CHAPTER 8

INTRODUCTION TO ADVANCED ELASTICITY THEORY

8.1. Type of stress

Any element of material may be subjected to three independent types of stress. Two of these have been considered in detai previously, namely *direct stresses* and *shear stresses*, and need not be considered further here. The third type, however, has not been specifically mentioned previously although it has in fact been present in some of the loading cases considered in earlier chapters; these are the so-called *body-force stresses*. These body forces arise by virtue of the bulk of the material, typical examples being:

(a) gravitational force due to a component's own weight: this has particular significance in civil engineering applications, e.g. dam and chimney design;
(b) centrifugal force, depending on radius and speed of rotation, with particular significance in high-speed engine or turbine design;
(c) magnetic field forces.

In many practical engineering applications the only body force present is the gravitational one, and in the majority of cases its effect is minimal compared with the other applied forces due to mechanical loading. In such cases it is therefore normally neglected. In high-speed dynamic loading situations such as the instances quoted in (b) above, however, the centrifugal forces far exceed any other form of loading and are therefore the primary factor for consideration.

Unlike direct and shear stresses, body force stresses are defined as **force per unit volume,** and particular note must be taken of this definition in relation to the proofs of formulae which follow.

8.2. The cartesian stress components: notation and sign convention

Consider an element of material subjected to a complex stress system in three dimensions. Whatever the type of applied loading the resulting stresses can always be reduced to the nine components, i.e. three direct and six shear, shown in Fig. 8.1.

It will be observed that in this case a modified notation is used for the stresses. This is termed the double-suffix notation and it is particularly useful in the detailed study of stress problems since it indicates both the direction of the stress *and* the plane on which it acts.

The *first* suffix gives the *direction* of the stress.

The *second* suffix gives the *direction of the normal of the plane* on which the stress acts.

Thus, for example,

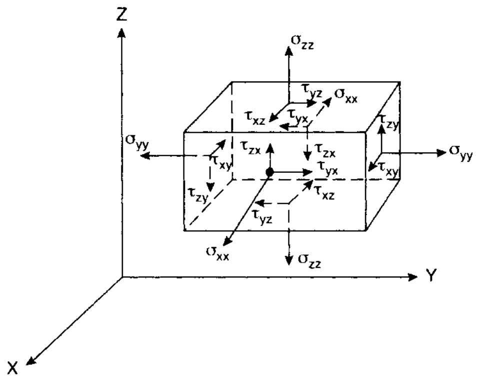

Fig. 8.1. The cartesian stress components.

σ_{xx} is the stress in the X direction on the X facing face (i.e. a *direct* stress). Common suffices therefore always indicate that the stress is a direct stress. Similarly, σ_{xy} is the stress in the X direction on the Y facing face (i.e. a *shear* stress). Mixed suffices always indicate the presence of shear stresses and thus allow the alternative symbols σ_{xy} or τ_{xy}. Indeed, the alternative symbol τ is not strictly necessary now since the suffices indicate whether the stress σ is a direct one or a shear.

8.2.1. Sign conventions

(a) *Direct stresses*. As always, direct stresses are assumed positive when tensile and negative when compressive.

(b) *Shear stresses*. Shear stresses are taken to be positive if they act in a positive cartesian (X, Y or Z) direction whilst acting on a plane whose outer normal points also in a positive cartesian direction.

Thus positive shear is assumed with + direction and + facing face.

Alternatively', positive shear is also given with – direction *and* – facing face (a double negative making a positive, as usual).

A careful study of Fig. 8.1 will now reveal that all stresses shown are positive in nature.

The *cartesian stress components* considered here relate to the three mutually perpendicular axes X, Y and Z. In certain loading cases, notably those involving axial symmetry, this system of components is inconvenient and an alternative set known as *cylindrical components* is used. These involve the variables, radius r, angle θ and axial distance z, and will be considered in detail later.

8.3. The state of stress at a point

Consider any point Q within a stressed material, the nine cartesian stress components at Q being known. It is now possible to determine the normal, direct and resultant stresses which act on any plane through Q whatever its inclination relative to the cartesian axes. Suppose one such plane ABC has a normal n which makes angles nx, ny and nz with the YZ, XZ and XY planes respectively as shown in Figs. 8.2 and 8.3. (Angles between planes

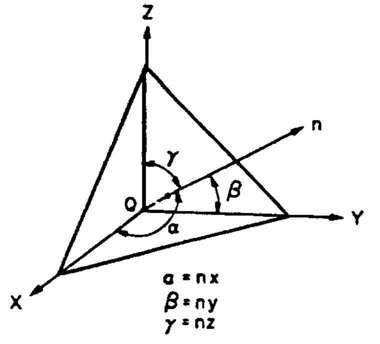

Fig. 8.2.

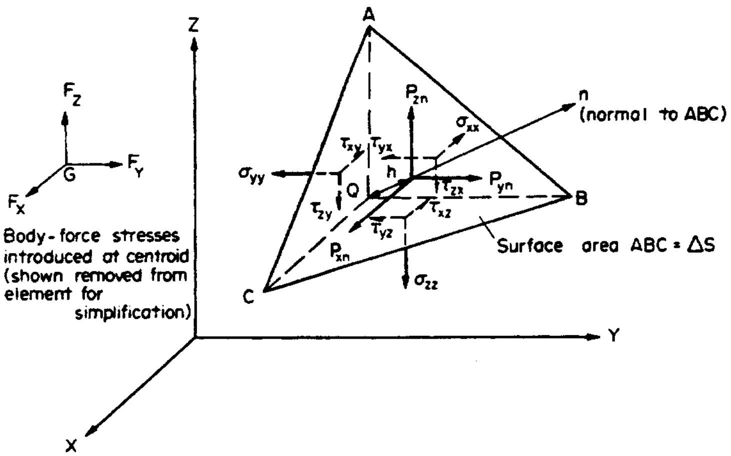

Fig. 8.3. The state of stress on an inclined plane through any given point in a three-dimensional cartesian stress system.

ABC and *YZ* are given by the angle between the normals to both planes n and x, etc.) For convenience, let the plane *ABC* initially be some perpendicular distance h from Q so that the cartesian stress components actually acting at Q can be shown on the sides of the tetrahdedron element *ABCQ* so formed (Fig. 8.3). In the derivation below the value of h will be reduced to zero so that the equations obtained will relate to the condition when *ABC* passes through Q.

In addition to the cartesian components, the unknown components of the stress on the plane *ABC*, i.e. p_{xn}, p_{yn} and p_{zn}, are also indicated, as are the body-force field stress components which act at the centre of gravity of the tetrahedron. (To improve clarity of the diagram they are shown displaced from the element.)

Since body-force stresses are defined as forces/unit volume, the components in the X, Y and Z directions are of the form

$$F \times \Delta S \frac{h}{3}$$

where $\Delta Sh/3$ is the volume of the tetrahedron. If the area of the surface *ABC*, i.e. ΔS, is assumed small then all stresses can be taken to be uniform and the component of force in the X direction due to σ_{xx} is given by

$$\sigma_{xx} \Delta S \cos nx$$

Stress components in the other axial directions will be similar in form.

Thus, for equilibrium of forces in the X direction,

$$p_{xn} \Delta S + F_x \Delta S \frac{h}{3} = \sigma_{xx} \Delta S \cos nx + \tau_{xy} \Delta S \cos ny + \tau_{xz} \Delta S \cos nz$$

As $h \to 0$ (i.e. plane ABC passes through Q), the second term above becomes very small and can be neglected. The above equation then reduces to

$$p_{xn} = \sigma_{xx} \cos nx + \tau_{xy} \cos ny + \tau_{xz} \cos nz \tag{8.1}$$

Similarly, for equilibrium of forces in the y and z directions,

$$\left.\begin{aligned} p_{yn} &= \sigma_{yy} \cos ny + \tau_{yx} \cos nx + \tau_{yz} \cos nz \quad (8.2) \\ p_{zn} &= \sigma_{zz} \cos nz + \tau_{zx} \cos nx + \tau_{zy} \cos ny \quad (8.3) \end{aligned}\right\}$$

The resultant stress p_n on the plane ABC is then given by

$$p_n = \sqrt{(p_{xn}^2 + p_{yn}^2 + p_{zn}^2)} \tag{8.4}$$

The normal stress σ_n is given by resolution perpendicular to the face ABC,

i.e. $$\sigma_n = p_{xn} \cos nx + p_{yn} \cos ny + p_{zn} \cos nz \tag{8.5}$$

and, by Pythagoras' theorem (Fig. 8.4), the shear stress τ_n is given by

$$\tau_n = \sqrt{(p_n^2 - \sigma_n^2)} \tag{8.6}$$

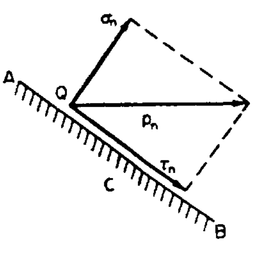

Fig. 8.4. Normal, shear and resultant stresses on the plane ABC.

It is often convenient and quicker to define the line of action of the resultant stress p_n by the *direction cosines*

$$l' = \cos(p_n x) = p_{xn}/p_n \tag{8.7}$$

$$m' = \cos(p_n y) = p_{yn}/p_n \tag{8.8}$$

$$n' = \cos(p_n z) = p_{zn}/p_n \tag{8.9}$$

The direction of the plane ABC being given by other direction cosines

$$l = \cos nx, \quad m = \cos ny, \quad n = \cos ny$$

It can be shown by simple geometry that

$$l^2 + m^2 + n^2 = 1 \quad \text{and} \quad (l')^2 + (m')^2 + (n')^2 = 1$$

Equations (8.1), (8.2) and (8.3) may now be written in two alternative ways.

(a) *Using the common symbol* σ for stress and relying on the double suffix notation to discriminate between shear and direct stresses:

$$p_{xn} = \sigma_{xx} \cos nx + \sigma_{xy} \cos ny + \sigma_{xz} \cos nz \quad (8.10)$$

$$p_{yn} = \sigma_{yx} \cos nx + \sigma_{yy} \cos ny + \sigma_{yz} \cos nz \quad (8.11)$$

$$p_{zn} = \sigma_{zx} \cos nx + \sigma_{zy} \cos ny + \sigma_{zz} \cos nz \quad (8.12)$$

In each of the above equations the first suffix is common throughout, the second suffix on the right-hand-side terms are in the order x, y, z throughout, and in each case the cosine term relates to the second suffix. These points should aid memorisation of the equations.

(b) *Using the direction cosine form*:

$$p_{xn} = \sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n \quad (8.13)$$

$$p_{yn} = \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n \quad (8.14)$$

$$p_{zn} = \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n \quad (8.15)$$

Memory is again aided by the notes above, but in this case it is the direction cosines, l, m and n which relate to the appropriate second suffices x, y and z.

Thus, provided that the direction cosines of a plane are known, together with the cartesian stress components at some point Q on the plane, the direct, normal and shear stresses on the plane at Q may be determined using, firstly, eqns. (8.13–15) and, subsequently, eqns. (8.4–6).

Alternatively the procedure may be carried out graphically as will be shown in §8.9.

8.4. Direct, shear and resultant stresses on an oblique plane

Consider again the oblique plane ABC having direction cosines l, m and n, i.e. these are the cosines of the angle between the normal to plane and the x, y, z directions.

In general, the resultant stress on the plane p_n will not be normal to the plane and it can therefore be resolved into two alternative sets of components.

(a) In the co-ordinate directions giving components p_{xn}, p_{yn} and p_{zn}, as shown in Fig. 8.5, with values given by eqns. (8.13), (8.14) and (8.15).

(b) Normal and tangential to the plane as shown in Fig. 8.6, giving components, of σ_n (normal or direct stress) and τ_n (shear stress) with values given by eqns. (8.5) and (8.6).

The value of the resultant stress can thus be obtained from either of the following equations:

$$p_n^2 = \sigma_n^2 + \tau_n^2 \quad (8.16)$$

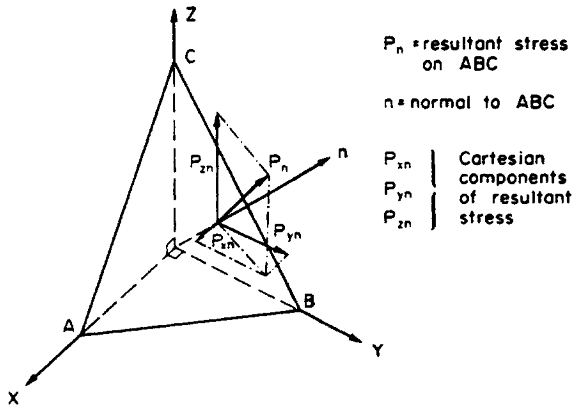

Fig. 8.5. Cartesian components of resultant stress on an inclined plane.

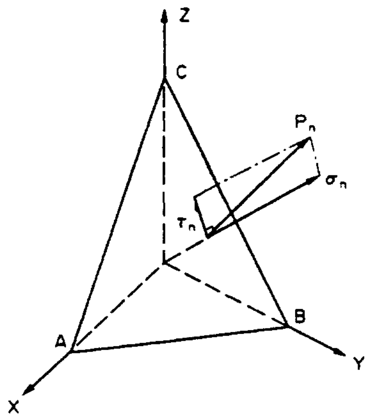

Fig. 8.6. Normal and tangential components of resultant stress on an inclined plane.

or
$$p_n^2 = p_{xn}^2 + p_{yn}^2 + p_{zn}^2 \tag{8.17}$$

these being alternative forms of eqns. (8.6) and (8.4) respectively.

From eqn. (8.5) the normal stress on the plane is given by:

$$\sigma_n = p_{xn} \cdot l + p_{yn} \cdot m + p_{zn} \cdot n$$

But from eqns. (8.13), (8.14) and (8.15)

$$p_{xn} = \sigma_{xx} \cdot l + \sigma_{xy} \cdot m + \sigma_{xz} \cdot n$$
$$p_{yn} = \sigma_{yx} \cdot l + \sigma_{yy} \cdot m + \sigma_{yz} \cdot n$$
$$p_{zn} = \sigma_{zx} \cdot l + \sigma_{zy} \cdot m + \sigma_{zz} \cdot n$$

∴ Substituting into eqn (8.5) and using the relationships $\sigma_{xy} = \sigma_{yx}$; $\sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$ which will be proved in §8.12

$$\sigma_n = \sigma_{xx} \cdot l^2 + \sigma_{yy} \cdot m^2 + \sigma_{zz} \cdot n^2 + 2\sigma_{xy} \cdot lm + 2\sigma_{yz} \cdot mn + 2\sigma_{xz} \cdot ln. \tag{8.18}$$

and from eqn. (8.6) the shear stress on the plane will be given by

$$\tau_n^2 = p_{xn}^2 + p_{yn}^2 + p_{zn}^2 - \sigma_n^2 \tag{8.19}$$

In the particular case where plane ABC is a principal plane (i.e. no shear stress):

$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$$

and $$\sigma_{xx} = \sigma_1, \quad \sigma_{yy} = \sigma_2 \quad \text{and} \quad \sigma_{zz} = \sigma_3$$

the above equations reduce to:

$$\sigma_n = \sigma_1 \cdot l^2 + \sigma_2 \cdot m^2 + \sigma_3 \cdot n^2 \tag{8.20}$$

and since $$p_{xn} = \sigma_1 \cdot l \quad p_{yn} = \sigma_2 \cdot m \quad \text{and} \quad p_{zn} = \sigma_3 \cdot n$$

$$\tau_n^2 = \sigma_1^2 \cdot l^2 + \sigma_2^2 \cdot m^2 + \sigma_3^2 \cdot n^2 - \sigma_n^2 \tag{8.21}$$

8.4.1. Line of action of resultant stress

As stated above, the resultant stress p_n is generally not normal to the plane ABC but inclined to the x, y and z axes at angles θ_x, θ_y and θ_z – see Fig. 8.7.

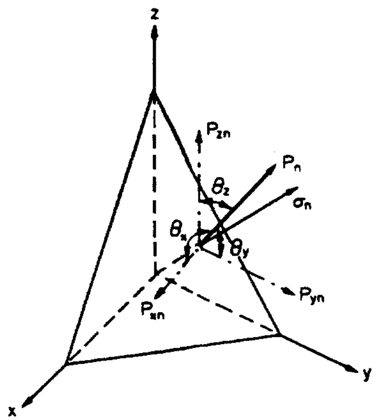

Fig. 8.7. Line of action of resultant stress.

The components of p_n in the x, y and z directions are then

$$\left.\begin{aligned} p_{xn} &= p_n . \cos\theta_x \\ p_{yn} &= p_n . \cos\theta_y \\ p_{zn} &= p_n . \cos\theta_z \end{aligned}\right\} \tag{8.22}$$

and the direction cosines which define the line of actions of the resultant stress are

$$\left.\begin{aligned} l' &= \cos\theta_x = p_{xn}/p_n \\ m' &= \cos\theta_y = p_{yn}/p_n \\ n' &= \cos\theta_z = p_{zn}/p_n \end{aligned}\right\} \tag{8.23}$$

8.4.2. Line of action of normal stress

By definition the normal stress is that which acts normal to the plane, i.e. the line of action of the normal stress has the same direction cosines as the normal to plane viz: l, m and n.

8.4.3. Line of action of shear stress

As shown in §8.4 the resultant stress p_n can be considered to have two components; one normal to the plane (σ_n) and one along the plane (the shear stress τ_n) – see Fig. 8.6.

Let the direction cosines of the line of action of this shear stress be l_s, m_s and n_s.

The alternative components of the resultant stress, p_{xn}, p_{yn} and p_{zn}, can then either be obtained from eqn (8.22) or by resolution of the normal and shear components along the x, y and z directions as follows:

$$\left.\begin{aligned} p_{xn} &= \sigma_n \cdot l + \tau_n \cdot l_s \\ p_{yn} &= \sigma_n \cdot m + \tau_n \cdot m_s \\ p_{zn} &= \sigma_n \cdot n + \tau_n \cdot n_s \end{aligned}\right\} \tag{8.24}$$

Thus the direction cosines of the line of action of the shear stress τ_n are:

$$\left.\begin{aligned} l_s &= \frac{p_{xn} - l \cdot \sigma_n}{\tau_n} \\ m_s &= \frac{p_{yn} - m \cdot \sigma_n}{\tau_n} \\ n_s &= \frac{p_{zn} - n \cdot \sigma_n}{\tau_n} \end{aligned}\right\} \tag{8.25}$$

8.4.4. Shear stress in any other direction on the plane

Let ϕ be the angle between the direction of the shear stress τ_n and the required direction. Then, since the angle between any two lines in space is given by,

$$\cos\phi = l_s \cdot l_\phi + m_s \cdot m_\phi + n_s \cdot n_\phi \tag{8.26}$$

where l_ϕ, m_ϕ, n_ϕ are the direction cosines of the new shear stress direction, it follows that the required magnitude of the shear stress on the "ϕ" plane will be given by

$$\tau_\phi = \tau_n \cdot \cos\phi \tag{8.27}$$

Alternatively, resolving the components of the resultant stress (p_{xn}, p_{yn} and p_{zn}) along the new direction we have:

$$\tau_\phi = p_{xn} \cdot l_\phi + p_{yn} \cdot m_\phi + p_{zn} \cdot n_\phi \tag{8.28}$$

and substituting eqns. (8.13), (8.14) and (8.15)

$$\begin{aligned} \tau_\phi = \sigma_{xx} \cdot ll_\phi + \sigma_{yy} \cdot mm_\phi + \sigma_{zz} \cdot nn_\phi + \sigma_{xy}(lm_\phi + l_\phi \cdot m) \\ + \sigma_{xz}(ln_\phi + nl_\phi) + \sigma_{yz}(mn_\phi + nm_\phi) \end{aligned} \tag{8.29}$$

Whilst eqn. (8.28) has been derived for the shear stress τ_ϕ it will, in fact, apply equally for any type of stress (i.e. shear or normal) which acts on the plane ABC in the ϕ direction.

In the case of the shear stress, however, its line of action must always be perpendicular to the normal to the plane so that

$$ll_\phi + mm_\phi + nn_\phi = 0.$$

In the case of a normal stress the relationship between the direction cosines is simply

$$l = l_\phi, m = m_\phi \text{ and } n = n_\phi$$

since the stress and the normal to the plane are in the same direction. Eqn. (8.29) then reduces to that found previously, viz. eqn. (8.18).

8.5. Principal stresses and strains in three dimensions – Mohr's circle representation

The procedure used for constructing Mohr's circle representation for a three-dimensional principal *stress* system has previously been introduced in §13.7[†]. For convenience of reference the resulting diagram is repeated here as Fig. 8.8. A similar representation for a three-dimensional principal *strain* system is shown in Fig. 8.9.

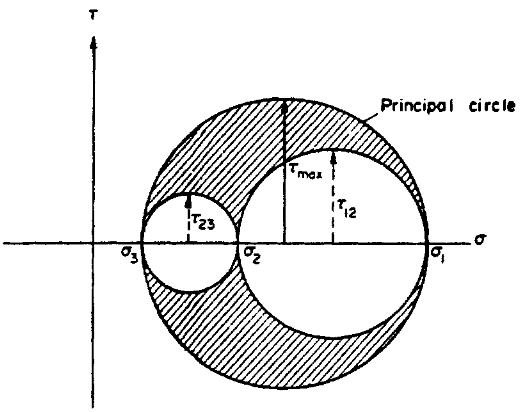

Fig. 8.8. Mohr circle representation of three-dimensional stress state showing the principal circle, the radius of which is equal to the greatest shear stress present in the system.

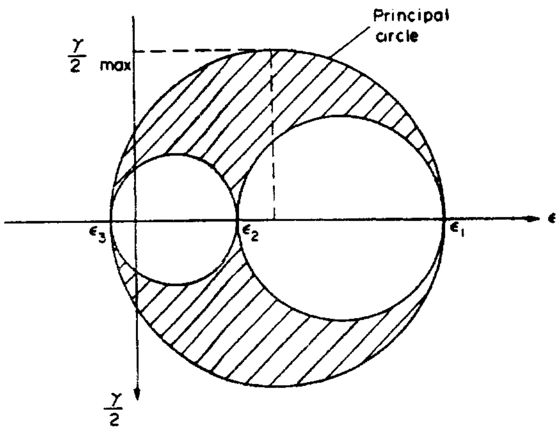

Fig. 8.9. Mohr representation for a three-dimensional principal strain system.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

In both cases the **principal circle** is indicated, the radius of which gives the maximum shear stress and *half* the maximum shear strain, respectively, in the three-dimensional system.

This form of representation utilises different diagrams for the stress and strain systems. An alternative procedure uses a single *combined diagram* for both cases and this is described in detail §§8.6 and 8.7.

8.6. Graphical determination of the direction of the shear stress τ_n on an inclined plane in a three-dimensional principal stress system

As before, let the inclined plane have direction cosines l, m and n. A true representation of this plane is given by constructing a so-called "true shape triangle" the ratio of the lengths of its sides being the ratio of the direction cosines–Fig. 8.10.

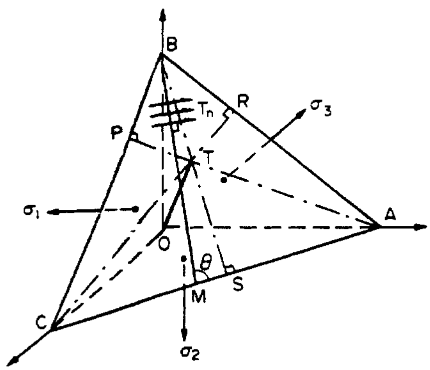

Fig. 8.10. Graphical determination of direction of shear stress on an inclined plane.

If lines are drawn perpendicular to each side from the opposite vertex, meeting the sides at points P, R and S, they will intersect at point T the "orthocentre". This is also the point through which the normal to the plane from O passes.

If σ_1, σ_2 and σ_3 are the three principal stresses then point M is positioned on AC such that

$$\frac{CM}{CA} = \frac{(\sigma_2 - \sigma_3)}{(\sigma_1 - \sigma_2)}$$

The required direction of the shear stress is then perpendicular to the line *BD*.

The equivalent procedure on the Mohr circle construction is as follows (see Fig. 8.11).

Construct the three stress circles corresponding to the three principal stresses σ_1, σ_2 and σ_3.
Set off line AB at an angle $\alpha = \cos^{-1} l$ to the left of the vertical through A.
Set off line CB at an angle $\gamma = \cos^{-1} n$ to the right of the vertical through C to meet AB at B.
Mark the points where these lines cut the principal circle R and P respectively.
Join AP and CR to cut at point T.
Join BT and extend to cut horizontal axis AC at S.
With point M the σ_2 position, join BM.

The required shear stress direction is then perpendicular to the line *BM*.

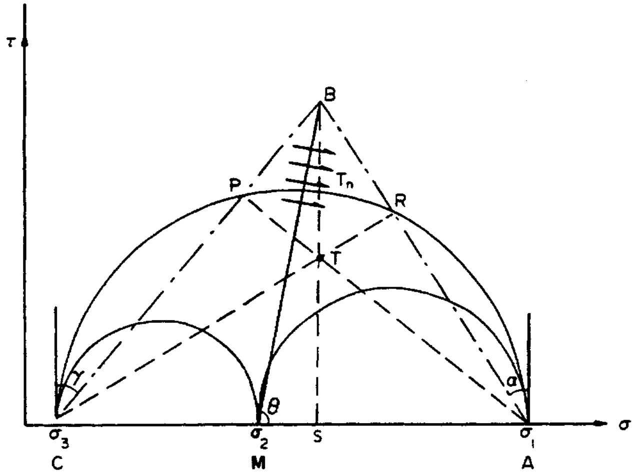

Fig. 8.11. Mohr circle equivalent procedure to that of Fig. 8.10.

