(\theta_2 - \theta_1)}{(R_1 + y)\theta_1}$$

and, since $\theta_2 = R_1\theta_1/R_2$.

$$\varepsilon = \frac{y\theta_1\left(\dfrac{R_1}{R_2} - 1\right)}{(R_1 + y)\theta_1} = \frac{\mathbf{y(R_1 - R_2)}}{\mathbf{R_2(R_1 + y)}} \qquad (12.1)$$

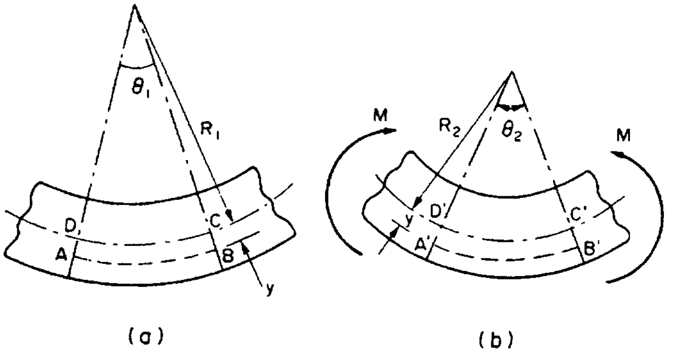


Fig. 12.1. Bending of beam with initial curvature (a) before bending, (b) after bending to new radius of curvature R_2.

For the case of slender, beams with y small in comparison with R_1 (i.e. when y can be neglected in comparison with R_1), the equation reduces to:

$$\varepsilon = y\frac{(R_1 - R_2)}{R_2R_1} = y\left[\frac{1}{R_2} - \frac{1}{R_1}\right] \tag{12.2}$$

The strain is thus directly proportional to y the distance from the neutral axis and, as for the case of straight beams, the stress and strain distribution across the beam section will be linear and the neutral axis will pass through the centroid of the section. Equation (12.2) can therefore be incorporated into a modified form of the "simple bending theory" thus:

$$\frac{M}{I} = \frac{\sigma}{y} = E\left[\frac{1}{R_2} - \frac{1}{R_1}\right] \tag{12.3}$$

For initially straight beams R_1 is infinite and eqn. (12.2) reduces to:

$$\varepsilon = \frac{y}{R_2} = \frac{y}{R}$$

(b) Deep beams with high initial curvature (i.e. small radius of curvature)

For deep beams where y can no longer be neglected in comparison with R_1 eqn. (12.1) must be fully applied. As a result, the strain distribution is no longer directly proportional to y and hence the stress and strain distributions across the beam section will be non-linear as shown in Fig. 12.2 and the neutral axis will not pass through the centroid of the section.

From eqn. (12.1) the stress at any point in the beam cross-section will be given by:

$$\sigma = E\varepsilon = \frac{Ey(R_1 - R_2)}{R_2(R_1 + y)} \tag{12.4}$$

For equilibrium of transverse forces across the section in the absence of applied end load $\int \sigma dA$ must be zero.

$$\therefore \quad \int \frac{Ey(R_1 - R_2)}{R_2(R_1 + y)}dA = \frac{E(R_1 - R_2)}{R_2}\int \frac{y}{(R_1 + y)} \cdot dA = 0 \tag{12.5}$$

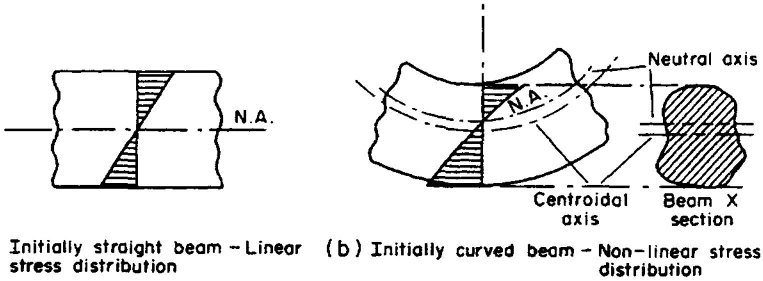


Fig. 12.2. Stress distributions across beams in bending. (a) Initially straight beam linear stress distribution; (b) initially curved deep beam-non-linear stress distribution.

i.e.
$$\int \frac{y}{(R_1+y)} \cdot dA = 0 \tag{12.6}$$

Unlike the case of bending of straight beams, therefore, it will be seen by inspection that the above integral no longer represents the first moment of area of the section about the centroid. Thus, *the centroid and the neutral axis can no longer coincide.*

The bending moment on the section will be given by:

$$M = \int \sigma \cdot dA \cdot y = \frac{E(R_1 - R_2)}{R_2} - \frac{y_2}{(R_1 + y)} \cdot dA \tag{12.7}$$

but
$$\int \frac{y^2}{(R_1 + y)} \cdot dA = \int \frac{y[(R_1 + y) - R_1]dA}{(R_1 + y)}$$
$$= \int y \cdot dA - R_1 \int \frac{y \cdot dA}{(R_1 + y)}$$

and from eqn. (12.5) the second integral term reduces to zero for equilibrium of transverse forces.

$\therefore$
$$\int \frac{y_2}{(R_1 + y)} \cdot dA = \int y \cdot dA = A\overline{y} = Ah$$

where h is the distance of the neutral axis from the centroid axis, see Fig. 12.3. Substituting in eqn. (12.7) we have:

$$M = \frac{E(R_1 - R_2)}{R_2} \cdot hA \tag{12.8}$$

From eqn. (12.4)

$$\frac{\sigma}{y}(R_1 + y) = \frac{E}{R_2}(R_1 - R_2)$$

$\therefore$
$$M = \frac{\sigma}{y}(R_1 + y)hA \tag{12.9}$$

i.e.
$$\frac{\sigma}{y} = \frac{M}{hA(R_1 + y)} \tag{12.10}$$

or
$$\sigma = \frac{My}{hA(R_1 + y)} = \frac{My}{hAR_0} \tag{12.11}$$

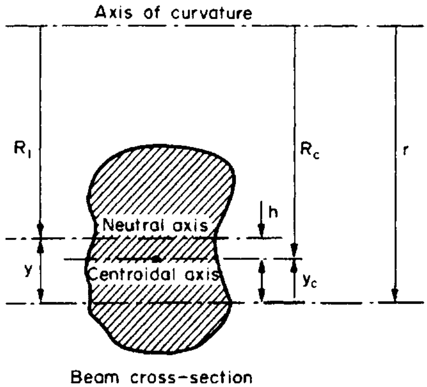


Fig. 12.3. Relative positions of neutral axis and centroidal axis.

On the opposite side of the neutral axis, where y will be negative, the stress becomes:

$$\sigma = -\frac{My}{hA(R_1 - y)} = -\frac{My}{hAR_i} \tag{12.12}$$

These equations show that the stress distribution follows a hyperbolic form. Equation (12.12) can be seen to be similar in form to the "simple bending" equation†.

$$\frac{\sigma}{y} = \frac{M}{I}$$

with the term $hA(R_1 + y)$ replacing the second moment of area I.

Thus in order to be able to calculate stresses in deep-section beams with high initial curvature, it is necessary to evaluate h and R_1, i.e. to locate the position of the neutral axis relative to the centroid or centroidal axis. This was shown above to be given by the condition:

$$\int \frac{y}{(R_1 + y)} \cdot dA = 0.$$

Now fibres distance y from the neutral axis will be some distance y_c from the centroidal axis as shown in Figs. 12.3 and 12.4 such that, in relation to the axis of curvature,

$$R_1 + y = R_c + y_c$$

with $$y = y_c + h$$

$\therefore$ from eqn. (12.5)

$$\int \frac{(y_c + h)}{(R_c + y_c)} \cdot dA = 0$$

Re-writing $$y_c + h = (R_c + y_c) - R_c + h = (R_c + y_c) - (R_c - h).$$

† Timoshenko and Roark both give details of correction factors which may be applied for standard cross-sectional shapes to be used in association with the simple straight beam equation. (S. Timoshenko, *Theory of Plates and Shells*, McGraw Hill, New York; R. J. Roark and W.C. Young, *Formulas for Stress and Strain*, McGraw Hill, New York).

