CHAPTER 4

RINGS, DISCS AND CYLINDERS SUBJECTED TO ROTATION AND THERMAL GRADIENTS

Summary

For *thin rotating rings and cylinders* of mean radius R, the tensile hoop stress set up is given by

$$\sigma_H = \rho\omega^2 R^2$$

The radial and hoop stresses at any radius r in a *disc of uniform thickness* rotating with an angular velocity ω rad/s are given by

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho\omega^2 r^2}{8}$$

where A and B are constants, ρ is the density of the disc material and ν is Poisson's ratio.

For a *solid disc* of radius R these equations give

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}(R^2 - r^2)$$

$$\sigma_H = \frac{\rho\omega^2}{8}\left[(3+\nu)R^2 - (1+3\nu)r^2\right]$$

At the centre of the solid disc these equations yield the maximum stress values

$$\sigma_{H_{max}} = \sigma_{r_{max}} = (3+\nu)\frac{\rho\omega^2 R^2}{8}$$

At the outside radius,

$$\sigma_r = 0$$

$$\sigma_H = (1-\nu)\frac{\rho\omega^2 R^2}{4}$$

For a *disc with a central hole*,

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right]$$

$$\sigma_H = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2}\right) - (1+3\nu)r^2\right]$$

the maximum stresses being

$$\sigma_{H_{\max}} = \frac{\rho\omega^2}{4}\left[(3+\nu)R_2^2 + (1-\nu)R_1^2\right] \quad \text{at the centre}$$

and

$$\sigma_{r_{\max}} = (3+\nu)\frac{\rho\omega^2}{8}[R_2 - R_1]^2 \quad \text{at } r = \sqrt{(R_1R_2)}$$

For *thick cylinders* or *solid shafts* the results can be obtained from those of the corresponding disc by replacing

$$\nu \text{ by } \nu/(1-\nu),$$

e.g. hoop stress at the centre of a rotating solid shaft is

$$\sigma_H = \left[3 + \frac{\nu}{(1-\nu)}\right]\frac{\rho\omega^2 r^2}{8}$$

Rotating thin disc of uniform strength

For uniform strength, i.e. $\sigma_H = \sigma_r = \sigma$ (constant over plane of disc), the disc thickness must vary according to the following equation:

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)}$$

4.1. Thin rotating ring or cylinder

Consider a thin ring or cylinder as shown in Fig. 4.1 subjected to a radial pressure p caused by the centrifugal effect of its own mass when rotating. The centrifugal effect on a unit length of the circumference is

$$p = m\omega^2 r$$

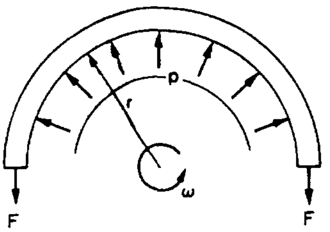

Fig. 4.1. Thin ring rotating with constant angular velocity ω.

Thus, considering the equilibrium of half the ring shown in the figure,

$$2F = p \times 2r \quad \text{(assuming unit length)}$$

$$F = pr$$

where F is the hoop tension set up owing to rotation.

The cylinder wall is assumed to be so thin that the centrifugal effect can be assumed constant across the wall thickness.

$$\therefore \qquad F = \text{mass} \times \text{acceleration} = m\omega^2 r^2 \times r$$

This tension is transmitted through the complete circumference and therefore is resisted by the complete cross-sectional area.

$$\therefore \qquad \text{hoop stress} = \frac{F}{A} = \frac{m\omega^2 r^2}{A}$$

where A is the cross-sectional area of the ring.

Now with unit length assumed, m/A is the mass of the material per unit volume, i.e. the density ρ.

$$\therefore \qquad \text{hoop stress} = \rho\omega^2 r^2$$

4.2. Rotating solid disc

(a) General equations

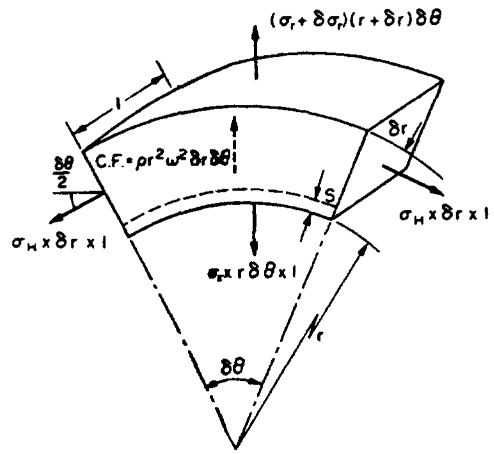

Fig. 4.2. Forces acting on a general element in a rotating solid disc.