8.7. The combined Mohr diagram for three-dimensional stress and strain systems

Consider any three-dimensional stress system with principal stresses σ_1, σ_2 and σ_3 (all assumed tensile). Principal strains are then related to the principal stresses as follows:

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3), \text{ etc.}$$

$$E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$$

$$= \sigma_1 - \nu(\sigma_1 + \sigma_2 + \sigma_3) + \nu\sigma_1 \quad (1)$$

Now the *hydrostatic, volumetric* or *mean stress* $\bar{\sigma}$ is defined as

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Therefore substituting in (1),

$$E\varepsilon_1 = \sigma_1(1 + \nu) - 3\nu\bar{\sigma} \quad (2)$$

But the volumetric stress $\bar{\sigma}$ may also be written in terms of the bulk modulus,

i.e. $$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

and

$$\text{volumetric strain} = \text{sum of the three linear strains}$$

$$= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \Delta$$

∴ $$K = \frac{\bar{\sigma}}{\Delta}$$

but $$E = 3K(1 - 2\nu)$$

∴ $$\bar{\sigma} = \Delta K = \Delta\frac{E}{3(1 - 2\nu)}$$

Substituting in (2),

$$E\varepsilon_1 = \sigma_1(1+\nu) - \frac{3\nu\Delta E}{3(1-2\nu)}$$

and, since $E = 2G(1+\nu)$,

$$2G(1+\nu)\varepsilon_1 = \sigma_1(1+\nu) - \frac{\nu\Delta 2G(1+\nu)}{(1-2\nu)}$$

$$\therefore \qquad \sigma_1 = 2G\left[\varepsilon_1 + \frac{\nu\Delta}{(1-2\nu)}\right]$$

But, mean strain

$$\bar{\varepsilon} = \tfrac{1}{3}(\varepsilon_1+\varepsilon_2+\varepsilon_3) = \tfrac{1}{3}\Delta$$

$$\therefore \qquad \boldsymbol{\sigma_1 = 2G\left[\varepsilon_1 + \frac{3\nu\bar{\varepsilon}}{(1-2\nu)}\right]} \qquad (8.30)$$

Alternatively, re-writing eqn. (8.16) in terms of ε_1,

$$\varepsilon_1 = \frac{\sigma_1}{2G} - \frac{3\nu}{(1-2\nu)}\bar{\varepsilon}$$

But $$\bar{\varepsilon} = \frac{\Delta}{3} = \frac{\bar{\sigma}}{3K}$$

But $$E = 2G(1+\nu) = 3K(1-2\nu)$$

i.e. $$3K = 2G\frac{(1+\nu)}{(1-2\nu)}$$

$$\therefore \qquad \bar{\varepsilon} = \frac{\bar{\sigma}(1-2\nu)}{2G(1+\nu)}$$

$$\therefore \qquad \varepsilon_1 = \frac{\sigma_1}{2G} + \frac{\bar{\sigma}}{2G}\frac{(1-2\nu)}{(1+\nu)}$$

i.e. $$\boldsymbol{\varepsilon_1 = \frac{1}{2G}\left[\sigma_1 - \frac{3\nu\bar{\sigma}}{(1+\nu)}\right]} \qquad (8.31)$$

In the above derivation the cartesian stresses σ_{xx}, σ_{yy} and σ_{zz} could have been used in place of the principal stresses σ_1, σ_2 and σ_3 to yield more general expressions but of identicial form. It therefore follows that the stress and associated strain in *any* given direction within a complex three-dimensional stress system is given by eqns. (8.30) and (8.31) which must satisfy the three-dimensional Mohr's circle construction.

Comparison of eqns. (8.30) and (8.31) indicates that

$$2G\varepsilon_1 = \sigma_1 - \frac{3\nu}{(1+\nu)}\bar{\sigma}$$

Thus, having constructed the three-dimensional Mohr's *stress* circle representations, the equivalent *strain* values may be obtained simply by reference to a new axis displaced a distance $(3\nu/(1+\nu))\bar{\sigma}$ as shown in Fig. 8.12 bringing the new axis origin to O'.

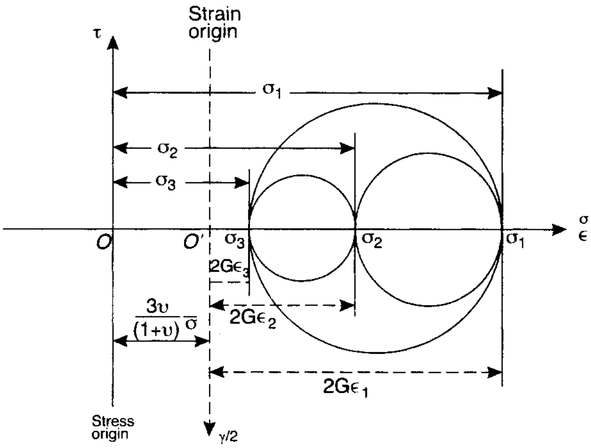

Fig. 8.12. The "combined Mohr diagram" for three-dimensional stress and strain systems.

Distances from the new axis to any principal stress value, e.g. σ_1, will then be $2G$ *times* the corresponding ε_1 principal strain value,

i.e. $$O'\sigma_1 \div 2G = \varepsilon_1$$

Thus the same circle construction will apply for both stresses and strains provided that:

(a) the shear strain axis is offset a distance $\dfrac{3\nu}{(1+\nu)}\bar{\sigma}$ to the right of the shear stress axis;
(b) a scale factor of $2G$, $[= E/(1+\nu)]$, is applied to measurements from the new axis.

8.8. Application of the combined circle to two-dimensional stress systems

The procedure of §14.13[†] uses a common set of axes and a common centre for Mohr's stress and strain circles, each having an appropriate radius and scale factor. An alternative procedure utilises the combined circle approach introduced above where a single circle can be used in association with two different origins to obtain both stress and strain values.

As in the above section the relationship between the stress and strain scales is

$$\frac{\text{stress scale}}{\text{strain scale}} = \frac{E}{(1+\nu)} = 2G$$

This is in fact the condition for both the stress and strain circles to have the same radius[‡] and should not be confused with the condition required in §14.13[†] of the alternative approach for the two circles to be concentric, when the ratio of scales is $E/(1-\nu)$.

[†] E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

[‡] For equal radii of both the stress and strain circles

$$\frac{(\sigma_1 - \sigma_2)}{2 \times \text{stress scale}} = \frac{(\varepsilon_1 - \varepsilon_2)}{2 \times \text{strain scale}}$$

$$\frac{\text{stress scale}}{\text{strain scale}} = \frac{(\sigma_1 - \sigma_2)}{(\varepsilon_1 - \varepsilon_2)} = \frac{(\sigma_1 - \sigma_2)}{(\sigma_1 - \sigma_2)}\frac{E}{(1+\nu)} = \frac{E}{(1+\nu)}$$

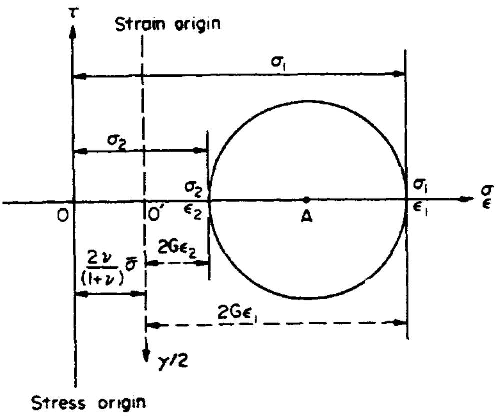

Fig. 8.13. Combined Mohr diagram for two-dimensional stress and strain systems.

With reference to Fig. 8.13 the two origins must then be positioned such that

$$OA = \frac{(\sigma_1 + \sigma_2)}{2 \times \text{strees scale}}$$

$$O'A = \frac{(\varepsilon_1 + \varepsilon_2)}{2 \times \text{strain scale}}$$

$$\therefore \quad \frac{OA}{O'A} = \frac{(\sigma_1 + \sigma_2)}{(\varepsilon_1 + \varepsilon_2)} \times \frac{\text{strain scale}}{\text{stress scale}}$$

$$= \frac{(\sigma_1 + \sigma_2)}{(\varepsilon_1 + \varepsilon_2)} \times \frac{(1 + \nu)}{E}$$

But

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

$$\therefore \quad \varepsilon_1 + \varepsilon_2 = \frac{1}{E}(\sigma_1 + \sigma_2)(1 - \nu)$$

$$\therefore \quad \frac{OA}{O'A} = \frac{(\sigma_1 + \sigma_2)}{(\sigma_1 + \sigma_2)} \frac{E}{(1 - \nu)} \frac{(1 + \nu)}{E} = \frac{(1 + \nu)}{(1 - \nu)}$$

Thus the distance between the two origins is given by

$$OO' = OA - O'A = OA - \frac{(1 - \nu)}{(1 + \nu)} OA$$

$$= \frac{(\sigma_1 + \sigma_2)}{2} \left[1 - \frac{(1 - \nu)}{(1 + \nu)}\right]$$

$$= \frac{(\sigma_1 + \sigma_2)(2\nu)}{2(1 + \nu)} = \frac{\nu}{(1 + \nu)}(\sigma_1 + \sigma_2)$$

$$= \frac{2\nu}{(1 + \nu)}\bar{\sigma} \qquad (8.32)$$

where $\overline{\sigma}$ is the mean stress in the two-dimensional stress system $= \frac{1}{2}(\sigma_1 + \sigma_2) =$ position of centre of stress circle.

The relationship is thus identical in form to the three-dimensional equivalent with 2 replacing 3 for the *two*-dimensional system.

Again, therefore, the *single-circle construction applies for both stresses and strain provided that the axes are offset by the appropriate amount and a scale factor for strains of 2G is applied.*

8.9. Graphical construction for the state of stress at a point

The following procedure enables the determination of the direct (σ_n) and shear (τ_n) stresses at any point on a plane whose direction cosines are known and, in particular, on the *octahedral planes* (see §8.19).

The construction procedure for Mohr's circle representation of three-dimensional stress systems has been introduced in §8.4. Thus, for a given state of stress producing principal stress σ_1, σ_2 and σ_3, Mohr's circles are as shown in Fig. 8.8.

For a given plane S characterised by direction cosines l, m and n the remainder of the required construction proceeds as follows (Fig. 8.14). (Only half the complete Mohr's circle representation is shown since this is sufficient for the execution of the construction procedure.)

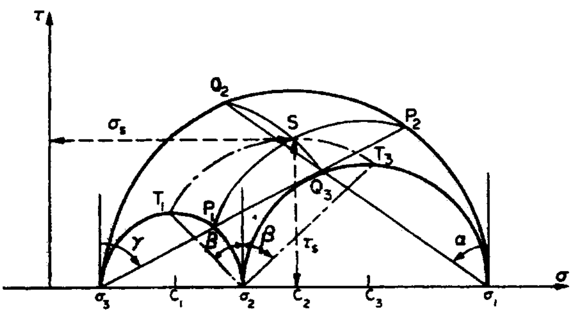

Fig. 8.14. Graphical construction for the state of stress on a general stress plane.

(1) Set off angle $\alpha = \cos^{-1} l$ from the vertical at σ_1 to cut the circles in Q_2 and Q_3.

(2) With centre C_1 (centre of σ_2, σ_3 circle) draw arc Q_2Q_3.

(3) Set off angle $\gamma = \cos^{-1} n$ from the vertical at σ_3 to cut the circles at P_1 and P_2.

(4) With centre C_3 (centre of σ_1, σ_2 circle) draw arc P_1P_2.

(5) The position S representing the required plane is then given by the point where the two arcs Q_2Q_3 and P_1P_2 intersect. *The stresses on this plane are then σ_s and τ_s* as shown. Careful study of the above construction procedure shows that the suffices of points considered in each step always complete the grouping 1, 2, 3. This should aid memorisation of the procedure.

(6) As a check on the accuracy of the drawing, set off angles $\beta = \cos^{-1} m$ on either side of the vertical through σ_2 to cut the $\sigma_2\sigma_3$ circle in T_1 and the $\sigma_1\sigma_2$ circle in T_3.

(7) With centre C_2 (centre of the $\sigma_1\sigma_3$ circle) draw arc T_1T_3 which should then pass through S if all steps have been carried out correctly and the diagram is accurate. The construction is very much easier to follow if all steps connected with points P, Q and T are carried out in different colours.

8.10. Construction for the state of strain on a general strain plane

The construction detailed above for determination of the state of *stress* on a general stress plane applies equally to the determination of *strains* when the symbols σ_1, σ_2 and σ_3 are replaced by the principal *strain* values ε_1, ε_2 and ε_3.

Thus, having constructed the three-dimensional Mohr representation of the principal strains as described in §8.4, the general plane is located as described above and illustrated in Fig. 8.15.

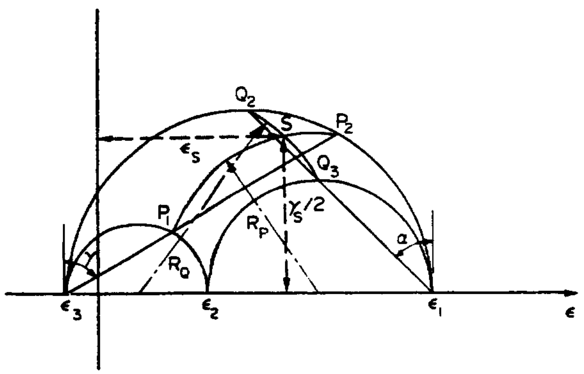

Fig. 8.15. Graphical construction for the state of strain on a general strain plane.

8.11. State of stress–tensor notation

The state of stress equations for any three-dimensional system of cartesian stress components have been obtained in §8.3 as:

$$p_{xn} = \sigma_{xx} \cdot l + \sigma_{xy} \cdot m + \sigma_{xz} \cdot n$$

$$p_{yn} = \sigma_{yx} \cdot l + \sigma_{yy} \cdot m + \sigma_{yz} \cdot n$$

$$p_{zn} = \sigma_{zx} \cdot l + \sigma_{zy} \cdot m + \sigma_{zz} \cdot n$$

The cartesian stress components within this equation can then be remembered conveniently in *tensor notation* as:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (\textit{general stress tensor}) \tag{8.33}$$

For a *principal stress system*, i.e. no shear, this reduces to:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \text{(principal stress tensor)} \tag{8.34}$$

and a special case of this is the so-called "*hydrostatic*" stress system with equal principal stresses in all three directions, i.e. $\sigma_1 = \sigma_2 = \sigma_3 = \overline{\sigma}$, and the tensor becomes:

$$\begin{bmatrix} \overline{\sigma} & 0 & 0 \\ 0 & \overline{\sigma} & 0 \\ 0 & 0 & \overline{\sigma} \end{bmatrix} \text{(hydrostatic stress tensor)} \tag{8.35}$$

As shown in §23.16 it is often convenient to divide a general stress into two parts, one due to a hydrostatic stress $\overline{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$, the other due to shearing deformations.

Another convenient tensor notation is therefore that for pure shear, ie $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ giving the tensor:

$$\begin{bmatrix} 0 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & 0 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & 0 \end{bmatrix} \text{(pure shear tensor)} \tag{8.36}$$

The general stress tensor (8.33) is then the combination of the hydrostatic stress tensor and the pure shear tensor.

i.e. *General three-dimensional stress state = hydrostatic stress state + pure shear state.*

This approach is utilised in other sections of this text, notably: §8.16, §8.19 and §8.20.

It therefore follows that an alternative method of presentation of a *pure shear state of stress* is, in tensor form:

$$\begin{bmatrix} (\sigma_1 - \overline{\sigma}) & 0 & 0 \\ 0 & (\sigma_2 - \overline{\sigma}) & 0 \\ 0 & 0 & (\sigma_3 - \overline{\sigma}) \end{bmatrix} \tag{8.37}$$

N.B.: It can be shown that the condition for a state of stress to be one of pure shear is that the first stress invariant is zero.

i.e. $$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0 \qquad \text{(see 8.15)}$$

8.12. The stress equations of equilibrium

(a) In cartesian components

In all the previous work on complex stress systems it has been assumed that the stresses acting on the sides of any element are constant. In many cases, however, a general system of direct, shear and body forces, as encountered in practical engineering applications, will produce stresses of variable magnitude throughout a component. Despite this, however, the distribution of these stresses must always be such that overall equilibrium both of the component, and of any element of material within the component, is maintained, and it is a consideration of the conditions necessary to produce this equilibrium which produces the so-called *stress equations of equilibrium.*

Consider, therefore, a body subjected to such a general system of forces resulting in the cartesian stress components described in §8.2 together with the body-force stresses $\boldsymbol{F_x}$,

F_y and F_z. The element shown in Fig. 8.16 then displays, for simplicity, only the stress components in the X direction together with the body-force stress components. It must be realised, however, that similar components act in the Y and Z directions and these must be considered when deriving equations for equilibrium in these directions: they, of course, have no effect on equilibrium in the X direction.

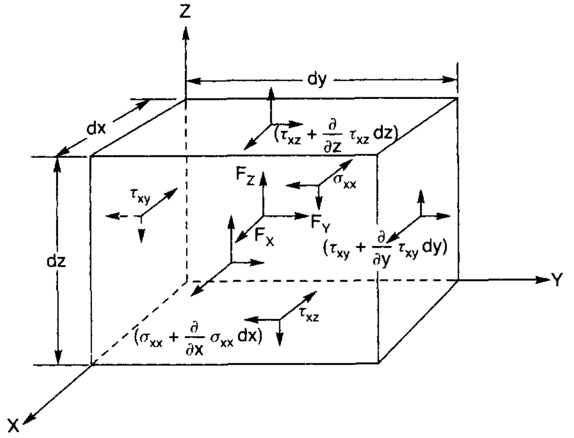

Fig. 8.16. Small element showing body force stresses and other stresses in the X direction only.

It will be observed that on each pair of opposite faces the stress changes in magnitude in the following manner,

e.g. stress on one face $= \sigma_{xx}$

stress on opposite face $= \sigma_{xx} +$ change in stress

$= \sigma_{xx} +$ rate of change $\times$ distance between faces

Now the rate of change of σ_{xx} with x is given by $\partial\sigma_{xx}/\partial x$, partial differentials being used since σ_{xx} may well be a function of y and z as well as of x.

Therefore

$$\text{stress on opposite face} = \sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}dx$$

Multiplying by the area $dy\,dz$ of the face on which this stress acts produces the force in the X direction.

Thus, for equilibrium of *forces* in the X direction,

$$\left[\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\,dx - \sigma_{xx}\right]dy\,dz + \left[\tau_{xy} + \frac{\partial}{\partial y}\tau_{xy}\,dy - \tau_{xy}\right]dx\,dz$$
$$+\left[\tau_{xz} + \frac{\partial}{\partial z}\tau_{xz}\,dz - \tau_{xz}\right]dx\,dy + F_x\,dx\,dy\,dz = 0$$

(The body-force term being defined as a stress per unit volume is multiplied by the volume $(dx\,dy\,dz)$ to obtain the corresponding force.)

Dividing through by $dx\,dy\,dz$ and simplifying,

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+F_x=0$$

Similarly, for equilibrium in the Y direction,

$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}+F_Y=0 \qquad (8.38)$$

and in the Z direction,

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+F_z=0$$

these equations being termed the *general stress equations of equilibrium.*

Bearing in mind the comments of §8.2, the symbol τ in the above equations may be replaced by σ, the mixed suffix denoting the fact that it is a shear stress, and the above equations can be remembered quite easily using a similar procedure to that used in §8.2 based on the suffices, i.e. first suffices and body-force terms are constant for each horizontal row and in the normal order x, y and z.

	X	Y	Z	
X	$\frac{xx}{\partial x}$	$\frac{xy}{\partial y}$	$\frac{xz}{\partial z}$	$+F_X=0$
Y	$\frac{yx}{\partial x}$	$\frac{yy}{\partial y}$	$\frac{yz}{\partial z}$	$+F_Y=0$
Z	$\frac{zx}{\partial x}$	$\frac{zy}{\partial y}$	$\frac{zz}{\partial z}$	$+F_Z=0$

The above equations have been derived by consideration of equilibrium of *forces* only, and this does not represent a complete check on the equilibrium of the system. This can only be achieved by an additional consideration of the *moments of the forces* which must also be in balance.

Consider, therefore, the element shown in Fig. 8.17 which, again for simplicity, shows only the stresses which produce moments about the Y axis. For convenience the origin of the cartesian coordinates has in this case been chosen to coincide with the centroid of the element. In this way the direct stress and body-force stress terms will be eliminated since the forces produced by these will have no moment about axes through the centroid.

It has been assumed that shear stresses τ_{xy}, τ_{yz} and τ_{xz} act on the coordinate planes passing through G so that they will each increase and decrease on either side of these planes as described above.

Thus, for equilibrium of moments about the Y axis,

$$\left[\tau_{xz}+\frac{\partial}{\partial z}(\tau_{xz})\frac{dz}{2}\right]dx\,dy\frac{dz}{2}+\left[\tau_{xz}-\frac{\partial}{\partial z}(\tau_{xz})\frac{dz}{2}\right]dx\,dy\frac{dz}{2}$$
$$-\left[\tau_{zx}+\frac{\partial}{\partial x}(\tau_{zx})\frac{dx}{2}\right]dy\,dz\frac{dx}{2}-\left[\tau_{zx}-\frac{\partial}{\partial x}(\tau_{zx})\frac{dx}{2}\right]dy\,dz\frac{dx}{2}=0$$

Dividing through by $(dx\ dy\ dz)$ and simplifying, this reduces to

$$\tau_{xz}=\tau_{zx}$$

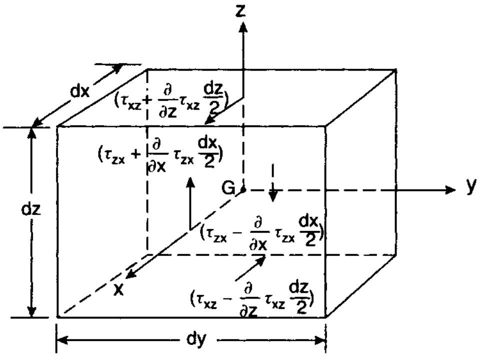

Fig. 8.17. Element showing only stresses which contribute to a moment about the Y axis.

Similarly, by consideration of the equilibrium of moments about the X and Z axes,

$$\tau_{zy} = \tau_{yz}$$

$$\tau_{xy} = \tau_{yx}$$

Thus the shears and complementary shears on adjacent faces are equal as in the simple two-dimensional case. The nine cartesian stress components thus reduce to *six independent* values,

i.e.
$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

(b) In cylindrical coordinates

The equations of equilibrium derived above in cartesian components are very useful for components and stress systems which can easily be referred to a set of three mutually perpendicular axes. There are many cases, however, e.g. those components with *axial symmetry*, where other coordinate axes prove far more convenient. One such set of axes is the *cylindrical coordinate* system with variables r, θ and z as shown in Fig. 8.18.

Consider, therefore, the equilibrium in a *radial* direction of the element shown in Fig. 8.19(a). Again, for simplicity, only those stresses which produce force components in this direction are indicated. It must be observed, however, that in this case the $\sigma_{\theta\theta}$ terms will also produce components in the radial direction as shown by Fig. 8.19(b). The body-force stress components are denoted by F_R, F_Z and F_θ.

Therefore, resolving forces radially,

$$\left[\sigma_{rr} + \frac{\partial}{\partial r}(\sigma_{rr})\,dr\right](r+dr)\,d\theta\,dz - \sigma_{rr}\,r\,d\theta\,dz + \left[\sigma_{r\theta} + \frac{\partial}{\partial\theta}(\sigma_{r\theta})\,d\theta\right]dr\,dz\cos\frac{d\theta}{2}$$

$$-\sigma_{r\theta}\,dr\,dz\cos\frac{d\theta}{2} + \left[\left(\sigma_{rz} + \frac{\partial(\sigma_{rz})}{\partial z}dz\right) - \sigma_{rz}\right]\left(r+\frac{dr}{2}\right)d\theta\,dr$$

$$-\sigma_{\theta\theta}\,dr\,dz\sin\frac{d\theta}{2} - \left[\sigma_{\theta\theta} + \frac{\partial}{\partial\theta}(\sigma_{\theta\theta})\,d\theta\right]dr\,dz\sin\frac{d\theta}{2} + F_R\,r\,dr\,d\theta\,dz = 0$$

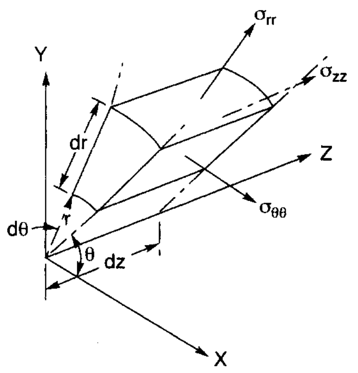

Fig. 8.18. Cylindrical coordinates.