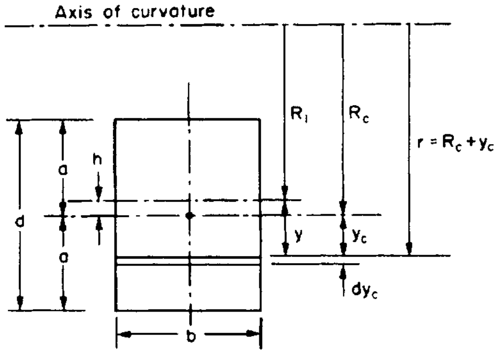


Fig. 12.4.

$$\int \frac{(y_c + h)}{(R_c + y_c)} = \int \frac{(R_c + y_c)}{(R_c + y_c)} \cdot dA - (R_c - h) \int \frac{1}{(R_c + y_c)} \cdot dA$$

$$= A - (R_c - h) \int \frac{1}{(R_c + y_c)} \cdot dA = 0$$

$\therefore$
$$h = R_c - \frac{A}{\displaystyle\int \frac{dA}{(R_c + y_c)}} = R_c - \frac{A}{\displaystyle\int \frac{dA}{r}} \tag{12.13}$$

and
$$R_1 = R_c - h = \frac{A}{\displaystyle\int \frac{dA}{(R_c + y_c)}} = \frac{A}{\displaystyle\int \frac{dA}{r}} \tag{12.14}$$

Examples 12.1 and 12.2 show how the theory may be applied and Table 12.1 gives some useful equations for $\int \frac{dA}{r}$ for standard shapes of beam cross-section.

Note

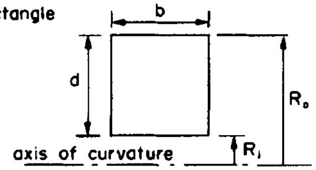
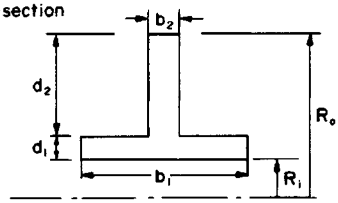
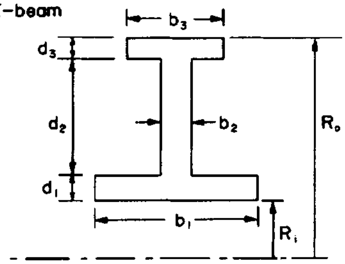
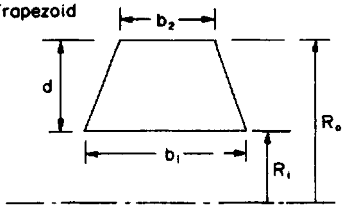
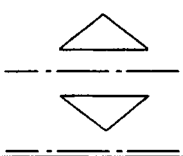
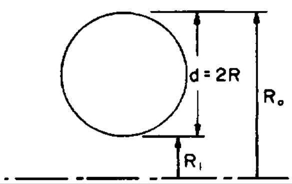
Before applying the above theory for bending of initially curved members it is perhaps appropriate to consider the benefits to be gained over that of an approximate solution using the simple bending theory.

Provided that the curvature is not large then the simple theory is reasonably accurate; for example, for a radius to beam depth ratio R_c/d of as low as 5 the error introduced in the maximum stress value is only of the order of 7%. The error then rises steeply, however, as curvature increases to a figure of approx. 30% at $R_c/d = 1.5$.

(c) Initially curved beams subjected to bending and additional direct load

In many practical engineering applications such as chain links, crane hooks, G-clamps etc., the component cross-sections will be subjected to both bending and additional direct load, whereas the equations derived in the previous sections have all been derived on the assumption of pure bending only. It is therefore necessary in such cases to obtain a solution by the application of the principle of superposition i.e. by resolving the loading system into

Table 12.1. Values of $\int \frac{dA}{r}$ for curved bars.

Cross-section	$\int \frac{dA}{r}$
(a) Rectangle 	$b \log_e \left(\frac{R_0}{R_i}\right)$ (N.B. The two following cross-sections are simply produced by the addition of terms of this form for each rectangular portion)
(b) T-section 	$b_1 \log_e \left(\frac{R_i + d_1}{R_i}\right) + b_2 \log_e \left(\frac{R_0}{R_i + d_1}\right)$
(c) I-beam 	$b_1 \log_e \left(\frac{R_i + d_1}{R_i}\right) + b_2 \log_e \left(\frac{R_0 - d_3}{R_i + d_1}\right) + b_3 \log_e \left(\frac{R_0}{R_0 - d_3}\right)$
(d) Trapezoid 	$\left[\frac{(b_1 R_0 - b_2 R_i)}{d} \log_e \left(\frac{R_0}{R_i}\right)\right] - b_1 + b_2$
(e) Triangle 	As above (d) with $b_2 = 0$ As above (d) with $b_1 = 0$
	$2\pi\{(R_i + R) - [(R_i + R)^2 - R^2]^{1/2}\}$

its separate bending, normal (and perhaps shear) loads on the section and combining the stress values obtained from the separate stress calculations. Normal and bending stresses may be added algebraically and combined with the shearing stresses using two- or three-dimensional complex stress equations or Mohr's circle.

Care must always be taken to consider the direction in which the moment is applied. In the derivation of the equations in the previous sections it has been shown acting in a direction to increase the initial curvature of the beam (Fig. 12.1) producing tensile bending stresses on the outside (convex) surface and compression on the inner (concave) surface. In the practical cases mentioned above, however, e.g. the chain link or crane hook, the moment which is usually applied will tend to straighten the beam and hence reduce its curvature. In these cases, therefore, tensile stresses will be set up on the inner surface and these will add to the tensile stresses produced by the direct load across the section to produce a maximum tensile (and potentially critical) stress condition on this surface – see Fig. 12.5.

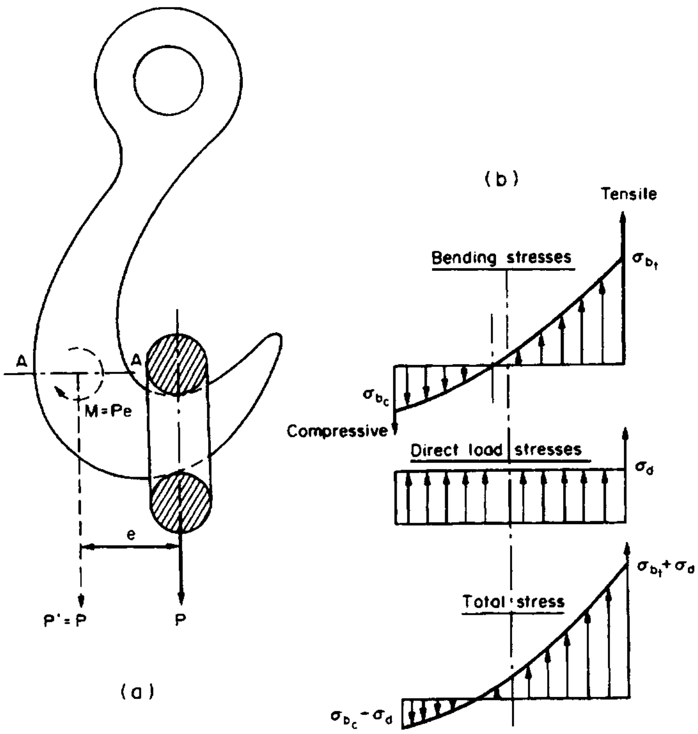


Fig. 12.5. Loading of a crane hook. (a) Load effect on section *AA* is direct load $P' = P$ plus moment $M = Pe$; (b) stress distributions across the section *AA*.

12.2. Bending of wide beams

The equations derived in *Mechanics of Materials 1* for the stress and deflection of beams subjected to bending relied on the assumption that the beams were narrow in relation to their depths in order that expansions or contractions in the lateral (z) direction could take place relatively freely.