Consider an element of a disc at radius r as shown in Fig. 4.2. Assuming unit thickness:

$$\text{volume of element} = r\,\delta\theta \times \delta r \times 1 = r\,\delta\theta\delta r$$

$$\text{mass of element} = \rho r\,\delta\theta\delta r$$

Therefore centrifugal force acting on the element

$$= m\omega^2 r$$

$$= \rho r\delta\theta\delta r\omega^2 r = \rho r^2\omega^2\delta\theta\,\delta r$$

Now for equilibrium of the element radially

$$2\sigma_H \delta r \sin\frac{\delta\theta}{2} + \sigma_r r\delta\theta - (\sigma_r + \delta\sigma_r)(r + \delta r)\delta\theta = \rho r^2\omega^2\delta\theta\,\delta r$$

If $\delta\theta$ is small,

$$\sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2} \text{ radian}$$

Therefore in the limit, as $\delta r \to 0$ (and therefore $\delta\sigma_r \to 0$) the above equation reduces to

$$\boldsymbol{\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2\omega^2} \tag{4.1}$$

If there is a radial movement or "shift" of the element by an amount s as the disc rotates, the radial strain is given by

$$\varepsilon_r = \frac{ds}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_H) \tag{4.2}$$

Now it has been shown in §9.1.3(a)† that the diametral strain is equal to the circumferential strain.

$$\therefore \qquad \frac{s}{r} = \frac{1}{E}(\sigma_H - \nu\sigma_r) \tag{4.3}$$

$$s = \frac{1}{E}(\sigma_H - \nu\sigma_r)$$

Differentiating,

$$\frac{ds}{dr} = \frac{1}{E}(\sigma_H - \nu\sigma_r) + \frac{r}{E}\left[\frac{d\sigma_H}{dr} - \frac{\nu d\sigma_r}{dr}\right] \tag{4.4}$$

Equating eqns. (4.2) and (4.4) and simplifying,

$$(\sigma_H - \sigma_r)(1 + \nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} = 0 \tag{4.5}$$

Substituting for $(\sigma_H - \sigma_r)$ from eqn. (4.1),

$$\left(r\frac{d\sigma_r}{dr} + \rho r^2\omega^2\right)(1 + \nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} = 0$$

$$\therefore \qquad \frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\rho r\omega^2(1 + \nu)$$

Integrating,

$$\sigma_H + \sigma_r = -\frac{\rho r^2\omega^2}{2}(1 + \nu) + 2A \tag{4.6}$$

where $2A$ is a convenient constant of integration.
Subtracting eqn. (4.1),

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\frac{\rho r^2\omega^2}{2}(3 + \nu) + 2A$$

But

$$2\sigma_r + r\frac{d\sigma_r}{dr} = \frac{d}{dr}\left[(r^2\sigma_r)\right] \times \frac{1}{r}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

$$\frac{d}{dr}(r^2\sigma_r) = r\left[-\frac{\rho r^2\omega^2}{2}(3+\nu) + 2A\right]$$

$$r^2\sigma_r = -\frac{\rho r^4\omega^2}{8}(3+\nu) + \frac{2Ar^2}{2} - B$$

where $-B$ is a second convenient constant of integration,

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho\omega^2 r^2}{8} \tag{4.7}$$

and from eqn. (4.5),

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho\omega^2 r^2}{8} \tag{4.8}$$

For a solid disc the stress at the centre is given when $r = 0$. With r equal to zero the above equations will yield infinite stresses whatever the speed of rotation unless B is also zero,

i.e. $B = 0$ and hence $B/r^2 = 0$ gives the only finite solution.

Now at the outside radius R the radial stress must be zero since there are no external forces to provide the necessary balance of equilibrium if σ_r were not zero.

Therefore from eqn. (4.7),

$$\sigma_r = 0 = A - (3+\nu)\frac{\rho\omega^2 R^2}{8}$$

$$\therefore \qquad A = (3+\nu)\frac{\rho\omega^2 R^2}{8}$$

Substituting in eqns. (4.7) and (4.8) the hoop and radial stresses at any radius r in a solid disc are given by

$$\sigma_H = (3+\nu)\frac{\rho\omega^2 R^2}{8} - (1+3\nu)\frac{\rho\omega^2 r^2}{8}$$

$$= \frac{\rho\omega^2}{8}\left[(3+\nu)R^2 - (1+3\nu)r^2\right] \tag{4.9}$$

$$\sigma_r = (3+\nu)\frac{\rho\omega^2 R^2}{8} - (3+\nu)\frac{\rho\omega^2 r^2}{8}$$

$$= (3+\nu)\frac{\rho\omega^2}{8}[R^2 - r^2] \tag{4.10}$$

(b) Maximum stresses

At the *centre* of the disc, where $r = 0$, the above equations yield equal values of hoop and radial stress which may also be seen to be the maximum stresses in the disc, i.e. maximum hoop and radial stress (at the centre)

$$= (3+\nu)\frac{\rho\omega^2 R^2}{8} \tag{4.11}$$

At the *outside* of the disc, at $r = R$, the equations give

$$\sigma_r = 0 \quad \text{and} \quad \sigma_H = (1 - \nu)\frac{\rho\omega^2 R^2}{4} \tag{4.12}$$

The complete distributions of radial and hoop stress across the radius of the disc are shown in Fig. 4.3.

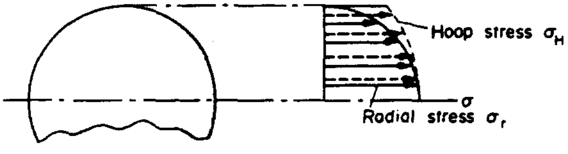

Fig. 4.3. Hoop and radial stress distributions in a rotating solid disc.

