

CHAPTER 7

SHEAR STRESS DISTRIBUTION

Summary

The *shear stress* in a beam at any transverse cross-section in its length, and at a point a vertical distance y from the neutral axis, *resulting from bending* is given by

$$\tau = \frac{QA\bar{y}}{Ib} \quad \text{or} \quad \tau = \frac{Q}{Ib} \int ydA$$

where Q is the applied vertical shear force at that section; A is the area of cross-section “above” y , i.e. the area between y and the outside of the section, which may be above or below the neutral axis (N.A.); \bar{y} is the distance of the centroid of area A from the N.A.; I is the second moment of area of the complete cross-section; and b is the breadth of the section at position y .

For **rectangular sections**,

$$\tau = \frac{6Q}{bd^3} \left[\frac{d^2}{4} - y^2 \right] \quad \text{with} \quad \tau_{\max} = \frac{3Q}{2bd} \quad \text{when} \quad y = 0$$

For **I-section beams** the *vertical shear* in the web is given by

$$\tau = \frac{Q}{2I} \left[\frac{h^2}{4} - y^2 \right] + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right]$$

with a *maximum* value of

$$\tau_{\max} = \frac{Qh^2}{8I} + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right]$$

The *maximum* τ of the *horizontal shear* in the flanges is

$$\tau_{\max} = \frac{Qb}{4I} (d - t_1)$$

For **circular sections**

$$\tau = \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right]$$

with a *maximum* value of

$$\tau_{\max} = \frac{4Q}{3\pi R^2}$$

The *shear centre* of a section is that point, in or outside the section, through which load must be applied to produce zero twist of the section. Should a section have two axes of symmetry, the point where they cross is automatically the shear centre.

The shear centre of a **channel section** is given by

$$e = \frac{k^2 h^2 t}{4I}$$

Introduction

If a horizontal beam is subjected to vertical loads a shearing force (S.F.) diagram can be constructed as described in Chapter 3 to determine the value of the vertical S.F. at any section. This force tends to produce relative sliding between adjacent vertical sections of the beam, and it will be shown in Chapter 13, §13.2, that it is always accompanied by *complementary shears* which in this case will be horizontal. Since the concept of complementary shear is sometimes found difficult to appreciate, the following explanation is offered.

Consider the case of two rectangular-sectioned beams lying one on top of the other and supported on simple supports as shown in Fig. 7.1. If some form of vertical loading is applied the beams will bend as shown in Fig. 7.2, i.e. if there is negligible friction between the mating surfaces of the beams each beam will bend independently of the other and as a result the lower surface of the top beam will slide relative to the upper surface of the lower beam.

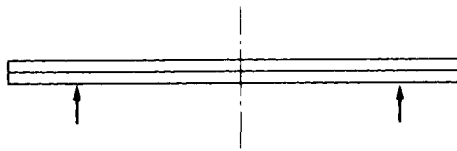


Fig. 7.1. Two beams (unconnected) on simple supports prior to loading.

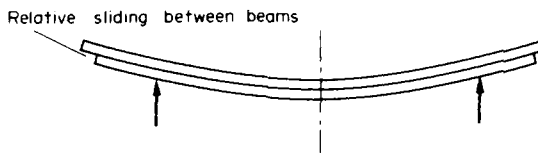


Fig. 7.2. Illustration of the presence of shear (relative sliding) between adjacent planes of a beam in bending.

If, therefore, the two beams are replaced by a single solid bar of depth equal to the combined depths of the initial two beams, then there must be some internal system of forces, and hence stresses, set up within the beam to prevent the above-mentioned sliding at the central fibres as bending takes place. Since the normal bending theory indicates that direct stresses due to bending are zero at the centre of a rectangular section beam, the prevention of sliding can only be achieved by horizontal shear stresses set up by the bending action.

Now on any element it will be shown in §13.2 that applied shears are always accompanied by complementary shears of equal value but opposite rotational sense on the perpendicular faces. Thus the *horizontal shears* due to bending are always associated with *complementary vertical shears* of equal value. For an element at either the top or bottom surface, however,

there can be no vertical shears if the surface is “free” or unloaded and hence the horizontal shear is also zero. It is evident, therefore, that, for beams in bending, shear stresses are set up both vertically and horizontally varying from some as yet undetermined value at the centre to zero at the top and bottom surfaces.

The method of determination of the remainder of the shear stress distribution across beam sections is considered in detail below.

7.1. Distribution of shear stress due to bending

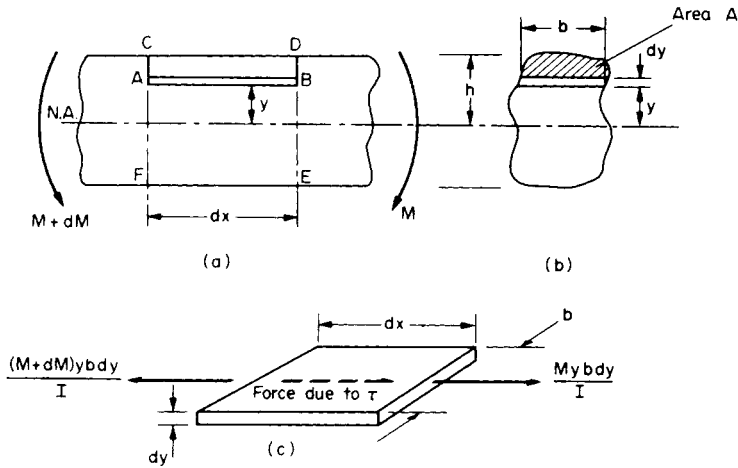


Fig. 7.3.