For beams that are very wide in comparison with their depth – see Fig. 12.6 – lateral deflections are constrained, particularly towards the centre of the beam, and such beams become stiffer than predicted by the simple theory and deflections are correspondingly reduced. In effect, therefore, the bending of narrow beams is a plane stress problem whilst that of wide beams becomes a plane strain problem – see §8.22.
For the beam of Fig. 12.6 the strain in the z direction is given by eqn. (12.6) as:

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y).$$

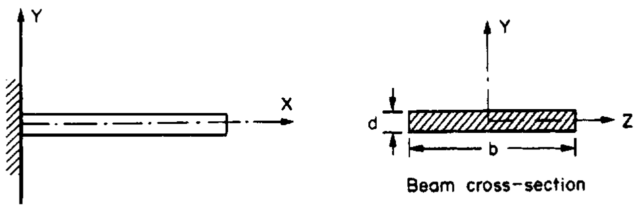


Fig. 12.6. Bending of wide beams ($b \gg d$)

Now for thin beams $\sigma_y = 0$ and, for total constraint of lateral (z) deformation at $z = 0$, $\varepsilon_z = 0$.

$\therefore$
$$0 = \frac{1}{E}(\sigma_z - \nu\sigma_x)$$

i.e.
$$\sigma_z = \nu\sigma_x$$

Thus, the strain in the longitudinal x direction will be:

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)\\ &= \frac{1}{E}(\sigma_x - 0 - \nu(\nu\sigma_x))\\ &= \frac{1}{E}(1-\nu^2)\sigma_x \qquad (12.15)\\ &= \frac{(1-\nu^2)}{E}\cdot\frac{My}{I} \qquad (12.16)\end{aligned}$$

Compared with the narrow beam case where $\varepsilon_x = \sigma_x/E$ there is thus a reduction in strain by the factor $(1 - \nu^2)$ and this can be introduced into the deflection equation to give:

$$\boldsymbol{\frac{d^2y}{dx^2} = (1-\nu^2)\frac{M}{EI}} \qquad (12.17)$$

Thus, all the formulae derived in Book 1 including those of the summary table, may be used for wide beams *provided that they are multiplied by* $(1 - \nu^2)$.

12.3. General expression for stresses in thin-walled shells subjected to pressure or self-weight

Consider the general shell or "surface of revolution" of arbitrary (but thin) wall thickness shown in Fig. 12.7 subjected to internal pressure. The stress system set up will be three-dimensional with stresses σ_1 (hoop) and σ_2 (meridional) in the plane of the surface and σ_3 (radial) normal to that plane. Strictly, all three of these stresses will vary in magnitude through the thickness of the shell wall but provided that the thickness is less than approximately one-tenth of the major, i.e. smallest, radius of curvature of the shell surface, this variation can be neglected as can the radial stress (which becomes very small in comparison with the hoop and meridional stresses).

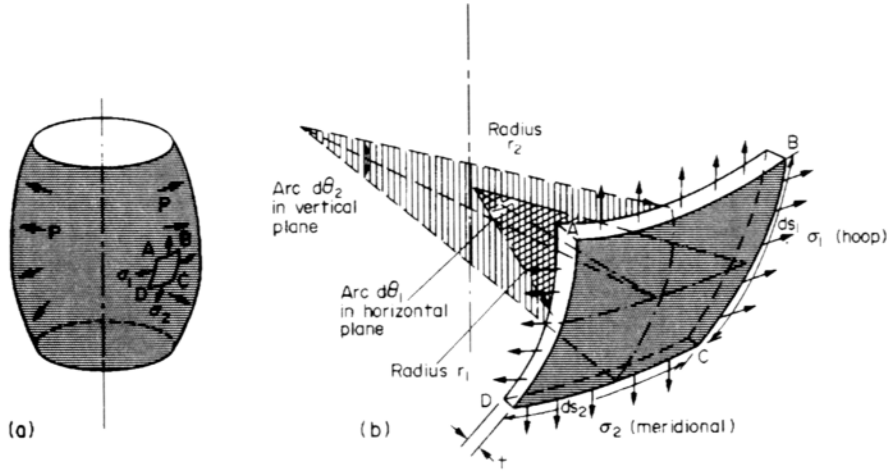


Fig. 12.7. (a) General surface of revolution subjected to internal pressure p; (b) element of surface with radii of curvature r_1 and r_2 in two perpendicular planes.

Because of this limitation on thickness, which makes the system statically determinate, the shell can be considered as a membrane with little or no resistance to bending. The stresses set up on any element are thus only the so-called "*membrane stresses*" σ_1 and σ_2 mentioned above, no additional bending stresses being required.

Consider, therefore, the equilibrium of the element ABCD shown in Fig. 12.7(b) where r_1 is the radius of curvature of the element in the horizontal plane and r_2 is the radius of curvature in the vertical plane.

The forces on the "vertical" and "horizontal" edges of the element are $\sigma_1 t ds_1$ and $\sigma_2 t ds_2$, respectively, and each are inclined relative to the radial line through the centre of the element, one at an angle $d\theta_1/2$ the other at $d\theta_2/2$.

Thus, resolving forces along the radial line we have, for an internal pressure p:

$$2(\sigma_1 t\, ds_1 \cdot \sin\frac{d\theta_1}{2} + \sigma_2 t\, ds_2 \cdot \sin\frac{d\theta_2}{2} = p \cdot ds_1 \cdot ds_2$$

Now for small angles $\sin d\theta/2 = d\theta/2$ radians

$$\therefore \quad 2\left(\sigma_1 t\, ds_1 \cdot \frac{d\theta_1}{2} + \sigma_2 t\, ds_2 \;\; \frac{d\theta_2}{2}\right) = p ds_1 \cdot ds_2$$

Also $ds_1 = r_2\,d\theta_2$ and $ds_2 = r_1\,d\theta_1$

$$\therefore \qquad \sigma_1\,t\,ds_1 \cdot \frac{ds_2}{r_1} + \sigma_2\,t\,ds_2 \frac{ds_1}{r_2} = p \cdot ds_1 \cdot ds_2$$

and dividing through by $ds_1 \cdot ds_2 \cdot t$ we have:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \frac{p}{t} \tag{12.18}$$

For a general shell of revolution, σ_1 and σ_2 will be unequal and a second equation is required for evaluation of the stresses set up. In the simplest application, i.e. that of the sphere, however, $r_1 = r_2 = r$ and symmetry of the problem indicates that $\sigma_1 = \sigma_2 = \sigma$. Equation (12.18) thus gives:

$$\sigma = \frac{pr}{2t}$$

In some cases, e.g. concrete domes or dishes, the self-weight of the vessel can produce significant stresses which contribute to the overall failure consideration of the vessel and to the decision on the need for, and amount of, reinforcing required. In such cases it is necessary to consider the vertical equilibrium of an element of the dome in order to obtain the required second equation and, bearing in mind that self-weight does not act radially as does applied pressure, eqn. (12.18) has to be modified to take into account the vertical component of the forces due to self-weight.

Thus for a dome of subtended arc 2θ with a force per unit area q due to self-weight, eqn. (12.18) becomes:

$$\frac{\sigma_1}{r_1} + \frac{\sigma_2}{r_2} = \pm\frac{q\cos\theta}{t} \tag{12.19}$$

Combining this equation with one obtained from vertical equilibrium considerations yields the required values of σ_1 and σ_2.

12.4. Bending stresses at discontinuities in thin shells

It is normally assumed that thin shells subjected to internal pressure show little resistance to bending so that only membrane (direct) stresses are set up. In cases where there are changes in geometry of the shell, however, such as at the intersection of cylindrical sections with hemispherical ends, the "incompatibility" of displacements caused by the membrane stresses in the two sections may give rise to significant local bending effects. At times these are so severe that it is necessary to introduce reinforcing at the junction locations.