4.3. Rotating disc with a central hole

(a) General equations

The general equations for the stresses in a rotating hollow disc may be obtained in precisely the same way as those for the solid disc of the previous section,

i.e.
$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu)\frac{\rho\omega^2 r^2}{8}$$
$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu)\frac{\rho\omega^2 r^2}{8}$$

The only difference to the previous treatment is the conditions which are required to evaluate the constants A and B since, in this case, B is not zero.

The above equations are similar in form to the Lamé equations for pressurised thick rings or cylinders with modifying terms added. Indeed, should the condition arise in service where a rotating ring or cylinder is also pressurised, then the pressure and rotation boundary conditions may be substituted simultaneously to determine appropriate values of the constants A and B.

However, returning to the rotation only case, the required boundary conditions are zero radial stress at both the inside and outside radius,

i.e. at $r = R_1$, $\sigma_r = 0$

$\therefore$
$$0 = A - \frac{B}{R_1^2} - (3 + \nu)\frac{\rho\omega^2 R_1^2}{8}$$

and at $r = R_2$, $\sigma_r = 0$

$\therefore$
$$0 = A - \frac{B}{R_2^2} - (3 + \nu)\frac{\rho\omega^2 R_2^2}{8}$$

Subtracting and simplifying,

$$B = (3+\nu)\frac{\rho\omega^2 R_1^2 R_2^2}{8}$$

and

$$A = (3+\nu)\frac{\rho\omega^2 (R_1^2 + R_2^2)}{8}$$

Substituting in eqns. (4.7) and (4.8) yields the final equation for the stresses

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right] \tag{4.13}$$

$$\sigma_H = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2}\right) - (1+3\nu)r^2\right] \tag{4.14}$$

(b) Maximum stresses

The *maximum hoop stress* occurs at the inside radius where $r = R_1$,

i.e.

$$\sigma_{H_{max}} = \frac{\rho\omega^2}{8}\left[(3+\nu)(R_1^2 + R_2^2 + R_2^2) - (1+3\nu)R_1^2\right]$$

$$= \frac{\rho\omega^2}{4}\left[(3+\nu)R_2^2 + (1-\nu)R_1^2\right] \tag{4.15}$$

As the value of the inside radius approaches zero the maximum hoop stress value approaches

$$\frac{\rho\omega^2}{4}(3+\nu)R_2^2$$

This is **twice** the value obtained at the centre of a solid disc rotating at the same speed. Thus the drilling of even a very small hole at the centre of a solid disc will double the maximum hoop stress set up owing to rotation.

At the outside of the disc when $r = R_2$

$$\sigma_{H_{min}} = \frac{\rho\omega^2}{4}\left[(3+v)R_1^2 + (1-v)R_2^2\right]$$

The *maximum radial stress* is found by consideration of the equation

$$\sigma_r = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right] \tag{4.13)(bis}$$

This will be a maximum when $\dfrac{d\sigma_r}{dr} = 0$,

i.e.

$$\text{when } 0 = \frac{d}{dr}\left[R_1^2 + R_2^2 - \frac{R_1^2 R_2^2}{r^2} - r^2\right]$$

$$0 = R_1^2 R_2^2 \frac{2}{r^3} - 2r$$

$$r^4 = R_1^2 R_2^2$$

$$r = \sqrt{(R_1 R_2)} \tag{4.16}$$

Substituting for r in eqn. (4.13).

$$\sigma_{r_{\max}} = (3+\nu)\frac{\rho\omega^2}{8}\left[R_1^2 + R_2^2 - R_1R_2 - R_1R_2\right]$$

$$= (3+\nu)\frac{\rho\omega^2}{8}[R_2 - R_1]^2 \qquad (4.17)$$

The complete radial and hoop stress distributions are indicated in Fig. 4.4.

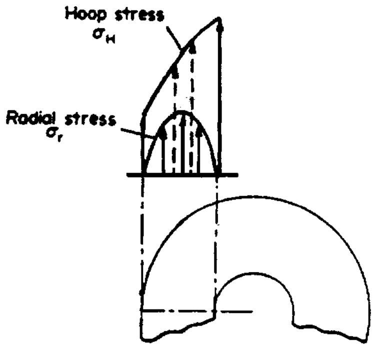

Fig. 4.4. Hoop and radial stress distribution in a rotating hollow disc.

