CHAPTER 9

THIN CYLINDERS AND SHELLS

Summary

The stresses set up in the walls of a *thin cylinder* owing to an internal pressure p are:

$$\text{circumferential or hoop stress } \sigma_H = \frac{pd}{2t}$$

$$\text{longitudinal or axial stress } \sigma_L = \frac{pd}{4t}$$

where d is the internal diameter and t is the wall thickness of the cylinder.

Then:

$$\text{longitudinal strain } \varepsilon_L = \frac{1}{E}[\sigma_L - \nu\sigma_H]$$

$$\text{hoop strain } \varepsilon_H = \frac{1}{E}[\sigma_H - \nu\sigma_L]$$

$$\text{change of internal volume of cylinder under pressure} = \frac{pd}{4tE}[5-4\nu]\,V$$

$$\text{change of volume of contained liquid under pressure} = \frac{pV}{K}$$

where K is the bulk modulus of the liquid.

For *thin rotating cylinders* of mean radius R the tensile hoop stress set up when rotating at ω rad/s is given by $\sigma_H = \rho\omega^2R^2$.

For *thin spheres*:

$$\text{circumferential or hoop stress } \sigma_H = \frac{pd}{4t}$$

$$\text{change of volume under pressure} = \frac{3pd}{4tE}[1-\nu]\,V$$

Effects of end plates and joints—add "joint efficiency factor" η to denominator of stress equations above.

9.1. Thin cylinders under internal pressure

When a thin-walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder material, namely the *circumferential* or *hoop* stress, the *radial* stress and the *longitudinal* stress. Provided that the ratio of thickness to inside diameter of the cylinder is less than 1/20, it is reasonably accurate to assume that the hoop and longitudinal stresses are constant across the wall thickness and that the magnitude of the radial stress set up is so small in comparison with the hoop and longitudinal stresses that it can be neglected. This is obviously an approximation since, in practice, it will vary from zero at the outside surface to a value equal to the internal pressure at the inside surface. For the purpose of the initial derivation of stress formulae it is also assumed that the ends of the cylinder and any riveted joints present have no effect on the stresses produced; in practice they will have an effect and this will be discussed later (§9.6).

9.1.1. Hoop or circumferential stress

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of half of the cylinder as shown in Fig. 9.1.

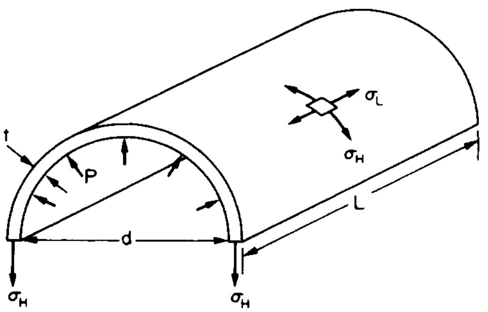

Fig. 9.1. Half of a thin cylinder subjected to internal pressure showing the hoop and longitudinal stresses acting on any element in the cylinder surface.

Total force on half-cylinder owing to internal pressure $= p \times$ projected area $= p \times dL$

Total resisting force owing to hoop stress σ_H set up in the cylinder walls

$$= 2\sigma_H \times Lt$$

$$\therefore \qquad 2\sigma_H Lt = pdL$$

$$\therefore \qquad \textbf{circumferential or hoop stress } \sigma_H = \frac{pd}{2t} \qquad (9.1)$$

9.1.2. Longitudinal stress

Consider now the cylinder shown in Fig. 9.2.

Total force on the end of the cylinder owing to internal pressure

$$= \text{pressure} \times \text{area} = p \times \frac{\pi d^2}{4}$$

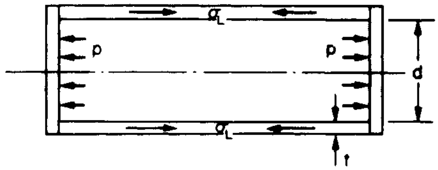

Fig. 9.2. Cross-section of a thin cylinder.

Area of metal resisting this force = πdt (approximately)

$$\therefore \qquad \text{stress set up} = \frac{\text{force}}{\text{area}} = p \times \frac{\pi d^2/4}{\pi dt} = \frac{pd}{4t}$$

i.e.

$$\textbf{longitudinal stress } \sigma_L = \frac{pd}{4t} \qquad (9.2)$$

9.1.3. *Changes in dimensions*

(a) Change in length

The change in length of the cylinder may be determined from the longitudinal strain, i.e. neglecting the radial stress.

$$\text{Longitudinal strain} = \frac{1}{E}[\sigma_L - \nu\sigma_H]$$

and

$$\text{change in length} = \text{longitudinal strain} \times \text{original length}$$

$$= \frac{1}{E}[\sigma_L - \nu\sigma_H]L$$

$$= \frac{pd}{4tE}[1-2\nu]L \qquad (9.3)$$

(b) Change in diameter

As above, the change in diameter may be determined from the strain on a diameter, i.e. the *diametral* strain.