Consider, therefore, such a situation as shown in Fig. 12.8 where both the cylindrical and hemispherical sections of the vessel are assumed to have uniform and equal thickness membrane stresses in the cylindrical portion are

$$\sigma_1 = \sigma_H = \frac{pr}{t} \quad \text{and} \quad \sigma_2 = \sigma_L = \frac{pr}{2t}$$

whilst for the hemispherical ends

$$\sigma_1 = \sigma_2 = \sigma_H = \frac{pr}{t}.$$

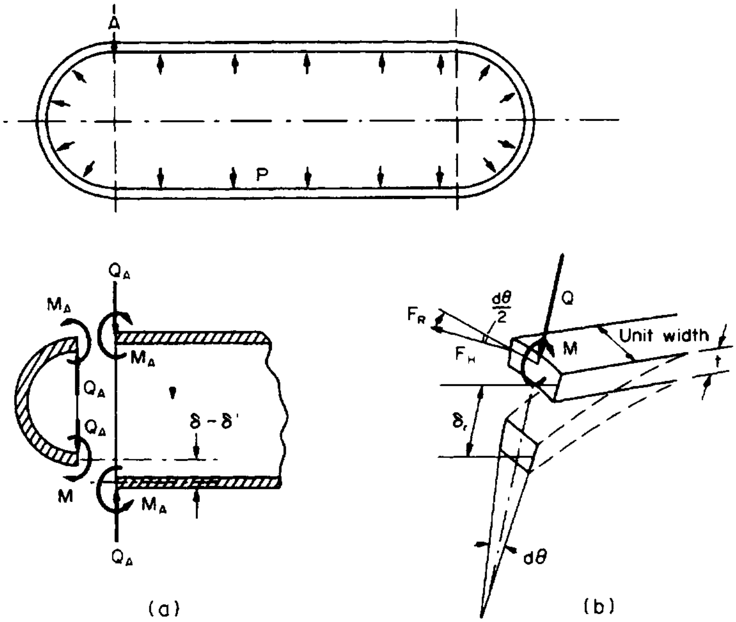


Fig. 12.8. Loading conditions at discontinuities in thin shells.

The radial displacements set up by these stress systems are; for the cylinder:

$$\delta = \frac{r}{E}(\sigma_H - \nu\sigma_L) = \frac{pr^2}{2tE}(2 - \nu)$$

and for the hemispherical ends:

$$\delta' = \frac{r}{E}(\sigma_H - \nu\sigma_H) = \frac{pr^2}{2tE}(1 - \nu).$$

There will thus be a difference in deformation radially of:

$$\delta - \delta' = \frac{pr^2}{2tE}[(2 - \nu) - 2(1 - \nu)]$$

$$= \frac{\nu pr^2}{2tE}$$

which can only be reacted by the introduction of shear forces and moments as shown in Fig. 12.8(a) where Q = shear force and M = moment, *both per unit length*.

Because of the total symmetry of the cylinder about its axis we may now consider bending of a small element of the cylinder of unit width as shown in Fig. 12.8 (b).

The shear stress Q produces inward bending of the elemental strip through a radial displacement δ_r and a compressive hoop or circumferential strain given by:

$$\varepsilon_H = \frac{\delta_r}{r}$$

with a corresponding hoop stress:

$$\sigma_H = \frac{E\delta_r}{r}$$

This stress sets up a force in the circumferential direction of

$$F_H = \sigma_H \times A = \frac{E\delta_r}{r} \times t \times 1.$$

This force has an outward radial component from both sides of the element of:

$$F_R = 2F_H \sin\frac{d\theta}{2} = 2F_H\frac{d\theta}{2} = \frac{2E\delta_r}{r}\frac{td\theta}{2}$$
$$= \frac{E\delta_r\ t\ d\theta}{r}$$

and since the strip is of unit width, $rd\theta = 1$

$$\therefore \qquad F_R = \frac{E\delta_r t}{r^2}$$

This force can be considered as a distributed load along the strip (since equal values will apply to all other unit lengths) and will act in opposition to the mis-match displacements caused by the membrane stresses.

It the strip were considered to be a simple beam then, the differential equation of bending would be:

$$\frac{EId^4y}{dx^4} = -\frac{E\delta_r t}{r^2}$$

but, as for the case of the deformation of circular plates in 7.2, the restraint on distortion produced by adjacent strips needs to be allowed for by replacing EI by the plate stiffness constant or flexural rigidity

$$D = \frac{Et^3}{12(1-\nu^2)}:$$

i.e.

$$D\frac{d^4y}{dx^4} = -\frac{E\delta_r t}{r^2}$$
$$= -\left[D \times 12\frac{(1-\nu^2)}{Et^3}\right]\frac{E\delta_r t}{r^2}$$
$$= -4D\beta^4\delta_r = -4D\beta^4 y \qquad (1)$$

where

$$\beta^4 = \frac{3(1-\nu^4)}{r^2t^2} \text{ and } y = \delta_r.$$

The solution to eqn. (1) is of the form:

$$y = \delta_r = e^{\beta x}(A_1\cos\beta x + A_2\sin\beta x) + e^{-\beta x}(A_3\cos\beta x + A_4\sin\beta x) \qquad (2)$$

Now as $x \to \infty$, $\delta_r \to \infty$ and $A_1 = A_2 = 0$.

At $x = 0$, $M = M_A$ and $D\dfrac{d^2y}{dx^2} = -M_A$.

At $x = 0$, $Q = Q_A$ and $D\dfrac{dy^3}{dx^3} = -Q_A$

Substituting these conditions into equation (2) gives:

$$A_3 = \frac{1}{2\beta^3 D}(Q_A - \beta M_A)$$

and

$$A_4 = \frac{M_A}{2\beta^2 D}$$

Substituting back into eqn. (2) we have:

$$y = \delta_r = \frac{e^{-\beta x}}{2\beta^3 D}[Q_A \cos\beta x - M_A\beta(\cos\beta x - \sin\beta x)] \tag{12.20}$$

which is the equation of a heavily damped oscillation, showing that significant values of σ_r, i.e. significant bending, will only be obtained at points local to the cylinder-end intersection. Any stiffening which is desired need, therefore, only to be local to the "joint".

In the special case where the material and the thickness are uniform throughout there will be no moment set up at the intersection A since the shear force Q_A will produce equal slopes and deflections in both the cylinder and the hemispherical end.

Bending stresses can be obtained from the normal relationship:

$$M = D\frac{d^2y}{dx^2}$$

i.e by differentiating equation (12.20) twice and by substitution of appropriate boundary conditions to determine the unknowns. For cases where the thickness is not constant throughout, and M therefore has a value, the conditions are:

(a) the sum of the deflections of the cylinder and the end at A must be zero,
(b) the slope or angle of rotation of the two parts at A must be equal.

12.5. Viscoelasticity

Certain materials, e.g., rubbers and plastics, exhibit behaviour which combines the characteristics of a viscous liquid and an elastic solid and the term which is used to describe this behaviour is "viscoelasticity". In the case of the elastic solid which follows Hooke's law (a "Hookean" solid) stress is linearily related to strain. For so-called "Newtonian" viscous liquids, however, stress is proportional to strain rate. If, therefore, a tensile test is carried out on a viscoelastic material the resulting stress-strain diagram will depend significantly on the rate of straining $\dot{\varepsilon}$, as shown in Fig. 12.9. Further, whilst the material may well recover totally from its strained position after release of loading it may do so along a different line from the loading line and stress will not be proportional to strain even within this "elastic" range.