4.4. Rotating thick cylinders or solid shafts

In the case of rotating thick cylinders the longitudinal stress σ_L must be taken into account and the longitudinal strain is assumed to be constant. Thus, writing the equations for the strain in three mutually perpendicular directions (see §4.2),

$$\varepsilon_L = \frac{1}{E}(\sigma_L - \nu\sigma_H - \nu\sigma_r) \qquad (4.18)$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_H - \nu\sigma_L) = \frac{ds}{dr} \qquad (4.19)$$

$$\varepsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_r - \nu\sigma_L) = \frac{s}{r} \qquad (4.20)$$

From eqn. (4.20)

$$Es = r[\sigma_H - \nu(\sigma_r + \sigma_L)]$$

Differentiating,

$$E\frac{ds}{dr} = r\left[\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_L}{dr}\right] + 1\,[\sigma_H - \nu\sigma_r - \nu\sigma_L]$$

Substituting for $E(ds/dr)$ in eqn. (4.19),

$$\sigma_r - \nu\sigma_H - \nu\sigma_L = r\left[\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} - \nu\frac{d\sigma_L}{dr}\right] + \sigma_H - \nu\sigma_r - \nu\sigma_L$$

$$\therefore \qquad 0 = (\sigma_H - \sigma_r)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} - \nu r\frac{d\sigma_L}{dr}$$

Now, since ε_L is constant, differentiating eqn. (4.18),

$$\frac{d\sigma_L}{dr} = \nu\left[\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr}\right]$$

$$\therefore \qquad 0 = (\sigma_H - \sigma_r)(1+\nu) + r(1-\nu^2)\frac{d\sigma_H}{dr} - \nu r(1+\nu)\frac{d\sigma_r}{dr}$$

Dividing through by $(1+\nu)$,

$$0 = (\sigma_H - \sigma_r) + r(1-\nu)\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr}$$

But the general equilibrium equation will be the same as that obtained in §4.2, eqn. (4.1),

i.e. $$\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho\omega^2 r^2$$

Therefore substituting for $(\sigma_H - \sigma_r)$,

$$0 = \rho\omega^2 r^2 + r\frac{d\sigma_r}{dr} + r(1-\nu)\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr}$$

$$0 = \rho\omega^2 r^2 + r(1-\nu)\left[\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr}\right]$$

$$\therefore \qquad \frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho\omega^2 r}{(1-\nu)}$$

Integrating,

$$\sigma_H + \sigma_r = -\frac{\rho\omega^2 r^2}{2(1-\nu)} + 2A$$

where $2A$ is a convenient constant of integration. This equation can now be compared with the equivalent equation of §4.2, when it is evident that similar results for σ_H and σ_r can be obtained if $(1+\nu)$ is replaced by $1/(1-\nu)$ or, alternatively, if ν is replaced by $\nu/(1-\nu)$, see §8.14.2. **Thus hoop and radial stresses in rotating thick cylinders can be obtained from the equations for rotating discs provided that Poisson's ratio ν is replaced by $\nu/(1-\nu)$**, e.g. the stress at the centre of a rotating solid shaft will be given by eqn. (4.11) for a solid disc modified as stated above,

i.e. $$\sigma_H = \left[3 + \frac{\nu}{(1-\nu)}\right]\frac{\rho\omega^2 R^2}{8} \qquad (4.21)$$

4.5. Rotating disc of uniform strength

In applications such as turbine blades rotating at high speeds it is often desirable to design for constant stress conditions under the action of the high centrifugal forces to which they are subjected.

Consider, therefore, an element of a disc subjected to equal hoop and radial stresses,

i.e. $$\sigma_H = \sigma_r = \sigma \quad \text{(Fig. 4.5)}$$

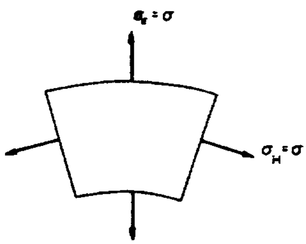

Fig. 4.5. Stress acting on an element in a rotating disc of uniform strength.

The condition of equal stress can only be achieved, as in the case of uniform strength cantilevers, by varying the thickness. Let the thickness be t at radius r and $(t + \delta t)$ at radius $(r + \delta r)$.

Then centrifugal force on the element

$$
\begin{aligned}
&= \text{mass} \times \text{acceleration} \\
&= (\rho t r \delta\theta \delta r)\omega^2 r \\
&= \rho t \omega^2 r^2 \delta\theta \delta r
\end{aligned}
$$

The equilibrium equation is then

$$\rho t \omega^2 r^2 \delta\theta \delta r + \sigma(r + \delta r)\delta\theta(t + \delta t) = 2\sigma t \delta r \sin \tfrac{1}{2}\delta\theta + \sigma_r t \delta\theta$$

i.e. in the limit

$$\sigma t dr = \rho\omega^2 r^2 t dr + \sigma t dr + \sigma r dt$$

$$\therefore \qquad \sigma r dt = -\rho\omega^2 r^2 t dr$$

$$\therefore \qquad \frac{dt}{dr} = -\frac{\rho\omega^2 r t}{\sigma}$$

Integrating,

$$\log_e t = -\frac{\rho\omega^2 r^2}{2\sigma} + \log_e A$$

where $\log_e A$ is a convenient constant.

$$\therefore \qquad t = Ae^{(-\rho\omega^2 r^2)/(2\sigma)}$$

where $r = 0$ $\qquad t = A = t_0$

i.e. *for uniform strength* the thickness of the disc must vary according to the following equation,

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)} \tag{4.22}$$

4.6. Combined rotational and thermal stresses in uniform discs and thick cylinders

If the temperature of any component is raised *uniformly* then, provided that the material is free to expand, expansion takes place without the introduction of any so-called thermal or temperature stresses. In cases where components, e.g. discs, are subjected to *thermal* *gradients*, however, one part of the material attempts to expand at a faster rate than another owing to the difference in temperature experienced by each part, and as a result stresses are developed. These are analogous to the differential expansion stresses experienced in compound bars of different materials and treated in §2.3.†