$$\text{Diametral strain} = \frac{\text{change in diameter}}{\text{original diameter}}$$

Now the change in diameter may be found from a consideration of the circumferential change. The stress acting around a circumference is the hoop or circumferential stress σ_H giving rise to the circumferential strain ε_H.

$$\text{Change in circumference} = \text{strain} \times \text{original circumference}$$

$$= \varepsilon_H \times \pi d$$

$$\text{New circumference} = \pi d + \pi d\varepsilon_H$$
$$= \pi d(1 + \varepsilon_H)$$

But this is the circumference of a circle of diameter $d(1 + \varepsilon_H)$

$$\therefore \quad \text{New diameter} = d(1 + \varepsilon_H)$$
$$\therefore \quad \text{Change in diameter} = d\varepsilon_H$$
$$\text{Diametral strain } \varepsilon_D = \frac{d\varepsilon_H}{d} = \varepsilon_H$$

i.e. **the diametral strain equals the hoop or circumferential strain** (9.4)

Thus
$$\text{change in diameter} = d\varepsilon_H = \frac{d}{E}[\sigma_H - \nu\sigma_L]$$
$$= \frac{pd^2}{4tE}[2 - \nu] \qquad (9.5)$$

(*c*) *Change in internal volume*

$$\text{Change in volume} = \text{volumetric strain} \times \text{original volume}$$

From the work of § 14.5, page 364.

$$\text{volumetric strain} = \text{sum of three mutually perpendicular direct strains}$$
$$= \varepsilon_L + 2\varepsilon_D$$
$$= \frac{1}{E}[\sigma_L - \nu\sigma_H] + \frac{2}{E}[\sigma_H - \nu\sigma_L]$$
$$= \frac{1}{E}[\sigma_L + 2\sigma_H - \nu(\sigma_H + 2\sigma_L)]$$
$$= \frac{pd}{4tE}[1 + 4 - \nu(2 + 2)]$$
$$= \frac{pd}{4tE}[5 - 4\nu]$$

Therefore with original internal volume V

$$\textbf{change in internal volume} = \frac{pd}{4tE}[5 - 4\nu]\,V \qquad (9.6)$$

9.2. Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig. 9.3 subjected to a radial pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length

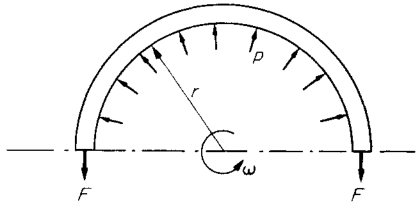

Fig. 9.3. Rotating thin ring or cylinder.

of the circumference is:

$$p = m\omega^2 r$$

Thus, considering the equilibrium of half the ring shown in the figure:

$$2F = p \times 2r$$

$$F = pr$$

where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be taken to be constant across the wall thickness.

$$\therefore \quad F = pr = m\omega^2 r^2$$

This tension is transmitted through the complete circumference and therefore is restricted by the complete cross-sectional area.

$$\therefore \quad \text{hoop stress} = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

where A is the cross-sectional area of the ring.

Now with unit length assumed, m/A is the mass of the ring material per unit volume, i.e. the density ρ.

$$\therefore \quad \textbf{hoop stress} = \boldsymbol{\rho\omega^2 r^2} \qquad (9.7)$$

9.3. Thin spherical shell under internal pressure

Because of the symmetry of the sphere the stresses set up owing to internal pressure will be two mutually perpendicular hoop or circumferential stresses of equal value and a radial stress. As with thin cylinders having thickness to diameter ratios less than 1 : 20, the radial stress is assumed negligible in comparison with the values of hoop stress set up. The stress system is therefore one of equal biaxial hoop stresses.

Consider, therefore, the equilibrium of the half-sphere shown in Fig. 9.4.

Force on half-sphere owing to internal pressure

$$= \text{pressure} \times \text{projected area}$$

$$= p \times \frac{\pi d^2}{4}$$

$$\text{Resisting force} = \sigma_H \times \pi dt \quad \text{(approximately)}$$

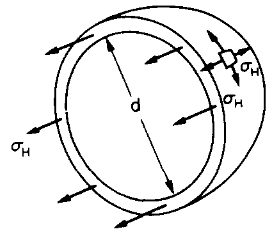

Fig. 9.4. Half of a thin sphere subjected to internal pressure showing uniform hoop stresses acting on a surface element.

$$\therefore \qquad p \times \frac{\pi d^2}{4} = \sigma_H \times \pi d t$$

or

$$\sigma_H = \frac{pd}{4t}$$

i.e. **circumferential or hoop stress** $= \dfrac{pd}{4t}$ (9.8)

9.3.1. Change in internal volume

As for the cylinder,

change in volume = original volume × volumetric strain

but

volumetric strain = sum of three mutually perpendicular strains (in this case all equal)

$$= 3\varepsilon_D = 3\varepsilon_H$$

$$= \frac{3}{E}[\sigma_H - \nu\sigma_H]$$

$$= \frac{3pd}{4tE}[1-\nu]$$

$\therefore$ **change in internal volume** $= \dfrac{3pd}{4tE}[1-\nu]V$ (9.9)

9.4. Vessels subjected to fluid pressure

If a fluid is used as the pressurisation medium the fluid itself will change in volume as pressure is increased and this must be taken into account when calculating the amount of fluid which must be pumped into the cylinder in order to raise the pressure by a specified amount, the cylinder being initially full of fluid at atmospheric pressure.