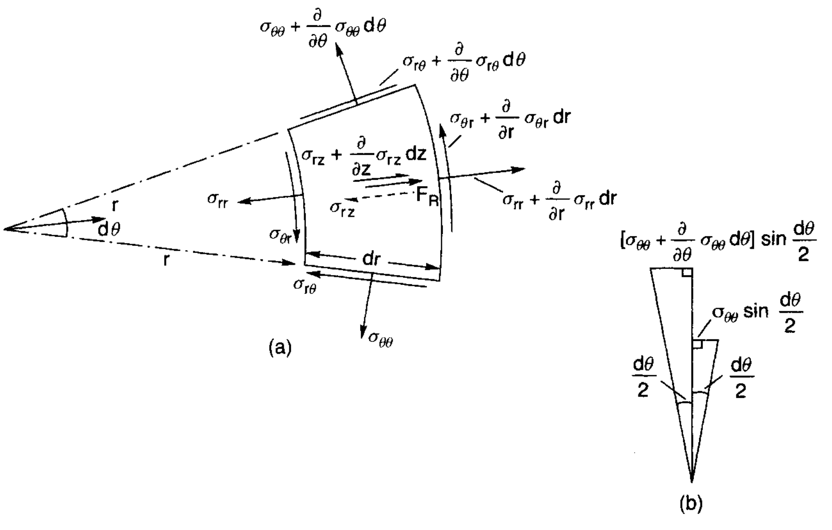

Fig. 8.19. (a) Element showing stresses which contribute to equilibrium in the radial and circumferential directions. (b) Radial components of hoop stresses.

With $\cos\dfrac{d\theta}{2} \simeq 1$ and $\sin\dfrac{d\theta}{2} \simeq \dfrac{d\theta}{2}$, this equation reduces to

$$\frac{\partial}{\partial r}(\sigma_{rr}) + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_{r\theta}) + \frac{\partial}{\partial z}(\sigma_{rz}) + \frac{(\sigma_{rr}-\sigma_{\theta\theta})}{r} + F_R = 0$$

Similarly, in the θ direction, the relevant equilibrium equation reduces to

$$\frac{\partial}{\partial r}(\sigma_{r\theta}) + \frac{1}{r}\frac{\partial(\sigma_{\theta\theta})}{\partial\theta} + \frac{\partial}{\partial z}(\sigma_{\theta z}) + \frac{2\sigma_{r\theta}}{r} + F_\theta = 0 \qquad (8.39)$$

and in the Z direction (Fig. 8.20)

$$\frac{\partial(\sigma_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(\sigma_{\theta z})}{\partial\theta} + \frac{\partial}{\partial z}(\sigma_{zz}) + \frac{\sigma_{rz}}{r} + F_z = 0$$

These are, then, the *stress equations of equilibrium in cylindrical coordinates* and in their most general form. Clearly these are difficult to memorise and, fortunately, very few problems arise in which the equations in this form are required. In many cases *axial symmetry* exists and circular sections remain concentric and circular throughout loading, i.e. $\sigma_{r\theta} = 0$.

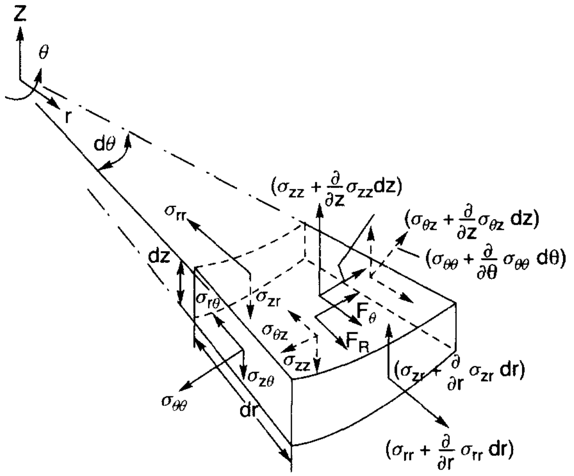

Fig. 8.20. Element indicating additional stresses which contribute to equilibrium in the axial (z) direction.

Thus **for axial symmetry** the equations reduce to

$$\left.\begin{aligned} \frac{\partial}{\partial r}(\sigma_{rr}) + \frac{\partial}{\partial z}(\sigma_{rz}) + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_R &= 0 \\ \frac{1}{r}\frac{\partial(\sigma_{\theta\theta})}{\partial \theta} + \frac{\partial(\sigma_{\theta z})}{\partial z} + F_\theta &= 0 \\ \frac{\partial}{\partial r}(\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial(\sigma_{zz})}{\partial z} + \frac{\sigma_{rz}}{r} + F_z &= 0 \end{aligned}\right\} \qquad (8.40)$$

Further simplification applies in cases where the **coordinate axes** can be selected to **coincide with principal stress directions** as in the case of thick cylinders subjected to uniform pressure or thermal gradients. In such cases there will be no shear, and in the absence of body forces the equations reduce to the relatively simple forms

$$\left.\begin{aligned} \frac{\partial}{\partial r}(\sigma_{rr}) + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} &= 0 \\ \frac{\partial(\sigma_{\theta\theta})}{\partial \theta} &= 0 \\ \frac{\partial(\sigma_{zz})}{\partial z} &= 0 \end{aligned}\right\} \qquad (8.41)$$

8.13. Principal stresses in a three-dimensional cartesian stress system

As an alternative to the graphical Mohr's circle procedures the principal stresses in three-dimensional complex stress systems can be determined analytically as follows.

The equations for the state of stress at a point derived in §8.3 may be combined to give the equation

$$\sigma_n^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_n^2 + (\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_n$$
$$- (\sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{xz}) = 0 \quad (8.42)$$

With a knowledge of the cartesian stress components this cubic equation can be solved for σ_n to produce the three principal stress values required. A general procedure for the solution of cubic equations is given below.

8.13.1. Solution of cubic equations

Consider the cubic equation

$$x^3 + ax^2 + bx + c = 0 \quad (1)$$

Substituting,
$$x = y - a/3 \quad (2)$$

with
$$p = b - a^2/3 \quad (3)$$

and
$$q = c - \frac{ab}{3} + \frac{2a^3}{27} \quad (4)$$

we obtain the modified equation

$$y^3 + py + q = 0 \quad (5)$$

Substituting,
$$y = rz \quad (6)$$

$$z^3 + \frac{pz}{r^2} + \frac{q}{r^3} = 0 \quad (7)$$

Now consider the standard trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \quad (8)$$

Rearranging and substituting $z = \cos\theta$, (9)

$$z^3 - \frac{3z}{4} - \frac{1}{4}\cos 3\theta = 0 \quad (10)$$

(7) and (10) are of similar form and will be identical provided that

$$r = \sqrt{-\frac{4p}{3}} \quad (11)$$

and
$$\cos 3\theta = -\frac{4q}{r^3} \quad (12)$$

Three values of θ may be obtained to satisfy (12),

i.e. $\theta, \theta + 120^\circ$ and $\theta + 240^\circ$

Then, from (9), three corresponding values of z are obtained, namely

$$z_1 = \cos\theta^\circ$$

$$z_2 = \cos(\theta + 120^\circ)$$

$$z_3 = \cos(\theta + 240^\circ)$$

(6) then yields appropriate values of y and hence the required values of x via (2).

8.14. Stress invariants; Eigen values and Eigen vectors

Consider the special case of the "stress at a point" tetrahedron Fig. 8.3 where plane ABC is a principal plane subjected to a principal stress σ_p and, by definition, zero shear stress. The normal stress is thus coincident with the resultant stress and both equal to σ_p.

If the direction cosines of σ_p (and hence of the principal plane) are l_p, m_p, n_p then:

$$p_{xn} = \sigma_p \cdot l_p$$

$$p_{yn} = \sigma_p \cdot m_p$$

$$p_{zn} = \sigma_p \cdot n_p$$

i.e. substituting in eqns. (8.13), (8.14) and (8.15) we have:

$$\sigma_p \cdot l_p = \sigma_{xx} \cdot l_p + \sigma_{xy} \cdot m_p + \sigma_{xz} \cdot n_p$$

$$\sigma_p \cdot m_p = \sigma_{yx} \cdot l_p + \sigma_{yy} \cdot m_p + \sigma_{yz} \cdot n_p$$

$$\sigma_p \cdot n_p = \sigma_{zx} \cdot l_p + \sigma_{zy} \cdot m_p + \sigma_{zz} \cdot n_p$$

or

$$\left.\begin{aligned} \mathbf{0} &= (\sigma_{xx} - \sigma_p) l_p + \sigma_{xy} \cdot m_p + \sigma_{xz} \cdot n_p \\ \mathbf{0} &= \sigma_{yx} l_p + (\sigma_{yy} - \sigma_p) m_p + \sigma_{yz} \cdot n_p \\ \mathbf{0} &= \sigma_{zx} l_p + \sigma_{zy} \cdot m_p + (\sigma_{zz} - \sigma_p) n_p \end{aligned}\right\} \quad (8.43)$$

Considering eqn. (8.43) as a set of three homogeneous linear equations in unknowns l_p, m_p and n_p, the direction cosines of the principal plane, one possible solution, viz. $l_p = m_p = n_p = 0$, can be dismissed since $l^2 + m^2 + n^2 = 1$ must always be maintained. The only other solution which gives real values for the direction cosines is that obtained by equating the determinant of the R.H.S. to zero:

i.e.

$$\begin{vmatrix} (\sigma_{xz} - \sigma_p) & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & (\sigma_{yy} - \sigma_p) & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & (\sigma_{zz} - \sigma_p) \end{vmatrix} = 0$$

Evaluating the determinant yields the so-called "*characteristic equation*"

$$\sigma_p^3 - (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\sigma_p^2 + [(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) - (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)]\sigma_p$$
$$- [\sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - (\sigma_{xx}\sigma_{yz}^2 + \sigma_{yy}\sigma_{zx}^2 + \sigma_{zz}\sigma_{xy}^2)] = 0 \quad (8.44)$$

Thus, for any given set of cartesian stress components in three dimensions a solution of this cubic equation is required before principal stress value can be determined; a graphical solution is not possible.

Eigen values

The solutions for the principal stresses σ_1, σ_2 and σ_3 from the characteristic equation are known as the **Eigen values** whilst the associated direction cosines l_p, m_p and n_p are termed the **Eigen vectors.**

One procedure for solution of the cubic characteristic equation is given in §8.10.

8.15. Stress invariants

If, for the same applied stress system, the stress components had been given relative to some other set of cartesian co-ordinates x', y' and z', the above equation would still apply (with x' replacing x, y' replacing y and z' replacing z) and would still produce the same principal stress values. It follows, therefore, that whatever axis system is chosen the coefficients of the various terms of the characteristics equation must have the same values, i.e. they are "non-varying quantities" or "*invariant*".

The equation can thus be re-written in the form:

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0 \tag{8.45}$$

with

$$\left.\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \\ I_2 &= (\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) - (\sigma_{xx}\cdot\sigma_{yy} + \sigma_{yy}\cdot\sigma_{zz} + \sigma_{zz}\cdot\sigma_{xx}) \\ I_3 &= \sigma_{xx}\cdot\sigma_{yy}\cdot\sigma_{zz} + 2\sigma_{xy}\cdot\sigma_{yz}\sigma_{zx} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2 \end{aligned}\right\} \tag{8.46}$$

the three quantities I_1, I_2 and I_3 being termed the **stress invariants.**

If the reference axes selected are the principal stress axes in the system then all shear components reduce to zero and the equations (8.46) reduce to:

$$\left.\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 &&= \Sigma\sigma_p \\ I_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) &&= \Sigma\sigma_p^2 \\ I_3 &= \sigma_1\sigma_2\sigma_3 &&= \Sigma\sigma_p^3 \end{aligned}\right\} \tag{8.47}$$

The first and second invariants are particularly important in development of the theory of plasticity since it is assumed that:

(a) I_1 has no influence on initial yielding
(b) $I_2 =$ constant can be taken as an important criterion of yielding.

For biaxial stress conditions, i.e. $\sigma_3 = 0$, the third stress invariant vanishes and the others reduce to

$$\left.\begin{aligned} I_1 &= \sigma_1 + \sigma_2 \\ I_2 &= \sigma_1\sigma_2 \end{aligned}\right\} \tag{8.48}$$

or, in the xy plane, from eqn. (8.46)

$$\left.\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} \\ I_2 &= \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned}\right\} \tag{8.49}$$

Now from eqn. (13.11)† the principal stresses in a two-dimensional stress system are given by:

$$\sigma_{1,2} = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \tfrac{1}{2}[(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2]^{\frac{1}{2}}$$

$$= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) \pm \tfrac{1}{2}[(\sigma_{xx} + \sigma_{yy})^2 - 4\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2]^{\frac{1}{2}}$$

which is the general solution of the following quadratic equation:

$$\sigma_p^2 = (\sigma_{xx} + \sigma_{yy})\sigma_p + (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) = 0$$

i.e.

$$\sigma_p^2 - I_1\sigma_p + I_2 = 0 \qquad (8.50)$$

The graphical solution of this equation is as follows (see Fig. 8.21):

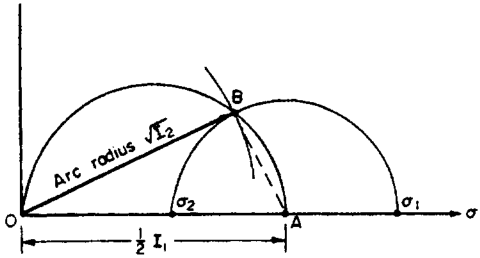

Fig. 8.21. Graphical determination of principal stresses in a two-dimensional stress system from known stress invariant I values (solution for positive I_2 value)

(i) On a horizontal (direct stress) axis mark off a length $OA = \frac{1}{2}I_1$.
(ii) Draw semi-circle on OA as diameter.
(iii) With centre O draw arc OB, radius $\sqrt{I_2}$, to cut the semi-circle at B.
(iv) With centre A and radius AB draw semi-circle to cut stress axis at σ_1 and σ_2 the required principal stress values.

N.B. If I_2 is negative (see §8.46), algebraically $\sqrt{I_2} > \frac{1}{2}I_1$ and the line OB cannot cut the semi-circle on OA as diameter and no solution can be obtained. In this case an alternative construction is required – see Fig. 8.22.

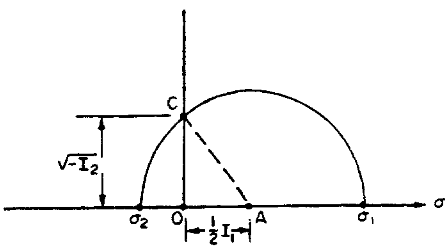

Fig. 8.22. As Fig. 8.21 but for negative I_2 value.

(i) Again mark off length $OA = \frac{1}{2}I_1$.
(ii) Erect perpendicular at O of length $OC = \sqrt{-I_2}$.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1977.

(iii) With centre A and radius AC draw a circle to cut OA (produced as necessary) at σ_1 and σ_2 the required principal stress values.

Returning to a three-dimensional principal stress system a further interesting graphical relationship is obtained from the 3D Mohr circle construction – see Fig. 8.23.*

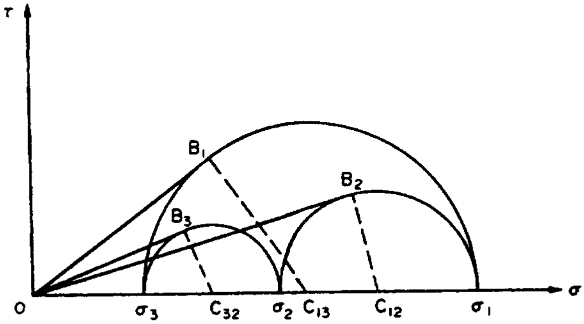

Fig. 8.23. Stress invariants for a three-dimensional stress system in terms of tangents to the Mohr stress circles $I_1 = \sigma_1 + \sigma_2 + \sigma_3, I_2 = OB_1^2 + OB_2^2 + OB_3^3, I_3 = OB_1 \cdot OB_2 \cdot OB_3$.

The three stress invariants are given in Fig. 8.23 in terms of the tangents to the three circles from the origin 0 as:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = OB_1^2 + OB_2^2 + OB_3^3$$

$$I_3 = \sigma_1\sigma_2\sigma_3 = OB_1 \times OB_2 \times OB_3$$

8.16. Reduced stresses

An alternative form of the cubic characteristic equation is obtained if a "*hydrostatic stress*" of $I_1/3$ is substracted from the original stress system to produce "*reduced stresses*" $\sigma' = \sigma - I_1/3$.

Thus, replacing σ_p by $(\sigma' + I_1/3)$ in eqn. (8.45) we have:

$$\sigma'^3 - \left(\frac{I_1^2 + 3I_2}{3}\right)\sigma' - \left(\frac{2I_1^3 + 9I_1I_2 + 27I_3}{27}\right) = 0$$

or

$$\boldsymbol{\sigma'^3 - J_1\sigma'^2 - J_2\sigma' - J_3 = 0} \qquad (8.51)$$

with

$$J_1 = 0$$

$$J_2 = \frac{1}{3}[I_1^2 + 3I_2]$$

$$J_3 = \frac{1}{27}[2I_1^3 + 9I_1I_2 + 27I_3]$$

* M.G. Derrington and W. Johnson, *The Defect of Mohr's Circle for Three-Dimensional Stress States*.

The terms J_1, J_2 and J_3 are termed the *invariants of reduced stress* and, again, have special significance in the consideration of yielding of metals and associated plastic theory.

It will be shown in §8.20 that the hydrostatic stress component does not affect the yield of metals and

$$\text{hydrostatic stress} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3}I_1$$

It therefore follows that first stress invariant I_1 also has no significance on yielding and since the principal stress system can be written, as above, in terms of reduced stresses $\sigma' = (\sigma - 1/3\ I_1)$ it also follows that it must be the reduced stress components which influence yielding.

(N.B.: "Reduced stresses" are synonymous with the deviatoric stresses introduced in §8.20.)

Other useful relationships which can be derived from the above equations are:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6J_2 \tag{8.52}$$

and $$(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2I_1^2 + 6I_2 \tag{8.53}$$

The left-hand sides of both equations are thus, in themselves, invariant and are useful in further considerations of strain energy, yielding and failure.

For example, the shear strain energy theory of elastic failure uses the criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 = \text{constant}$$

which, from eqn. (8.52), can be simply re-written as

$$J_2 = \text{constant}.$$

N.B.: It should be remembered that eqns. (8.52) and (8.53) are merely different ways of presenting the same information since:

$$6J_2 = 2I_1^2 + 6I_2.$$

8.17. Strain invariants

It has been shown in §14.10† that the basic transformation equations for stress and strain have identical form provided that ε is used in place of σ and $\gamma/2$ in place of τ. The equations derived above for the stress invariants will therefore apply equally for strain conditions provided that the same rules are followed.

8.18. Alternative procedure for determination of principal stresses (eigen values)

An alternative solution to the characteristic cubic equation expressed in stress invariant format, viz. eqn. (8.45), is as follows:

Given the basic equation:

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0 \qquad (8.45)\text{bis}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

the stress invariants may be calculated from:

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$
$$I_2 = -(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \qquad (8.46)\text{bis}$$
$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\tau_{xy}\tau_{yz}\tau zx - \sigma_{xx}\tau_{yz}^2 - \sigma_{yy}\tau_{zx}^2 - \sigma_{zz}\tau_{xy}^2$$

and the required principal stresses obtained from‡:

$$\boldsymbol{\sigma_{p_1} = 2S\cos(a/3) + I_1/3}$$
$$\boldsymbol{\sigma_{p_2} = 2S\cos[(a/3) + 120^\circ] + I_1/3} \qquad (8.54)$$
$$\boldsymbol{\sigma_{p_3} = 2S\cos[(a/3 + 240^\circ] + I_1/3}$$

with $S = (R/3)^{1/2}$ and $\alpha = \cos^{-1}(-Q/2T)$

and
$$R = \frac{1}{3}I_1^2 - I_2$$
$$Q = \frac{1}{3}I_1I_2 - I_3 - \frac{2}{27}I_1^3$$
$$T = \left(\frac{1}{27}R^3\right)^{1/2}$$

After calculation of the three principal stress values, they can be placed in their normal conventional order of magnitude, viz. σ_1, σ_2 and σ_3.

The procedure is, in effect, the same as that of §8.13 but carried out in terms of the stress invariants.

8.18.1. Evaluation of direction cosines for principal stresses (eigen vectors)

Having determined the three principal stress values for a given three-dimensional complex stress state using the procedures of §8.13.1 or §8.18, above, a complete solution of the problem generally requires a determination of the directions in which these stresses act–as given by their respective direction cosines or eigen vector values.

The relationship between a particular principal stress σ_p and the cartesian stress components is given by eqn (8.43)

i.e.
$$(\sigma_{xx} - \sigma_p)l + \tau_{xy}\cdot m + \tau_{xz}\cdot n = 0$$
$$\tau_{xy}\cdot l + (\sigma_{yy} - \sigma_p)m + \tau_{yz}n = 0$$
$$\tau_{xz}\cdot l + \tau_{yz}\cdot m + (\sigma_{zz} - \sigma_p)n = 0$$

If one of the known principal stress values, say σ_1, is substituted in the above equations together with the given cartesian stress components, three equations result in the three unknown direction cosines for that principal stress i.e. l_1, m_1 and n_1.

However, only two of these are independent equations and the additional identity $l_1^2 + m_1^2 + n_1^2 = 1$ is required in order to evaluate $l_1 m_1$ and n_1.

‡ E.E. Messal, "Finding true maximum shear stress", *Machine Design*, Dec. 1978.

The procedure can then be repeated substituting the other principal stress values σ_2 and σ_3, in turn, to produce eigen vectors for these stresses but it is tedious and an alternative matrix approach is recommended as follows:

Equation (8.43) above can be expressed in matrix form, thus:

$$\begin{bmatrix} (\sigma_{xx} - \sigma_p) & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & (\sigma_{zz} - \sigma_p) \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = 0$$

with cofactors of the determinant on the elements of the first row of:

$$a = \begin{vmatrix} (\sigma_{yy} - \sigma_p) & \tau_{yz} \\ \tau_{yz} & (\sigma_{zz} - \sigma_p) \end{vmatrix}$$

$$b = -\begin{vmatrix} \tau_{xy} & \tau_{yz} \\ \tau_{xz} & (\sigma_{zz} - \sigma_p) \end{vmatrix}$$

$$c = \begin{vmatrix} \tau_{xy} & (\sigma_{yy} - \sigma_p) \\ \tau_{xz} & \tau_{yz} \end{vmatrix}$$

with the direction cosines or eigen vectors of the principal stresses given by:

$$l_p = ak \qquad m_p = bk \qquad n_p = ck$$

with
$$k = \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

thus satisfying the identity $l_p^2 + m_p^2 + n_p^2 = 1$.

Substitution of any principal stress value, again say σ_1, into the above equations together with the given cartesian stress components allows solution of the determinants and yields values for a_1, b_1 and c_1, hence k_1 and hence l_1, m_1 and n_1, the desired eigen vectors. The process can then be repeated for the other principal stress values $\sigma_2 + \sigma_3$.

8.19. Octahedral planes and stresses

Any complex three-dimensional stress system produces three mutually perpendicular principal stresses σ_1, σ_2, and σ_3. Associated with this stress state are so-called *octahedral planes* each of which cuts across the corners of a principal element such as that shown in Fig. 8.24 to produce the octahedron (8-sided figure) shown in Fig. 8.25. The stresses acting on the octahedral planes have particular significance.

The normal stresses acting on each of the octahedral planes are equal in value and tend to compress or enlarge the octahedron without distorting its shape. They are thus said to be *hydrostatic* stresses and have values given by

$$\sigma_{\text{oct}} = \tfrac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3] = \bar{\sigma} \tag{8.55}$$

Similarly, the shear stresses acting on each of the octahedral planes are also identical and tend to distort the octahedron without changing its volume.

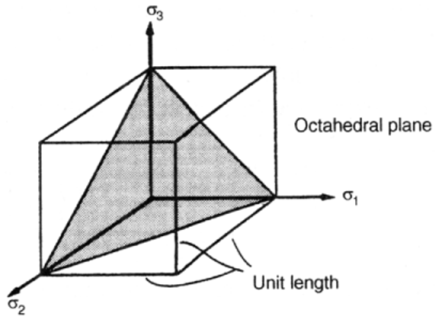

Fig. 8.24. Cubical principal stress element showing one of the octahedral planes.

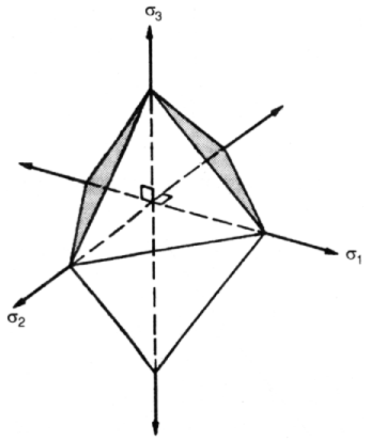

Fig. 8.25. Principal stress system showing the eight octahedral planes forming an octahedron.

The value of the *octahedral shear stresses*† is given by

$$\tau_{oct} = \tfrac{1}{3} - [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$
$$= \tfrac{2}{3} - [\tau_{12}^2 + \tau_{23}^2 + \tau_{13}^2]^{1/2}$$

τ_{12}, τ_{23} and τ_{13} being the maximum shear stresses in the 1–2, 2–3 and 1–3 planes respectively.