Consider, therefore, a disc initially unstressed and subjected to a temperature rise T. Then, for a radial movement s of any element, eqns. (4.2) and (4.3) may be modified to account for the strains due to temperature thus:

$$\frac{ds}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_H + E\alpha T) \tag{4.23}$$

and

$$\frac{s}{r} = \frac{1}{E}(\sigma_H - \nu\sigma_r + E\alpha T) \tag{4.24}$$

where α is the coefficient of expansion of the disc material (see §2.3)†

From eqn. (4.24),

$$\frac{ds}{dr} = \frac{1}{E}\left[(\sigma_H - \nu\sigma_r + E\alpha T) + r\left(\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} + E\alpha\frac{dT}{dr}\right)\right]$$

Therefore from eqn. (4.23),

$$\frac{1}{E}(\sigma_r - \nu\sigma_H + E\alpha T) = \frac{1}{E}\left[(\sigma_H - \nu\sigma_r + E\alpha T) - r\left(\frac{d\sigma_H}{dr} - \nu\frac{d\sigma_r}{dr} + E\alpha\frac{dT}{dr}\right)\right]$$

$$\therefore \quad (\sigma_H - \sigma_r)(1+\nu) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} + E\alpha r\frac{dT}{dr} = 0 \tag{4.25}$$

but, from the equilibrium eqn. (4.1),

$$\sigma_H - \sigma_r - r\frac{d\sigma_r}{dr} = \rho r^2\omega^2$$

Therefore substituting for $(\sigma_H - \sigma_r)$ in eqn. (4.25),

$$(1+\nu)\left(\rho r^2\omega^2 + r\frac{d\sigma_r}{dr}\right) + r\frac{d\sigma_H}{dr} - \nu r\frac{d\sigma_r}{dr} + E\alpha r\frac{dT}{dr} = 0$$

$$(1+\nu)\rho r^2\omega^2 + r\frac{d\sigma_r}{dr} + r\frac{d\sigma_H}{dr} + E\alpha r\frac{dT}{dr} = 0$$

$$\frac{d\sigma_H}{dr} + \frac{d\sigma_r}{dr} = -(1+\nu)\rho r\omega^2 - E\alpha\frac{dT}{dr}$$

Integrating,

$$\sigma_H + \sigma_r = -(1+\nu)\frac{\rho r^2\omega^2}{2} - E\alpha T + 2A \tag{4.26}$$

where, again, $2A$ is a convenient constant.

Subtracting eqn. (4.1),

$$2\sigma_r + r\frac{d\sigma_r}{dr} = -\frac{\rho r^2\omega^2}{2}(3+\nu) - E\alpha T + 2A$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

But $$2\sigma_r + r\frac{d\sigma_r}{dr} = \frac{d}{dr}\left[(r^2\sigma_r) \times \frac{1}{r}\right]$$

$$\therefore \quad \frac{d}{dr}(r^2\sigma_r) = r\left[-\frac{\rho r^2\omega^2}{2}(3+\nu) - E\alpha T + 2A\right]$$

Integrating, $$r^2\sigma_r = -\frac{\rho r^4\omega^2}{8}(3+\nu) - E\alpha\int Tr\,dr + \frac{2Ar^2}{2} - B$$

where, as in eqn. (4.7), $-B$ is a second convenient constant of integration.

$$\therefore \quad \sigma_r = A - \frac{B}{r^2} - \frac{\rho r^2\omega^2}{8}(3+\nu) - \frac{Ea}{r^2}\int Tr\,dr \tag{4.27}$$

Then, from eqn. (4.26),

$$\sigma_H = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho r^2\omega^2}{8} - EaT + \frac{Ea}{r^2}\int Tr\,dr \tag{4.28}$$

i.e. the expressions obtained for the hoop and radial stresses are those of the standard Lamé equations for simple pressurisation with (a) modifying terms for rotational effects as obtained in previous sections of this chapter, and (b) modifying terms for thermal effects.

A solution to eqns. (4.27) and (4.28) *for discs* may thus be obtained provided that the way in which T varies with r is known. Because of the form of the equations it is clear that, if required, pressure, rotational and thermal effects can be considered simultaneously and the appropriate values of A and B determined.