Consider the portion of a beam of length dx , as shown in Fig. 7.3a, and an element AB distance y from the N.A. Under any loading system the B.M. across the beam will change from M at B to $(M + dM)$ at A . Now as a result of bending,

$$\text{longitudinal stress } \sigma = \frac{My}{I}$$

$$\text{longitudinal stress at } A = \frac{(M + dM)y}{I}$$

and
$$\text{longitudinal stress at } B = \frac{My}{I}$$

$$\therefore \text{longitudinal force on the element at } A = \sigma A = \frac{(M + dM)y}{I} \times bdy$$

and
$$\text{longitudinal force on the element at } B = \frac{My}{I} \times bdy$$

The force system on the element is therefore as shown in Fig. 7.3c with a net out-of-balance force to the left

$$\begin{aligned}
 &= \frac{(M + dM)y}{I} bdy - \frac{My}{I} bdy \\
 &= \frac{dM}{I} ybdy
 \end{aligned}$$

Therefore total out-of-balance force from all sections above height y

$$= \int_y^h \frac{dM}{I} ybdy$$

For equilibrium, this force is resisted by a shear force set up on the section of length dx and breadth b , as shown in Fig. 7.4.

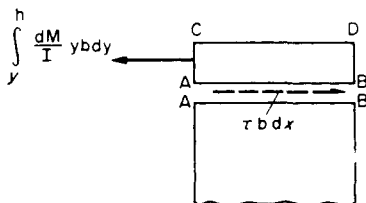


Fig. 7.4.

Thus if the shear stress is τ , then

$$\tau b dx = \frac{dM}{I} \int_y^h ybdy \quad (7.1)$$

But $\int_y^h ybdy =$ first moment of area of shaded portion of Fig. 7.3b about the N.A.
 $= A\bar{y}$

where A is the area of shaded portion and \bar{y} the distance of its centroid from the N.A.

Also $\frac{dM}{dx} =$ rate of change of the B.M.

$=$ S.F. Q at the section

$$\therefore \tau = \frac{QA\bar{y}}{Ib} \quad (7.2)$$

or, alternatively,

$$\tau = \frac{Q}{Ib} \int_y^h ydA \quad \text{where } dA = bdy \quad (7.3)$$

7.2. Application to rectangular sections

Consider now the rectangular-sectioned beam of Fig. 7.5 subjected at a given transverse cross-section to a S.F. Q .

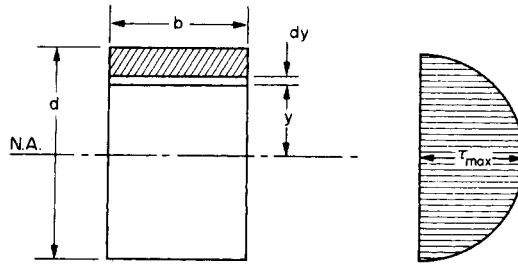


Fig. 7.5. Shear stress distribution due to bending of a rectangular section beam.

$$\begin{aligned} \tau &= \frac{QA\bar{y}}{Ib} \\ &= \frac{Q}{Ib} \times b \left(\frac{d}{2} - y\right) \times \left[y + \frac{\left(\frac{d}{2} - y\right)}{2} \right] \\ &= \frac{Q \times 12}{bd^3 \times b} \times b \left(\frac{d}{2} - y\right) \frac{\left(\frac{d}{2} + y\right)}{2} \\ &= \frac{6Q}{bd^3} \left[\frac{d^2}{4} - y^2 \right] \quad (\text{i.e. a parabola}) \end{aligned} \tag{7.4}$$

Now
$$\tau_{\max} = \frac{6Q}{bd^3} \times \frac{d^2}{4} = \frac{3Q}{2bd} \quad \text{when } y = 0 \tag{7.5}$$

and average
$$\tau = \frac{Q}{bd} \tag{7.6}$$

$$\therefore \tau_{\max} = \frac{3}{2} \times \tau_{\text{average}} \tag{7.6}$$

7.3. Application to I-section beams

Consider the I-section beam shown in Fig. 7.6.

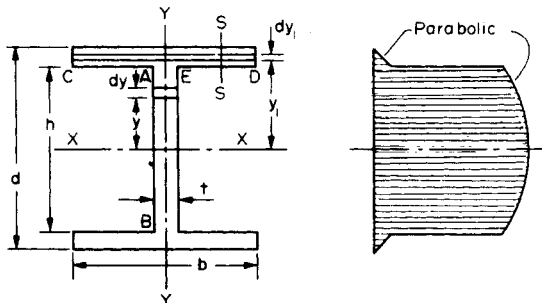


Fig. 7.6. Shear stress distribution due to bending of an I-section beam.

7.3.1. Vertical shear in the web

The distribution of shear stress due to bending at any point in a given transverse cross-section is given, in general, by eqn. (7.3)

$$\tau = \frac{Q}{Ib} \int_y^{d/2} y dA$$

In the case of the I-beam, however, the width of the section is not constant so that the quantity dA will be different in the web and the flange. Equation (7.3) must therefore be modified to

$$\begin{aligned} \tau &= \frac{Q}{It} \int_y^{h/2} ty dy + \frac{Q}{It} \int_{h/2}^{d/2} by_1 dy_1 \\ &= \frac{Q}{2I} \left[\frac{h^2}{4} - y^2 \right] + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \end{aligned}$$

As for the rectangular section, the first term produces a parabolic stress distribution. The second term is a constant and equal to the value of the shear stress at the top and bottom of the web, where $y = h/2$,

$$\text{i.e.} \quad \tau_A = \tau_B = \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \quad (7.7)$$

The *maximum shear* occurs at the N.A., where $y = 0$,

$$\tau_{\max} = \frac{Qh^2}{8I} + \frac{Qb}{2It} \left[\frac{d^2}{4} - \frac{h^2}{4} \right] \quad (7.8)$$

7.3.2. Vertical shear in the flanges

(a) *Along the central section YY*

The vertical shear in the flange where the width of the section is b is again given by eqn. (7.3) as

$$\begin{aligned} \tau &= \frac{Q}{Ib} \int_{y_1}^{d/2} y_1 dA \\ &= \frac{Q}{Ib} \int_{y_1}^{d/2} y_1 b dy_1 = \frac{Q}{I} \left[\frac{d^2}{8} - \frac{y_1^2}{2} \right] \end{aligned} \quad (7.9)$$

The *maximum* value is that at the bottom of the flange when $y_1 = h/2$,

$$\tau_{\max} = \frac{Q}{I} \left[\frac{d^2}{8} - \frac{h^2}{8} \right] = \frac{Q}{8I} [d^2 - h^2] \quad (7.10)$$

this value being considerably smaller than that obtained at the top of the web.