One starting point for the mathematical consideration of the behaviour of viscoelastic materials is the derivation of a linear differential equation which, in its most general form, can be written as:

$$A\sigma = B\varepsilon$$

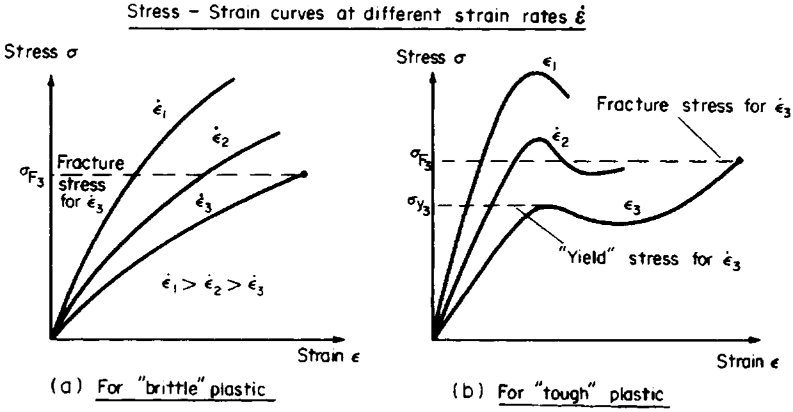


Fig. 12.9. Stress-strain curves at different strain rates $\dot{\varepsilon}$.

with A and B linear differential operators with respect to time, or as:

$$A_0\sigma + A_1\frac{d\sigma}{dt} + A_2\frac{d^2\sigma}{dt^2} + \ldots = B_0\varepsilon + B_1\frac{d\varepsilon}{dt} + B_2\frac{d^2\varepsilon}{dt^2} + \ldots \quad (12.21)$$

In most cases this equation can be simplified to two terms on either side of the expression, the first relating to stress (or strain) the second to its first differential. This will be shown below to be equivalent to describing viscoelastic behaviour by mechanical models composed of various configurations of springs and dashpots. The simplest of these models contain one spring and one dashpot only and are due to Voigt/Kelvin and Maxwell.

(a) Voigt–Kelvin Model

The behaviour of Hookean solids can be simply represented by a spring in which stress is directly and linearly related to strain,

i.e. $$\sigma_s = E\varepsilon_s$$

The Newtonian liquid, however, needs to be represented by a dashpot arrangement in which a piston is moved through the Newtonian fluid. The constant of proportionality relating stress to strain rate is then the coefficient of viscosity η of the fluid.

i.e. $$\sigma_D = \eta\dot{\varepsilon}_D \quad (12.22)$$

In order to represent a viscoelastic material, therefore, it is necessary to consider a suitable combination of spring and dashpot. One such arrangement, known as the *Voigt–Kelvin model*, combines the spring and dashpot in parallel as shown in Fig. 12.10.

The response of this model, i.e. the relationship between stress σ, strain ε and strain rate $\dot{\varepsilon}$ is given by:

$$\sigma = \sigma_s + \sigma_D$$

and since the strain is common to both parts of the parallel model $\varepsilon_S = \varepsilon_D = \varepsilon$

∴ $$\sigma = E\varepsilon + \eta\dot{\varepsilon} \quad (12.23)$$

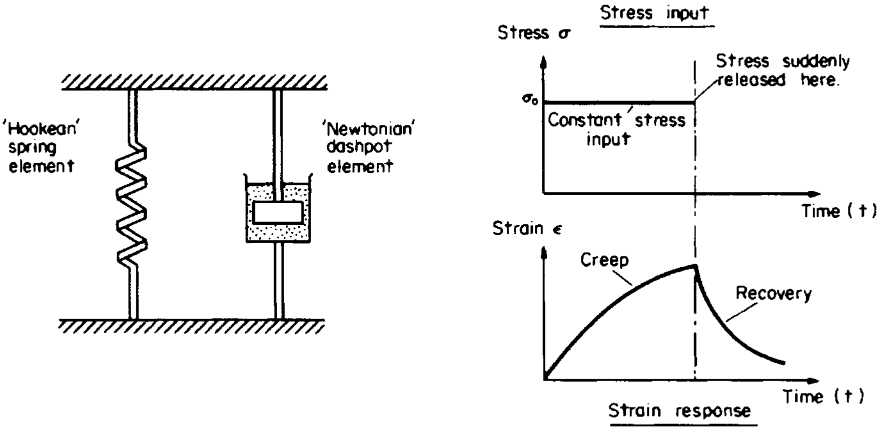


Fig. 12.10. Voigt–Kelvin spring/dashpot model with elements in parallel.

with the stress σ, in effect, shared between the two components of the model (the spring and the dashpot) as for any system of components in parallel.

The inclusion of the strain rate term $\dot{\varepsilon}$ makes the stress response time-dependent and this represents the principal difference in behaviour from that of elastic solids.

If a stress σ_0 is applied to the model, held constant for a time t and then released the strain response will be that indicated in Fig. 12.10. The first part of the response, i.e. the change in strain at constant stress is termed the *creep* of the material, the second part, when stress is removed, is termed the *recovery*.

For *stress relaxation*, i.e. relaxation of stress at constant strain

$$\varepsilon = \text{constant} \quad \text{and} \quad \frac{d\varepsilon}{dt} = 0$$

Equation (12.23) then gives

$$\sigma = E\varepsilon$$

indicating that, according to the Voigt–Kelvin model, the material behaves as an elastic solid under these conditions–clearly an inaccurate representation of viscoelastic behaviour in general.

For creep under constant stress $\sigma = \sigma_0$, however, eqn. (12.23) now gives;

$$\sigma_0 = E\varepsilon + \eta\frac{d\varepsilon}{dt}$$

from which it can be shown that

$$\varepsilon = \frac{\sigma_0}{E}[1 - e^{-Et/\eta}] \tag{12.24}$$

In the special case where $\sigma = \sigma_0 = 0$, the so-called "*recovery*" stage, this reduces to:

$$\varepsilon = \varepsilon_0 e^{-Et/\eta} = \varepsilon_0 e^{-t/t'} \tag{12.25}$$

and this equation indicates that the strain recovers exponentially with time, with t' a characteristic time constant known as the "*retardation time*".

(b) Maxwell model

An alternative model for viscoelastic behaviour proposed by Maxwell again uses a combination of a spring and dashpot but this time in series as shown in Fig. 12.11.

Whereas in the Voigt–Kelvin (parallel) model the stress is shared between the components, in the Maxwell (series) model the stress is common to both elements.

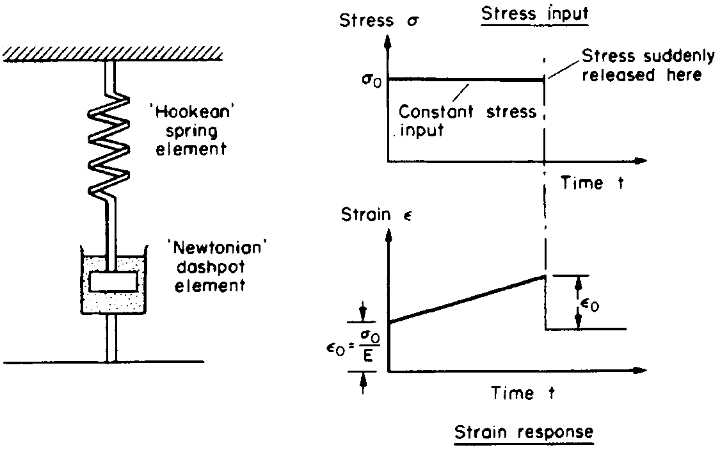


Fig. 12.11. Maxwell model with elements in series.

The strain, however, will be the sum of the strains of the two parts, i.e., the strain of the spring ε_S plus the strain of the dashpot ε_D

$\therefore$ $$\varepsilon = \varepsilon_S + \varepsilon_D$$

Differentiating:

$$\dot{\varepsilon} = \dot{\varepsilon}_S + \dot{\varepsilon}_D \tag{1}$$

Now $\sigma_S = E\varepsilon_S \quad \therefore$ $$\dot{\varepsilon}_S = \frac{\dot{\sigma}_S}{E}$$

and $\sigma_D = \eta\dot{\varepsilon}_D \quad \therefore$ $$\dot{\varepsilon}_D = \frac{\sigma_D}{\eta}$$

Now, for the series model, $$\sigma_S = \sigma_D = \sigma$$

$\therefore$ substituting in (1) we obtain the basic response equation for the Maxwell model.

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{12.26}$$

The response of this model to a stress σ_0 held constant over a time t and released, is shown in Fig. 12.11.

Let us now consider the response of the Maxwell model to the "standard" relaxation and recovery stages as was carried out previously for the Voigt–Kelvin model.