Now the *bulk modulus* of a fluid is defined as follows:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

where, in this case, volumetric stress = pressure p

and volumetric strain $= \dfrac{\text{change in volume}}{\text{original volume}} = \dfrac{\delta V}{V}$

$$\therefore \quad K = \frac{p}{\delta V/V} = \frac{pV}{\delta V}$$

i.e. **change in volume of fluid under pressure** $= \dfrac{pV}{K}$ (9.10)

The extra fluid required to raise the pressure must, therefore, take up this volume together with the increase in internal volume of the cylinder itself.

$\therefore$ extra fluid required to raise *cylinder* pressure by p

$$= \frac{pd}{4tE}[5-4\nu]V + \frac{pV}{K} \tag{9.11}$$

Similarly, for *spheres*, the extra fluid required is

$$= \frac{3pd}{4tE}[1-\nu]V + \frac{pV}{K} \tag{9.12}$$

9.5. Cylindrical vessel with hemispherical ends

Consider now the vessel shown in Fig. 9.5 in which the wall thickness of the cylindrical and hemispherical portions may be different (this is sometimes necessary since the hoop stress in the cylinder is twice that in a sphere of the same radius and wall thickness). For the purpose of the calculation the internal diameter of both portions is assumed equal.
From the preceding sections the following formulae are known to apply:

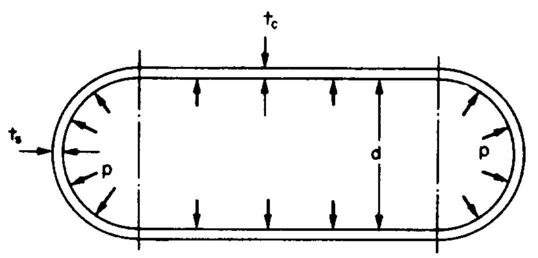

Fig. 9.5. Cross-section of a thin cylinder with hemispherical ends.

(a) *For the cylindrical portion*

$$\text{hoop or circumferential stress} = \sigma_{H_c} = \frac{pd}{2t_c}$$

$$\text{longitudinal stress} = \sigma_{L_c} = \frac{pd}{4t_c}$$

$$\therefore \qquad \text{hoop or circumferential strain} = \frac{1}{E}[\sigma_{H_c} - \nu\sigma_{L_c}]$$

$$= \frac{pd}{4t_cE}[2-\nu]$$

(b) *For the hemispherical ends*

$$\text{hoop stress} = \sigma_{H_s} = \frac{pd}{4t_s}$$

$$\therefore \qquad \text{hoop strain} = \frac{1}{E}[\sigma_{H_s} - \nu\sigma_{H_s}]$$

$$= \frac{pd}{4t_sE}[1-\nu]$$

Thus equating the two strains in order that there shall be no distortion of the junction,

$$\frac{pd}{4t_cE}[2-\nu] = \frac{pd}{4t_sE}[1-\nu]$$

i.e.

$$\frac{t_s}{t_c} = \frac{(1-\nu)}{(2-\nu)} \qquad (9.13)$$

With the normally accepted value of Poisson's ratio for general steel work of 0.3, the thickness ratio becomes

$$\frac{t_s}{t_c} = \frac{0.7}{1.7}$$

i.e. the thickness of the cylinder walls must be approximately 2.4 times that of the hemispherical ends for no distortion of the junction to occur. In these circumstances, because of the reduced wall thickness of the ends, the maximum stress will occur in the ends. For *equal maximum stresses* in the two portions the thickness of the cylinder walls must be twice that in the ends but some distortion at the junction will then occur.

9.6. Effects of end plates and joints

The preceding sections have all assumed uniform material properties throughout the components and have neglected the effects of endplates and joints which are necessary requirements for their production. In general, the strength of the components will be reduced by the presence of, for example, riveted joints, and this should be taken into account by the introduction of a *joint efficiency factor* η into the equations previously derived.

Thus, for *thin cylinders*:

$$\text{hoop stress} = \frac{pd}{2t\eta_L}$$

where η_L is the efficiency of the longitudinal joints,

$$\text{longitudinal stress} = \frac{pd}{4t\eta_C}$$

where η_C is the efficiency of the circumferential joints.