At the outside of the flanges, where $y_1 = d/2$, the vertical shear (and the complementary horizontal shear) are zero. At intermediate points the distribution is again parabolic producing the total stress distribution indicated in Fig. 7.6. As a close approximation, however, the distribution across the flanges is often taken to be linear since its effect is minimal compared with the values in the web.

(b) *Along any other section SS, removed from the web*

At the general section *SS* in the flange the shear stress at both the upper and lower edges must be zero. The distribution across the thickness of the flange is then the same as that for a rectangular section of the same dimensions.

The discrepancy between the values of shear across the free surfaces *CA* and *ED* and those at the web-flange junction indicate that the distribution of shear at the junction of the web and flange follows a more complicated relationship which cannot be investigated by the elementary analysis used here. Advanced elasticity theory must be applied to obtain a correct solution, but the values obtained above are normally perfectly adequate for general design work particularly in view of the following comments.

As stated above, the vertical shear stress in the flanges is very small in comparison with that in the web and is often neglected. Thus, in girder design, it is normally assumed that the web carries all the vertical shear. Additionally, the thickness of the web t is often very small in comparison with b such that eqns. (7.7) and (7.8) are nearly equal. The distribution of shear across the web in such cases is then taken to be uniform and equal to the total shear force Q divided by the cross-sectional area (th) of the web alone.

7.3.3. Horizontal shear in the flanges

The proof of §7.1 considered the equilibrium of an element in a vertical section of a component similar to element *A* of Fig. 7.9. Consider now a similar element *B* in the horizontal flange of the channel section (or I section) shown in Fig. 7.7.

The element has dimensions dz , t and dx comparable directly to the element previously treated of dy , b and dx . The proof of §7.1 can be applied in precisely the same way to this flange element giving an out-of-balance force on the element, from Fig. 7.9(b),

$$\begin{aligned} &= \frac{(M + dM)}{I} y \cdot t dz - \frac{My \cdot t dz}{I} \\ &= \frac{dM}{I} y \cdot t dz \end{aligned}$$

with a total out-of-balance force for the sections between z and L

$$= \int_z^L \frac{dM}{I} y \cdot t dz.$$

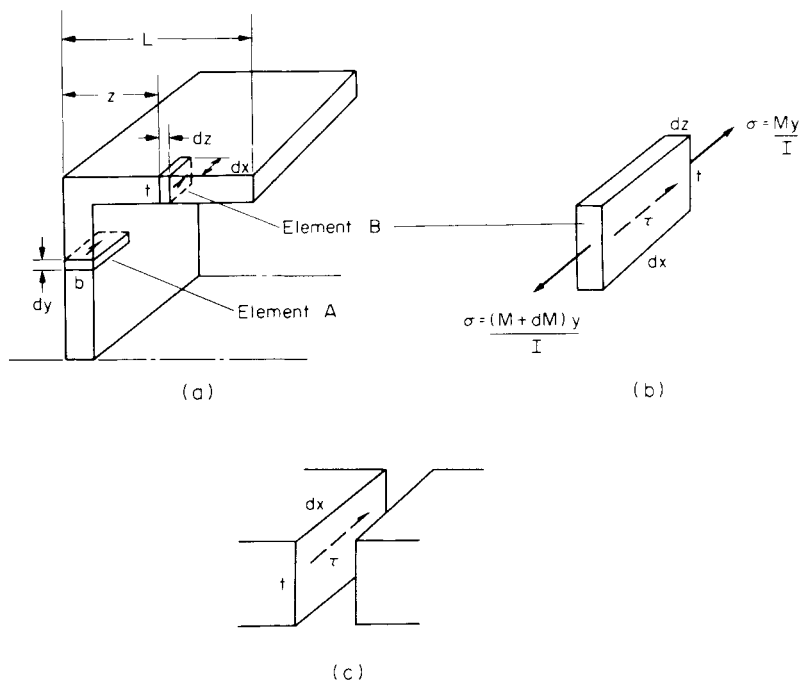


Fig. 7.7. Horizontal shear in flanges.

This force being reacted by the shear on the element shown in Fig. 7.9(c),

$$= \tau t dx$$

$$\therefore \tau t dx = \int_z^L \frac{dM}{I} \cdot y t dz$$

and
$$\tau = \frac{dM}{dx} \cdot \frac{1}{It} \int_z^L t dz \cdot y$$

But
$$\int_z^L t dz \cdot y = A \bar{y} \quad \text{and} \quad \frac{dM}{dx} = Q.$$

$$\therefore \tau = \frac{QA \bar{y}}{It} \tag{7.11}$$

Thus the same form of expression is obtained to that of eqn (7.2) but with the breadth b of the web replaced by thickness t of the flange: I and y still refer to the N.A. and A is the area of the flange 'beyond' the point being considered.

Thus the horizontal shear stress distribution in the flanges of the I section of Fig. 7.8 can

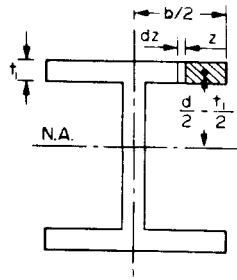


Fig. 7.8.

now be obtained from eqn. (7.11):

$$\tau = \frac{QA\bar{y}}{It}$$

with
$$\bar{y} = \frac{d}{2} - \frac{t_1}{2} = \frac{1}{2}(d - t_1)$$

$$A = t_1 dz$$

$$t = t_1$$

Thus
$$\tau = \frac{Q}{It_1} \int_0^{b/2} \frac{1}{2}(d - t_1)t_1 dz = \frac{Q}{2I}(d - t_1)[z]_0^{b/2}$$

The distribution is therefore linear from zero at the free ends of the flange to a maximum value of

$$\tau_{\max} = \frac{Qb}{4I}(d - t_1) \quad \text{at the centre} \quad (7.12)$$

7.4. Application to circular sections

In this case it is convenient to use the alternative form of eqn. (7.2), namely (7.1),

$$\tau b dx = \frac{dM}{I} \int_z^h b y dy$$

$$\tau = \frac{Q}{Ib} \int_z^h b y dy$$

Consider now the element of thickness dz and breadth b shown in Fig. 7.9.