For *thick cylinders* with an axial length several times the outside diameter the above plane stress equations may be modified to the equivalent plane strain equations (see §8.14.2) by replacing ν by $\nu/(1-\nu)$, E by $E/(1-\nu^2)$ and α by $(1+\nu)\alpha$.

i.e. $E\alpha$ **becomes** $E\alpha(1-\nu)$

In the absence of rotation the equations simplify to

$$\sigma_r = A - \frac{B}{r^2} - \frac{Ea}{r^2}\int Tr\,dr \tag{4.29}$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{Ea}{r^2}\int Tr\,dr - EaT \tag{4.30}$$

With a *linear variation of temperature* from $T = 0$ at $r = 0$,

i.e. with $$T = Kr$$

$$\sigma_r = A - \frac{B}{r^2} - \frac{EaKr}{3} \tag{4.31}$$

$$\sigma_H = A + \frac{B}{r^2} - 2\frac{EaKr}{3} \tag{4.32}$$

With a *steady heat flow*, for example, in the case of thick cylinders when $E\alpha$ becomes $E\alpha/(1-\nu)$–see p. 125.

$$\frac{r\,dT}{dr} = \text{constant} = b$$

$$\therefore \qquad \frac{dT}{dr} = \frac{b}{r} \quad \text{and} \quad T = a + b\log_e r$$

and the equations become

$$\sigma_r = A - \frac{B}{r^2} - \frac{EaT}{2(1-\nu)} \tag{4.33}$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{EaT}{2(1-\nu)} - \frac{Eab}{2(1-\nu)} \tag{4.34}$$

In practical applications where the temperature is higher on the inside of the disc or thick cylinder than the outside, the thermal stresses are tensile on the outside surface and compressive on the inside. They may thus be considered as favourable in pressurised thick cylinder applications where they will tend to reduce the high tensile stresses on the inside surface due to pressure. However, in the chemical industry, where endothermic reactions may be contained within the walls of a thick cylinder, the reverse situation applies and the two stress systems add to provide a potentially more severe stress condition.

Examples

Example 4.1

A steel ring of outer diameter 300 mm and internal diameter 200 mm is shrunk onto a solid steel shaft. The interference is arranged such that the radial pressure between the mating surfaces will not fall below 30 MN/m^2 whilst the assembly rotates in service. If the maximum circumferential stress on the inside surface of the ring is limited to 240 MN/m^2, determine the maximum speed at which the assembly can be rotated. It may be assumed that no relative slip occurs between the shaft and the ring.

For steel, $\rho = 7470$ kg/m^3, $\nu = 0.3$, $E = 208$ GN/m^2.

Solution

From eqn. (4.7)

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \tag{1}$$

Now when $r = 0.15$, $\qquad \sigma_r = 0$

$$\therefore \qquad 0 = A - \frac{B}{0.15^2} - \frac{3.3}{8}\rho\omega^2(0.15)^2 \tag{2}$$

Also, when $r = 0.1$, $\qquad \sigma_r = -30$ MN/m^2

$$\therefore \qquad -30 \times 10^6 = A - \frac{B}{0.1^2} - \frac{3.3}{8}\rho\omega^2(0.1)^2 \tag{3}$$

(2)–(3), $$30 \times 10^6 = B(100 - 44.4) - \frac{3.3}{8}\rho\omega^2(0.0225 - 0.01)$$

$$\therefore \qquad B = \frac{30 \times 10^6}{55.6} + 3.3 \times \frac{0.0125 \times 7470}{8 \times 55.6}\omega^2$$

$$B = 0.54 \times 10^6 + 0.693\omega^2$$

and from (3),

$$A = 100(0.54 \times 10^6 + 0.693\omega^2) + \frac{3.3 \times 7470 \times 0.01\omega^2}{8} - 30 \times 10^6$$
$$= 54 \times 10^6 + 69.3\omega^2 + 30.8\omega^2 - 30 \times 10^6$$
$$= 24 \times 10^6 + 100.1\omega^2$$

But since the maximum hoop stress at the inside radius is limited to 240 MN/m^2, from eqn. (4.8)

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

i.e.

$$240 \times 10^6 = (24 \times 10^6 + 100.1\omega^2) + \frac{(0.54 \times 10^6 + 0.693\omega^2)}{0.1^2} - \frac{1.9}{8} \times 7470 \times 0.01\omega^2$$

$$240 \times 10^6 = 78 \times 10^6 + 169.3\omega^2 - 17.7\omega^2$$

$$\therefore \quad 151.7\omega^2 = 162 \times 10^6$$

$$\omega^2 = \frac{162 \times 10^6}{151.7} = 1.067 \times 10^6$$

$$\omega = 1033 \text{ rad/s} = \mathbf{9860\ rev/min}$$

Example 4.2

A steel rotor disc which is part of a turbine assembly has a uniform thickness of 40 mm. The disc has an outer diameter of 600 mm and a central hole of 100 mm diameter. If there are 200 blades each of mass 0.153 kg pitched evenly around the periphery of the disc at an effective radius of 320 mm, determine the rotational speed at which yielding of the disc first occurs according to the maximum shear stress criterion of elastic failure.

For steel, $E = 200$ GN/m^2, $\nu = 0.3$, $\rho = 7470$ kg/m^3 and the yield stress σ_y in simple tension $= 500$ MN/m^2.

Solution

Total mass of blades $= 200 \times 0.153 = 30.6$ kg

Effective radius $= 320$ mm

Therefore centrifugal force on the blades $= m\omega^2 r = 30.6 \times \omega^2 \times 0.32$

Now the area of the disc rim $= \pi dt = \pi \times 0.6 \times 0.004 = 0.024\pi$m^2

The centrifugal force acting on this area thus produces an effective radial stress acting on the outside surface of the disc since the blades can be assumed to produce a uniform loading around the periphery.