For *thin spheres*:

$$\text{hoop stress} = \frac{pd}{4t\eta}$$

Normally the joint efficiency is stated in percentage form and this must be converted into equivalent decimal form before substitution into the above equations.

9.7. Wire-wound thin cylinders

In order to increase the ability of thin cylinders to withstand high internal pressures without excessive increases in wall thickness, and hence weight and associated material cost, they are sometimes wound with high tensile steel tape or wire under tension. This subjects the cylinder to an initial hoop, compressive, stress which must be overcome by the stresses owing to internal pressure before the material is subjected to tension. There then remains at this stage the normal pressure capacity of the cylinder before the maximum allowable stress in the cylinder is exceeded.

It is normally required to determine the tension necessary in the tape during winding in order to ensure that the maximum hoop stress in the cylinder will not exceed a certain value when the internal pressure is applied.

Consider, therefore, the half-cylinder of Fig. 9.6, where σ_H denotes the hoop stress in the cylinder walls and σ_t the stress in the rectangular-sectioned tape. Let conditions before pressure is applied be denoted by suffix 1 and after pressure is applied by suffix 2.

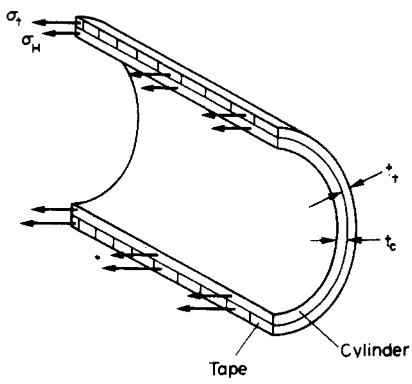

Fig. 9.6. Section of a thin cylinder with an external layer of tape wound on with a tension.

Now

$$\text{force owing to tape} = \sigma_{t_1} \times \text{area} = \sigma_{t_1} \times 2Lt_t$$

$$\text{resistive force in the cylinder material} = \sigma_{H_1} \times 2Lt_c$$

i.e. for equilibrium

$$\sigma_{t_1} \times 2Lt_t = \sigma_{H_1} \times 2Lt_c$$

or

$$\sigma_{t_1} \times t_t = \sigma_{H_1} \times t_c$$

so that the *compressive* hoop stress set up in the cylinder walls after winding and before pressurisation is given by

$$\sigma_{H_1} = \sigma_{t_1} \times \frac{t_t}{t_c} \text{ (compressive)} \qquad (9.14)$$

This equation will be modified if wire of circular cross-section is used for the winding process in preference to rectangular-sectioned tape. The area carrying the stress σ_{t_1} will then be $2na$ where a is the cross-sectional area of the wire and n is the number of turns along the cylinder length.

After pressure has been applied another force is introduced

$$= \text{pressure} \times \text{projected area} = pdL$$

Again, equating forces for equilibrium of the half-cylinder,

$$pdL = (\sigma_{H_2} \times 2Lt_c) + (\sigma_{t_2} \times 2Lt_t) \qquad (9.15)$$

where σ_{H_2} is the hoop stress in the cylinder after pressurisation and σ_{t_2} is the final stress in the tape after pressurisation.

Since the limiting value of σ_{H_2} is known for any given internal pressure p, this equation yields the value of σ_{t_2}.

Now the change in strain on the outside surface of the cylinder must equal that on the inside surface of the tape if they are to remain in contact.

$$\text{Change in strain in the tape} = \frac{\sigma_{t_2} - \sigma_{t_1}}{E_t}$$

where E_t is Young's modulus of the tape.

In the absence of any internal pressure originally there will be no longitudinal stress or strain so that the original strain in the cylinder walls is given by σ_{H_1}/E_c, where E_c is Young's modulus of the cylinder material. When pressurised, however, the cylinder will be subjected to a longitudinal strain so that the final strain in the cylinder walls is given by

$$\frac{1}{E_c}[\sigma_{H_2} - \nu\sigma_L] = \frac{1}{E_c}\left[\sigma_{H_2} - \nu\frac{pd}{4t_c}\right]$$

$$\therefore \quad \text{change in strain on the cylinder} = \frac{1}{E_c}\left[\sigma_{H_2} - \nu\frac{pd}{4t_c} - \sigma_{H_1}\right]$$

$$\therefore \quad \frac{1}{E_t}[\sigma_{t_2} - \sigma_{t_1}] = \frac{1}{E_c}\left[\sigma_{H_2} - \nu\frac{pd}{4t_c} - \sigma_{H_1}\right]$$

Thus with σ_{H_1} obtained in terms of σ_{t_1} from eqn. (9.14), p and σ_{H_2} known, and σ_{t_2} found from eqn. (9.15) the only unknown σ_{t_1} can be determined.

Examples

Example 9.1

A thin cylinder 75 mm internal diameter, 250 mm long with walls 2.5 mm thick is subjected to an internal pressure of $7\,MN/m^2$. Determine the change in internal diameter and the change in length.