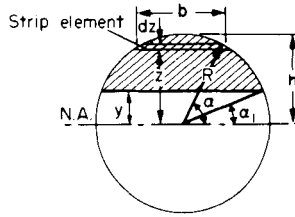


Fig. 7.9.

Now $b = 2R \cos \alpha$, $y = z = R \sin \alpha$ and $dz = R \cos \alpha d\alpha$ and, at section distance y from the N.A., $b = 2R \cos \alpha_1$,

$$\begin{aligned}
 \therefore \tau &= \frac{Q}{I \times 2R \cos \alpha_1} \int_{\alpha_1}^{\pi/2} 2R \cos \alpha R \sin \alpha R \cos \alpha d\alpha \\
 &= \frac{Q \times 4}{2R \cos \alpha_1 \times \pi R^4} \int_{\alpha_1}^{\pi/2} 2R^3 \cos^2 \alpha \sin \alpha d\alpha \quad \text{since } I = \frac{\pi R^4}{4} \\
 &= \frac{4Q}{\pi R^2 \cos \alpha_1} \left[-\frac{\cos^3 \alpha}{3} \right]_{\alpha_1}^{\pi/2} \\
 &= \frac{4Q}{3\pi R^2 \cos \alpha_1} \left[\cos^3 \alpha_1 \right]_{\pi/2}^{\alpha_1} = \frac{4Q \cos^2 \alpha_1}{3\pi R^2} \\
 &= \frac{4Q}{3\pi R^2} [1 - \sin^2 \alpha_1] = \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right] \tag{7.13}
 \end{aligned}$$

i.e. a parabola with its maximum value at $y = 0$.

Thus $\tau_{\max} = \frac{4Q}{3\pi R^2}$

Now mean stress $= \frac{Q}{\pi R^2}$

$$\therefore \frac{\text{maximum shear stress}}{\text{mean shear stress}} = \frac{\frac{4Q}{3\pi R^2}}{\frac{Q}{\pi R^2}} = \frac{4}{3} \tag{7.14}$$

Alternative procedure

Using eqn. (7.2), namely $\tau = \frac{QA\bar{y}}{Ib}$, and referring to Fig. 7.9,

$$\frac{b}{2} = (R^2 - z^2)^{1/2} = R \cos \alpha \quad \text{and} \quad \sin \alpha = \frac{z}{R}$$

$$\begin{aligned}
 A\bar{y} \text{ for the shaded segment} &= \int_{R \sin \alpha}^R A\bar{y} \quad \text{for strip element} \\
 &= \int_{R \sin \alpha}^R b dz z \\
 &= 2 \int_{R \sin \alpha}^R (R^2 - z^2)^{1/2} z dz \\
 &= \frac{2}{3} [(R^2 - z^2)^{3/2}]_{R \sin \alpha}^R \\
 &= \frac{2}{3} [R^2(1 - \sin^2 \alpha)]^{3/2} \\
 &= \frac{2}{3} R^3 (\cos^2 \alpha)^{3/2} = \frac{2}{3} R^3 \cos^3 \alpha
 \end{aligned}$$

$$\therefore \tau = \frac{QA\bar{y}}{Ib} = \frac{Q \times \frac{2}{3} R^3 \cos^3 \alpha}{\frac{\pi R^4}{4} \times 2R \cos \alpha}$$

$$\text{since } I = \frac{\pi R^4}{4}$$

$$\begin{aligned}
 \therefore \tau &= \frac{4Q}{3\pi R^2} \cos^2 \alpha = \frac{4Q}{3\pi R^2} [1 - \sin^2 \alpha] \\
 &= \frac{4Q}{3\pi R^2} \left[1 - \left(\frac{y}{R} \right)^2 \right] \quad (7.13) \text{ bis.}
 \end{aligned}$$

7.5. Limitation of shear stress distribution theory

There are certain practical situations where eq. (7.2) leads to an incomplete solution and it is necessary to consider other conditions, such as equilibrium at a free surface, before a valid solution is obtained. Take, for example, the case of the bending of a bar or beam having a circular cross-section as shown in Fig. 7.10(a).

The shear stress distribution across the section owing to bending is given by eqn. (7.2) as:

$$\tau = \frac{Q}{3I} \left[\left(\frac{d}{2} \right)^2 - y^2 \right] = \tau_{yz} \quad (7.15)$$

At some horizontal section AB , therefore, the shear stresses will be as indicated in Fig. 7.10, and will be equal along AB , with a value given by eqn. (7.15).

Now, for an element at A , for example, there should be no component of stress normal to the surface since it is unloaded and equilibrium would not result. The vertical shear given by the equation above, however, clearly would have a component normal to the surface. A valid solution can only be obtained therefore if a secondary shear stress τ_{xz} is set up in the z direction which, together with τ_{yz} , produces a resultant shear stress tangential to the free surface—see Fig. 7.10(b).

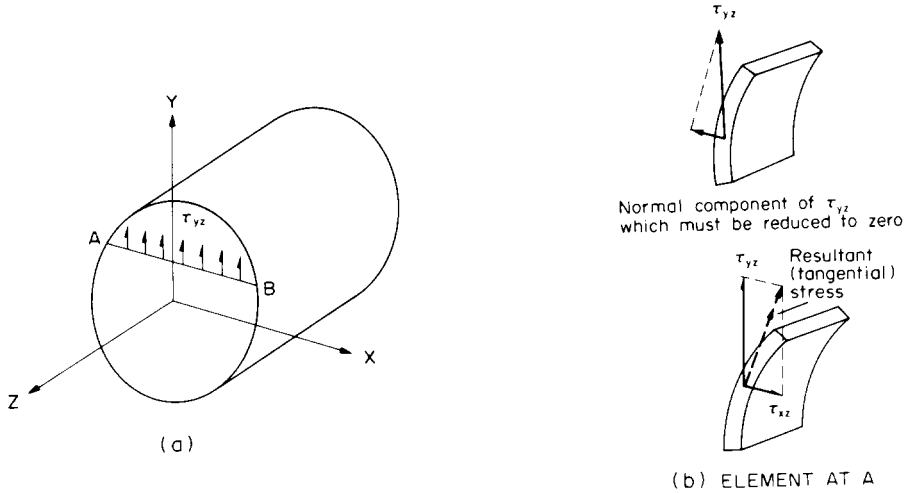


Fig. 7.10.