Therefore radial stress at outside surface

$$= \frac{30.6 \times \omega^2 \times 0.32}{0.024\pi} = 130\omega^2 \text{ N/m}^2 \quad \text{(tensile)}$$

Now eqns. (4.7) and (4.8) give the general form of the expressions for hoop and radial stresses set up owing to rotation,

i.e.
$$\sigma_r = A - \frac{B}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2 \qquad (1)$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2 \qquad (2)$$

When $r = 0.05$, $\sigma_r = 0$

$$\therefore \qquad 0 = A - 400B - \frac{3.3}{8}\rho\omega^2(0.05)^2 \qquad (3)$$

When $r = 0.3$, $\sigma_r = +130\omega^2$

$$\therefore \qquad 130\omega^2 = A - 11.1B - \frac{3.3}{8}\rho\omega^2(0.3)^2 \qquad (4)$$

(4)–(3),
$$130\omega^2 = 388.9B - \frac{3.3}{8}\rho\omega^2(9 - 0.25)10^{-2}$$

$$130\omega^2 = 388.9B - 270\omega^2$$

$$B = \frac{(130+270)}{388.9}\omega^2 = 1.03\omega^2$$

Substituting in (3),

$$A = 412\omega^2 + \frac{3.3}{8} \times 7470(0.05)^2\omega^2$$

$$= 419.7\omega^2 = 420\omega^2$$

Therefore substituting in (2) and (1), the stress conditions at the inside surface are

$$\sigma_H = 420\omega^2 + 412\omega^2 - 4.43\omega^2 = 827\omega^2$$

with $\sigma_r = 0$

and at the outside $\sigma_H = 420\omega^2 + 11.42\omega^2 - 159\omega^2 = 272\omega^2$

with $\sigma_r = 130\omega^2$

The most severe stress conditions therefore occur at the inside radius where the maximum shear stress is greatest

i.e.
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{827\omega^2 - 0}{2}$$

Now the maximum shear stress theory of elastic failure states that failure is assumed to occur when this stress equals the value of τ_{max} at the yield point in simple tension,

i.e.
$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y - 0}{2} = \frac{\sigma_y}{2}$$

Thus, for failure according to this theory,

$$\frac{\sigma_y}{2} = \frac{827\omega^2}{2}$$

i.e.
$$827\omega^2 = \sigma_y = 500 \times 10^6$$

$\therefore$
$$\omega^2 = \frac{500}{827} \times 10^6 = 0.604 \times 10^6$$
$$\omega = 780 \text{ rad/s} = \mathbf{7450 \text{ rev/min}}$$

Example 4.3

The cross-section of a turbine rotor disc is designed for uniform strength under rotational conditions. The disc is keyed to a 60 mm diameter shaft at which point its thickness is a maximum. It then tapers to a minimum thickness of 10 mm at the outer radius of 250 mm where the blades are attached. If the design stress of the shaft is 250 MN/m^2 at the design speed of 12000 rev/min, what is the required maximum thickness? For steel $\rho = 7470$ kg/m^3.

Solution

From eqn. (4.22) the thickness of a uniform strength disc is given by

$$t = t_0 e^{(-\rho\omega^2 r^2)/(2\sigma)} \qquad (1)$$

where t_0 is the thickness at $r = 0$.

Now at $r = 0.25$,

$$\frac{\rho\omega^2 r^2}{2\sigma} = \frac{7470}{2 \times 250 \times 10^6}\left(12\,000 \times \frac{2\pi}{60}\right)^2 \times 0.25^2 = 1.47$$

and at $r = 0.03$,

$$\frac{\rho\omega^2 r^2}{2\sigma} = \frac{7470}{2 \times 250 \times 10^6}\left(12\,000 \times \frac{2\pi}{60}\right)^2 \times 0.03^2$$
$$= 1.47 \times \frac{9 \times 10^{-4}}{625 \times 10^{-4}} = 0.0212$$

But at $r = 0.25$, $\quad t = 10$ mm

Therefore substituting in (1),

$$0.01 = t_0 e^{-1.47} = 0.2299\, t_0$$
$$t_0 = \frac{0.01}{0.2299} = 0.0435\text{m} = 43.5 \text{ mm}$$

Therefore at $r = 0.03$

$$t = 0.0435 e^{-0.0212} = 0.0435 \times 0.98$$
$$= 0.0426 \text{ m} = \mathbf{42.6 \text{ mm}}$$

Example 4.4

(a) Derive expressions for the hoop and radial stresses developed in a solid disc of radius R when subjected to a thermal gradient of the form $T = Kr$. Hence determine the position and magnitude of the maximum stresses set up in a steel disc of 150 mm diameter when the temperature rise is 150°C. For steel, $\alpha = 12 \times 10^{-6}$ per °C and $E = 206.8$ GN/m^2.

(b) How would the values be changed if the temperature at the centre of the disc was increased to 30°C, the temperature rise across the disc maintained at 150°C and the thermal gradient now taking the form $T = a + br$?