If, in addition to the internal pressure, the cylinder is subjected to a torque of 200 N m, find the magnitude and nature of the principal stresses set up in the cylinder. $E = 200\,GN/m^2$. $\nu = 0.3$.

Solution

(a) From eqn. (9.5), change in diameter $= \dfrac{pd^2}{4tE}(2-\nu)$

$$= \frac{7 \times 10^6 \times 75^2 \times 10^{-6}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9}(2-0.3)$$

$$= 33.4 \times 10^{-6}\,m$$

$$= \mathbf{33.4\,\mu m}$$

(b) From eqn. (9.3), change in length $= \dfrac{pdL}{4tE}(1-2\nu)$

$$= \frac{7 \times 10^6 \times 75 \times 10^{-3} \times 250 \times 10^{-3}}{4 \times 2.5 \times 10^{-3} \times 200 \times 10^9}(1-0.6)$$

$$= \mathbf{26.2\,\mu m}$$

(c) Hoop stress $\sigma_H = \dfrac{pd}{2t} = \dfrac{7 \times 10^6 \times 75 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}}$

$$= \mathbf{105\,MN/m^2}$$

Longitudinal stress $\sigma_L = \dfrac{pd}{4t} = \dfrac{7 \times 10^6 \times 75 \times 10^{-3}}{2 \times 2.5 \times 10^{-3}}$

$$= \mathbf{52.5\,MN/m^2}$$

In addition to these stresses a shear stress τ is set up.

From the torsion theory,

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \quad \tau = \frac{TR}{J}$$

Now $$J = \frac{\pi}{32}\frac{(80^4 - 75^4)}{10^{12}} = \frac{\pi}{32}\frac{(41-31.6)}{10^6} = 0.92 \times 10^{-6}\,m^4$$

Then shear stress $$\tau = \frac{200 \times 20 \times 10^{-3}}{0.92 \times 10^{-6}} = 4.34\,MN/m^2.$$

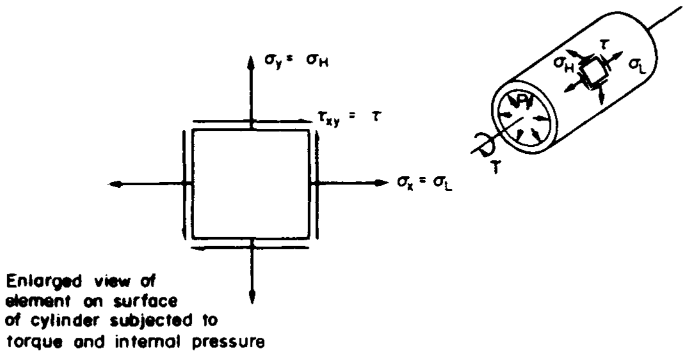

Fig. 9.7. Enlarged view of the stresses acting on an element in the surface of a thin cylinder subjected to torque and internal pressure.

The stress system then acting on any element of the cylinder surface is as shown in Fig. 9.7. The principal stresses are then given by eqn. (13.11),

$$\begin{aligned}\sigma_1 \text{ and } \sigma_2 &= \tfrac{1}{2}(\sigma_x+\sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}{}^2]}\\ &= \tfrac{1}{2}(105+52.5) \pm \tfrac{1}{2}\sqrt{[(105-52.5)^2+4(4.34)^2]}\\ &= \tfrac{1}{2}\times 157.5 \pm \tfrac{1}{2}\sqrt{(2760+75.3)}\\ &= 78.75 \pm 26.6\end{aligned}$$

Then $\sigma_1 = 105.35\ \text{MN/m}^2$ and $\sigma_2 = 52.15\ \text{MN/m}^2$

The principal stresses are

105.4 MN/m² and **52.2 MN/m²** both tensile.

Example 9.2

A cylinder has an internal diameter of 230 mm, has walls 5 mm thick and is 1 m long. It is found to change in internal volume by 12.0×10^{-6} m³ when filled with a liquid at a pressure p. If $E = 200\ \text{GN/m}^2$ and $\nu = 0.25$, and assuming rigid end plates, determine:

(a) the values of hoop and longitudinal stresses;
(b) the modifications to these values if joint efficiencies of 45% (hoop) and 85% (longitudinal) are assumed;
(c) the necessary change in pressure p to produce a further increase in internal volume of 15%. The liquid may be assumed incompressible.