Thus the general state of stress may be represented on octahedral planes as shown in Fig. 8.26, the *direction cosines* of the octahedral planes being given by

$$l = m = n = \pm 1/\sqrt{1^2 + 1^2 + 1^2} = \pm 1/\sqrt{3} \tag{8.58}$$

The values of the octahedral shear and direct stresses may also be obtained by the graphical construction of §8.9 since they are represented by a point in the shaded area of the three-dimensional Mohr's circle construction of Figs. 8.8 and 8.9.

† A.J. Durelli, E.A. Phillips and C.H. Tsao. *Analysis of Stress and Strain*, chap. 3. p. 26, McGraw-Hill, New York. 1958.

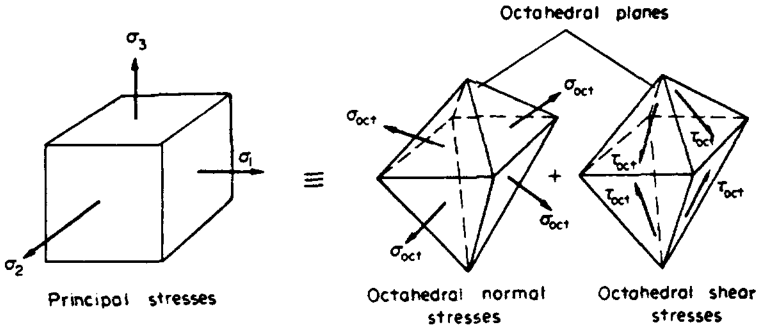

Fig. 8.26. Representation of a general state of stress on the octahedral planes.

The octahedral shear stress has a particular significance in relation to the elastic failure of materials. Whilst its value is always smaller than the greatest numerical (principal) shear stress, it nevertheless has a value which is influenced by all three principal stress values and has been shown to be a reliable criterion for predicting yielding under complex loading conditions.

The **maximum octahedral shear stress theory of elastic failure** thus assumes that yield or failure under complex stress conditions will occur when the octahedral shear stress has a value equal to that obtained in the simple tensile test at yield.

Now for uniaxial tension, $\sigma_2 = \sigma_3 = 0$ and $\sigma_1 = \sigma_y$ and from eqn. (8.56)

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y$$

Therefore the criterion of failure becomes

$$\frac{\sqrt{2}}{3}\sigma_y = \frac{1}{3}\,[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

i.e.
$$2\sigma_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \qquad (8.59)$$

This is clearly the same criterion as that referred to earlier as the Maxwell/von Mises *distortion* or *shear strain energy* theory.

8.20. Deviatoric stresses

It is sometimes convenient to consider stresses with reference to some false zero, i.e. to measure their values above or below some selected datum stress value, and not their absolute values. This is particularly useful in advanced analysis using the theory of plasticity.

The selected datum stress $\bar{\sigma}$ or "false zero" is taken to be that stress which produces only a change in volume. This is the stress which acts equally in all directions and is referred to earlier (page 251) as the *hydrostatic* or *dilatational* stress. This is defined in terms of the principal stresses or the cartesian stresses as follows:

$$\bar{\sigma} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) \qquad (8.60)$$

i.e. $\bar{\sigma}$ = mean of the three principal stress values.

The principal stresses in any three-dimensional complex stress system may now be written in the form

$$\sigma_1 = \text{mean stress} + \text{deviation from the mean}$$

$$= \textit{hydrostatic} \text{ stress} + \textit{deviatoric} \text{ stress}$$

Thus the additional terms required to make up any stress value from the datum to the absolute value are termed the *deviatoric stresses* and written with a prime superscript,

i.e. $$\sigma_1 = \overline{\sigma} + \sigma', \quad \text{etc.}$$

Cartesian stresses σ_{xx}, σ_{yy} and σ_{zz} can now be referred to the new datum as follows:

$$\left.\begin{aligned} \sigma'_{xx} &= \sigma_{xx} - \overline{\sigma} = \tfrac{1}{3}(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}) \\ \sigma'_{yy} &= \sigma_{yy} - \overline{\sigma} = \tfrac{1}{3}(2\sigma_{yy} - \sigma_{xx} - \sigma_{zz}) \\ \sigma'_{zz} &= \sigma_{zz} - \overline{\sigma} = \tfrac{1}{3}(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy}) \end{aligned}\right\} \quad (8.61)$$

All the above values then represent deviatoric stresses.

It may be observed that the system used for representing stresses in terms of the datum stress and the deviation from the datum is, in effect, a consideration of the normal and shear stresses respectively, on the octahedral planes, since the octahedral and deviatoric planes are equally inclined to all three axes ($l = m = n = \pm 1/\sqrt{3}$) and the selected datum stress

$$\overline{\sigma} = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

is also the octahedral normal stress value.

As stated earlier when discussing octahedral stresses, this has a particular relevance to the yield behaviour of materials.

Whilst any detailed study of the theory of plasticity is beyond the scope of this text, the fundamental requirements of the theory should be understood. These are:

(a) the volume of material remains constant under plastic deformation;
(b) the hydrostatic stress component $\overline{\sigma}$ does not cause yielding of the material;
(c) the hydrostatic stress component $\overline{\sigma}$ does not influence the point at which yielding occurs.

From these points it is clear that it is therefore **the deviatoric or octahedral shear stresses which must govern the yield behaviour of materials**. This is supported by the accuracy of the octahedral shear stress (distortion energy) theory and, to a lesser extent, the maximum shear stress theory, in predicting the elastic failure of *ductile* materials. Both theories involve stress differences, i.e. shear stresses, and are therefore independent of the hydrostatic stress as indicated by (b) above.

The representation of a principal stress system in terms of the octahedral and deviatoric stresses may thus be shown as in Fig. 8.27.

It should now be clear that the terms *hydrostatic, volumetric, mean, dilational and octahedral normal stresses* all indicate the same quantity.

The standard elastic stress–strain relationships of eqn. (8.71)

$$\varepsilon_{xx} = \tfrac{1}{E}[\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}]$$

$$\varepsilon_{yy} = \tfrac{1}{E}[\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}]$$

$$\varepsilon_{zz} = \tfrac{1}{E}[\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}]$$

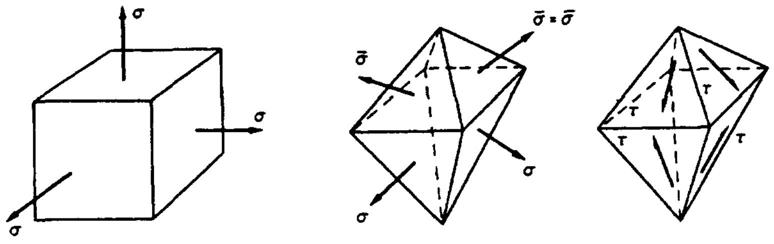

Fig. 8.27. Representation of a principal stress system in terms of octahedral and deviatoric stresses.

may be re-written in a form which distinguishes between those parts which contribute only to a change in volume and those producing a change of shape.

Thus, for a hydrostatic or mean stress $\sigma_m = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$ and remembering the relationship between the elastic constants $E = 2G(1 + \nu)$

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}(1 - 2\nu)\sigma_m + \frac{1}{2G}(\sigma_{xx} - \sigma_m) \\ \varepsilon_{yy} &= \frac{1}{E}(1 - 2\nu)\sigma_m + \frac{1}{2G}(\sigma_{yy} - \sigma_m) \\ \varepsilon_{zz} &= \frac{1}{E}(1 - 2\nu)\sigma_m + \frac{1}{2G}(\sigma_{zz} - \sigma_m) \end{aligned}\right\} \quad (8.62)$$

with $\gamma_{xy} = \tau_{xy}/2G$; $\gamma_{yz} = \tau_{yz}/2G$; $\gamma_{zx} = \tau_{zx}/2G$.
The terms $(\sigma_{xx} - \sigma_m)$, $(\sigma_{yy} - \sigma_m)$ and $(\sigma_{zz} - \sigma_m)$ are the *deviatoric* components of stress.

The volumetric strain ε_m associated with the hydrostatic or mean stress σ_m is then:

$$\varepsilon_m = \frac{\sigma_m}{K} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

where K is the bulk modulus.

8.21. Deviatoric strains

As for the deviatoric stresses the *deviatoric strains* are also defined with reference to some selected "false zero" or datum value,

$$\bar{\varepsilon} = \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \quad (8.63)$$

= mean of the three principal strain values.

Thus, referred to the new datum, the principal strain values become

$$\varepsilon_1' = \varepsilon_1 - \bar{\varepsilon} = \varepsilon_1 - \tfrac{1}{3}(\varepsilon_1 + \varepsilon 2 + \varepsilon_3)$$

$$\therefore \qquad \left.\begin{aligned} \varepsilon'_1 &= \tfrac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \\ \text{Similarly,} \qquad \varepsilon'_2 &= \tfrac{1}{3}(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3) \\ \varepsilon'_3 &= \tfrac{1}{3}(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2) \end{aligned}\right\} \qquad (8.64)$$

and these are the so-called *deviatoric strains*. It may now be observed that the following relationship applies:

$$\varepsilon'_1 + \varepsilon'_2 + \varepsilon'_3 = 0$$

It can also be shown that the deviatoric strains are related to the principal strains as follows:

$$(\varepsilon'_1)^2 + (\varepsilon'_2)^2 + (\varepsilon'_3)^2 = \tfrac{1}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2] \qquad (8.66)$$

8.22. Plane stress and plane strain

If a body consists of two parallel planes a constant thickness apart and bounded by any closed surface as shown in Fig. 8.28, it is said to be a *plane body*. Associated with this type of body there is a particular class of problems within the general theory of elasticity which are termed *plane elastic* problems, and these allow a number of simplifying assumptions in their treatment.

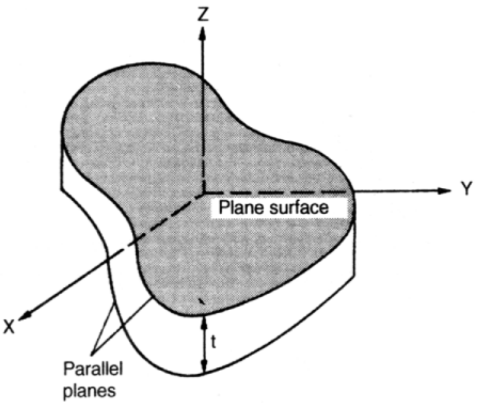

Fig. 8.28. A plane element.

In order to qualify for these simplifications, however, there are a number of restrictions which must be placed on the applied load system:

(1) no loads may be applied to the top and bottom plane surfaces (in practice there is often a uniform stress in the Z direction on the planes but this can always be reduced to zero by superimposing a suitable stress σ_{zz} of opposite sign);

(2) the loads on the lateral boundaries (and the surface shears) must be in the plane of the body and must be uniformly distributed across the thickness;

(3) similarly, body forces in the X and Y directions directions must be uniform across the thickness and the bodv force in the Z direction must be zero.

There is no limitation on the thickness of the plane body and, indeed, the thickness serves as a means of classification within the general type of problem. Normally a *plane stress* approach is applied to members which are relatively thin in relation to their other dimensions, whereas *plane strain* methods are employed for relatively thick members. The terms plane stress and plane strain are defined in detail below.

The plane elastic type of problem may thus be defined as one in which stresses and strains do not vary in the Z direction. Additionally, lines parallel to the Z axis remain straight and parallel to the axis throughout loading.

i.e.
$$\gamma_{zx} = \gamma_{zy} = 0$$

(The problem of torsion provides an exception to this rule.)

8.22.1. Plane stress

A plane stress problem is taken to be one in which σ_{zz} is zero. As stated above, cases where a uniform stress is applied to the plane surfaces can easily be reduced to this condition by application of a suitable σ_{zz} stress of opposite sign. Shear components in the Z direction must also be zero.

i.e.
$$\tau_{zx} = \tau_{zy} = 0 \tag{8.67}$$

Under these conditions the stress equations of equilibrium in cartesian coordinates reduce to

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y &= 0 \end{aligned}\right\} \tag{8.68}$$

The following stress and strain relationships then apply:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \qquad \varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$$

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}[\varepsilon_{xx} + \nu\varepsilon_{yy}] \qquad \sigma_{yy} = \frac{E}{(1-\nu^2)}[\varepsilon_{yy} + \nu\varepsilon_{xx}]$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

Plane stress systems are often referred to as *two-dimensional* or *bi-axial* stress systems, a typical example of which is the case of thin plates loaded at their edges with forces applied in the plane of the plate.

8.22.2. Plane strain

Plane strain problems are normally defined as those in which the strains in the Z direction are zero. Again, problems with a uniform strain in the Z direction at all points on the plane surface can be reduced to the above case by the addition of a suitable uniform stress σ_{zz}, the additional lateral strains and displacements so introduced being easily calculated.
Thus

$$\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0 \tag{8.69}$$

Also, from the basic assumptions of plane elastic problems,

$$\tau_{zy} = \tau_{zx} = 0$$

The equations of stress equilibrium in this case reduce to

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + Y = 0 \end{aligned}\right\} \qquad (8.70)$$

The stress–strain relations are then as follows:

$$\varepsilon_{xx} = \frac{(1-\nu^2)}{E}\left[\sigma_{xx} - \frac{\nu}{(1-\nu)}\sigma_{yy}\right] \quad \text{and} \quad \varepsilon_{yy} = \frac{(1-\nu^2)}{E}\left[\sigma_{yy} - \frac{\nu}{(1-\nu)}\sigma_{xx}\right]$$

$$\sigma_{xx} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{xx} + \frac{\nu}{(1-\nu)}\varepsilon_{yy}\right]$$

$$\sigma_{yy} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_{yy} + \frac{\nu}{(1-\nu)}\varepsilon_{xx}\right]$$

Also $$\tau_{xy} = G\gamma_{xy}$$

It should be noted that the plane strain equations can be derived simply from the plane stress equations by replacing

$$\nu \text{ by } \frac{\nu}{(1-\nu)} \quad \text{and} \quad E \text{ by } \frac{E}{(1-\nu^2)}$$

A typical example of plane strain is the pressurisation of long cylinders where the above equations given accurate results, particularly in the middle portion of the cylinder, whether the end conditions are free, partially fixed or rigidly fixed.

An example of the transfer of a plane stress to a corresponding plane strain solution is given when the relevant equations for the hoop and radial stresses present in rotating thick cylinders are readily obtained from those of rotating thin discs by use of the substitution $\nu/(1-\nu)$ in place of ν (see §4.4).

8.23. The stress–strain relations

The following formulae form a useful summary of the relationships which exist between the stresses and strains in a general three-dimensional stress system.

(a) Strains in terms of stress

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})] \qquad & \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} = \frac{\tau_{xy}}{G} \\ \varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx}+\sigma_{zz})] \qquad & \gamma_{yz} &= \frac{2(1+\nu)}{E}\tau_{yz} = \frac{\tau_{yz}}{G} \\ \varepsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})] \qquad & \gamma_{zx} &= \frac{2(1+\nu)}{E}\tau_{zx} = \frac{\tau_{zx}}{G} \end{aligned}\right\} \qquad (8.71)$$

(b) Stresses in terms of strains

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_{xx} + \nu(\varepsilon_{yy} + \varepsilon_{zz} - \varepsilon_{xx}) \qquad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}\\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_{yy} + \nu(\varepsilon_{xx} + \varepsilon_{zz} - \varepsilon_{yy}) \qquad \tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} = G\gamma_{yz}\\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_{zz} + \nu(\varepsilon_{xx} + \varepsilon_{yy} - \varepsilon_{zz}) \qquad \tau_{zx} = \frac{E}{2(1+\nu)}\gamma_{zx} = G\gamma_{zx}
\end{aligned}\right\} \quad (8.72)$$

with $$E = 2G(1+\nu) \quad \text{and} \quad E = 3K(1-2\nu)$$

hence $$K = \frac{2G(1+\nu)}{3(1-2\nu)}$$

(c) For biaxial stress conditions:

(a) *Strains in terms of stresses*

$$\varepsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}]$$

$$\varepsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}] \quad \text{and} \quad \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}[\sigma_{xx} + \sigma_{yy}]$$

(b) *Stresses in terms of strains*

$$\sigma_{xx} = \frac{E}{(1-\nu^2)}[\varepsilon_{xx} + \nu\varepsilon_{yy}] \quad \text{and} \quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}[\varepsilon_{yy} + \nu\varepsilon_{xx}]$$

Equivalent expressions apply for polar coordinates with r, θ and z replacing x, y and z respectively.

8.24. The strain–displacement relationships

Consider the deformation of a cubic element of material as load is applied. Any corner of the element, e.g. P, will then move to some position P', the movement having components u, v and w in the X, Y and Z directions respectively as shown in Fig. 8.29. Other points in the cube will also be displaced but generally by different amounts.

The movement in the X direction will be given by

$$u = \left(\frac{\partial u}{\partial x}\right)\delta x$$

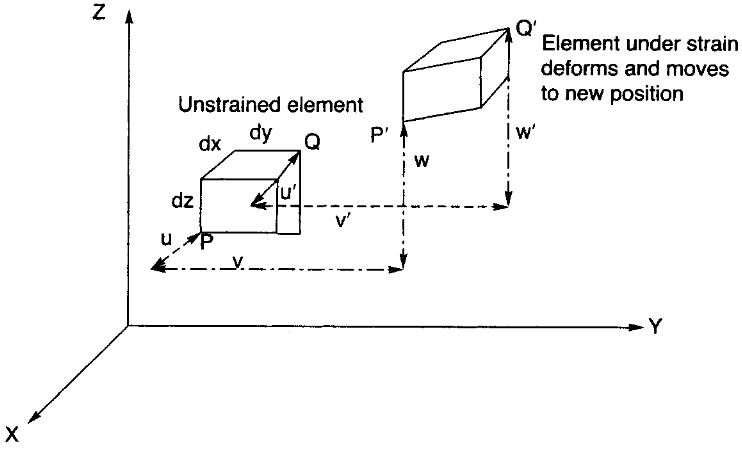

Fig. 8.29. Deformation of a cubical element under load.

The *strain* in the X direction will then be

$$\varepsilon_{xx} = \frac{\text{change in length}}{\text{original length}} = \frac{\left(\dfrac{\partial u}{\partial x}\right)\delta x}{\delta x}$$

i.e.

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{aligned}\right\} \qquad (8.73)$$

Similarly,

Consider now Fig. 8.30 which shows the deformations in the XY plane enlarged.

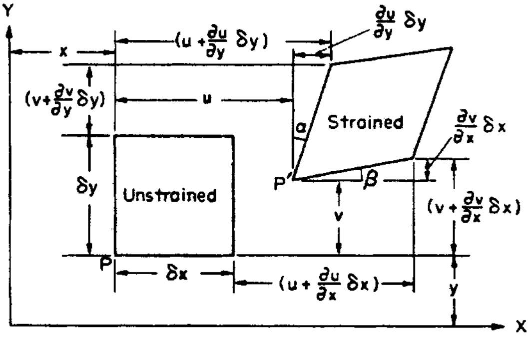

Fig. 8.30. Deformations under load in the XY plane.

Shear strains are defined as angles of deformation or changes in angles between two perpendicular segments. Thus γ_{xy} is the change in angle between two perpendicular segments in the XY plane as load is applied,

i.e.

$$\gamma_{xy} = \alpha + \beta = \frac{\left(\frac{\partial v}{\partial x}\right)\delta x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)\delta y}{\delta y}$$

∴

$$\left.\begin{aligned} \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{aligned}\right\} \tag{8.74}$$

Similarly, and

Summary of the strain–displacement equations

(a) *In cartesian coordinates* with displacements u, v and w along x, y and z respectively.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

(b) *In polar coordinates* with displacements u_r, u_θ and u_z along r, θ and z respectively: these equations become:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\cdot\frac{\partial u_\theta}{\partial \theta}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

with

$$\gamma_{r\theta} = \frac{1}{r}\cdot\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$\gamma_{\theta z} = \frac{1}{r}\cdot\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}$$

$$\gamma_{zr} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

8.25. The strain equations of transformation

Using the experimental or theoretical procedures described in earlier sections it is possible to derive the values of the direct and shear stresses acting at a point on a body. These are normally obtained with reference to some convenient set of X, Y coordinates which, for example, may be parallel to the edges of the component considered. Sometimes, however, it may be more convenient to refer the values obtained to some other set of axes $X'Y'$ at an angle θ to the original axes.

In this case the two-dimensional versions of eqns. (8.73) and (8.74) apply equally well to the new axes (Fig. 8.31),

i.e. $$\varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} \quad \varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} \quad \text{and} \quad \gamma_{x'y'} = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}$$

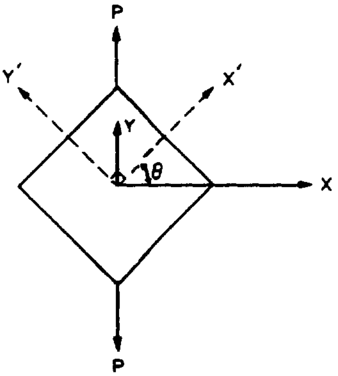

Fig. 8.31. Alternative coordinates to which strains may be referred.

Now, using the partial differentiation chain rule,

$$\begin{aligned}\frac{\partial u'}{\partial x'} &= \left[\frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'}\right] u' \\ &= \left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right](u\cos\theta + v\sin\theta) \\ &= \cos^2\theta\frac{\partial u}{\partial x} + \sin^2\theta\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\sin\theta\cos\theta\end{aligned}$$

$\therefore$ $$\varepsilon_{x'x'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

Or, in terms of the double angle 2θ,

$$\boldsymbol{\varepsilon_{x'x'} = \tfrac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy}) + \tfrac{1}{2}(\varepsilon_{xx} - \varepsilon_{yy})\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta} \tag{8.75}$$

This is the same as eqn. (14.14) obtained in §14.10† *for the normal strain on any plane in terms of the coordinate strains.* Indeed, the above represents an alternative proof for what are really similar requirements.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

8.26. Compatibility

Equations (8.73) and (8.74) relate the six components of strain (three direct and three shear) to the equivalent displacements under a three-dimensional stress system. If, however, the situation arises where the six strain components are known, as they could well be following some theoretical or experimental strain analysis, then the above equations represent three in excess of that required for solution of the three unknown displacements (three unknowns require only three equations for solution). Thus, unless the solution obtained from any three equations satisfies the other three equations, then the values cannot be accepted as a valid solution. Certain specific relations must therefore be satisfied before a valid solution is obtained and these are termed *the compatibility relations*.

The problem can be considered physically as follows: consider a body divided into a large number of small cubic elements. When load is applied the elements deform and simple measurements of length and angle changes will yield the direct and shear strains in each element. These can be summated to produce the overall component strains if required. If, however, the deformed elements are separated and provided in their deformed shapes as a jigsaw puzzle, the puzzle can only be completed, i.e. the elements fully assembled without voids or discontinuities, if each element is correctly strained or deformed. The procedure used to check this condition then represents the compatibility equations. The compatibility relationships in terms of strain are derived as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \therefore \quad \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \therefore \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 v}{\partial x^2 \partial y}$$

But

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Therefore differentiating once with respect to x and once with respect to y,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

i.e.

$$\left.\begin{aligned} \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \\ \text{Similarly,}\quad \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} &= \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \\ \text{and}\quad \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} &= \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} \end{aligned}\right\} \qquad (8.76)$$

These are three of the compatibility equations.