Solutions for the value of τ_{xz} and its effect on τ_{yz} are beyond the scope of this text† but the principal outlined indicates a limitation of the shear stress distribution theory which should be appreciated.

7.6. Shear centre

Consider the channel section of Fig. 7.11 under the action of a shearing force Q at a distance e from the centre of the web. The shearing stress at any point on the cross-section of the channel is then given by the equation $\tau = \frac{QA\bar{y}}{Ib}$. The distribution in the rectangular web

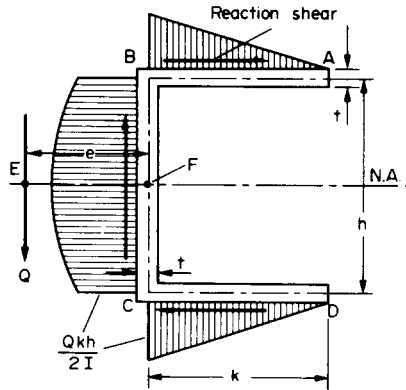


Fig. 7.11.

† I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd edition (McGraw Hill, New York, 1956).

will be parabolic, as previously found, but will not reduce to zero at each end because of the presence of the flange areas.

When the stress in the flange is being determined the breadth b is replaced by the thickness t , but I and \bar{y} still refer to the N.A.

$$\text{Thus from eqn. 7.11, } \tau_B = \frac{QA\bar{y}}{It} = \frac{Q \times kt}{It} \times \frac{h}{2} = \frac{Qkh}{2I}$$

$$\text{and } \tau_A = 0 \quad \text{since area beyond } A = 0$$

Between A and B the distribution is linear, since τ is directly proportional to the distance along AB (Q , t , h and I all being constant). An exactly similar distribution will be obtained for CD .

The stresses in the flanges give rise to forces represented by

$$\text{average stress} \times \text{area} = \frac{1}{2} \times \frac{Qkh}{2I} \times kt = \frac{Qk^2ht}{4I}$$

These produce a torque about F which must equal the applied torque, stresses in the web producing forces which have no moment about F .

Equating torques about F for equilibrium:

$$Q \times e = \frac{Qk^2ht}{4I} \times h$$

$$\therefore e = \frac{k^2 h^2 t}{4I} \quad (7.16)$$

Thus if a force acts on the axis of symmetry, distance e from the centre of the web, there will be no tendency for the section to twist since moments will be balanced. The point E is then termed the *shear centre* of the section.

The *shear centre* of a section is therefore defined as that point through which load must be applied for zero twist of the section. With loads applied at the shear centre of beam sections, stresses will be produced due to pure bending, and evaluation of the stresses produced will be much easier than would be the case if torsion were also present.

It should be noted that if a section has two axes of symmetry the point where they cross is automatically the shear centre.

Examples

Example 7.1

At a given position on a beam of uniform I-section the beam is subjected to a shear force of 100 kN. Plot a curve to show the variation of shear stress across the section and hence determine the ratio of the maximum shear stress to the mean shear stress.

Solution

Consider the I-section shown in Fig. 7.12. By symmetry, the centroid of the section is at

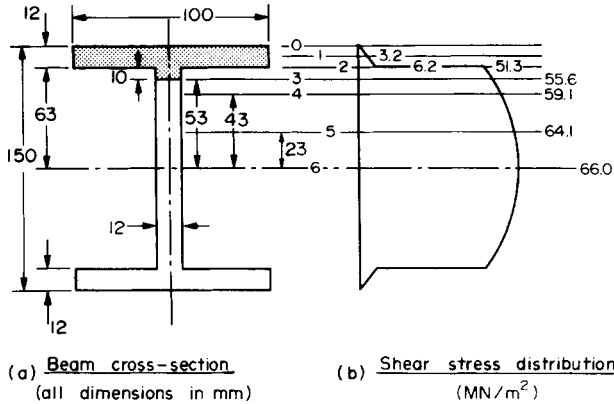


Fig. 7.12.

mid-height and the neutral axis passes through this position. The second moment of area of the section is then given by

$$I = \frac{100 \times 150^3 \times 10^{-12}}{12} - \frac{88 \times 126^3 \times 10^{-12}}{12}$$

$$= (28.125 - 14.67)10^{-6} = 13.46 \times 10^{-6} \text{ m}^4$$

The distribution of shear stress across the section is

$$\tau = \frac{QA\bar{y}}{Ib} = \frac{100 \times 10^3 A\bar{y}}{13.46 \times 10^{-6} b} = 7.43 \times 10^9 \frac{A\bar{y}}{b}$$

The solution of this equation is then best completed in tabular form as shown below. In this case, because of symmetry, only sections above the N.A. need be considered since a similar distribution will be obtained below the N.A.

It should be noted that two values of shear stress are required at section 2 to take account of the change in breadth at this section. The values of A and \bar{y} for sections 3, 4, 5 and 6 are those of a T-section beam and may be found as shown for section 3 (shaded area of Fig. 7.12a).

Section	$A \times 10^{-6}$ (m ²)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = \frac{QA\bar{y}}{Ib}$ (MN/m ²)
0	0	—	—	—
1	$100 \times 6 = 600$	72	100	3.2
2	$100 \times 12 = 1200$	69	100	6.2
2	1200	69	12	51.3
3	1320	68	12	55.6
4	1440	66.3	12	59.1
5	1680	61.6	12	64.1
6	1956	54.5	12	66.0

For section 3:

Taking moments about the top edge (Fig. 7.12a),

$$(100 \times 12 \times 6)10^{-9} + (10 \times 12 \times 17)10^{-9} = (100 \times 12 + 10 \times 12)h \times 10^{-9}$$

where h is the centroid of the shaded T-section,

$$7200 + 2040 = (1200 + 120)h$$

$$h = \frac{9240}{1320} = 7 \text{ mm}$$

$$\therefore \bar{y}_3 = 75 - 7 = 68 \text{ mm}$$

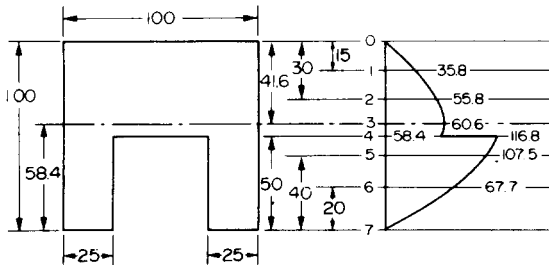
The distribution of shear stress due to bending is then shown in Fig. 7.12b, giving a maximum shear stress of $\tau_{\max} = 66 \text{ MN/m}^2$.