For stress relaxation $d\varepsilon/dt = 0$, and from eqn. (12.26)

$$0 = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

i.e.
$$\frac{d\sigma}{\sigma} = -\frac{E}{\eta} \cdot dt$$

If, at $t = 0$, $\sigma = \sigma_0$, the initial stress, this equation can be integrated to yield

$$\sigma = \sigma_0 e^{-Et/\eta} = \sigma_0 e^{-t/t''} \qquad (12.27)$$

This is analogous to the strain "recovery" equation (12.25) showing that, in this case, stress relaxes from its initial value σ_0 exponentially with time dependent upon the relaxation time t''.

For the creep recovery stage from a constant level of stress, $d\sigma/dt = 0$ and eqn. (12.26) gives

$$\dot{\varepsilon} = \frac{\sigma}{\eta} \qquad (12.28)$$

the basic equation of pure Newtonian flow. Generally, however, the creep behaviour of viscoelastic materials is far more complex and, once again, the model does not adequately represent both recovery and relaxation situations. More accurate model representations can only be obtained, therefore, by suitable combinations of the Voigt–Kelvin and Maxwell models (see Figs. 12.12 and 12.13).

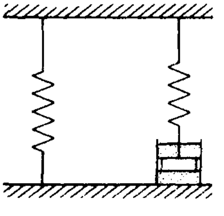


Fig. 12.12. The "standard linear solid" model.

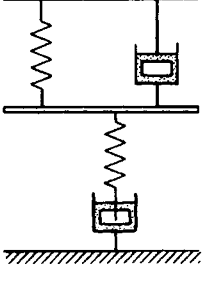


Fig. 12.13. Maxwell and Voigt–Kelvin models in series.

(c) Linear and non-linear viscoelasticity

Both the Voigt–Kelvin and Maxwell models represent so-called *linear viscoelasticity* (which must not be interpreted as meaning that stress is proportional to strain as indicated earlier). Linear viscoelasticity is said to occur when, as a result of a series of creep tests at constant stress levels, the ratios of strain to stress are plotted against time either in the form:

$$\varepsilon = \sigma f(t) \text{ or } \varepsilon = f_1(\sigma) f_2(t).$$

The strain to stress ratio in such tests is termed the *creep compliance*.

Neither the Voigt–Kelvin nor the Maxwell model, will fully represent the behaviour of polymers although the combination of the two, in series, as shown in Fig. 12.13, will give a reasonable approximation of polymer linear viscoelastic behaviour. Unfortunately, however, the range of strain over which linear viscoelasticity is exhibited by polymers is very small.

Non-linear viscoelasticity occurs when the creep compliance–time curve follows an equation of the form:

$$\varepsilon = f'(\sigma, t)$$

This form of viscoelasticity can only be modelled using non-linear springs and dashpots, and the analysis of such systems can become extremely complex.

A convenient approximate solution[1,2] for the design of components constructed from polymers employs the use of "*isochronous*" *stress–strain curves* and a "*secant modulus*" $E_s(t)$. If a series of creep tests are carried out to produce a set of strain-time curves at various stress levels a number of constant time sections can be taken through the curves to enable isochronous (constant time) stress–strain diagrams to be plotted in Fig. 12.14. Such results may be obtained under tensile, compressive or shear loading. Alternatively these data may be obtained from manufacturers' data sheets. One of these isochronous curves can then be selected on the basis of the known lifetime requirement of the component and used for the determination of the secant modulus.

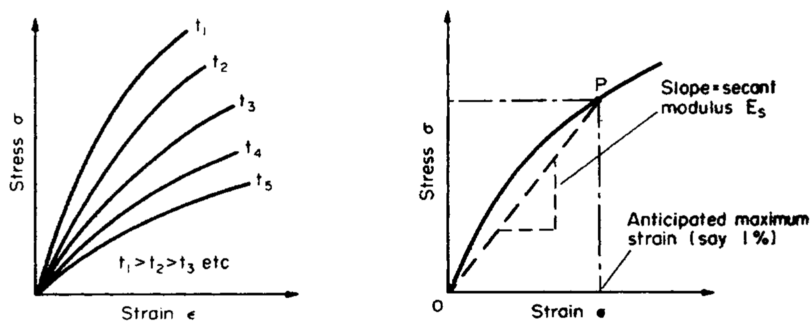


Fig. 12.14. Use of isochronous curves for design.

Defining a point P on the isochronous curve, be it either the expected maximum stress or strain (usually taken as 1%), allows a straight line to be drawn from P to the origin O, the slope of which gives the secant modulus. As stated above, this modulus may be as a result of tension, compression or shear and the appropriate value can then be used to replace E and G in the standard elastic formulae derived in other chapters of this text. If such formulae also contains Poisson's ratio ν this must also be replaced by its equivalent under creep conditions, the so-called "*creep contraction*" or "*lateral strain ratio*" $\nu(t)$. See Example 12.2.

References

1. Benham, P. P. and McCammond, D., "Approximate creep analysis for thermoplastic beams and struts", *J.S.A.*, **6**, 1, 1971.
2. Benham, P. P. and McCammond, D., "A study of design stress analysis problems for thermoplastics using time dependent data", *Plastics and Polymers*, Oct. 1969.

Examples

Example 12.1

The gantry shown in Fig. 12.15 is constructed from 100 mm × 50 mm rectangular cross-section and, under service conditions, supports a maximum load P of 20 kN. Determine the maximum distance d at which P can be safely applied if the maximum tensile and compressive stresses for the material used are limited to 30 MN/m^2 and 100 MN/m^2 respectively.

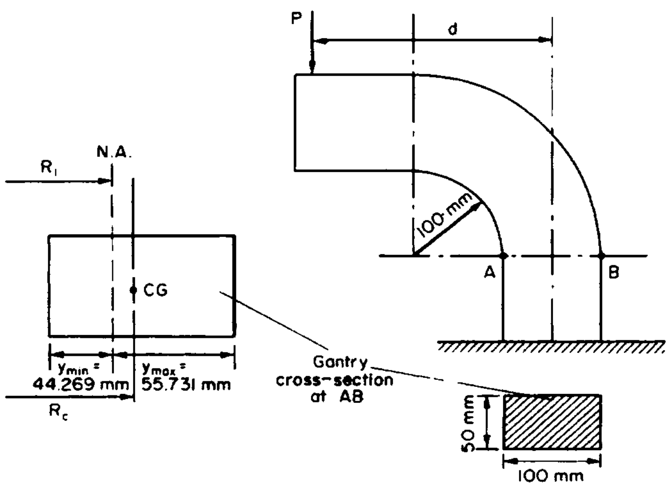


Fig. 12.15.

How would this value change if the cross-section were circular, but of the same cross-sectional area?

Solution

For the gantry and cross-section of Fig. 12.15 the following values are obtained by inspection:

$$R_c = 150 \text{ mm} \quad R_i = 100 \text{ mm} \quad R_o = 200 \text{ mm} \quad b = 50 \text{ mm}.$$

$\therefore$ From Table 12.1(a)

$$\int \frac{dA}{r} = b \log_e \left(\frac{R_o}{R_i}\right) = 50 \log_e \left(\frac{200}{100}\right)$$

$$= 34.6574 \text{ mm}$$

$$\therefore \quad R_1 = \frac{A}{\int \frac{dA}{r}} = \frac{50 \times 100}{34.6574} = 144.269 \text{ mm}$$

$$\therefore \quad h = R_c - R_1 = 150 - 144.269 = 5.731 \text{ mm.}$$

$$\text{Direct stress (compressive) due to } P = \frac{P}{A} = \frac{20 \times 10^3}{(100 \times 50)10^{-6}} = 4 \text{ MN/m}^2$$

Thus, for maximum tensile stress of 30 MN/m^2 to be reached at B the bending stress (tensile) must be $30 + 4 = 34$ MN/m^2.

$$\text{Now } y_{max} = 50 + 5.731$$

$$= 55.731 \text{ at } B$$

$$\text{and bending stress at } B = \frac{My}{hA(R_1 + y)} = 34 \text{ MN/m}^2,$$

$$\therefore \quad \frac{(40 \times 10^3 d) \times 55.731 \times 10^{-3}}{(5.731 \times 10^{-3})(50 \times 100 \times 10^{-6})(200 \times 10^{-3})} = 34 \times 10^6$$

$$\therefore \quad d = 174.69 \text{ mm.}$$

For maximum compressive stress of 100 MN/m^2 at A the compressive bending stress must be limited to $100 - 4 = 96$ MN/m^2 in order to account for the additional direct load effect.

$$\therefore \quad \text{At } A, \text{ with } y_{min} = 50 - 5.731 = 44.269$$

$$\text{bending stress} = \frac{(20 \times 10^3)44.269 \times 10^{-3}}{(5.731 \times 10^{-3})(50 \times 100 \times 10^{-6})(100 \times 10^{-3})} = 96 \times 10^6$$

$$\therefore \quad d = 310.7 \text{ mm.}$$

The critical condition is therefore on the tensile stress at B and the required maximum value of d is **174.69 mm**.