It can also be shown† that a further three compatibility relationships apply, namely

† A.E.H. Love, *Treatise on the Mathematical Theory of Elasticity*, 4th edn., Dover Press, New York, 1944.

$$\left.\begin{aligned}2\frac{\partial^2\varepsilon_{xx}}{\partial y\,\partial z}&=\frac{\partial}{\partial x}\left[\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{yz}}{\partial x}\right]\\2\frac{\partial^2\varepsilon_{yy}}{\partial z\,\partial x}&=\frac{\partial}{\partial y}\left[\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{yz}}{\partial x}\right]\\2\frac{\partial^2\varepsilon_{zz}}{\partial x\,\partial y}&=\frac{\partial}{\partial z}\left[-\frac{\partial\gamma_{xy}}{\partial z}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{yz}}{\partial x}\right]\end{aligned}\right\}\qquad(8.77)$$

The compatibility equations can also be written in terms of stress as follows:
Consider the first of the strain compatibility relationships given in eqn. (8.41).

i.e. $$\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

For plane strain conditions (and a similar derivation shows that the equation derived is equally appropriate for plane stress) we have:

$$\varepsilon_{xx}=\frac{1}{E}(\sigma_{xx}-v\sigma_{yy})$$

$$\varepsilon_{yy}=\frac{1}{E}(\sigma_{yy}-v\sigma_{xx})$$

and $$\gamma_{xy}=\frac{2(1+v)}{E}\tau_{xy}$$

Substituting:

$$\frac{1}{E}\frac{\partial^2\sigma_{xx}}{\partial y^2}-\frac{v}{E}\frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{1}{E}\frac{\partial^2\sigma_{yy}}{\partial x^2}-\frac{v}{E}\frac{\partial^2\sigma_{xx}}{\partial x^2}=\frac{2(1+v)}{E}\frac{\partial^2\tau_{xy}}{\partial x\partial y}\qquad(1)$$

Now from the equilibrium equations assuming plane stress and zero body force stresses we have:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0\qquad(2)$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0\qquad(3)$$

Differentiating (2) with respect to x and (3) with respect to y and adding we have:

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}=-2\frac{\partial^2\tau_{xy}}{\partial x\partial y}\qquad(4)$$

Eliminating τ_{xy} between eqns (4) and (1) we obtain:

$$\frac{1}{E}\frac{\partial^2\sigma_{xx}}{\partial y^2}-\frac{v}{E}\frac{\partial^2\sigma_{yy}}{\partial y^2}+\frac{1}{E}\frac{\partial^2\sigma_{yy}}{\partial x^2}-\frac{v}{E}\frac{\partial^2\sigma_{xx}}{\partial x^2}=-\frac{(1+v)}{E}\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}$$

i.e. $$\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial x^2}+\frac{\partial^2\sigma_{xx}}{\partial y^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}=0$$

or $$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right](\sigma_x+\sigma_y)=0\qquad(8.78)$$

A similar development for cylindrical coordinates yields the stress equation of compability

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r}\right] (\sigma_{rr} + \sigma_{\theta\theta}) = 0 \tag{8.79}$$

which in the case of axial symmetry (where stresses are independent of θ) reduces to:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] (\sigma_{rr} + \sigma_{\theta\theta}) = 0. \tag{8.80}$$

8.27. The stress function concept

From the earlier work of this chapter it should now be evident that in elastic stress analysis there are generally fifteen unknown quantities to be determined; six stresses, six strains and three displacements. These are functions of the independent variables x, y and z (in cartesian coordinates) or r, θ and z (in cylindrical polar coordinates). A quick look at the governing equations presented earlier in the chapter will convince the reader that the equations are difficult to solve for these unknowns, except for a number of relatively simple problems.

In order to extend the range of useful solutions several techniques are available. In the first instance one may make certain assumptions about the physical problem in an effort to simplify the equations. For example, are the loading and boundary conditions such that:

(i) the plane stress assumption is adequate – as in a thin-walled pressure vessel? or,
(ii) does plane strain exist – as in the case of a pressurised thick cylinder?

If we can convince ourselves that these assumptions are valid we reduce the three-dimensional problem to the two-dimensional case.

Having simplified the governing differential equations one must then devise techniques to solve, or further reduce, their complexity. One such concept was that proposed by Sir George B. Airy.[†]. His approach was to assume that the stresses in the two-dimensional problem σ_{xx}, σ_{yy} and τ_{xy} could be described by a single function of x and y. This function ϕ is referred to as a "*stress function*" (later the "*Airy stress function*") and it appears to be the first time that such a concept was used. Airy's approach was later generalised for the three-dimensional case by Clerk Maxwell.[‡]

Airy proposed that the stresses be derived from a particular function ϕ such that:

$$\left.\begin{aligned} \sigma_{xx} &= \frac{\partial^2\phi}{\partial y^2} \\ \sigma_{yy} &= \frac{\partial^2\phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x \partial y} \end{aligned}\right\} \tag{8.81}$$

[†] G.B. Airy, *Brit. Assoc. Advancement of Sci. Rep.* 1862; *Phil. Trans. Roy. Soc.* **153** (1863), 49–80
[‡] J.C. Maxwell *Edinburgh Roy. Soc. Trans.*, **26** (1872), 1–40.

It should be noted that these equations satisfy the two-dimensional versions of equilibrium equations (8.38):

i.e.

$$\left.\begin{aligned}\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0\end{aligned}\right\} \tag{8.82}$$

It is also necessary that the stress function ϕ must not only satisfy the equilibrium conditions of the problem but must also satisfy the compatibility relationships, i.e. eqn. 8.76. For the two-dimensional case these reduce to:

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\frac{\partial^2\varepsilon_{xx}}{\partial y^2}+\frac{\partial^2\varepsilon_{yy}}{\partial x^2} \qquad (8.76)\text{bis.}$$

This equation can be written in terms of stress using the appropriate constitutive (stress–strain) relations. To illustrate the procedure the *plane strain* case will be considered. In this the relevant equations are:

$$\left.\begin{aligned}\varepsilon_{xx}&=\frac{(1-v^2)}{E}\left[\sigma_{xx}-\frac{v}{(1-v)}\sigma_{yy}\right]\\ \varepsilon_{yy}&=\frac{(1-v^2)}{E}\left[\sigma_{yy}-\frac{v}{(1-v)}\sigma_{xx}\right]\\ \gamma_{xy}&=\frac{\tau_{xy}}{G}=\frac{2(1+v)}{E}\tau_{xy}\end{aligned}\right\} \tag{8.83}$$

By substituting these into the compatibility equation (8.76) the following is obtained:

$$2\frac{\partial^2\tau_{xy}}{\partial x\partial y}=\frac{\partial^2}{\partial y^2}[(1-v)\sigma_{xx}-v\sigma_{yy}]+\frac{\partial^2}{\partial x^2}[(1-v)\sigma_{yy}-v\sigma_{xx}]$$

From the equilibrium eqn. (8.70) we get:

$$2\frac{\partial^2\tau_{xy}}{\partial x\partial y}=-\left(\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right)-\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)$$

Combining these equations to eliminate the shear stress τ_{xy}, gives:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})=-\frac{1}{(1-v)}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right) \tag{8.84}$$

A similar equation can be obtained for the *plane stress* case, namely:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})=-(1+v)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right) \tag{8.85}$$

If the body forces X and Y have constant values the same equation holds for both plane stress and plane strain, namely:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_{xx}+\sigma_{yy})=\nabla^2(\sigma_{xx}+\sigma_{yy})=0 \tag{8.86}$$

This equation is known as the "*Laplace differential equation*" or the "*harmonic differential equation.*" The function $(\sigma_{xx} + \sigma_{yy})$ is referred to as a "harmonic" function. It is interesting to note that the Laplace equation, which of course incorporates all the previous equations, does not contain the elastic constants of the material. **This is an important conclusion for the experimentalist since, providing there exists geometric similarity, material isotropy and linearity and similar applied loading of both model and prototype, then the stress distribution per unit load will be identical in each.** The stress function, previously defined, must satisfy the 'Laplace equation' (8.86). Thus:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$$

or,

$$\frac{\partial^4 \phi}{\partial x^4} + \frac{2\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (8.87)$$

Alternatively, this can be re-written in the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \phi = 0$$

or abbreviated to

$$\nabla^4 \phi = 0 \qquad (8.88)$$

indicating that the stress function must be a biharmonic function. Equation (8.87) is often referred to as the "*biharmonic equation*" with ϕ known as the "*Airy stress function*".

It is worth noting, at this point in the development, that the problem of plane strain, or plane stress, has been reduced to seeking a solution of the biharmonic equation (8.87) such that the stress components satisfy the boundary conditions of the problem.

Thus, provided that a suitable polynomial expression in x and y (or r and θ) is used for the stress function ϕ then both equilibrium and compatibility are automatically assured. Consideration of the boundary conditions associated with any particular stress system will then yield the appropriate coefficients of the various terms of the polynomial and a complete solution is obtained.

8.27.1. Forms of Airy stress function in Cartesian coordinates

The stress function concept was developed by Airy initially to investigate the bending theory of straight rectangular beams. It was thus natural that a rectangular cartesian coordinate system be used. As an introduction to this topic, therefore, forms of stress function in cartesian coordinates will be explored and applied to a number of fairly simple beam problems. It is hoped that the reader will gain confidence in using the approach and be able to tackle a range of more interesting problems where cylindrical polars (r, θ) is an appropriate alternative coordinate system.

(a) The eqns. (8.81) which define the stress function imply that the most simple function of ϕ to produce a stress is $\boldsymbol{\phi = Ax^2}$, since the lower orders when differentiated twice give a zero result. Substituting this into eqns. (8.81) gives:

$$\sigma_{xx} = 0, \quad \sigma_{yy} = 2A \quad \text{and} \quad \tau_{xy} = 0$$

Thus a stress function of the form $\phi = Ax^2$ can be used to describe a condition of constant stress $2A$ in the y direction over the entire region of a component, e.g. uniform tension or compression testing

(b) $$\boldsymbol{\phi = By^3}.$$

For this stress function

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6By$$

with σ_{yy} and τ_{xy} zero.

Thus σ_{xx} is a linear function of vertical dimension y, a situation typical of beam bending.

(c) $$\boldsymbol{\phi = Ax^2 + Bxy + Cy^2}.$$

In this case

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2C \quad \text{(a constant)}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 2A \quad \text{(a constant)}$$

$$\tau_{xy} = \frac{\partial^2\phi}{\partial x \partial y} = -B \quad \text{(a constant)}$$

and the stress function is suitable for any uniform plane stress state.

(d) $$\boldsymbol{\phi = Ax^3 + Bx^2y + Cxy^2 + Dy^3}.$$

Then

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2Cx + 6Dy$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 6Ax + 2By$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} = -2Bx - 2Cy$$

and all stresses may be seen to vary linearly with x and y.

For the particular case where $A = B = C = 0$ the situation resolves itself into that of case (b) i.e. suitable for pure bending.

For many problems an extension of the above function to a comprehensive polynomial expression is found to be rather useful. An appropriate technique is to postulate a general form which will adequately represent the applied loading and boundary conditions. The form of this could be:

$$\begin{aligned}\phi &= Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 \\ &\quad + Hx^4 + Jx^3y + Kx^2y^2 + Lxy^3 + My^4 + Nx^5 + Px^4y \\ &\quad + Qx^3y^2 + Rx^2y^3 + Sxy^4 + Ty^5 + \cdots \end{aligned} \tag{8.89}$$

Any term containing x or y up to the third power will automatically satisfy the biharmonic equation $\nabla^4(\phi) = 0$. However, terms containing x^4 or y^4, or higher powers, will appear in the biharmonic equation. Relations of the associated coefficients can thereby be found which will satisfy $\nabla^4(\phi) = 0$.

Although beyond the scope of the present text, it is worth noting that the polynomial approach has severe limitations when applied to cases with discontinuous loads on the boundary. For such cases, a stress function in the form of a trigonometric series – a Fourier series for example – should be used.

8.27.2. Case 1 – Bending of a simply supported beam by a uniformly distributed loading

An end-supported beam of length $2L$, depth $2d$ and unit width is loaded with a uniformly distributed load w/unit length as shown in Fig. 8.32. From the work of Chapter 4† the reader will be aware of the solution of this problem using the simple bending theory sometimes known as "engineers bending". Using this simple approach it is possible to obtain values for the longitudinal stress σ_{xx} and the shear stress τ_{xy}. However, the stress function provides the stress analyst with information about *all* the two-dimensional stresses and thereby the regions of applicability where the more straightforward methods can be used with confidence.
The boundary conditions of this problem are:

(i) at $y = +d$; $\sigma_{yy} = 0$ for all values of x,
(ii) at $y = -d$; $\sigma_{yy} = -w$ for all values of x,
(iii) at $y = \pm d$; $\tau_{xy} = 0$ for all values of x.

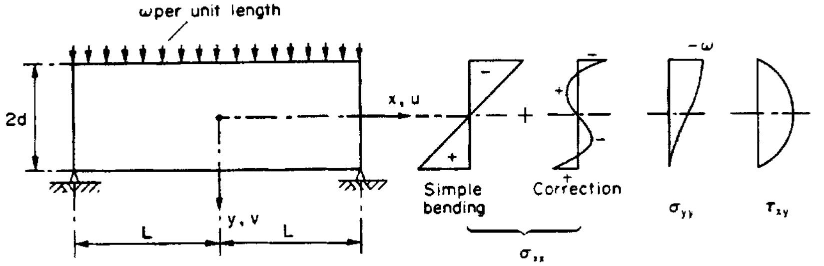

Fig. 8.32. The bending of a simply supported beam by a uniformly distributed load w/unit length.

The overall equilibrium requirements are: –

(iv) $\int_{-d}^{d} \sigma_{xx} y \cdot dy = w(L^2 - x^2)/2$ for the equilibrium of moments at any position x,
(v) $\int_{-d}^{d} \sigma_{xx} dy = 0$ for the equilibrium of forces at any position x.

The biharmonic equation:

(vi) $\nabla^4(\phi) = 0$ must also be satisfied.

To deal with these conditions it is necessary to use the 5th-order polynomial as given in eqn. (8.89) containing eighteen coefficients A to T.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

From eqn. (8.81)

$$\left.\begin{aligned}\sigma_{xx} &= \frac{\partial^2\phi}{\partial y^2} = 2C + 2Fx + 6Gy + 2Kx^2 + 6Lxy + 12My^2 + 2Qx^3 + 6Rx^2y \\ &\qquad + 12Sxy^2 + 20Ty^3 \\ \sigma_{yy} &= \frac{\partial^2\phi}{\partial x^2} = 2A + 6Dx + 2Ey + 12Hx^2 + 6Jxy + 2Ky^2 + 20Nx^3 + 12Px^2y \\ &\qquad + 6Qxy^2 + 2Ry^3 \\ \tau_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} = -[B + 2Ex + 2Fy + 3Jx^2 + 4Kxy + 3Ly^2 + 4Px^3 + 6Qx^2y \\ &\qquad + 6Rxy^2 + 4Sy^3]\end{aligned}\right\} \qquad (8.90)$$

Using the conditions (i) to (vi) it is possible to set up a series of algebraic equations to determine the values of the eighteen coefficients A to T. Since these conditions must be satisfied for all x values it is appropriate to equate the coefficients of the x terms, for example x^3, x^2, x and the constants, on both sides of the equations. In the case of the biharmonic equation, condition (vi), all x and y values must be satisfied. This procedure gives the following results:

$A = -w/4$	$G = (wL^2/8d^3) - w/20d$	$N = 0$
$B = 0$	$H = 0$	$P = 0$
$C = 0$	$J = 0$	$Q = 0$
$D = 0$	$K = 0$	$R = -w/8d^3$
$E = 3w/8d$	$L = 0$	$S = 0$
$F = 0$	$M = 0$	$T = w/40d^3$

The stress function ϕ can thus be written:

$$\phi = -\frac{w}{4}x^2 + \frac{3w}{8d}x^2y + \left(\frac{wL^2}{8d^3} - \frac{w}{20d}\right)y^3 - \frac{w}{8d^3}x^2y^3 + \frac{5}{40d^3}y^5 \qquad (8.91)$$

The values for the stresses follow using eqn. (8.90) with $I = 2d^3/3$

$$\left.\begin{aligned}\sigma_{xx} &= \frac{w(L^2 - x^2)y}{2I} + \frac{w}{2I}\left(-\frac{2}{5}d^2y + \frac{2}{3}y^3\right) \\ \sigma_{yy} &= -\frac{w}{2I}\left(\frac{2}{3}d^3 - d^2y + \frac{y^3}{3}\right) \\ \tau_{xy} &= -\frac{wx}{2I}(d^2 - y^2)\end{aligned}\right\} \qquad (8.92a–c)$$

These stresses are plotted in Fig. 8.32. The longitudinal stress σ_{xx} consists of two parts. The first term $w(L^2 - x^2)y/2I$ is that given by simple bending theory ($\sigma_{xx} = My/I$). The second term may be considered as a correction term which arises because of the effect of the σ_{yy} compressive stress between the longitudinal fibres. The term is independent of x and therefore constant along the beam. It thus has a value on the ends of the beam given by $x = \pm L$. The expression for σ_{xx} in eqn. (8.92a) is, therefore, only an exact solution if normal forces on the end exist and are distributed in such a manner as to produce the σ_{xx} values given by eqn. (8.92a) at $x = \pm L$. That is as shown by the correction term in Fig. 8.32. However, conditions (iv) and (v) have guaranteed that forces and moments are in equilibrium at the ends $x = \pm L$ and thus, from Saint-Venant's principle, one could conclude that at distance larger than, say, the depth of the beam, the stress distribution given by eqn. (8.92a) is accurate even when the ends are free. Such correction stresses are, however, of small magnitude compared with the simple bending terms when the span of the beam is large in comparison with its depth.

The equation for the shear stress (8.92c) predicts a parabolic distribution of τ_{xy} on every section x. This implies that at the ends $x = \pm L$ the beam must be supported in such a way that these shear stresses are developed. The values predicted by eqn. (8.92c) coincide with the simple solution. The σ_{yy} stress decreases from its maximum on the top surface to zero at the bottom edge. This again is of small magnitude compared to σ_{xx} in a thin beam type component. However, these stresses can be of importance in a deep beam, or a slab arrangement.

Derivation of the displacements in the beam

From the strain displacement relations, the constitutive relations and the derived stresses it is possible to obtain the displacements in the beam. Although this approach is not really part of the stress function concept, it is included for interest at this point in the development. The procedure is as follows:

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial v}{\partial x} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \tau_{xy}/G \end{aligned}\right\} \qquad (8.93\text{a-c})$$

Substituting for σ_{xx} and σ_{yy} from eqns (8.92a,b) and integrating (8.93a,b) the following is obtained:

$$u = \frac{w}{2EI}\left[\left(L^2x - \frac{x^3}{3}\right)y + \left(\frac{2}{3}y^3 - \frac{2}{5}d^2y\right)x + \nu x\left(\frac{1}{3}\cdot y^3 - d^3y + \frac{2}{3}d^3\right)\right] + u_0(y) \qquad (8.94)$$

where $u_0(y)$ is a function of y,

$$v = -\frac{w}{2EI}\left[\frac{y^4}{12} - \frac{d^2y^2}{2} + \frac{2d^3y}{3} + \frac{v}{2}(L^2 - x^2)y^2 + \frac{v}{6}y^4 - \frac{v}{5}d^2y\right] + v_0(x) \qquad (8.95)$$

where $v_0(x)$ is a function of x.

From eqns (8.92c) and (8.93c)

$$\gamma_{xy} = -\frac{w(1+v)}{EI}(d^2 - y^2)x \qquad (8.96)$$

Differentiating u with respect to y and v with respect to x and adding as in eqn. (8.93c) one can equate the result to the right hand side of eqn. 8.96. After simplifying:

$$\frac{w}{2EI}\left[L^2x-\frac{x^3}{3}-\frac{2}{5}xd^2-\nu xd^2\right]+\frac{\partial u_0(y)}{\partial y}+\frac{\partial v_0(x)}{\partial x}=-\frac{w(1+\nu)}{EI}d^2x \tag{8.97}$$

In eqn. (8.97) some terms are functions of x alone and some are functions of y alone. There is no constant term. Denoting the functions of x and y by $F(x)$ and $G(y)$ respectively, we have:

$$F(x)=\frac{w}{2EI}\left[L^2x-\frac{x^3}{3}-\frac{2}{5}xd^2-\nu xd^2\right]+\frac{w(1+\nu)d^2x}{EI}+\frac{\partial v_0(x)}{\partial x}$$

$$G(y)=\frac{\partial u_0(y)}{\partial y}.$$

Equation (8.97) is thus written

$$F(x)+G(y)=0$$

If such an equation is to apply for all values of x and y then the functions $F(x)$ and $G(y)$ must themselves be constants and they must be equal in value but opposite in sign. That is in this case, $F(x)=A_1$ and $G(y)=-A_1$.

Thus: $$\frac{\partial u_0(y)}{\partial y}=-A_1 \quad \therefore\ u_0(y)=-A_1y+B_1 \tag{8.98}$$

$$\frac{\partial v_0(x)}{\partial x}=-\frac{w}{2EI}\left[L^2x-\frac{x^3}{3}-\frac{2}{5}xd^2-\nu xd^2+2(1+\nu)d^2x\right]+A_1$$

$$\therefore\quad v_0(x)=-\frac{w}{2EI}\left[L^2\frac{x^2}{2}-\frac{x^4}{12}-\frac{1}{5}x^2d^2-\frac{\nu}{2}x^2d^2+(1+\nu)d^2x^2\right]+A_1x+C_1 \tag{8.99}$$

Using the boundary conditions of the problem:

at $x=0,\ y=0,\ u=0$: substituting eqn. (8.98) into (8.94) gives $B_1=0$,

at $x=0,\ y=0,\ v=\delta$: substituting eqn. (8.99) into (8.95) gives $C_1=\delta$,

at $x=0,\ y=0,\ \dfrac{\partial v}{\partial x}=0$ thus $A_1=0$.

Thus:

$$\left.\begin{aligned}u&=\frac{w}{2EI}\left[\left(L^2x-\frac{x^3}{3}\right)y+x\left(\frac{2}{3}y^2-\frac{2}{5}d^2y\right)+\nu x\left(\frac{y^3}{3}-d^2y+\frac{2}{3}d^3\right)\right]\\ v&=-\frac{w}{2EI}\left[\frac{y^4}{12}-\frac{d^2y^2}{2}+\frac{2}{3}d^3y+\nu(L^2-x^2)\frac{y^2}{2}+\frac{\nu}{6}y^4-\frac{\nu}{5}d^2y\right.\\ &\left.\quad+L^2\frac{x^2}{2}-\frac{x^4}{12}-\frac{d^2x^2}{5}+\left(1+\frac{\nu}{2}\right)d^2x^2\right]+\delta\end{aligned}\right\} \tag{8.100}$$

To determine the vertical deflection of the central axis we put $y=0$ in the above equation, that is:

$$v_{y=0}=\delta-\frac{w}{2EI}\left[L^2\frac{x^2}{2}-\frac{x^4}{12}-\frac{d^2x^2}{5}+\left(1+\frac{\nu}{2}\right)d^2x^2\right]$$

Using the fact that $v = 0$ at $x = \pm L$ we find that the central deflection δ is given by:

$$\delta = \frac{5}{24}\frac{wL^4}{EI}\left[1 + \frac{d^2}{L^2}\frac{12}{5}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \tag{8.101}$$

The first term is the central deflection predicted by the simple bending theory. The second term is the correction to include deflection due to shear. As indicated by the form of eqn. (8.101) the latter is small when the span/depth ratio is large, but is more significant for deep beams. By combining equations (8.100) and (8.101) the displacements u and v can be obtained at any point (x, y) in the beam.

8.27.3. The use of polar coordinates in two dimensions

Many engineering components have a degree of axial symmetry, that is they are either rotationally symmetric about a central axis, as in a circular ring, disc and thick cylinder, or contain circular holes which dominate the stress field, or yet again are made up from parts of hollow discs, like a curved bar. In such cases it is advantageous to use cylindrical polar coordinates (r, θ, z), where r and θ are measured from a fixed origin and axis, respectively and z is in the axial direction. The equilibrium equations for this case are given in eqns. (8.40) and (8.41).

The form of applied loading for these components need not be restricted to the simple rotationally symmetric cases dealt with in earlier chapters. In fact the great value of the stress function concept is that complex loading patterns can be adequately represented by the use of either $\cos n\theta$ and/or $\sin n\theta$, where n is the harmonic order.

A two-dimensional stress field $(\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta})$ is again used for these cases. That is plane stress or plane strain is assumed to provide an adequate approximation of the three-dimensional problem. The next step is to transform the biharmonic eqn. (8.87) to the relevant polar form, namely:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0 \tag{8.102}$$

The stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are related to the stress function ϕ in a similar manner to σ_{xx} and σ_{yy}. The resulting values are:

$$\left.\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\\
\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2}\\
\tau_{r\theta} &= \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)
\end{aligned}\right\} \tag{8.103}$$

The derivation of these from the corresponding cartesian coordinate values is a worthwhile exercise for a winter evening.

8.27.4. Forms of stress function in polar coordinates

In cylindrical polars the stress function is, in general, of the form:

$$\phi = f(r)\cos n\theta \quad \text{or} \quad \phi = f(r)\sin n\theta \tag{8.104}$$

where $f(r)$ is a function of r alone and n is an integer.

In exploring the form of ϕ in polars one can avoid the somewhat tedious polynomial expression used for the cartesian coordinates, by considering the following three cases:

(a) *The axi-symmetric case when* $n = 0$ (independent of θ), $\phi = f(r)$. Here the biharmonic eqn. (8.102) reduces to:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 \phi = 0 \tag{8.105}$$

and the stresses in eqn. (8.103) to:

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr}, \quad \sigma_{\theta\theta} = \frac{d^2\phi}{dr^2}, \quad \tau_{r\theta} = 0 \tag{8.106}$$

Equation (8.105) has a general solution:

$$\phi = Ar^2 \ln r + Br^2 + C\ln r + D \tag{8.107}$$

(b) *The asymmetric case* $n = 1$

$$\phi = f_1(r)\sin\theta \quad \text{or} \quad \phi = f_1(r)\cos\theta.$$

Equation (8.102) has the solution for

$$f_1(r) = A_1 r^3 + B_1/r + C_1 r + D_1 r\ln r \tag{8.108}$$

i.e. $$\phi = (A_1 r^3 + B_1/r + C_1 r + D_1 r\ln r)\sin\theta \quad (\text{or } \cos\theta)$$

(c) *The asymmetric cases* $n \geqslant 2$.

$$\phi = f_n(r)\sin n\theta \quad \text{or} \quad \phi = f_n(r)\cos n\theta$$

$$f_n(r) = A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \tag{8.109}$$

i.e. $$\phi = (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2})\sin n\theta \quad (\text{or} \cos n\theta)$$

Other useful solutions are $\quad \phi = Cr\sin\theta \quad$ or $\quad \phi = Cr\cos\theta \quad$ (8.110)

In the above A, B, C and D are constants of integration which enable formulation of the various problems.