Now the mean shear stress across the section is:

$$\begin{aligned} \tau_{\text{mean}} &= \frac{\text{shear force}}{\text{area}} = \frac{100 \times 10^3}{3.912 \times 10^{-3}} \\ &= 25.6 \text{ MN/m}^2 \\ \frac{\text{max. shear stress}}{\text{mean shear stress}} &= \frac{66}{25.6} = 2.58 \end{aligned}$$

Example 7.2

At a certain section a beam has the cross-section shown in Fig. 7.13. The beam is simply supported at its ends and carries a central concentrated load of 500 kN together with a load of 300 kN/m uniformly distributed across the complete span of 3 m. Draw the shear stress distribution diagram for a section 1 m from the left-hand support.



(a) Beam cross-section (mm) (b) Shear stress distribution (MN/m²)

Fig. 7.13.

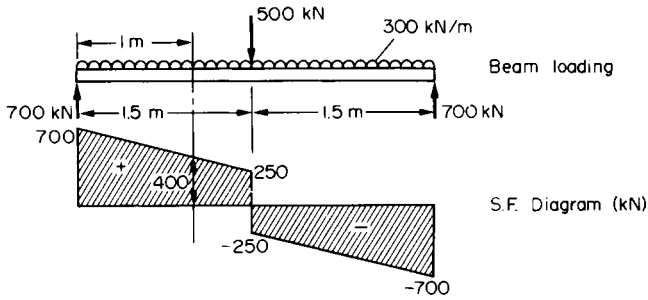


Fig. 7.14.

Solution

From the S.F. diagram for the beam (Fig. 7.14) it is evident that the S.F. at the section 1 m from the left-hand support is 400 kN,

i.e. $Q = 400 \text{ kN}$

To find the position of the N.A. of the beam section of Fig. 7.13(a) take moments of area about the base.

$$(100 \times 100 \times 50)10^{-9} - (50 \times 50 \times 25)10^{-9} = (100 \times 100 - 50 \times 50) \bar{y} \times 10^{-9}$$

$$500000 - 62500 = (10000 - 2500) \bar{y}$$

$$\bar{y} = \frac{437500}{7500} = 58.4 \text{ mm}$$

Then $I_{NA} = \left[\frac{100 \times 41.6^3}{3} + 2 \left(\frac{25 \times 58.4^3}{3} \right) + \left(\frac{50 \times 8.4^3}{3} \right) \right] 10^{-12}$

$$= (2.41 + 3.3 + 0.0099)10^{-6} = 5.72 \times 10^{-6} \text{ m}^4$$

$\therefore \tau = \frac{QA\bar{y}}{Ib} = \frac{400 \times 10^3}{5.72 \times 10^{-6}} \frac{A\bar{y}}{b} = 7 \times 10^{10} \frac{A\bar{y}}{b}$

Section	$A \times 10^{-6}$ (m ²)	$\bar{y} \times 10^{-3}$ (m)	$b \times 10^{-3}$ (m)	$\tau = 7 \times 10^4 \frac{A\bar{y}}{b}$ (MN/m ²)
0	0	-	-	0
1	1500	34.1	100	35.8
2	3000	26.6	100	55.8
3	4160	20.8	100	60.6
4	2500	33.4	100	58.4
4	2500	33.4	50	116.8
5	2000	38.4	50	107.5
6	1000	48.4	50	67.7
7	0	-	-	0

The shear stress distribution across the section is shown in Fig. 7.13b.

Example 7.3

Determine the values of the shear stress owing to bending at points *A*, *B* and *C* in the beam cross-section shown in Fig. 7.15 when subjected to a shear force of $Q = 140$ kN. Hence sketch the shear stress distribution diagram.

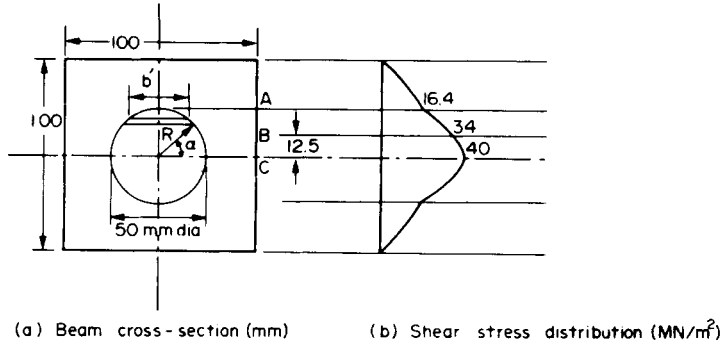


Fig. 7.15.

Solution

By symmetry the centroid will be at the centre of the section and

$$I_{NA} = \frac{100 \times 100^3}{12} \times 10^{-12} - \frac{\pi \times 50^4}{64} \times 10^{-12}$$

$$= (8.33 - 0.31)10^{-6} = 8.02 \times 10^{-6} \text{ m}^4$$

At *A*:

$$\tau_A = \frac{QA\bar{y}}{Ib} = \frac{140 \times 10^3 \times (100 \times 25 \times 37.5)10^{-9}}{8.02 \times 10^{-6} \times 100 \times 10^{-3}} = 16.4 \text{ MN/m}^2$$

At *B*:

Here the required $A\bar{y}$ is obtained by subtracting $A\bar{y}$ for the portion of the circle above *B* from that of the rectangle above *B*. Now for the circle

$$A\bar{y} = \int_{12.5}^{25} b'y dy \quad (\text{Fig. 7.15a})$$

But, when $y = 12.5$, $\sin \alpha = \frac{12.5}{25} = \frac{1}{2} \quad \therefore \alpha = \pi/6$

and when $y = 25$, $\sin \alpha = \frac{25}{25} = 1 \quad \therefore \alpha = \pi/2$

Also $b' = 2R \cos \alpha$, $y = R \sin \alpha$ and $dy = R \cos \alpha d\alpha$

$$\begin{aligned}
 \therefore \text{ for circle portion, } A\bar{y} &= \int_{\pi/6}^{\pi/2} 2R \cos \alpha R \sin \alpha R \cos \alpha d\alpha \\
 &= \int_{\pi/6}^{\pi/2} 2R^3 \cos^2 \alpha \sin \alpha d\alpha \\
 &= 2R^3 \left[-\frac{\cos^3 \alpha}{3} \right]_{\pi/6}^{\pi/2} \\
 &= \frac{2 \times 25^3 \times 10^{-9}}{3} \left[\left(\frac{\sqrt{3}}{2} \right)^3 \right] = 6.75 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

\therefore for complete section above B

$$\begin{aligned}
 A\bar{y} &= 100 \times 37.5 \times 31.25 \times 10^{-9} - 6.75 \times 10^{-6} \\
 &= 110.25 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

and $b' = 2R \cos \pi/6 = 2 \times 25 \times 10^{-3} \times \sqrt{3}/2 = 43.3 \times 10^{-3} \text{ m}$

$\therefore b = (100 - 43.3)10^{-3} = 56.7 \times 10^{-3} \text{ m}$

$\therefore \tau_B = \frac{QA\bar{y}}{Ib} = \frac{140 \times 10^3 \times 110.25 \times 10^{-6}}{8.02 \times 10^{-6} \times 56.7 \times 10^{-3}} = 34 \text{ MN/m}^2$

At C :

$$\begin{aligned}
 A\bar{y} \text{ for semicircle} &= 2R^3 \left[-\frac{\cos^3 \alpha}{3} \right]_0^{\pi/2} \\
 &= \frac{2 \times 25^3 \times 10^{-9}}{3} [-(0 - 1)] = 10.41 \times 10^{-6} \text{ m}^3
 \end{aligned}$$

$A\bar{y}$ for section above C

$$= (100 \times 50 \times 25)10^{-9} - 10.41 \times 10^{-6} = 114.59 \times 10^{-6} \text{ m}^3$$

and $b = (100 - 50)10^{-3} = 50 \times 10^{-3} \text{ m}$

$\therefore \tau_c = \frac{140 \times 10^3 \times 114.59 \times 10^{-6}}{8.02 \times 10^{-6} \times 50 \times 10^{-3}} = 40 \text{ MN/m}^2$

The total shear stress distribution across the section is then sketched in Fig. 7.15b.

Example 7.4

A beam having the cross-section shown in Fig. 7.16 is constructed from material having a constant thickness of 1.3 mm. Through what point must vertical loads be applied in order that there shall be no twisting of the section? Sketch the shear stress distribution.

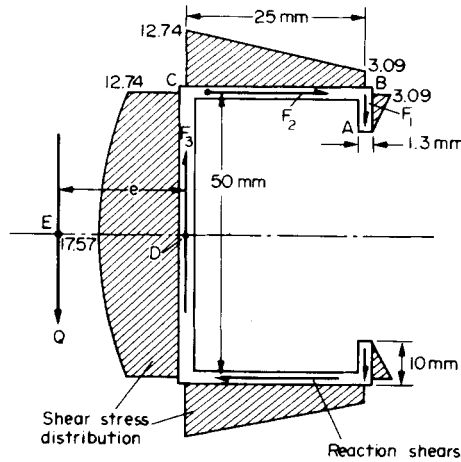


Fig. 7.16.

Solution

Let a load of QN be applied through the point E , distance e from the centre of the web.

$$\begin{aligned}
 I_{\text{N.A.}} &= \left[\frac{1.3 \times 50^3}{12} + 2 \left(\frac{25 \times 1.3^3}{12} + 25 \times 1.3 \times 25^2 \right) \right. \\
 &\quad \left. + 2 \left(\frac{1.3 \times 10^3}{12} + 1.3 \times 10 \times 20^2 \right) \right] \times 10^{-12} \\
 &= [1.354 + 2(0.00046 + 2.03) + 2(0.011 + 0.52)] 10^{-8} \\
 &= 6.48 \times 10^{-8} \text{ m}^4
 \end{aligned}$$

Shear stress

$$\begin{aligned}
 \tau_A &= \frac{QA\bar{y}}{Ib} = \frac{Q \times 0}{Ib} = 0 \\
 \tau_B &= \frac{Q \times (10 \times 1.3 \times 20) 10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}} = 3.09Q \text{ kN/m}^2 \\
 \tau_C &= 3.09Q + \frac{Q(25 \times 1.3 \times 25) 10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}} \\
 &= 3.09Q + 9.65Q = 12.74Q \text{ kN/m}^2 \\
 \tau_D &= 12.74Q + \frac{Q(25 \times 1.3 \times 12.5) 10^{-9}}{6.48 \times 10^{-8} \times 1.3 \times 10^{-3}} \\
 &= 12.74Q + 4.83Q = 17.57Q \text{ kN/m}^2
 \end{aligned}$$

The shear stress distribution is then sketched in Fig. 7.16. It should be noted that whilst the distribution is linear along BC it is not strictly so along AB . For ease of calculation of the shear centre, however, it is usually assumed to be linear since the contribution of this region to

moments about D is small (the shear centre is the required point through which load must be applied to produce zero twist of the section).

Thus taking moments of forces about D for equilibrium,

$$\begin{aligned} Q \times e \times 10^{-3} &= 2F_1 \times 25 \times 10^{-3} + 2F_2 \times 25 \times 10^{-3} \\ &= 50 \times 10^{-3} \left[\frac{1}{2} \times 3.09 Q \times 10^3 \times (10 \times 1.3 \times 10^{-6}) \right. \\ &\quad \left. + 10^3 \frac{(3.09Q + 12.74Q)}{2} (25 \times 1.3 \times 10^{-6}) \right] \\ &= 13.866Q \times 10^{-3} \\ e &= 13.87 \text{ mm} \end{aligned}$$

Thus, loads must be applied through the point E , 13.87 mm to the left of the web centre-line for zero twist of the section.