Solution

(a) From eqn. (9.6)

$$\text{change in internal volume} = \frac{pd}{4tE}(5-4\nu)V$$

original volume $V = \frac{\pi}{4} \times 230^2 \times 10^{-6} \times 1 = 41.6 \times 10^{-3}\,m^3$

Then change in volume $= 12 \times 10^{-6} = \dfrac{p \times 230 \times 10^{-3} \times 41.6 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 200 \times 10^9}(5-1)$

Thus
$$p = \frac{12 \times 10^{-6} \times 4 \times 5 \times 10^{-3} \times 200 \times 10^9}{230 \times 10^{-3} \times 41.6 \times 10^{-3} \times 4}$$
$$= \mathbf{1.25\,MN/m^2}$$

Hence,
$$\text{hoop stress} = \frac{pd}{2t} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3}}$$
$$= \mathbf{28.8\,MN/m^2}$$
$$\text{longitudinal stress} = \frac{pd}{4t} = \mathbf{14.4\,MN/m^2}$$

(b) Hoop stress, acting on the longitudinal joints (§9.6)
$$= \frac{pd}{2t\eta_L} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{2 \times 5 \times 10^{-3} \times 0.85}$$
$$= \mathbf{33.9\,MN/m^2}$$

Longitudinal stress (acting on the circumferential joints)
$$= \frac{pd}{4t\eta_c} = \frac{1.25 \times 10^6 \times 230 \times 10^{-3}}{4 \times 5 \times 10^{-3} \times 0.45}$$
$$= \mathbf{32\,MN/m^2}$$

(c) Since the change in volume is directly proportional to the pressure, the necessary 15% increase in volume is achieved by increasing the pressure also by 15%.

$$\text{Necessary increase in } p = 0.15 \times 1.25 \times 10^6$$
$$= \mathbf{1.86\,MN/m^2}$$

Example 9.3

(a) A sphere, 1 m internal diameter and 6 mm wall thickness, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of 3 MN/m^2 gauge? For water, $K = 2.1\,GN/m^2$.

(b) The sphere is now placed in service and filled with gas until there is a volume change of $72 \times 10^{-6}\,m^3$. Determine the pressure exerted by the gas on the walls of the sphere.

(c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure?

For the material of the sphere $E = 200\,GN/m^2$, $\nu = 0.3$ and the yield stress σ_y in simple tension $= 280\,MN/m^2$.

Solution

(a) Bulk modulus $K = \dfrac{\text{volumetric stress}}{\text{volumetric strain}}$

Now volumetric stress = pressure $p = 3\,\text{MN/m}^2$

and volumetric strain = change in volume ÷ original volume

i.e.
$$K = \frac{p}{\delta V/V}$$

$\therefore$ change in volume of water $= \dfrac{pV}{K} = \dfrac{3 \times 10^6}{2.1 \times 10^9} \times \dfrac{4\pi}{3}(0.5)^3$

$$= \mathbf{0.748 \times 10^{-3}\ m^3}$$

(b) From eqn. (9.9) the change in volume is given by

$$\delta V = \frac{3pd}{4tE}(1-\nu)V$$

$$\therefore \quad 72 \times 10^{-6} = \frac{3p \times 1 \times \frac{4}{3}\pi(0.5)^3(1-0.3)}{4 \times 6 \times 10^{-3} \times 200 \times 10^9}$$

$$\therefore \quad p = \frac{72 \times 10^{-6} \times 4 \times 6 \times 200 \times 10^6 \times 3}{3 \times 4\pi(0.5)^3 \times 0.7}$$

$$= 314 \times 10^3\ \text{N/m}^2 = \mathbf{314\ kN/m^2}$$

(c) The maximum stress set up in the sphere will be the hoop stress,

i.e.
$$\sigma_1 = \sigma_H = \frac{pd}{4t}$$

Now, according to the maximum principal stress theory (see §15.2) failure will occur when the maximum principal stress equals the value of the yield stress of a specimen subjected to simple tension,

i.e. when
$$\sigma_1 = \sigma_y = 280\ \text{MN/m}^2$$

Thus
$$280 \times 10^6 = \frac{pd}{4t}$$

$$p = \frac{280 \times 10^6 \times 4 \times 6 \times 10^{-3}}{1}$$

$$= 6.72 \times 10^6\ \text{N/m}^2 = \mathbf{6.7\ MN/m^2}$$

The sphere would therefore yield at a pressure of $6.7\ \text{MN/m}^2$.

Example 9.4

A closed thin copper cylinder of 150 mm internal diameter having a wall thickness of 4 mm is closely wound with a single layer of steel tape having a thickness of 1.5 mm, the tape being

wound on when the cylinder has no internal pressure. Estimate the tensile stress in the steel tape when it is being wound to ensure that when the cylinder is subjected to an internal pressure of 3.5 MN/m^2 the tensile hoop stress in the cylinder will not exceed 35 MN/m^2. For copper, Poisson's ratio $\nu = 0.3$ and $E = 100\,GN/m^2$; for steel, $E = 200\,GN/m^2$.

Solution

Let σ_t be the stress in the tape and let conditions before pressure is applied be denoted by suffix 1 and after pressure is applied by suffix 2.