As in the case of the cartesian coordinate system these stress functions must satisfy the compatibility relation embodied in the biharmonic equation (8.102). Although the reader is assured that they are satisfactory functions, checking them is always a beneficial exercise.

In those cases when it is not possible to adequately represent the form of the applied loading by a single term, say $\cos 2\theta$, then a Fourier series representation using eqn. (8.109) can be used. Details of this are given by Timoshenko and Goodier.[†]

[†] S. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, 1951.

In the presentation that follows examples of these cases are given. It will be appreciated that the scope of these are by no means exhaustive but a number of worthwhile solutions are given to problems that would otherwise be intractable. Only the stress values are presented for these cases, although the derivation of the displacements is a natural extension.

8.27.5. Case 2 – Axi-symmetric case: solid shaft and thick cylinder radially loaded with uniform pressure

This obvious case will be briefly discussed since the Lamé equations which govern this problem are so well known and do provide a familiar starting point.

Substituting eqn. (8.107) into the stress equations (8.106) results in

$$\left.\begin{aligned}\sigma_{rr} &= A(1+2\ln r)+2B+C/r^2\\ \sigma_{\theta\theta} &= A(3+2\ln r)+2B-C/r^2\\ \tau_{r\theta} &= 0\end{aligned}\right\} \tag{8.111}$$

When a *solid shaft* is loaded on the external surface, the constants A and C must vanish to avoid the singularity condition at $r=0$. Hence $\sigma_{rr}=\sigma_{\theta\theta}=2B$. That is uniform tension, or compression over the cross section.

In the case of the *thick cylinder*, three constants, A, B, and C have to be determined. The constant A is found by examining the form of the tangential displacement v in the cylinder. The expression for this turns out to be a multi-valued expression in θ, thus predicting a different displacement every time θ is increased to $\theta+2\pi$. That is every time we scan one complete revolution and arrive at the same point again we get a different value for v. To avoid this difficulty we put $A=0$. Equations (8.111) are thus identical in form to the Lamé eqns. (10.3 and 10.4).† The two unknown constants are determined from the applied load conditions at the surface.

8.27.6. Case 3 – The pure bending of a rectangular section curved beam

Consider a circular arc curved beam of narrow rectangular cross-section and unit width, bent in the plane of curvature by end couples M (Fig. 8.33). The beam has a constant cross-section and the bending moment is constant along the beam. In view of this one would expect that the stress distribution will be the same on each radial cross-section, that is, it will be independent of θ. The axi-symmetric form of ϕ, as given in eqn. (8.107), can thus be used:-

i.e. $$\phi = Ar^2\ln r + Br^2 + C\ln r + D$$

The corresponding stress values are those of eqns (8.111)

$$\begin{aligned}\sigma_{rr} &= A(1+2\ln r)+2B+C/r^2\\ \sigma_{\theta\theta} &= A(3+2\ln r)+2B-C/r^2\\ \tau_{r\theta} &= 0\end{aligned}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

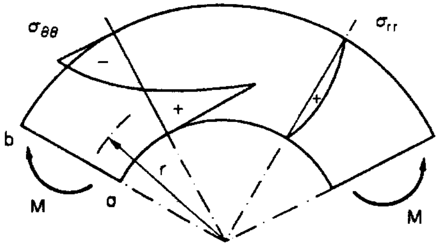

Fig. 8.33. Pure bending of a curved beam.

The boundary conditions for the curved beam case are:

(i) $\sigma_{rr} = 0$ at $r = a$ and $r = b$ (a and b are the inside and outside radii, respectively);
(ii) $\int_a^b \sigma_{\theta\theta} = 0$, for the equilibrium of forces, over any cross-section;
(iii) $\int_a^b \sigma_{\theta\theta}\, r\, dr = -M$, for the equilibrium of moments, over any cross-section;
(iv) $\tau_{r\theta} = 0$, at the boundary $r = a$ and $r = b$.

Using these conditions the constants A, B and C can be determined. The final stress equations are as follows:

$$\left.\begin{aligned} \sigma_{rr} &= \frac{4M}{Q}\left(\frac{a^2b^2}{r^2}\ln\frac{b}{a} - a^2\ln\frac{r}{a} - b^2\ln\frac{b}{r}\right) \\ \sigma_{\theta\theta} &= \frac{4M}{Q}\left(-\frac{a^2b^2}{r^2}\ln\frac{b}{a} - a^2\ln\frac{r}{a} - b^2\ln\frac{b}{r} + b^2 - a^2\right) \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \tag{8.112}$$

where $Q = 4a^2b^2\left(\ln\dfrac{b}{a}\right)^2 - (b^2 - a^2)^2$

The distributions of these stresses are shown on Fig. 8.33. Of particular note is the nonlinear distribution of the $\sigma_{\theta\theta}$ stress. This predicts a higher inner fibre stress than the simple bending ($\sigma = My/I$) theory.

8.27.7. Case 4. Asymmetric case $n = 1$. Shear loading of a circular arc cantilever beam

To illustrate this form of stress function the curved beam is again selected; however, in this case the loading is a shear loading as shown in Fig. 8.34.

As previously the beam is of narrow rectangular cross-section and unit width. Under the shear loading P the bending moment at any cross-section is proportional to $\sin\theta$ and, therefore it is reasonable to assume that the circumferential stress $\sigma_{\theta\theta}$ would also be associated with $\sin\theta$. This points to the case $n = 1$ and a stress function given in eqn. (8.108).

i.e.
$$\phi = (A_1r^3 + B_1/r + C_1r + D_1r\ln r)\sin\theta \tag{8.113}$$

Using eqns. (8.103) the three stresses can be written

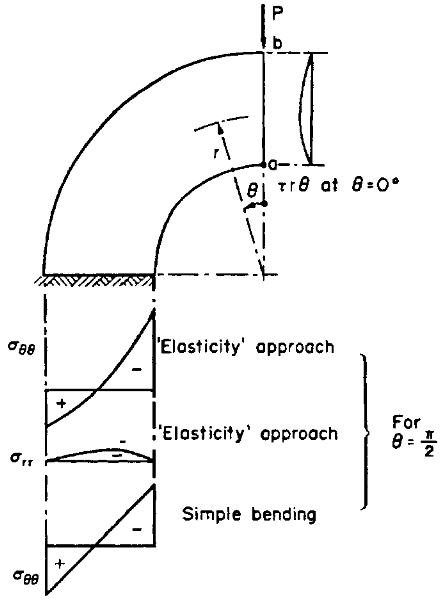

Fig. 8.34. Shear loading of a curved cantilever.

$$\left.\begin{aligned}\sigma_{rr} &= (2A_1 r - 2B_1/r^3 + D_1/r)\sin\theta \\ \sigma_{\theta\theta} &= (6A_1 r + 2B_1/r^3 + D_1/r)\sin\theta \\ \tau_{r\theta} &= -(2A_1 r - 2B_1/r^3 + D_1/r)\cos\theta\end{aligned}\right\} \quad (8.114)$$

The boundary conditions are:

(i) $\sigma_{rr} = \tau_{r\theta} = 0$, for $r = a$ and $r = b$.
(ii) $\int_a^b \tau_{r\theta}\, dr = P$, for equilibrium of vertical forces at $\theta = 0$.

Using these conditions the constants A_1, B_1 and D_1 can be determined. The final stress values are:

$$\left.\begin{aligned}\sigma_{rr} &= \frac{P}{S}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\sin\theta \\ \sigma_{\theta\theta} &= \frac{P}{S}\left(3r - \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\sin\theta \\ \tau_{r\theta} &= -\frac{P}{S}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\cos\theta\end{aligned}\right\} \quad (8.115)$$

where $s = a^2 - b^2 + (a^2 + b^2)\ln b/a$.
It is noted from these equations that at the load point $\theta = 0$,

$$\left.\begin{aligned}\sigma_{rr} &= \sigma_{\theta\theta} = 0 \\ \tau_{r\theta} &= -\frac{P}{S}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\end{aligned}\right\} \quad (8.116)$$

As in the previous cases the load P must be applied to the cantilever according to eqn. (8.116) – see Fig. 8.34.

At the fixed end, $\theta = \frac{\pi}{2}$;

$$\left.\begin{aligned} \sigma_{rr} &= \frac{P}{S}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right) \\ \sigma_{\theta\theta} &= \frac{P}{S}\left(3r - \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right) \\ \tau_{r\theta} &= 0 \end{aligned}\right\} \quad (8.117)$$

The distributions of these stresses are shown in Fig. 8.34. They are similar to that for the pure moment application. The simple bending ($\sigma = My/I$) result is also shown. As in the previous case it is noted that the simple approach underestimates the stresses on the inner fibre.

8.27.8. Case 5–The asymmetric cases $n \geqslant 2$–stress concentration at a circular hole in a tension field

The example chosen to illustrate this category concerns the derivation of the stress concentration due to the presence of a circular hole in a tension field. A large number of stress concentrations arise because of geometric discontinuities–such as holes, notches, fillets, etc., and the derivation of the peak stress values, in these cases, is clearly of importance to the stress analyst and the designer.

The distribution of stress round a small circular hole in a flat plate of unit thickness subject to a uniform tension σ_{xx}, in the x direction was first obtained by Prof. G. Kirsch in 1898.† The width of the plate is considered large compared with the diameter of the hole as shown in Fig. 8.35. Using the Saint-Venant's‡ principle the small central hole will not affect the

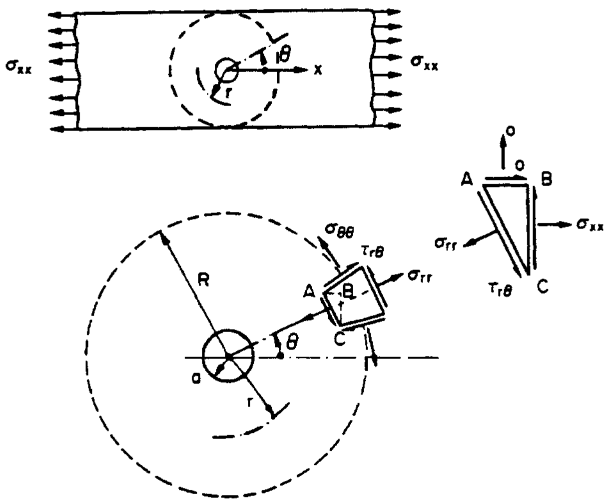

Fig. 8.35. Elements in a stress field some distance from a circular hole.

† G. Kirsch Verein Deutsher Ingenieure (V.D.I.) *Zeitschrift*, **42** (1898), 797–807.
‡ B. de Saint-Venant, *Mem. Acad. Sc. Savants E'trangers*, **14** (1855), 233–250.

stress distribution at distances which are large compared with the diameter of the hole–say the width of the plate. Thus on a circle of large radius R the stress in the x direction, on $\theta = 0$ will be σ_{xx}. Beyond the circle one can expect that the stresses are effectively the same as in the plate without the hole.

Thus at an angle θ, equilibrium of the element ABC, at radius $r = R$, will give

$$\sigma_{rr}.AC = \sigma_{xx}BC\cos\theta, \quad \text{and since, } \cos\theta = BC/AC$$

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta,$$

or

$$\sigma_{rr} = \frac{\sigma_{xx}}{2}(1 + \cos 2\theta).$$

Similarly,

$$\tau_{r\theta}.AC = -\sigma_{xx}BC\sin\theta$$

$$\therefore \tau_{r\theta} = -\sigma_{xx}\cos\theta\sin\theta = -\frac{\sigma_{xx}}{2}\sin 2\theta.$$

Note the sign of $\tau_{r\theta}$ indicates a direction opposite to that shown on Fig. 8.35.

Kirsch noted that the total stress distribution at $r = R$ can be considered in two parts:

(a) a constant radial stress $\sigma_{xx}/2$

(b) a condition varying with 2θ, that is; $\sigma_{rr} = \frac{\sigma_{xx}}{2}\cos 2\theta$, $\tau_{r\theta} = -\frac{\sigma_{xx}}{2}\sin 2\theta$.

The final result is obtained by combining the distributions from (a) and (b). *Part (a), shown in Fig. 8.36*, can be treated using the Lamé equations; The boundary conditions are:

$$\text{at } r = a \quad \sigma_{rr} = 0$$

$$r = R \quad \sigma_{rr} = \sigma_{xx}/2$$

Using these in the Lamé equation, $\sigma_{rr} = A + B/r^2$

gives,

$$A = \frac{\sigma_{xx}}{2}\left(\frac{R^2}{R^2 - a^2}\right) \quad \text{and} \quad B = -\frac{\sigma_{xx}}{2}\left(\frac{R^2a^2}{R^2 - a^2}.\right)$$

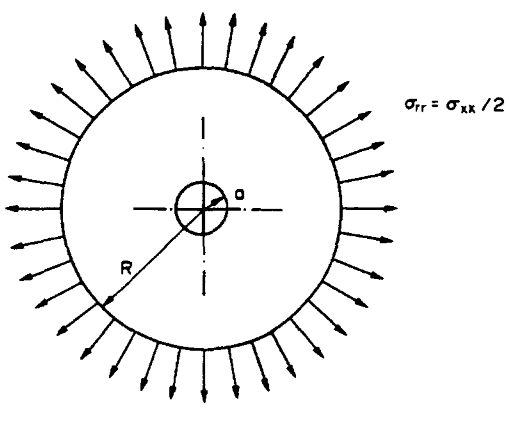

Fig. 8.36. A circular plate loaded at the periphery with a uniform tension.

When $R \gg a$ these can be modified to $A = \frac{\sigma_{xx}}{2}$ and $B = -\frac{\sigma_{xx}}{2}a^2$

Thus
$$\left.\begin{aligned}\sigma_{rr} &= \frac{\sigma_{xx}}{2}\left(1 - \frac{a^2}{r^2}\right)\\ \sigma_{\theta\theta} &= \frac{\sigma_{xx}}{2}\left(1 + \frac{a^2}{r^2}\right)\\ \tau_{r\theta} &= 0\end{aligned}\right\} \quad (8.118)$$

Part (b), shown in Fig 8.37 is a new case with normal stresses varying with $\cos 2\theta$ and shear stresses with $\sin 2\theta$.

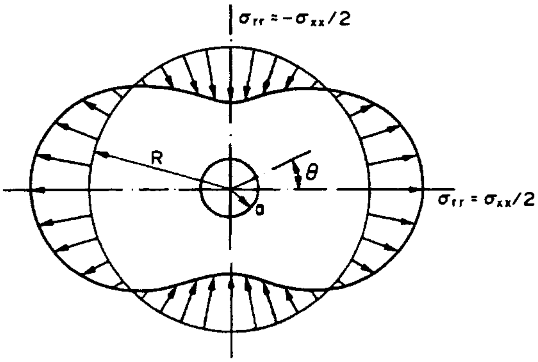

Fig. 8.37. A circular plate loaded at the periphery with a radial stress $= \frac{\sigma_{xx}}{2}\cos 2\theta$ (shown above) and a shear stress $= -\frac{\sigma_{xx}}{2}\sin 2\theta$.

This fits into the category of $n = 2$ with a stress function eqn. (8.109);

i.e.
$$\phi = (A_2 r^2 + B_2/r^2 + C_2 r^4 + D_2)\cos 2\theta \quad (8.119)$$

Using eqns. (8.103) the stresses can be written:

$$\left.\begin{aligned}\sigma_{rr} &= -(2A_2 + 6B_2/r^4 + 4D_2/r^2)\cos 2\theta\\ \sigma_{\theta\theta} &= (2A_2 + 6B_2/r^4 + 12C_2 r^2)\cos 2\theta\\ \tau_{r\theta} &= (2A_2 - 6B_2/r^4 + 6C_2 r^2 - 2D_2/r^2)\sin 2\theta\end{aligned}\right\} \quad (8.120)$$

The four constants are found such that σ_{rr} and $\tau_{r\theta}$ satisfy the boundary conditions:

$$\text{at } r = a, \quad \sigma_{rr} = \tau_{r\theta} = 0$$
$$\text{at } r = R \to \infty, \quad \sigma_{rr} = \frac{\sigma_{xx}}{2}\cos 2\theta, \quad \tau_{r\theta} = -\frac{\sigma_{xx}}{2}\sin 2\theta$$

From these,

$$A_2 = -\sigma_{xx}/4, \quad B_2 = -\sigma_{xx}a^4/4$$
$$C_2 = 0, \quad D_2 = \sigma_{xx}a^2/2$$

Thus:

$$\left.\begin{aligned}\sigma_{rr} &= \frac{\sigma_{xx}}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta \\ \sigma_{\theta\theta} &= -\frac{\sigma_{xx}}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= -\frac{\sigma_{xx}}{2}(1 + 2a^2/r^2 - 3a^4/r^4)\sin 2\theta\end{aligned}\right\} \quad (8.121)$$

The sum of the stresses given by eqns. (8.120) and (8.121) is that proposed by Kirsch. At the edge of the hole σ_{rr} and $\tau_{r\theta}$ should be zero and this can be verified by substituting $r = a$ into these equations.

The distribution of $\sigma_{\theta\theta}$ round the hole, i.e. $r = a$, is obtained by combining eqns. (8.120) and (8.121):

i.e.

$$\sigma_{\theta\theta} = \sigma_{xx}(1 - 2\cos 2\theta) \quad (8.122)$$

and is shown on Fig. 8.38(a).

When $\theta = 0$; $\sigma_{\theta\theta} = -\sigma_{xx}$ and when $\theta = \frac{\pi}{2}$; $\sigma_{\theta\theta} = 3\sigma_{xx}$.

The stress concentration factor (S.C.F) defined as Peak stress/Average stress, gives an S.C.F. = 3 for this case.

The distribution across the plate from point A $\left(\theta = \frac{\pi}{2}\right)$ is:

$$\sigma_{\theta\theta} = \frac{\sigma_{xx}}{2}\left(2 + \frac{a^2}{r^2} + \frac{3a^4}{r^4}\right) \quad (8.123)$$

This is shown in Fig. 8.38(b), which indicates the rapid way in which $\sigma_{\theta\theta}$ approaches σ_{xx} as r increases. Although the solution is based on the fact that $R \gg a$, it can be shown that even when $R = 4a$, that is the width of the plate is four times the diameter of the hole, the error in the S.C.F. is less than 6%.

Using the stress distribution derived for this case it is possible, using superposition, to obtain S.C.F. values for a range of other stress fields where the circular hole is present, see problem No. 8.52 for solution at the end of this chapter.

A similar, though more complicated, analysis can be carried out for an elliptical hole of major diameter $2a$ across the plate and minor diameter $2b$ in the stress direction. In this case the S.C.F. $= 1 + 2a/b$ (see also §8.3). Note that for the circular hole $a = b$, and the S.C.F. = 3, as above.

8.27.9. Other useful solutions of the biharmonic equation

(a) Concentrated line load across a plate

The way in which an elastic medium responds to a concentrated line of force is the final illustrative example to be presented in this section. In practice it is neither possible to apply a genuine line load nor possible for the plate to sustain a load without local plastic deformation. However, despite these local perturbations in the immediate region of the load, the rest of the plate behaves in an elastic manner which can be adequately represented by the governing equations obtained earlier. It is thus possible to use the techniques developed above to analyse the concentrated load problem.

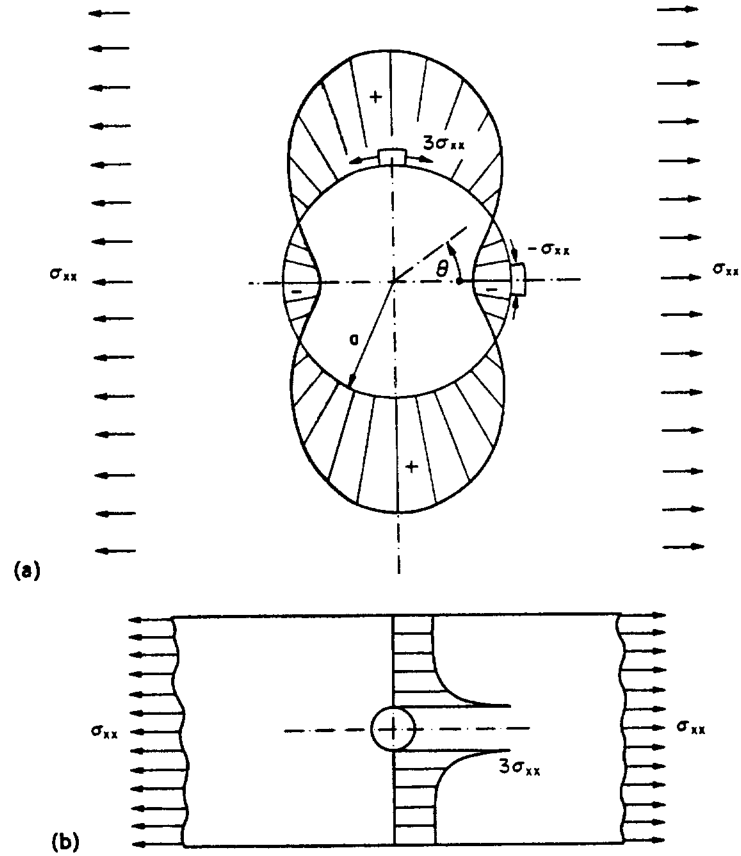

Fig. 8.38. (a) Distribution of circumferential stress $\sigma_{\theta\theta}$ round the hole in a tension field; (b) distribution of circumferential stress $\sigma_{\theta\theta}$ across the plate.

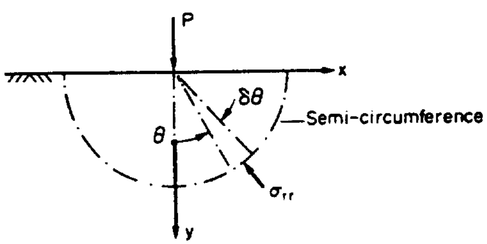

Fig. 8.39. Concentrated load on a semi-infinite plate.

Consider a force P *per unit width* of the plate applied as a line load normal to the surface – see Fig. 8.39. The plate will be considered as equivalent to a semi-infinite solid, that is, one that extends to infinity in the x and y directions below the horizon, $\theta = \pm\frac{\pi}{2}$. The plate is assumed to be of unit width. It is convenient to use cylindrical polars again for this problem.

Using Boussinesq's solutions† for a semi-infinite body, Alfred-Aimé Flamant obtained (in 1892)‡ the stress distribution for the present case. He showed that on any semi-circumference round the load point the stress is entirely radial, that is: $\sigma_{\theta\theta} = \tau_{r\theta} = 0$ and σ_{rr} will be a principal stress. He used a stress function of the type given in eqn. (8.110), namely: $\phi = Cr\theta\sin\theta$ which predicts stresses:

$$\sigma_{rr} = \frac{2C}{r}\cos\theta,\ \sigma_{\theta\theta} = \tau_{r\theta} = 0$$

Applying overall equilibrium to this case it is noted that the resultant vertical force over any semi-circle, of radius r, must equal the applied force P:

$$P = -\int_{-\pi/2}^{\pi/2} (\sigma_{rr} \cdot r\,d\theta)\cos\theta = -\int_{-\pi/2}^{\pi/2} (2C\cos^2\theta)\,d\theta = -C\pi$$

Thus $$\phi = -\frac{Pr\theta}{\pi}\sin\theta$$

and $$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r} \tag{8.124}$$

This can be transformed into x and y coordinates:

$$\left.\begin{aligned} \sigma_{yy} &= \sigma_{rr}\cos^2\theta \\ \sigma_{xx} &= \sigma_{rr}\sin^2\theta \\ \tau_{xy} &= \sigma_{rr}\sin\theta\cos\theta \end{aligned}\right\} \tag{8.125}$$

See also §8.3.3 for further transformation of these equations.

This type of solution can be extended to consider the wedge problem, again subject to a line load as shown in Figs. 8.40(a) and (b).

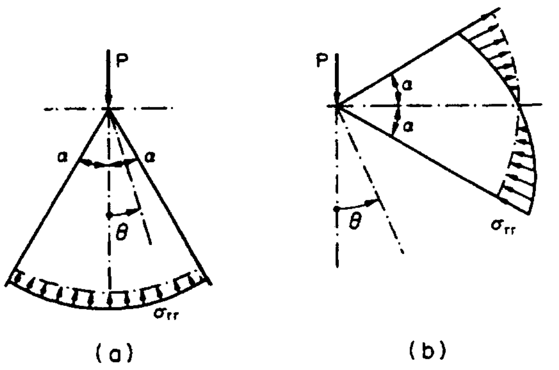

Fig. 8.40. Forces on a wedge.

† J. Boussinesq, *Application de potentiels a l'étude de l'equilibre*! Paris, 1885; also *Comptes Rendus Acad Sci.*, **114** (1892), 1510–1516.

‡ Flamant AA *Comptes Rendus Acad. Sci.*, **114** (1892), 1465–1468.