If a circular section were used of radius R and of equal cross-sectional area to the rectangular section then $\pi R^2 = 100 \times 50$ and $R = 39.89$ mm.

$\therefore$ From Table 12.1 assuming R_c remains at 150 mm

$$\int \frac{dA}{r} = 2\pi\{(R_i + R) - \sqrt{(R_1 + R)^2 - R^2}\}$$

$$= 2\pi\{150 - \sqrt{150^2 - 39.894^2}\}$$

$$= 2\pi \times 5.4024 = 33.944 \text{ mm.}$$

$$\therefore \quad R_1 = \frac{A}{\int \frac{dA}{r}} = \frac{50 \times 100}{33.944} = 147.301 \text{ mm,}$$

with $h = R_c - R_1 = 150 - 147.3 = 2.699$ mm.

$\therefore$ For critical tensile stress at B with $y = 39.894 + 2.699 = 42.593$.

$$\frac{My}{hA(R_1 + y)} = 34 \text{ MN/m}^2.$$

$$\therefore \quad \frac{(20 \times 10^3 \times d)(42.593 \times 10^{-3})}{2.699 \times 10^{-3} \times (\pi \times 39.894^2 \times 10^{-6})(150 + 39.894)} = 34 \times 10^6$$

$$\mathbf{d = 102.3}$$

i.e. Use of the circular section reduces the limit of d within which the load P can be applied.

Example 12.2

A constant time section of 1000 h taken through a series of strain–time creep curves obtained for a particular polymer at various stress levels yields the following isochronous stress–strain data.

σ(kN/m^2)	1.0	2.25	3.75	5.25	6.54	7.85	9.0
ε(%)	0.23	0.52	0.85	1.24	1.68	2.17	2.7

The polymer is now used to manufacture:

(a) a disc of thickness 6 mm, which is to rotate at 500 rev/min continuously,
(b) a diaphragm of the same thickness which is to be subjected to a uniform lateral pressure of 16 N/m^2 when clamped around its edge.

Determine the radius required for each component in order that a limiting stress of 6 kN/m^2 is not exceeded after 1000 hours of service. Hence find the maximum deflection of the diaphragm after this 1000 hours of service.

The lateral strain ratio for the polymer may be taken as 0.45 and its density as 1075 kN/m^3.

Solution

(a) From eqn. (4.11) the maximum stress at the centre of a solid rotating disc is given by:

$$\sigma_{r_{\max}} = \sigma_{\theta_{\max}} = (3 + \nu)\frac{\rho\omega^2 R^2}{8}$$

For the limiting stress condition, therefore, with Poissons ratio ν replaced by the lateral strain ratio:

$$6 \times 10^3 = 3.45 \times 1075 \times \frac{(500 \times 2\pi)^2}{60} \times \frac{R^2}{8}$$

From which $R^2 = 0.00472$

and $R = 0.0687 \text{ m} = \mathbf{68.7\ mm}.$

(b) For the diaphragm with clamped edges the maximum stress is given by eqn. (22.24) as:

$$\sigma_{r_{max}} = \frac{3qR^2}{4t^2}$$

$$\therefore \quad 6 \times 10^3 = \frac{3 \times 16 \times R^2}{4 \times (6 \times 10^{-3})^2}$$

From which $R = 0.134$ m = **134 mm**.

The maximum deflection of the diaphragm is then given in Table 7.1 as:

$$\delta_{max} = \frac{3qR^4}{16Et^3}(1 - v^2)$$

Here it is necessary to replace Young's modulus E by the secant modulus obtained from the isochronous curve data and Poisson's ratio by the lateral strain ratio.

The 1000 hour isochronous curve has been plotted from the given data in Fig. 12.16 producing a secant modulus of 405 kN/m^2 at the stated limiting stress of 6 kN/m^2; this being the slope of the line from the origin to the 6 kN/m^2 point on the isochronous curve.

$$\therefore \quad \delta_{max} = \frac{3 \times 16 \times (134 \times 10^{-3})^4 \times (1 - 0.45^2)}{16 \times 405 \times 10^3 \times (6 \times 10^{-3})^3}$$

$$= 0.0088 \text{ m} = \mathbf{8.8\ mm}.$$

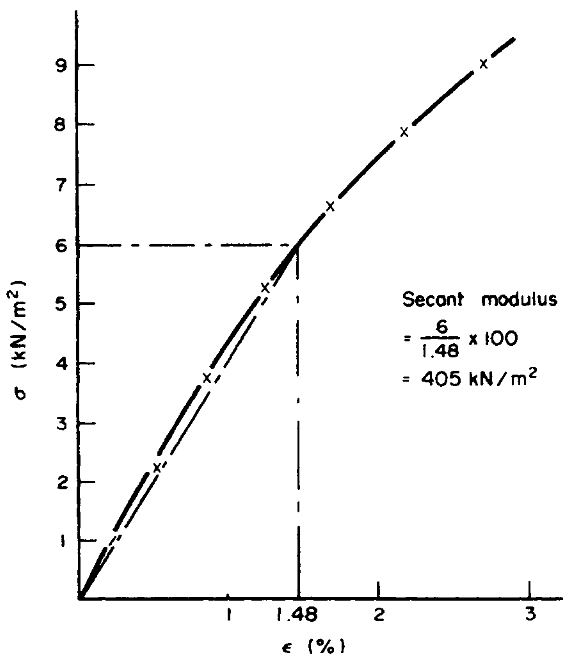


Fig. 12.16.

Problems

12.1 (B). The bracket shown in Fig. 12.17 is constructed from material with 50 mm × 25 mm rectangular cross-section and it supports a vertical load of 10 kN at *C*. Determine the magnitude of the stresses set up at *A* and *B*.

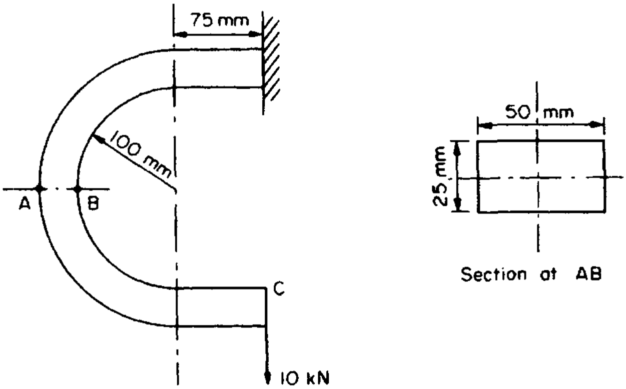


Fig. 12.17.

What percentage error would be obtained if the simple bending theory were applied?