Consider the half-cylinder shown (before pressure is applied) in Fig. 9.6 (see page 206):

$$\text{force owing to tension in tape} = \sigma_{t_1} \times \text{area}$$

$$= \sigma_{t_1} \times 1.5 \times 10^{-3} \times L \times 2$$

$$\text{resistive force in the material of cylinder wall} = \sigma_{H_1} \times 4 \times 10^{-3} \times L \times 2$$

$$\therefore \quad 2\sigma_{H_1} \times 4 \times 10^{-3} \times L = 2\sigma_{t_1} \times 1.5 \times 10^{-3} \times L$$

$$\therefore \quad \sigma_{H_1} = \frac{1.5}{4}\sigma_{t_1} = 0.375\,\sigma_{t_1} \text{(compressive)} \qquad (1)$$

After pressure is applied another force is introduced

$$= \text{pressure} \times \text{projected area}$$

$$= p(dL)$$

Equating forces now acting on the half-cylinder,

$$pdL = (\sigma_{H_2} \times 2 \times 4 \times 10^{-3} \times L) + (\sigma_{t_2} \times 2 \times 1.5 \times 10^{-3} \times L)$$

but

$$p = 3.5 \times 10^6 \text{ N/m}^2 \quad \text{and} \quad \sigma_{H_2} = 35 \times 10^6 \text{ N/m}^2$$

$$\therefore \quad 3.5 \times 10^6 \times 150 \times 10^{-3} L = (35 \times 10^6 \times 2 \times 4 \times 10^{-3} L) + (\sigma_{t_2} \times 2 \times 1.5 \times 10^{-3} \times L)$$

$$\therefore \quad 525 \times 10^6 = 280 \times 10^6 + 3\sigma_{t_2}$$

$$\therefore \quad \sigma_{t_2} = \frac{(525 - 280)}{3} 10^6$$

$$\sigma_{t_2} = 82 \text{ MN/m}^2$$

The change in strain on the outside of the cylinder and on the inside of the tape must be equal:

$$\text{change in strain in tape} = \frac{\sigma_{t_2} - \sigma_{t_1}}{E_s}$$

$$\text{original strain in cylinder walls} = \frac{\sigma_{H_1}}{E_c}$$

(Since there is no pressure in the cylinder in the original condition there will be no longitudinal stress.)

Final strain in cylinder (after pressurising)

$$= \frac{\sigma_{H_2}}{E_c} - \frac{\nu\sigma_L}{E_c}$$

$$= \frac{1}{Ec}\left(\sigma_{H_2} - \frac{\nu pd}{4t}\right)$$

Then change in strain in cylinder

$$= \frac{1}{E_c}\left(\sigma_{H_2} - \frac{\nu pd}{4t} - \sigma_{H_1}\right)$$

Then
$$\frac{1}{E_s}(\sigma_{t_2} - \sigma_{t_1}) = \frac{1}{E_c}\left(\sigma_{H_2} - \frac{\nu pd}{4t} - \sigma_{H_1}\right)$$

Substituting for σ_{H_1} from eqn. (1)

$$\frac{82 \times 10^6 - \sigma_{t_1}}{200 \times 10^9} = \frac{1}{100 \times 10^9}\left[35 \times 10^6 - \frac{0.3 \times 3.5 \times 10^6 \times 154 \times 10^{-3}}{4 \times 4 \times 10^{-3}} - 0.375\,\sigma_{t_1}\right]$$

$$82 \times 10^6 - \sigma_{t_1} = 2(35 \times 10^6 - 10.1 \times 10^6 - 0.375\,\sigma_{t_1})$$

$$= 49.8 \times 10^6 - 0.75\,\sigma_{t_1}$$

Then
$$1.75\,\sigma_{t_1} = (82.0 - 49.8)10^6$$

$$\sigma_{t_1} = \frac{32.2 \times 10^6}{1.75}$$

$$= \mathbf{18.4\,MN/m^2}$$

Problems

9.1 (A). **Determine the hoop and longitudinal stresses set up in a thin boiler shell of circular cross-section, 5 m long and of 1.3 m internal diameter when the internal pressure reaches a value of 2.4 bar (240 kN/m²). What will then** be its change in diameter? The wall thickness of the boiler is 25 mm. $E = 210\,GN/m^2$; $\nu = 0.3$.
[6.24, 3.12 MN/m²; 0.033 mm.]

9.2 (A). Determine the change in volume of a thin cylinder of original volume $65.5 \times 10^{-3}\,m^3$ and length 1.3 m if its wall thickness is 6 mm and the internal pressure 14 bar (1.4 MN/m²). For the cylinder material $E = 210\,GN/m^2$; $\nu = 0.3$.
[$17.5 \times 10^{-6}\,m^3$.]

9.3 (A). What must be the wall thickness of a thin spherical vessel of diameter 1 m if it is to withstand an internal pressure of 70 bar (7 MN/m²) and the hoop stresses are limited to 270 MN/m²? [12.96 mm.]