Problems

7.1 (A/B). A uniform I-section beam has flanges 150 mm wide by 8 mm thick and a web 180 mm wide and 8 mm thick. At a certain section there is a shearing force of 120 kN. Draw a diagram to illustrate the distribution of shear stress across the section as a result of bending. What is the maximum shear stress? [86.7 MN/m².]

7.2 (A/B). A girder has a uniform T cross-section with flange 250 mm × 50 mm and web 220 mm × 50 mm. At a certain section of the girder there is a shear force of 360 kN.

Plot neatly to scale the shear-stress distribution across the section, stating the values:

- (a) where the web and the flange of the section meet;
- (b) at the neutral axis.

[B.P.] [7.47, 37.4, 39.6 MN/m².]

7.3 (A/B). A beam having an inverted T cross-section has an overall width of 150 mm and overall depth of 200 mm. The thickness of the crosspiece is 50 mm and of the vertical web 25 mm. At a certain section along the beam the vertical shear force is found to be 120 kN. Draw neatly to scale, using 20 mm spacing except where closer intervals are required, a shear-stress distribution diagram across this section. If the mean stress is calculated over the whole of the cross-sectional area, determine the ratio of the maximum shear stress to the mean shear stress.

[B.P.] [3.37.]

7.4 (A/B). The channel section shown in Fig. 7.17 is simply supported over a span of 5 m and carries a uniformly distributed load of 15 kN/m run over its whole length. Sketch the shearing-stress distribution diagram at the point of maximum shearing force and mark important values. Determine the ratio of the maximum shearing stress to the average shearing stress.

[B.P.] [3, 9.2, 9.3 MN/m²; 2.42.]

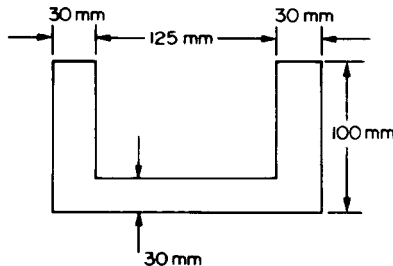


Fig. 7.17.

7.5 (A/B). Fig. 7.18 shows the cross-section of a beam which carries a shear force of 20 kN. Plot a graph to scale which shows the distribution of shear stress due to bending across the cross-section.

[I.Mech.E.] [21.7, 5.2, 5.23 MN/m²]

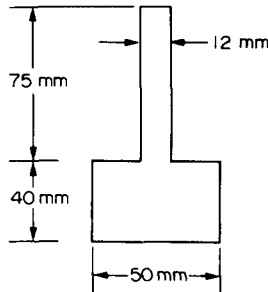


Fig. 7.18.

7.6 (B). Show that the difference between the maximum and mean shear stress in the web of an I-section beam is $\frac{Qh^2}{24I}$ where Q is the shear force on the cross-section, h is the depth of the web and I is the second moment of area of the cross-section about the neutral axis of bending. Assume the I-section to be built of rectangular sections, the flanges having width B and thickness t and the web a thickness b . Fillet radii are to be ignored. [I.Mech.E.]

7.7 (B). Deduce an expression for the shearing stress at any point in a section of a beam owing to the shearing force at that section. State the assumptions made.

A simply supported beam carries a central load W . The cross-section of the beam is rectangular of depth d . At what distance from the neutral axis will the shearing stress be equal to the mean shearing stress on the section?

[U.L.C.I.] [$d/\sqrt{12}$]

7.8 (B). A steel bar rolled to the section shown in Fig. 7.19 is subjected to a shearing force of 200 kN applied in the direction YY . Making the usual assumptions, determine the average shearing stress at the sections A , B , C and D , and find the ratio of the maximum to the mean shearing stress in the section. Draw to scale a diagram to show the variation of the average shearing stress across the section.

[U.L.] [Clue: treat as equivalent section similar to that of Example 7.3.] [7.2, 12.3, 33.6, 43.8 MN/m², 3.93]

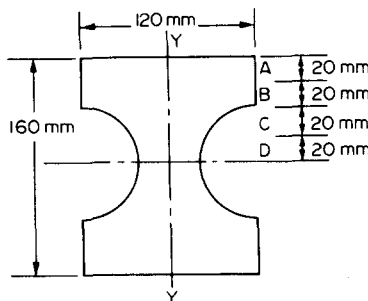


Fig. 7.19.

7.9 (C). Using customary notation, show that the shear stress over the cross-section of a loaded beam is given by $\tau = \frac{QA\bar{y}}{Ib}$.

The cross-section of a beam is an isosceles triangle of base B and height H , the base being arranged in a horizontal plane. Find the shear stress at the neutral axis owing to a shear force Q acting on the cross-section and express it in terms of the mean shear stress.

(The second moment of area of a triangle about its base is $BH^3/12$.)

[U.L.C.I.] [$\frac{8Q}{3BH}; \frac{4}{3}\tau_{\text{mean}}$]

7.10 (C). A hollow steel cylinder, of 200 mm external and 100 mm internal diameter, acting as a beam, is subjected to a shearing force $Q = 10$ kN perpendicular to the axis. Determine the mean shearing stress across the section and, making the usual assumptions, the average shearing stress at the neutral axis and at sections 25, 50 and 75 mm from the neutral axis as fractions of the mean value.

Draw a diagram to show the variation of average shearing stress across the section of the cylinder.

[U.L.] [0.425 MN/m²; 1.87, 1.65, 0.8, 0.47 MN/m².]

7.11 (C). A hexagonal-cross-section bar is used as a beam with its greatest dimension vertical and simply supported at its ends. The beam carries a central load of 60 kN. Draw a stress distribution diagram for a section of the beam at quarter span. All sides of the bar have a length of 25 mm.

($I_{N.A.}$ for triangle = $bh^3/36$ where b = base and h = height.)

[0, 9.2, 14.8, 25.9 MN/m² at 12.5 mm intervals above and below the N.A.]