[−161.4 MN/m², + 169.4 MN/m², 19%, 13.6%]

12.2 (B). A crane hook is constructed from trapezoidal cross-section material. At the critical section *AB* the dimensions are as shown in Fig. 12.18. The hook supports a vertical load of 25 kN with a line of action 40 mm from *B* on the inside face. Calculate the values of the stresses at points *A* and *B* taking into account both bending and direct load effects across the section. [129.2 MN/m², −80.3 MN/m²]

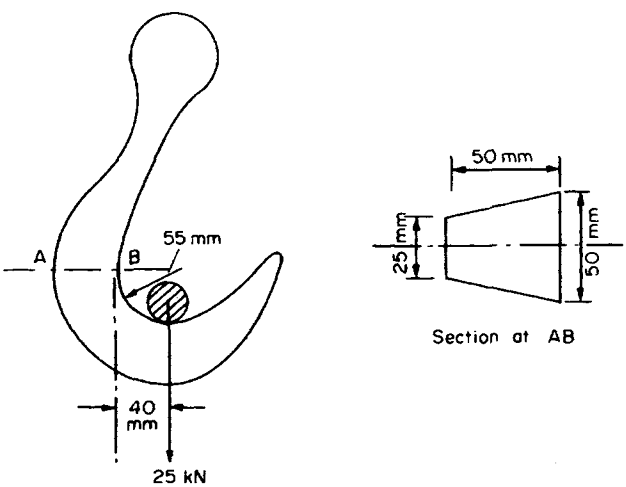


Fig. 12.18.

12.3 (B). A G-clamp is constructed from I-section material as shown in Fig. 12.19. Determine the maximum stresses at the central section *AB* when a clamping force of 2 kN is applied.

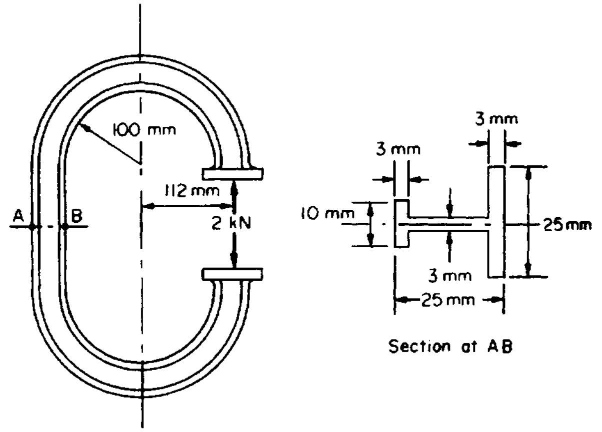


Fig. 12.19.

How do these values compare with those which would be obtained using simple bending theory applied to a straight beam of the same cross-section? [267 MN/m^2, -347.5 MN/m^2, 240 MN/m^2, 380.7 MN/m^2]

12.4 (B). Part of the frame of a machine tool can be considered to be of the form shown in Fig. 12.20. A decision is required whether to construct the frame from T or rectangular section material of the dimensions shown.

Compare the critical stresses set up at section *AB* for each of the cross-sections when the frame is subjected to a peak load of 5 kN and discuss the results obtained in relation to the decision required.

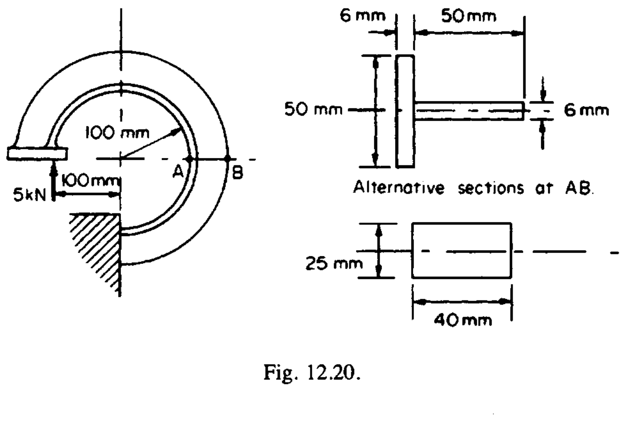


Fig. 12.20.

Plot diagrams of the stress distribution across *AB* for each cross-sectional shape.

[122.7 MN/m^2, -198 MN/m^2, 193.4 MN/m^2, -143.2 MN/m^2]

12.5 (B). (a) By consideration of the Maxwell model, derive an expression for the internal stress after time t of a polymer held under constant strain conditions and hence show that the relaxation time is equal to η/G where η is the coefficient of viscosity and G is the shear modulus.

(b) A shear stress of 310 MN/m^2 is applied to a polymer which is then held under fixed strain conditions. After 1 year the internal stress decreases to a value of 207 MN/m^2. Calculate the value to which the stress will fall after 2 years, assuming the polymer behaves according to the Maxwell model. [$\tau = \tau_0 e^{-Gt/\eta}$; 138 MN/m^2]

12.6 (B). (a) Spring and dashpot arrangements are often used to represent the mechanical behaviour of polymers. Analyse the mathematical stress strain relationship for the Maxwell and Kelvin–Voigt models under conditions of (i) constant stress, (ii) constant strain, (iii) recovery, and draw the appropriate strain–time, stress–time diagrams, commenting upon their suitability to predict behaviour of real polymers.

(b) Maxwell and Kelvin–Voigt models are to be set up to simulate the behaviour of a plastic. The elastic and viscous constants for the Kelvin–Voigt model are 2×10^9 N/m^2 and 100×10^9 Ns/m^2 respectively and the viscous constant for the Maxwell model is 272×10^9 Ns/m^2. Calculate a value for the elastic constant for the Maxwell model if both models are to predict the same strain after 100 seconds when subjected to the same stress.
[15.45×10^9 N/m^2]

12.7 (B). The model shown in Fig. 12.21 is frequently used to simulate the mechanical behaviour of polymers:

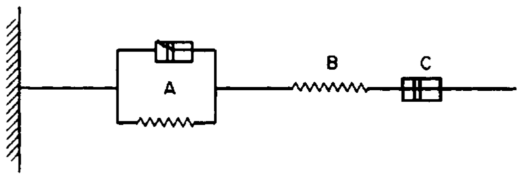


Fig. 12.21.

(a) With reference to Figure 12.21, state what components of total strain the elements A, B and C represent.
(b) Sketch a typical strain–time graph for the model when the load F is applied and then removed. Clearly label those parts of the graph corresponding to the strain components ε_1, ε_2 and ε_3.
(c) A certain polymer may be modelled on such a system by using the following constants for the elements:

Dashpot A:	viscosity = 10^6 Ns/m^2
Dashpot B:	viscosity = 100×10^6 Ns/m^2
Spring A:	shear modulus = 50×10^3 N/m^2
Spring B:	shear modulus = 10^9 N/m^2

This polymer is subjected to a direct stress of 6×10^3 N/m^2 for 30 seconds ONLY.
Determine the strain in the polymer after 30 seconds, 60 seconds and 2000 seconds.
[3.17×10^{-2}, 0.75×10^{-2}, 0.06×10^{-2}]

12.8 (C). For each of the following typical engineering components and loading situations sketch and dimension the components and allocate appropriate loadings. As a preliminary step towards finite element analysis of each case, select and sketch a suitable analysis region, specify complete boundary conditions and add an appropriate element mesh. Make use of symmetry and St. Venant's criteria wherever possible.

(a) A shelf support bracket welded to a vertical upright.
(b) An engine con-rod with particular attention paid to shoulder fillet radii for weight reduction purposes (see Fig. 6.1)
(c) A washing machine agitator cross-section (see Fig. 5.14), bar-tube fillet radii and relative thicknesses of particular concern.
(d) The extruded alloy section of Fig. 1.21. Model to be capable of consideration of varying lines of action of applied force.
(e) A circular pipe flange used to connect two internally pressurised pipes. Model to be capable of including the effect of bolt tensions and external moments on the joint. You may assume that the pipe is free to expand axially.
(f) A C.T.S. (compact test specimen) for brittle fracture compliance testing. Stress distributions at the crack tip are required.
(g) A square storage hopper fabricated from thin rectangular plates welded together and supported by means of welded angle around the upper edge. It may be assumed that the hopper is full with an equivalent hydrostatic pressure p throughout. The supporting frame can be assumed rigid.
(h) A four-point beam bending test rig with plastic beam mounted on steel pads over steel knife edges. The degree of indentation of the plastic and deformation of the steel pad are required.
(i) Thick cylinder with flat ends and sharp fillet radii subjected to internal pressure. The model should be capable of assessing the effect of different end plate thicknesses.
(j) A pressurised thick cylinder containing a 45° nozzle entry. Stress concentrations at the nozzle entry are required.