9.4 (A/B). A steel cylinder 1 m long, of 150 mm internal diameter and plate thickness 5 mm, is subjected to an internal pressure of 70 bar (7 MN/m²); the increase in volume owing to the pressure is $16.8 \times 10^{-6}\,m^3$. Find the values of Poisson's ratio and the modulus of rigidity. Assume $E = 210\,GN/m^2$. [U.L.] [0.299; 80.8 GN/m².]

9.5 (B). Define bulk modulus K, and show that the decrease in volume of a fluid under pressure p is pV/K. Hence derive a formula to find the extra fluid which must be pumped into a thin cylinder to raise its pressure by an amount p.

How much fluid is required to raise the pressure in a thin cylinder of length 3 m, internal diameter 0.7 m, and wall thickness 12 mm by 0.7 bar (70 kN/m²)? $E = 210\,GN/m^2$ and $\nu = 0.3$ for the material of the cylinder and $K = 2.1\,GN/m^2$ for the fluid.
[$5.981 \times 10^{-3}\,m^3$.]

9.6 (B). A spherical vessel of 1.7 m diameter is made from 12 mm thick plate, and it is to be subjected to a hydraulic test. Determine the additional volume of water which it is necessary to pump into the vessel, when the vessel is initially just filled with water, in order to raise the pressure to the proof pressure of 116 bar (11.6 MN/m²). The bulk modulus of water is 2.9 GN/m². For the material of the vessel, $E = 200\,GN/m^2$, $\nu = 0.3$.
[$26.14 \times 10^{-3}\,m^3$.]

9.7 (B). A thin-walled steel cylinder is subjected to an internal fluid pressure of 21 bar (2.1 MN/m^2). The boiler is of 1 m inside diameter and 3 m long and has a wall thickness of 30 mm. Calculate the hoop and longitudinal stresses present in the cylinder and determine what torque may be applied to the cylinder if the principal stress is limited to 150 MN/m^2. [35, 17.5 MN/m^2; 6 MN m.]

9.8 (B). A thin cylinder of 300 mm internal diameter and 12 mm thickness is subjected to an internal pressure p while the ends are subjected to an external pressure of $\frac{1}{2}p$. Determine the value of p at which elastic failure will occur according to (a) the maximum shear stress theory, and (b) the maximum shear strain energy theory, if the limit of proportionality of the material in simple tension is 270 MN/m^2. What will be the volumetric strain at this pressure? $E = 210\,GN/m^2$; $\nu = 0.3$ [21.6, 23.6 MN/m^2, 2.289×10^{-3}, 2.5×10^{-3}.]

9.9 (C). A brass pipe has an internal diameter of 400 mm and a metal thickness of 6 mm. A single layer of high-tensile wire of diameter 3 mm is wound closely round it at a tension of 500 N. Find (a) the stress in the pipe when there is no internal pressure; (b) the maximum permissible internal pressure in the pipe if the working tensile stress in the brass is 60 MN/m^2; (c) the stress in the steel wire under condition (b). Treat the pipe as a thin cylinder and neglect longitudinal stresses and strains. $E_S = 200\,GN/m^2$; $E_B = 100\,GN/m^2$.

[U.L.] [27.8, 3.04 MN/m^2; 104.8 MN/m^2.]

9.10 (B). A cylindrical vessel of 1 m diameter and 3 m long is made of steel 12 mm thick and filled with water at 16°C. The temperature is then raised to 50°C. Find the stresses induced in the material of the vessel given that over this range of temperature water increases 0.006 per unit volume. (Bulk modulus of water = 2.9 GN/m^2; E for steel = 210 GN/m^2 and $\nu = 0.3$.) Neglect the expansion of the steel owing to temperature rise.

[663, 331.5 MN/m^2.]

9.11 (C). A 3 m long aluminium-alloy tube, of 150 mm outside diameter and 5 mm wall thickness, is closely wound with a single layer of 2.5 mm diameter steel wire at a tension of 400 N. It is then subjected to an internal pressure of 70 bar (7 MN/m^2).

(a) Find the stress in the tube before the pressure is applied.

(b) Find the final stress in the tube.

$E_A = 70\,GN/m^2$; $\nu_A = 0.28$; $E_S = 200\,GN/m^2$ [−32, 20.5 MN/m^2.]

9.12 (B). (a) Derive the equations for the circumferential and longitudinal stresses in a thin cylindrical shell.

(b) A thin cylinder of 300 mm internal diameter, 3 m long and made from 3 mm thick metal, has its ends blanked off. Working from first principles, except that you may use the equations derived above, find the change in capacity of this cylinder when an internal fluid pressure of 20 bar is applied. $E = 200\,GN/m^2$; $\nu = 0.3$. [$201 \times 10^{-6}\,m^3$.]

9.13 (A/B). Show that the tensile hoop stress set up in a thin rotating ring or cylinder is given by:

$$\sigma_H = \rho\omega^2 r^2.$$

Hence determine the maximum angular velocity at which the disc can be rotated if the hoop stress is limited to 20 MN/m^2. The ring has a mean diameter of 260 mm. [3800 rev/min.]