(b) The wedge subject to an axial load – Figure 8.40(a)

For this case,

$$P = -\int_{-\alpha}^{\alpha} (\sigma_{rr} \cdot r\, d\theta) \cos\theta$$

$$P = -\int_{-\alpha}^{\alpha} 2C \cdot \cos^2\theta\, d\theta$$

$$P = -C(2\alpha + \sin 2\alpha)$$

Thus,

$$\sigma_{rr} = -\frac{2P\cos\theta}{r(2\alpha + \sin 2\alpha)} \tag{8.126}$$

(c) The wedge subject to a normal end load – Figure 8.40(a)

Here,

$$P = -\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} (\sigma_{rr} \cdot r\, d\theta) \cos\theta$$

$$P = -\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} 2C \cdot \cos^2\theta\, d\theta$$

$$P = -C(2\alpha - 2\sin 2\alpha).$$

Thus,

$$\sigma_{rr} = -\frac{2p\cos\theta}{r(2\alpha - \sin 2\alpha)} \tag{8.127}$$

From a combination of these cases any inclination of the load can easily be handled.

(d) Uniformly distributed normal load on part of the surface – Fig. 8.41

The result for σ_{rr} obtained in eqn. (8.124) can be used to examine the case of a uniformly distributed normal load q per unit length over part of a surface-say $\theta = \frac{\pi}{2}$. It is required to find the values of the normal and shear stresses (σ_{xx}, σ_{yy}, τ_{xy}) at the point A situated as indicated in Fig. 8.41. In this case the load is divided into a series of discrete lengths δx over which the load is δP, that is $\delta P = q\delta x$. To make use of eqn. (8.124) we must transform this into polars (r, θ). That is

$$dx = r\,d\theta/\cos\theta. \quad \text{Thus, } dP = q \cdot r\,d\theta/\cos\theta \tag{8.128}$$

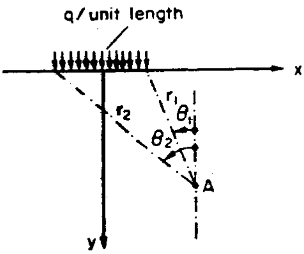

Fig. 8.41. A distributed force on a semi-infinite plate.

Then from eqn. (8.124)

$$d\sigma_{rr} = -\frac{2}{\pi r} dP \cos\theta$$

Substituting eqn. (8.128): $$d\sigma_{rr} = -\frac{2}{\pi r} \cdot q \cdot r d\theta = -\frac{2q}{\pi} d\theta$$

Making use of eqns. (8.125): $$d\sigma_{yy} = -\frac{2q}{\pi} \cos^2\theta \, d\theta$$

$$d\sigma_{xx} = -\frac{2q}{\pi} \sin^2\theta \, d\theta$$

$$d\tau_{xy} = -\frac{2q}{\pi} \sin\theta \cos\theta$$

The total stress values at the point A due to all the discrete loads over θ_1 to θ_2 can then be written,

$$\left.\begin{aligned} \sigma_{yy} &= -\frac{2q}{\pi} \int_{\theta_1}^{\theta_2} \cos^2\theta \, d\theta \\ &= -\frac{q}{2\pi}[2(\theta_2 - \theta_1) + (\sin 2\theta_2 - \sin 2\theta_1)] \\ \sigma_{xx} &= -\frac{q}{2\pi}[2(\theta_2 - \theta_1) - (\sin 2\theta_2 - \sin 2\theta_1)] \\ \tau_{xy} &= -\frac{q}{2\pi}[\cos 2\theta_1 - \cos 2\theta_2] \end{aligned}\right\} \quad (8.129)$$

Closure

The stress function concept described above was developed over 100 years ago. Despite this, however, the ideas contained are still of relevance today in providing a series of classical solutions to otherwise intractable problems, particularly in the study of plates and shells.

Examples

Example 8.1

At a point in a material subjected to a three-dimensional stress system the cartesian stress coordinates are:

$$\sigma_{xx} = 100 \text{ MN/m}^2 \quad \sigma_{yy} = 80 \text{ MN/m}^2 \quad \sigma_{zz} = 150 \text{ MN/m}^2$$

$$\sigma_{xy} = 40 \text{ MN/m}^2 \quad \sigma_{yz} = -30 \text{ MN/m}^2 \quad \sigma_{zx} = 50 \text{ MN/m}^2$$

Determine the normal, shear and resultant stresses on a plane whose normal makes angles of 52° with the X axis and 68° with the Y axis.

Solution

The direction cosines for the plane are as follows:

$$l = \cos 52^\circ = 0.6157$$

$$m = \cos 68^\circ = 0.3746$$

and, since $l^2 + m^2 + n^2 = 1$,

$$n^2 = 1 - (0.6157^2 + 0.3746^2)$$
$$= 1 - (0.3791 + 0.1403) = 0.481$$
$$\therefore \quad n = 0.6935$$

Now from eqns. (8.13–15) the components of the resultant stress on the plane in the X,Y and Z directions are given by

$$p_{xn} = \sigma_{xx}l + \sigma_{xy}m + \sigma_{xz}n$$
$$p_{yn} = \sigma_{yy}m + \sigma_{yx}l + \sigma_{yz}n$$
$$p_{zn} = \sigma_{zz}n + \sigma_{zx}l + \sigma_{zy}m$$
$$p_{xn} = (100 \times 0.6157) + (40 \times 0.3746) + (50 \times 0.6935) = 111.2 \text{ MN/m}^2$$
$$p_{yn} = (80 \times 0.3746) + (40 \times 0.6157) + (-30 \times 0.6935) = 33.8 \text{ MN/m}^2$$
$$p_{zn} = (150 \times 0.6935) + (50 \times 0.6157) + (-30 \times 0.3746) = 123.6 \text{ MN/m}^2$$

Therefore from eqn. (8.4) the resultant stress p_n is given by

$$p_n = \left[p_{xn}^2 + p_{yn}^2 + p_{zn}^2\right]^{1/2} = \left[111.2^2 + 33.8^2 + 123.6^2\right]^{1/2}$$
$$= \mathbf{169.7\ MN/m^2}$$

The normal stress σ_n is given by eqn. (8.5),

$$\sigma_n = p_{xn}l + p_{yn}m + p_{zn}n$$
$$= (111.2 \times 0.6157) + (33.8 \times 0.3746) + (123.6 \times 0.6935)$$
$$= \mathbf{166.8\ MN/m^2}$$

and the shear stress τ_n is found from eqn. (8.6),

$$\tau_n = \sqrt{(p_n^2 - \sigma_n^2)} = (28798 - 27830)^{1/2}$$
$$= \mathbf{31\ MN/m^2}$$

Example 8.2

Show how the equation of equilibrium in the radial direction of a cylindrical coordinate system can be reduced to the form

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0$$

for use in applications involving long cylinders of thin uniform wall thickness.

Hence show that for such a cylinder of internal radius R_0, external radius R and wall thickness T (Fig. 8.42) the radial stress σ_{rr} at any thickness t is given by

$$\sigma_{rr} = -p\frac{R_0}{T}\frac{(T - t)}{(R_0 + t)}$$

where p is the internal pressure, the external pressure being zero.

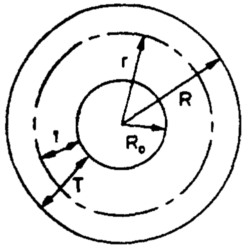

Fig. 8.42.

For thin-walled cylinders the circumferential stress $\sigma_{\theta\theta}$ can be assumed to be independent of radius.

What will be the equivalent expression for the circumferential stress?

Solution

The relevant equation of equilibrium is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_r = 0$$

Now for long cylinders plane strain conditions may be assumed,

i.e. $$\frac{\partial \sigma_{rz}}{\partial z} = 0$$

By symmetry, the stress conditions are independent of θ,

$\therefore$ $$\frac{\partial \sigma_{r\theta}}{\partial \theta} = 0$$

and, in the absence of body forces,

$$F_r = 0$$

Thus the equilibrium equation reduces to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} = 0$$

Since $\sigma_{\theta\theta}$ is independent of r this equation can be conveniently rearranged as follows:

$$\sigma_{rr} + r\frac{\partial \sigma_{rr}}{\partial r} = \sigma_{\theta\theta}$$

$$\frac{\partial}{\partial r}(r\sigma_{rr}) = \sigma_{\theta\theta}$$

Integrating,

$$r\sigma_{rr} = \sigma_{\theta\theta} r + C \quad (1)$$

Now at $r = R$, $\sigma_{rr} = 0$

$\therefore$ substituting in (1), $$0 = R\sigma_{\theta\theta} + C$$

$\therefore$ $$C = -R\sigma_{\theta\theta} \quad (2)$$

Also at $r = R_0$, $\sigma_{rr} = -p$,

$$\therefore \quad -R_0 p = R_0\sigma_{\theta\theta} + C$$
$$= -(R - R_0)\sigma_{\theta\theta}$$
$$\therefore \quad \sigma_{\theta\theta} = \frac{R_0 p}{(R - R_0)} \qquad (3)$$

Substituting in (1),

$$r\sigma_{rr} = \sigma_{\theta\theta} r - R\sigma_{\theta\theta} = -(R - r)\sigma_{\theta\theta}$$
$$\therefore \quad \sigma_{rr} = -\frac{(R - r)}{r} \times \frac{R_0 p}{(R - R_0)}$$
$$\sigma_{rr} = -\frac{(T - t)}{r} p \frac{R_0}{T}$$
$$= -\frac{pR_0}{T}\frac{(T - t)}{(R_0 + t)}$$

and from (3)

$$\sigma_{\theta\theta} = \frac{R_0 p}{(R - R_0)} = \frac{R_0 p}{T}$$

Example 8.3

A three-dimensional complex stress system has principal stress values of 280 MN/m^2, 50 MN/m^2 and $-$ 120 MN/m^2. Determine (a) analytically and (b) graphically:

(i) the limiting value of the maximum shear stress;
(ii) the values of the octahedral normal and shear stresses.

Solution (a): Analytical

(i) The limiting value of the maximum shear stress is the greatest value obtained in any plane of the three-dimensional system. In terms of the principal stresses this is given by

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3)$$
$$= \tfrac{1}{2}[280 - (-120) = \mathbf{200\ MN/m^2}$$

(ii) The octahedral normal stress is given by

$$\sigma_{oct} = \tfrac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3]$$
$$= \tfrac{1}{3}[280 + 50 + (-120)] = \mathbf{70\ MN/m^2}$$

(iii) The octahedral shear stress is

$$\tau_{oct} = \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$
$$= \tfrac{1}{3}\left[(280 - 50)^2 + (50 + 120)^2 + (-120 - 280)^2\right]^{1/2}$$

$$= \tfrac{1}{3}[52900 + 28900 + 160000]^{1/2}$$

$$= \mathbf{163.9\ MN/m^2}$$

Solution (b): Graphical

(i) The graphical solution is obtained by constructing the three-dimensional Mohr's representation of Fig. 8.43. The limiting value of the maximum shear stress is then equal to the radius of the principal circle.

i.e. $$\tau_{max} = \mathbf{200\ MN/m^2}$$

(ii) The direction cosines of the octahedral planes are

$$l = m = n = \frac{1}{\sqrt{3}} = 0.5774$$

i.e. $$\alpha = \beta = \gamma = \cos^{-1} 0.5774 = 54°52'$$

The values of the normal and shear stresses on these planes are then obtained using the procedures of §8.7.

By measurement, $$\sigma_{oct} = \mathbf{70\ MN/m^2}$$

$$\tau_{oct} = \mathbf{164\ MN/m^2}$$

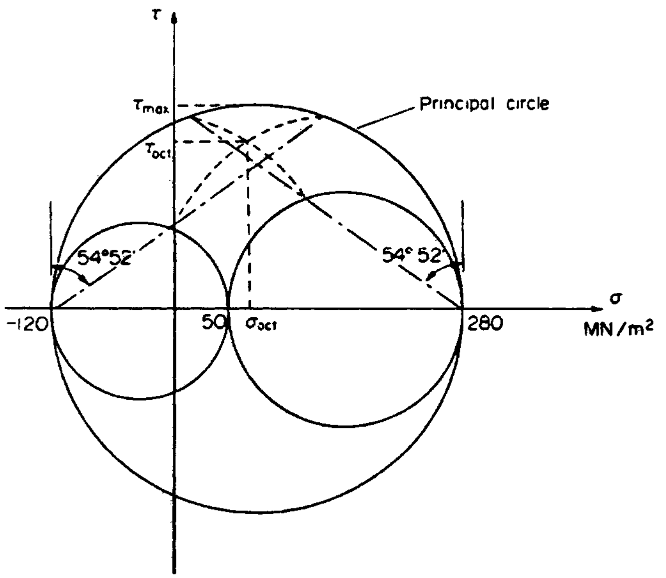

Fig. 8.43.

Example 8.4

A rectangular strain gauge rosette bonded at a point on the surface of an engineering component gave the following readings at peak load during test trials:

$$\varepsilon_0 = 1240 \times 10^{-6}, \quad \varepsilon_{45} = 400 \times 10^{-6}, \quad \varepsilon_{90} = 200 \times 10^{-6}$$

Determine the magnitude and direction of the principal stresses present at the point, and hence construct the full three-dimensional Mohr representations of the stress and strain systems present. $E = 210$ GN/m^2, $\nu = 0.3$.

Solution

The two-dimensional Mohr's strain circle representing strain conditions in the plane of the surface at the point in question is drawn using the procedure of §14.14† (Fig. 8.44).

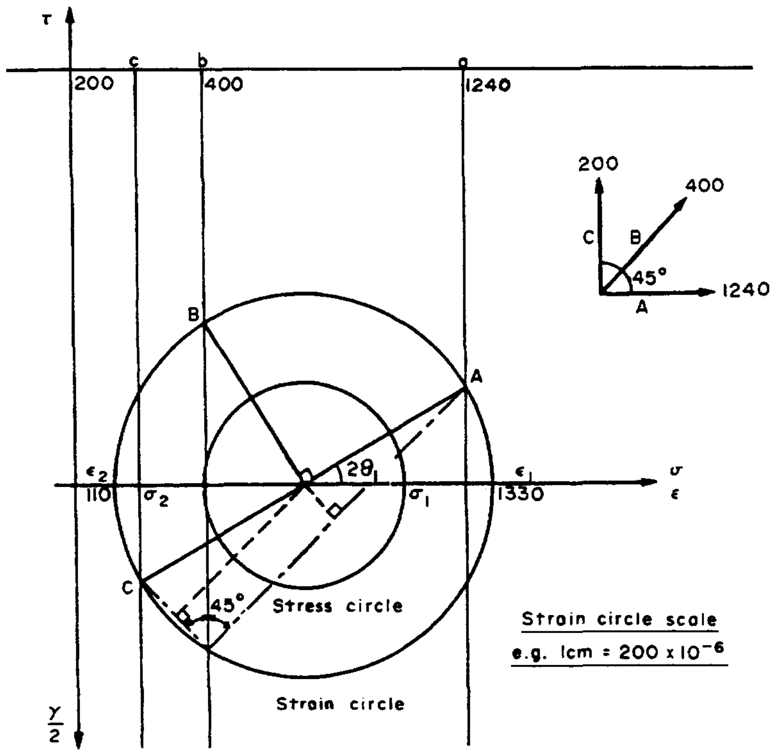

Fig. 8.44.

This establishes the values of the principal strains in the surface plane as 1330 $\mu\varepsilon$ and 110 $\mu\varepsilon$.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

The relevant two-dimensional *stress* circle can then be superimposed as described in §14.13 using the relationships:

$$\text{radius of stress circle} = \frac{(1-\nu)}{(1+\nu)} \times \text{radius of strain circle}$$
$$= \frac{0.7}{1.3} \times 3.05 = 1.64 \text{ cm}$$
$$\text{stress scale} = \frac{E}{(1-\nu)} \times \text{strain scale}$$
$$= \frac{210 \times 10^9}{0.7} \times 200 \times 10^{-6}$$
$$= 60 \text{ MN/m}^2$$

i.e. 1 cm on the stress diagram represents 60 MN/m^2.

The two principal stresses in the plane of the surface are then:

$$\sigma_1(= 5.25 \text{ cm}) = \mathbf{315\ MN/m^2}$$
$$\sigma_2(= 2.0 \text{ cm}) = \mathbf{120\ MN/m^2}$$

The third principal stress, normal to the free (unloaded) surface, is zero,

i.e. $$\sigma_3 = 0$$

The directions of the principal stresses are also obtained from the stress circle. With reference to the 0° gauge direction,

$$\sigma_1 \text{ lies at } \theta_1 = \mathbf{15^\circ\ clockwise}$$
$$\sigma_2 \text{ lies at } (15^\circ + 90^\circ) = \mathbf{105^\circ\ clockwise}$$

with σ_3 **normal to the surface** and hence to the plane of σ_1 and σ_2.

N.B. – These angles are the directions of the principal stresses (and strains) and they do not refer to the directions of the plane on which the stresses act, these being normal to the above directions.

It is now possible to determine the value of the third principal strain, i.e. that normal to the surface. This is given by eqn. (14.2) as

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu\sigma_1 - \nu\sigma_2]$$
$$= \frac{1}{210 \times 10^9}[0 - 0.3(315 + 120)]\, 10^6$$
$$= -621 \times 10^{-6} = \mathbf{-621\ \mu\varepsilon}$$

The complete Mohr's three-dimensional stress and strain representations can now be drawn as shown in Figs. 8.45 and 8.46.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

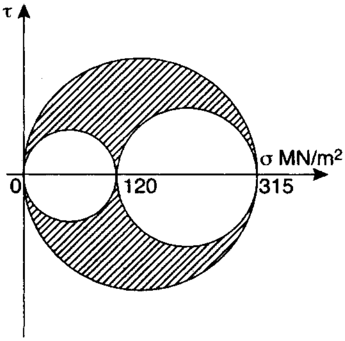

Fig. 8.45. Mohr stress circles.

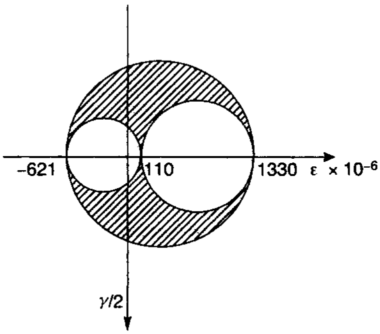

Fig. 8.46. Mohr strain circles.

Problems

8.1 (B). Given that the following strains exist at a point in a three-dimensional system determine the equivalent stresses which act at the point.

Take $E = 206\ \text{GN/m}^2$ and $\nu = 0.3$.

$$\varepsilon_{xx} = 0.0010 \qquad \gamma_{xy} = 0.0002$$
$$\varepsilon_{yy} = 0.0005 \qquad \gamma_{zx} = 0.0008$$
$$\varepsilon_{zz} = 0.0007 \qquad \gamma_{yz} = 0.0010$$

[420, 340, 372, 15.8, 63.4, 79.2 MN/m^2.]

8.2 (B). The following cartesian stresses act at a point in a body subjected to a complex loading system. If $E = 206\ \text{GN/m}^2$ and $\nu = 0.3$, determine the equivalent strains present.

$$\sigma_{xx} = 225\ \text{MN/m}^2 \qquad \sigma_{yy} = 75\ \text{MN/m}^2 \qquad \sigma_{zz} = 150\ \text{MN/m}^2$$
$$\tau_{xy} = 110\ \text{MN/m}^2 \qquad \tau_{yz} = 50\ \text{MN/m}^2 \qquad \tau_{zx} = 70\ \text{MN/m}^2$$

[764.6, 182, 291, 1388, 631, 883.5, all $\times 10^{-6}$.]

8.3 (B). Does a uniaxial stress field produce a uniaxial strain condition? Repeat Problem 8.2 for the following stress field:

$$\sigma_{xx} = 225\ \text{MN/m}^2$$
$$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

[No; 1092, -327.7, -327.7, 0, 0, 0, all $\times 10^{-6}$.]

8.4 (C). The state of stress at a point in a body is given by the following equations:

$$\sigma_{xx} = ax + by^2 + cz^3 \qquad \tau_{xy} = l + mz$$
$$\sigma_{yy} = dx + ey^2 + fz^3 \qquad \tau_{yz} = ny + pz$$
$$\sigma_{zz} = gx + hy^2 + kz^3 \qquad \tau_{zx} = qx^2 + sz^2$$

If equilibrium is to be achieved what equations must the body-force stresses X, Y and Z satisfy?

[$-(a + 2sz)$; $-(p + 2ey)$; $-(n + 2qx + 3kz^2)$.]

8.5 (C). At a point the state of stress may be represented in standard form by the following:

$(3x^2 + 3y^2 - z)$	$(z - 6xy - \frac{3}{4})$	$(x + y - \frac{3}{2})$
$(z - 6xy - \frac{3}{4})$	$3y^2$	0
$(x + y - \frac{3}{2}$	0	$(3x + y - z + \frac{5}{4})$

Show that, if body forces are neglected, equilibrium exists.

8.6 (C). The plane stress distribution in a flat plate of unit thickness is given by:

$$\sigma_{xx} = yx^3 - 2axy + by$$
$$\sigma_{yy} = xy^3 - 2x^3y$$
$$\sigma_{xy} = -\tfrac{3}{2}x^2y^2 + ay^2 + \frac{x^4}{2} + c$$

Show that, in the absence of body forces, equilibrium exists. The load on the plate is specified by the following boundary conditions:

$$\text{At } x = \pm\frac{w}{2}, \quad \sigma_{xy} = 0$$
$$\text{At } x = -\frac{w}{2}, \quad \sigma_{xx} = 0$$

where w is the width of the plate.

If the length of the plate is L, determine the values of the constants a,b and c and determine the total load on the edge of the plate, $x = w/2$.

[B.P.] $\left[\frac{3w^2}{8}, -\frac{w^3}{4}, -\frac{w^4}{32}, -\frac{w^3L^2}{4}.\right]$

8.7 (C). Derive the stress equations of equilibrium in cylindrical coordinates and show how these may be simplified for plane strain conditions.

A long, thin-walled cylinder of inside radius R and wall thickness T is subjected to an internal pressure p. Show that, if the hoop stresses are assumed independent of radius, the radial stress at any thickness t is given by

$$\sigma_{rr} = \frac{pR}{(R+t)}\left[\frac{t}{T} - 1\right]$$

8.8 (B). Prove that the following relationship exists between the direction cosines:

$$l^2 + m^2 + n^2 = 1$$

8.9 (C). The six cartesian stress components are given at a point P for three different loading cases as follows (all MN/m^2):

	Case 1	Case 2	Case 3
σ_{xx}	100	100	100
σ_{yy}	200	200	−200
σ_{zz}	300	100	100
τ_{xy}	0	300	200
τ_{yz}	0	100	300
τ_{zx}	0	200	300

Determine for each case the resultant stress at P on a plane through P whose normal is coincident with the X axis. [100, 374, 374 MN/m^2.]

8.10 (C). At a point in a material the stresses are:

$$\sigma_{xx} = 37.2 \text{ MN/m}^2 \qquad \sigma_{yy} = 78.4 \text{ MN/m}^2 \qquad \sigma_{zz} = 149 \text{ MN/m}^2$$

$$\sigma_{xy} = 68.0 \text{ MN/m}^2 \qquad \sigma_{yz} = -18.1 \text{ MN/m}^2 \qquad \sigma_{zx} = 32 \text{ MN/m}^2$$

Calculate the shear stress on a plane whose normal makes an angle of 48° with the X axis and 71° with the Y axis. [41.3 MN/m^2.]

8.11 (C). At a point in a stressed material the cartesian stress components are:

$$\sigma_{xx} = -40 \text{ MN/m}^2 \qquad \sigma_{yy} = 80 \text{ MN/m}^2 \qquad \sigma_{zz} = 120 \text{ MN/m}^2$$

$$\sigma_{xy} = 72 \text{ MN/m}^2 \qquad \sigma_{yz} = 46 \text{ MN/m}^2 \qquad \sigma_{zx} = 32 \text{ MN/m}^2$$

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 48° with the X axis and 61° with the Y axis. [135, 86.6, 161 MN/m^2.]

8.12 (C). Commencing from the equations defining the state of stress at a point, derive the general stress relationship for the normal stress on an inclined plane:

$$\sigma_n = \sigma_{xx}l^2 + \sigma_{zz}n^2 + \sigma_{yy}m^2 + 2\sigma_{xy}lm + 2\sigma_{yz}mn + 2\sigma_{zx}ln$$

Show that this relationship reduces for the plane stress system ($\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$) to the well-known equation

$$\sigma_n = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta$$

where $\cos\theta = l$.

8.13 (C). At a point in a material a resultant stress of value 14 MN/m^2 is acting in a direction making angles of 43°, 75° and 50°53′ with the coordinate axes X, Y and Z.

(a) Find the normal and shear stresses on an oblique plane whose normal makes angles of 67°13′, 30° and 71°34′, respectively, with the same coordinate axes.

(b) If $\sigma_{xy} = 1.5$ MN/m^2, $\sigma_{yz} = -0.2$ MN/m^2 and $\sigma_{xz} = 3.7$ MN/m^2 determine σ_{xx}, σ_{yy} and σ_{zz}.

[10, 9.8, 19.9, 3.58, 23.5 MN/m^2.]

8.14 (C). Three principal stresses of 250, 100 and −150 MN/m^2 act in a direction X, Y and Z respectively. Determine the normal, shear and resultant stresses which act on a plane whose normal is inclined at 30° to the Z axis, the projection of the normal on the XY plane being inclined at 55° to the XZ plane.

[−75.2, 134.5, 154.1 MN/m^2.]

8.15 (C). The following cartesian stress components exist at a point in a body subjected to a three-dimensional complex stress system:

$$\sigma_{xx} = 97 \text{ MN/m}^2 \qquad \sigma_{yy} = 143 \text{ MN/m}^2 \qquad \sigma_{zz} = 173 \text{ MN/m}^2$$

$$\sigma_{xy} = 0 \qquad \sigma_{yz} = 0 \qquad \sigma_{zx} = 102 \text{ MN/m}^2$$

Determine the values of the principal stresses present at the point. [233.8, 143.2, 35.8 MN/m^2.]

8.16 (C). A certain stress system has principal stresses of 300 MN/m^2, 124 MN/m^2 and 56 MN/m^2.

(a) What will be the value of the maximum shear stress?

(b) Determine the values of the shear and normal stresses on the octahedral planes.

(c) If the yield stress of the material in simple tension is 240 MN/m^2, will the above stress system produce failure according to the distortion energy and maximum shear stress criteria?

[122 MN/m^2; 104, 160 MN/m^2; No, Yes.]

8.17 (C). A pressure vessel is being tested at an internal pressure of 150 atmospheres (1 atmosphere = 1.013 bar). Strains are measured at a point on the inside surface adjacent to a branch connection by means of an equiangular strain rosette. The readings obtained are:

$$\varepsilon_0 = 0.23\% \qquad \varepsilon_{+120} = 0.145\% \qquad \varepsilon_{-120} = 0.103\%$$

Draw Mohr's circle to determine the magnitude and direction of the principal strains. $E = 208\ \text{GN/m}^2$ and $\nu = 0.3$.
Determine also the octahedral normal and shear strains at the point.

[0.235%, 0.083%, −0.142%, 9°28′; $\varepsilon_{oct} = 0.0589\%$, $\gamma_{oct} = 0.310\%$.]

8.18 (C). At a point in a stressed body the principal stresses in the X, Y and Z directions are:

$$\sigma_1 = 49\ \text{MN/m}^2 \qquad \sigma_2 = 27.5\ \text{MN/m}^2 \qquad \sigma_3 = -6.3\ \text{MN/m}^2$$

Calculate the resultant stress on a plane whose normal has direction cosines $l = 0.73$, $m = 0.46$, $n = 0.506$. Draw Mohr's stress plane for the problem to check your answer. [38 MN/m^2.]

8.19 (C). For the data of Problem 8.18 determine graphically, and by calculation, the values of the normal and shear stresses on the given plane.

Determine also the values of the octahedral direct and shear stresses. [30.3, 23 MN/m^2; 23.4, 22.7 MN/m^2.]

8.20 (C). During tests on a welded pipe-tee, internal pressure and torque are applied and the resulting distortion at a point near the branch gives rise to shear components in the r, θ and z directions.

A rectangular strain gauge rosette mounted at the point in question yields the following strain values for an internal pressure of 16.7 MN/m^2:

$$\varepsilon_0 = 0.0013 \qquad \varepsilon_{45} = 0.00058 \qquad \varepsilon_{90} = 0.00187$$

Use the Mohr diagrams for stress and strain to determine the state of stress on the octahedral plane. $E = 208\ \text{GN/m}^2$ and $\nu = 0.29$.

What is the direct stress component on planes normal to the direction of zero extension?

[$\sigma_{oct} = 310\ \text{MN/m}^2$; $\tau_{oct} = 259\ \text{MN/m}^2$; 530 MN/m^2.]

8.21 (C). During service loading tests on a nuclear pressure vessel the distortions resulting near a stress concentration on the inside surface of the vessel give rise to shear components in the r, θ and z directions. A rectangular strain gauge rosette mounted at the point in question gives the following strain values for an internal pressure of 5 MN/m^2.

$$\varepsilon_0 = 150 \times 10^{-6},\ \varepsilon_{45} = 220 \times 10^{-6} \text{ and } \varepsilon_{90} = 60 \times 10^{-6}$$

Use the Mohr diagrams for stress and strain to determine the principal stresses and the state of stress on the octahedral plane at the point. For the material of the pressure vessel $E = 210\ \text{GN/m}^2$ and $\nu = 0.3$.

[B.P.] [52.5, 13.8, −5 MN/m^2; $\sigma_{oct} = 21\ \text{MN/m}^2$, $\tau_{oct} = 24\ \text{MN/m}^2$.]

8.22 (C). From the construction of the Mohr strain plane show that the ordinate $\frac{1}{2}\gamma$ for the case of $\alpha = \beta = \gamma$ (octahedral shear strain) is

$$\tfrac{1}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2]^{1/2}$$

8.23 (C). A stress system has three principal values:

$$\sigma_1 = 154\ \text{MN/m}^2 \qquad \sigma_2 = 113\ \text{MN/m}^2 \qquad \sigma_3 = 68\ \text{MN/m}^2$$

(a) Find the normal and shear stresses on a plane with direction cosines of $l = 0.732$, $m = 0.521$ with respect to the σ_1 and σ_2 directions.
(b) Determine the octahedral shear and normal stresses for this system. Check numerically.

[126, 33.4 MN/m^2; 112, 35.1 MN/m^2.]

8.24 (C). A plane has a normal stress of 63 MN/m^2 inclined at an angle of 38° to the greatest principal stress which is 126 MN/m^2. The shear stress on the plane is 92 MN/m^2 and a second principal stress is 53 MN/m^2. Find the value of the third principal stress and the angle of the normal of the plane to the direction of stress.

[−95 MN/m^2; 60°.]

8.25 (C). The normal stress σ_n on a plane has a direction cosine l and the shear stress on the plane is τ_n. If the two smaller principal stresses are equal show that

$$\sigma_1 = \sigma_n + \frac{\tau_n}{l}\sqrt{(1 - l^2)} \quad \text{and} \quad \sigma_2 = \sigma_3 = \sigma_n - \tau_n \frac{1}{\sqrt{(1 - l^2)}}$$

If $\tau_n = 75\ \text{MN/m}^2$, $\sigma_n = 36\ \text{MN/m}^2$ and $l = 0.75$, determine, graphically σ_1 and σ_2. [102, −48 MN/m^2.]

8.26 (C). If the strains at a point are $\varepsilon = 0.0063$ and $\gamma = 0.00481$, determine the value of the maximum principal strain ε_1 if it is known that the strain components make the following angles with the three principal strain directions:

For ε:	$\alpha = 38.5°$	$\beta = 56°$	γ = positive	
For γ:	$\alpha' = 128°32'$	$\beta' = 45°10'$	γ' = positive	[0.0075.]

8.27 (C). What is meant by the term deviatoric strain as related to a state of strain in three dimensions? Show that the sum of three deviatoric strains ε_1', ε_2' and ε_3' is zero and also that they can be related to the principal strains ε_1, ε_2 and ε_3 as follows:

$$\varepsilon_1'^2 + \varepsilon_2'^2 + \varepsilon_3'^2 = \tfrac{1}{3}[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_1)^2 + (\varepsilon_3 - \varepsilon_1)^2] \qquad \text{[C.E.I.]}$$

8.28 (C). The readings from a rectangular strain gauge rosette bonded to the surface of a strained component are as follows:

$$\varepsilon_0 = 592 \times 10^{-6} \qquad \varepsilon_{45} = 308 \times 10^{-6} \qquad \varepsilon_{90} = -432 \times 10^{-6}$$

Draw the full three-dimensional Mohr's stress and strain circle representations and hence determine:
(a) the principal strains and their directions;
(b) the principal stresses;
(c) the maximum shear stress.

Take $E = 200$ GN/m^2 and $\nu = 0.3$.

[640×10^{-6}, -480×10^{-6}; at 12° and 102° to A, 109, −63.5, 86.25 MN/m^2]

8.29 (C). For a rectangular beam, unit width and depth $2d$, simple beam theory gives the longitudinal stress $\sigma_{xx} = CMy/I$ where

y = ordinate in depth direction (+ downwards)

M = BM in yx plane (+ sagging)

The shear force is Q and the shear stress τ_{xy} is to be taken as zero at top and bottom of the beam.

$\sigma_{yy} = 0$ at the bottom and $\sigma_{yy} = -w$/unit length, i.e. a distributed load, at the top.

$$\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$$

Using the equations of equilibrium in cartesian coordinates and without recourse to beam theory, find the distribution of σ_{yy} and σ_{xy}.

$$\text{[U.L.]} \left[\sigma_{yy} = \frac{w}{2I}\left(d^2 y - \frac{y^3}{3} - \frac{2d^3}{3}\right), \quad \sigma_{xy} = -\frac{Q}{2I}(d^2 - y^2).\right]$$

8.30 (C). Determine whether the following strain fields are compatible:

(a)
$\varepsilon_{xx} = 2x^2 + 3y^2 + z + 1$
$\varepsilon_{yy} = 2y^2 + x^2 + 3z + 2$
$\varepsilon_{zz} = 3x + 2y + z^2 + 1$
$\gamma_{xy} = 8xy$
$\gamma_{yz} = 0$
$\gamma_{zx} = 0$
[Yes]

(b)
$\varepsilon_{xx} = 3y^2 + xy$
$\varepsilon_{yy} = 2y + 4z + 3$
$\varepsilon_{zz} = 3zx + 2xy + 3yz + 2$
$\gamma_{xy} = 6xy$
$\gamma_{yz} = 2x$
$\gamma_{zx} = 2y$
[No]

8.31 (C). The normal stress σ_n on a plane has a direction cosine l and the shear stress on the plane is τ. If the two smaller principal stresses are equal show that

$$\sigma_1 = \sigma_n + \frac{\tau}{l}\sqrt{(1 - l^2)} \quad \text{and} \quad \sigma_2 = \sigma_3 = \sigma_n - \frac{\tau l}{\sqrt{(1 - l^2)}}$$

8.32 (C). (i) A long thin-walled cylinder of internal radius R_0, external radius R and wall thickness T is subjected to an internal pressure p, the external pressure being zero. Show that if the circumferential stress ($\sigma_{\theta\theta}$) is independent of the radius r then the radial stress (σ_{rr}) at any thickness t is given by

$$\sigma_{rr} = -p\frac{R_0}{T}\frac{(T - t)}{(R_0 + t)}$$

The relevant equation of equilibrium which may be used is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + F_r = 0$$

(ii) Hence determine an expression for $\sigma_{\theta\theta}$ in terms of T.
(iii) What difference in approach would you adopt for a similar treatment in the case of a thick-walled cylinder?
[B.P.] [$R_0 p/T$.]

8.33 (C). Explain what is meant by the following terms and discuss their significance:
(a) Octahedral planes and stresses.
(b) Hydrostatic and deviatoric stresses.
(c) Plastic limit design.
(d) Compatibility.
(e) Principal and product second moments of area. [B.P.]

8.34 (C). At a point in a stressed material the cartesian stress components are:

$$\sigma_{xx} = -40 \text{ MN/m}^2 \qquad \sigma_{yy} = 80 \text{ MN/m}^2 \qquad \sigma_{zz} = 120 \text{ MN/m}^2$$

$$\sigma_{xy} = 72 \text{ MN/m}^2 \qquad \sigma_{xz} = 32 \text{ MN/m}^2 \qquad \sigma_{yz} = 46 \text{ MN/m}^2$$

Calculate the normal, shear and resultant stresses on a plane whose normal makes an angle of 48° with the X axis and 61° with the Y axis. [B.P.] [135.3, 86.6, 161 MN/m^2.]

8.35 (C). The Cartesian stress components at a point in a three-dimensional stress system are those given in problem 8.33 above.
(a) What will be the directions of the normal and shear stresses on the plane making angles of 48° and 61° with the X and Y axes respectively?
[$l'm'n'$ = 0.1625, 0.7010, 0.6914; $l_s m_s n_s$ = −0.7375, 0.5451, 0.4053]
(b) What will be the magnitude of the shear stress on the octahedral planes where $l = m = n = 1/\sqrt{3}$?
[10.7 MN/m^2]

8.36 (C). Given that the cartesian stress components at a point in a three-dimensional stress system are:

$$\sigma_{xx} = 20 \text{ MN/m}^2, \qquad \sigma_{yy} = 5 \text{ MN/m}^2, \qquad \sigma_{zz} = -50 \text{ MN/m}^2$$

$$\tau_{xy} = 0, \qquad \tau_{yz} = 20 \text{ MN/m}^2, \qquad \tau_{zx} = -40 \text{ MN/m}^2$$

(a) Determine the stresses on planes with direction cosines 0.8165, 0.4082 and 0.4082 relative to the X, Y and Z axes respectively. [−14.2, 46.1, 43.8 MN/m^2]
(b) Determine the shear stress on these planes in a direction with direction cosines of 0, −0.707, 0.707.
[39 MN/m^2]

8.37 (C). In a finite element calculation of the stresses in a steel component, the stresses have been determined as follows, with respect to the reference directions X, Y and Z:

$$\sigma_{xx} = 10.9 \text{ MN/m}^2 \qquad \sigma_{yy} = 51.9 \text{ MN/m}^2 \qquad \sigma_{zz} = -27.8 \text{ MN/m}^2$$

$$\tau_{xy} = -41.3 \text{ MN/m}^2 \qquad \tau_{yz} = -8.9 \text{ MN/m}^2 \qquad \tau_{zx} = 38.5 \text{ MN/m}^2$$

It is proposed to change the material from steel to unidirectional glass-fibre reinforced polyester, and it is important that the direction of the fibres is the same as that of the maximum principal stress, so that the tensile stresses perpendicular to the fibres are kept to a minimum.

Determine the values of the three principal stresses, given that the value of the intermediate principal stress is 3.9 MN/m^2. [−53.8; 3.9; 84.9 MN/m^2]

Compare them with the safe design tensile stresses for the glass-reinforced polyester of: parallel to the fibres, 90 MN/m^2; perpendicular to the fibres, 10 MN/m^2.

Then take the direction cosines of the *major* principal stress as $l = 0.569$, $m = -0.781$, $n = 0.256$ and determine the maximum allowable misalignment of the fibres to avoid the risk of exceeding the safe design tensile stresses. (Hint: compression stresses can be ignored.) [15.9°]

8.38 (C). The stresses at a point in an isotropic material are:

$$\sigma_{xx} = 10 \text{ MN/m}^2 \qquad \sigma_{yy} = 25 \text{ MN/m}^2 \qquad \sigma_{zz} = 50 \text{ MN/m}^2$$

$$\tau_{xy} = 15 \text{ MN/m}^2 \qquad \tau_{yz} = 10 \text{ MN/m}^2 \qquad \tau_{zx} = 20 \text{ MN/m}^2$$

Determine the magnitudes of the maximum principal normal strain and the maximum principal shear strain at this point, if Young's modulus is 207 GN/m^2 and Poisson's ratio is 0.3. [280$\mu\varepsilon$; 419$\mu\varepsilon$]

8.39 (C). Determine the principal stresses in a three-dimensional stress system in which:

$$\sigma_{xx} = 40 \text{ MN/m}^2 \quad \sigma_{yy} = 60 \text{ MN/m}^2 \quad \sigma_{zz} = 50 \text{ MN/m}^2$$
$$\sigma_{xz} = 30 \text{ MN/m}^2 \quad \sigma_{xy} = 20 \text{ MN/m}^2 \quad \sigma_{yz} = 10 \text{ MN/m}^2$$

[90 MN/m^2, 47.3 MN/m^2, 12.7 MN/m^2]

8.40 (C). If the stress tensor for a three-dimensional stress system is as given below and one of the principal stresses has a value of 40 MN/m^2 determine the values of the three eigen vectors.

$$\begin{bmatrix} 30 & 10 & 10 \\ 10 & 0 & 20 \\ 10 & 20 & 0 \end{bmatrix}$$

[0.816, 0.408, 0.408]

8.41 (C). Determine the values of the stress invariants and the principal stresses for the cartesian stress components given in Problem 8.2. [450; 423.75; 556.25; 324.8; 109.5; 15.6 MN/m^2]

8.42 (C). The stress tensor for a three-dimensional stress system is given below. Determine the magnitudes of the three principal stresses and determine the eigen vectors of the major principal stress.

$$\begin{bmatrix} 80 & 15 & 10 \\ 15 & 0 & 25 \\ 10 & 25 & 0 \end{bmatrix}$$

[85.3, 19.8, −25.1 MN/m^2, 0.9592, 0.2206, 0.1771.]

8.43 (C). A hollow steel shaft is subjected to combined torque and internal pressure of unknown magnitudes. In order to assess the strength of the shaft under service conditions a rectangular strain gauge rosette is mounted on the outside surface of the shaft, the centre gauge being aligned with the shaft axis. The strain gauge readings recorded from this gauge are shown in Fig. 8.47.

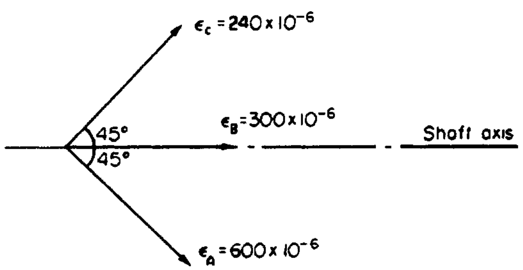

Fig. 8.47.

If E for the steel = 207 GN/m^2 and $\nu = 0.3$, determine:

(a) the principal strains and their directions;

(b) the principal stresses.

Draw complete Mohr's circle representations of the stress and strain systems present and hence determine the maximum shear stresses and maximum shear strain.

[636×10^{-6} at 16.8° to A, -204×10^{-6} at 106.8° to A, -360×10^{-6} perp. to plane; 159, −90, 0 MN/m^2; 79.5 MN/m^2, 996×10^{-6}.]

8.44 (C). At a certain point in a material a resultant stress of 40 MN/m^2 acts in a direction making angles of 45°, 70° and 60° with the coordinate axes X, Y and Z. Determine the values of the normal and shear stresses on an oblique plane through the point given that the normal to the plane makes angles of 80°, 54° and 38° with the same coordinate axes.

If $\sigma_{xy} = 25$ MN/m^2, $\sigma_{xz} = 18$ MN/m^2 and $\sigma_{yz} = -10$ MN/m^2, determine the values of σ_{xx}, σ_{yy} and σ_{zz} which act at the point. [28.75, 27.7 MN/m^2; −3.5, 29.4, 28.9 MN/m^2.]

8.45 (C). The plane stress distribution in a flat plate of unit thickness is given by

$$\sigma_{xx} = x^3 y - 2y^3 x$$

$$\sigma_{yy} = y^3 x - 2pxy + qx$$

$$\sigma_{xy} = \frac{y^4}{2} - \frac{3}{2}x^2 y^2 + px^2 + s$$

If body forces are neglected, show that equilibrium exists.

The dimensions of the plate are given in Fig. 8.48 and the following boundary conditions apply:

$$\text{at } y = \pm\frac{b}{2} \qquad \sigma_{xy} = 0$$

and

$$\text{at } y = -\frac{b}{2} \qquad \sigma_{yy} = 0$$

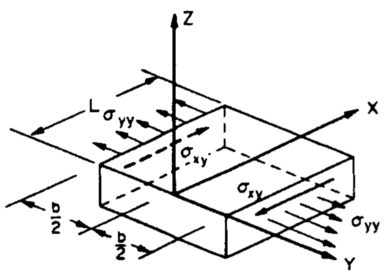

Fig. 8.48.

Determine:

(a) the values of the constants p, q and s;

(b) the total load on the edge $y = \pm b/2$. [B.P.] $\left[\frac{3b^2}{8}, \frac{-b^3}{4}, \frac{-b^4}{32}, \frac{b^3L^2}{4}.\right]$

8.46 (C). Derive the differential equation in cylindrical coordinates for radial equilibrium without body force of an element of a cylinder subjected to stresses σ_r, σ_θ.

A steel tube has an internal diameter of 25 mm and an external diameter of 50 mm. Another tube, of the same steel, is to be shrunk over the outside of the first so that the shrinkage stresses just produce a condition of yield at the inner surfaces of each tube. Determine the necessary difference in diameters of the mating surfaces before shrinking and the required external diameter of the outer tube. Assume that yielding occurs according to the maximum shear stress criterion and that no axial stresses are set up due to shrinking. The yield stress in simple tension or compression = 420 MN/m^2 and E = 208 GN/m^2. [C.E.I.] [0.126 mm, 100 mm.]

8.47 (C). For a particular plane strain problem the strain displacement equations in cylindrical coordinates are:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \varepsilon_z = \gamma_{r\theta} = \gamma_{\theta z} = \gamma_{zr} = 0$$

Show that the appropriate compatibility equation in terms of stresses is

$$\nu r\frac{\partial \sigma_r}{\partial r} - (1-\nu)r\frac{\partial \sigma_\theta}{\partial r} + \sigma_r - \sigma_\theta = 0$$

where ν is Poisson's ratio.

State the nature of a problem that the above equations can represent. [C.E.I.]

8.48 (C). A bar length L, depth d, thickness t is simply supported and loaded at each end by a couple C as shown in Fig. 8.49. Show that the stress function $\phi = Ay^3$ adequately represents this problem. Determine the value of the coefficient A in terms of the given symbols. [$A = 2C/td^3$]

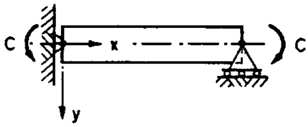

Fig. 8.49.

8.49 (C). A cantilever of unit width and depth $2d$ is loaded with forces at its free end as shown in Fig. 8.50. The stress function which satisfies the loading is found to be of the form:

$$\phi = ay^2 + by^3 + cxy^3 + exy$$

where the coordinates are as shown.

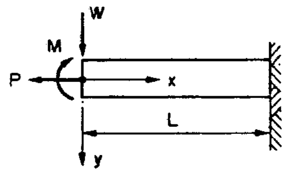

Fig. 8.50.

Determine the value of the constants a, b, c and e and hence show that the stresses are:

$$\sigma_{xx} = P/2d + 3My/2d^3 - 3Wxy/2d^3,$$

$$\sigma_{yy} = 0$$

$$\tau_{xy} = 3Wy^2/4d^3 - 3W/4d.$$

8.50 (C). A cantilever of unit width length L and depth $2a$ is loaded by a linearly distributed load as shown in Fig. 8.51, such that the load at distance x is qx per unit length. Proceeding from the sixth order polynomial derive the 25 constants using the boundary conditions, overall equilibrium and the biharmonic equation. Show that the stresses are:

$$\sigma_{xx} = \frac{qx^3y}{4a^3} + \frac{q}{4a^3}\left(-2xy^3 + \frac{6}{5}a^2xy\right)$$

$$\sigma_{yy} = -q\frac{x}{2} + qx\left(\frac{y^3}{4a^3} - \frac{3y}{4a}\right)$$

$$\tau_{xy} = \frac{3qx^2}{8a^3}(a^2 - y^2) - \frac{q}{8a^3}(a^4 - y^4) + \frac{3q}{20a}(a^2 - y^2)$$

Examine the state of stress at the free end ($x = 0$) and comment on the discrepancy of the shear stress. Compare the shear stress obtained from elementary theory, for $L/2a = 10$, with the more rigorous approach with the additional terms.

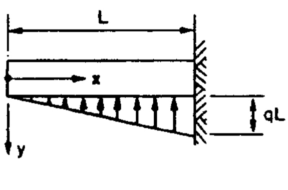

Fig. 8.51.

8.51 (C). Determine if the expression $\phi = (\cos^3\theta)/r$ is a permissible Airy stress function, that is, make sure it satisfies the biharmonic equation. Determine the radial and shear stresses (σ_{rr} and $\tau_{r\theta}$) and sketch these on the periphery of a circle of radius a.

$$\left[\sigma_{rr} = \frac{2}{r^3}\cos\theta(3 - 5\cos^2\theta),\ \tau_{r\theta} = -\frac{6}{r^3}\cos^2\theta\sin\theta.\right]$$

8.52 (C). The stress concentration factor due to a small circular hole in a flat plate subject to tension (or compression) in one direction is three. By superposition of the Kirsch solutions determine the stress concentration factors due to a hole in a flat plate subject to (a) pure shear, (b) two-dimensional hydrostatic tension. Show that the same result for case (b) can be obtained by considering the Lamé solution for a thick cylinder under external tension when the outside radius tends to infinity. [(a) 4; (b) 2.]

8.53 (C). Show that $\phi - Cr^2(\alpha - \theta + \sin\theta\cos\theta - \tan\alpha\cos^2\theta)$ is a permissible Airy stress function and derive expressions for the corresponding stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$.

These expressions may be used to solve the problem of a tapered cantilever beam of thickness carrying a uniformly distributed load q/unit length as shown in Fig. 8.52.

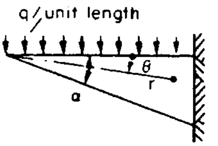

Fig. 8.52.

Show that the derived stresses satisfy every boundary condition along the edges $\theta = 0°$ and $\theta = \alpha$. Obtain a value for the constant C in terms of q and α and thus show that:

$$\sigma_{rr} = \frac{qr}{t(\tan\alpha - \alpha)} \quad \text{when } \theta = 0°$$

Compare this value with the longitudinal bending stress at $\theta = 0°$ obtained from the simple bending theory when $\alpha = 5°$ and $\alpha = 30°$. What is the percentage error when using simple bending?

$$\left[C = \frac{q}{2t(\tan\alpha - \alpha)},\quad -0.2\% \text{ and } -7.6\% \text{ (simple bending is lower)}\right]$$