Solution

(a) The hoop and radial stresses are given by eqns. (4.29) and (4.30) as follows:

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha E}{r^2}\int Tr\,dr \tag{1}$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha E}{r^2}\int Tr\,dr - \alpha ET \tag{2}$$

In this case

$$\int Tr\,dr = K\int r^2 dr = \frac{Kr^3}{3}$$

the constant of integration being incorporated into the general constant A.

$$\therefore \qquad \sigma_r = A - \frac{B}{r^2} - \frac{\alpha EKr}{3} \tag{3}$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha EKr}{3} - \alpha EKr \tag{4}$$

Now in order that the stresses at the centre of the disc, where $r = 0$, shall not be infinite, B must be zero and hence B/r^2 is zero. Also $\sigma_r = 0$ at $r = R$.

Therefore substituting in (3),

$$0 = A - \frac{\alpha EKR}{3} \text{ and } A = \frac{\alpha EKR}{3}$$

Substituting in (3) and (4) and rearranging,

$$\boldsymbol{\sigma_r = \frac{\alpha EK}{3}(R - r)}$$

$$\boldsymbol{\sigma_H = \frac{\alpha EK}{3}(R - 2r)}$$

The variation of both stresses with radius is linear and they will both have maximum values at the centre where $r = 0$.

$$\therefore \qquad \sigma_{r_{\max}} = \sigma_{H_{\max}} = \frac{\alpha EKR}{3}$$

$$= \frac{12 \times 10^{-6} \times 206.8 \times 10^9 \times K \times 0.075}{3}$$

Now $T = Kr$ and T must therefore be zero at the centre of the disc where r is zero. Thus, with a known temperature rise of 150°C, it follows that the temperature at the outside radius must be 150°C.

$$\therefore \qquad 150 = K \times 0.075$$

$$\therefore \qquad K = 2000^\circ/\text{m}$$

i.e. $$\sigma_{r_{max}} = \sigma_{H_{max}} = \frac{12 \times 10^{-6} \times 206.8 \times 10^9 \times 2000 \times 0.075}{3}$$

$$= \mathbf{124\ MN/m^2}$$

(b) With the modified form of temperature gradient,

$$\int Tr\,dr = \int (a+br)r\,dr = \int (ar + br^2)\,dr$$

$$= \frac{ar^2}{2} + \frac{br^3}{3}$$

Substituting in (1) and (2),

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha E}{r^2}\left[\frac{ar^2}{2} + \frac{br^3}{3}\right] \qquad (5)$$

$$\sigma_H = A + \frac{B}{r^2} + \frac{\alpha E}{r^2}\left[\frac{ar^2}{2} + \frac{br^3}{3}\right] - \alpha ET \qquad (6)$$

Now $T = a + br$

Therefore at the inside of the disc where $r = 0$ and $T = 30°C$,

$$30 = a + b(0) \qquad (7)$$

and $a = 30$

At the outside of the disc where $T = 180°C$,

$$180 = a + b(0.075) \qquad (8)$$

(8) − (7) $150 = 0.075b \quad \therefore\ b = 2000$

Substituting in (5) and (6) and simplifying,

$$\sigma_r = A - \frac{B}{r^2} - \alpha E(15 + 667r) \qquad (9)$$

$$\sigma_H = A + \frac{B}{r^2} + \alpha E(15 + 667r) - \alpha ET \qquad (10)$$

Now for finite stresses at the centre,

$$B = 0$$

Also, at $r = 0.075$, $\sigma_r = 0$ and $T = 180°C$

Therefore substituting in (9),

$$0 = A - 12 \times 10^{-6} \times 206.8 \times 10^9(15 + 667 \times 0.075)$$

$$0 = A - 12 \times 206.8 \times 10^3 \times 65$$

$$\therefore \quad A = 161.5 \times 10^6$$

From (9) and (10) the maximum stresses will again be at the centre where $r = 0$,

i.e. $\sigma_{r_{max}} = \sigma_{H_{max}} = A - \alpha ET = \mathbf{124\ MN/m^2}$, as before.

N.B. The same answers would be obtained for any linear gradient with a temperature difference of 150°C. Thus a solution could be obtained with the procedure of part (a) using the form of distribution $T = Kr$ with the value of T at the outside taken to be 150°C (the value at $r = 0$ being automatically zero).

Example 4.5

An initially unstressed short steel cylinder, internal radius 0.2 m and external radius 0.3 m, is subjected to a temperature distribution of the form $T = a + b\log_e r$ to ensure constant heat flow through the cylinder walls. With this form of distribution the radial and circumferential stresses at any radius r, where the temperature is T, are given by

$$\sigma_r = A - \frac{B}{r^2} - \frac{\alpha ET}{2(1-\nu)}$$

$$\sigma_H = A + \frac{B}{r^2} - \frac{\alpha ET}{2(1-\nu)} - \frac{E\alpha b}{2(1-\nu)}$$

If the temperatures at the inside and outside surfaces are maintained at 200°C and 100°C respectively, determine the maximum circumferential stress set up in the cylinder walls. For steel, $E = 207\ \text{GN/m}^2$, $\nu = 0.3$ and $\alpha = 11 \times 10^{-6}$ per °C.

Solution

$$T = a + b\log_e r$$

$\therefore$
$$200 = a + b\log_e 0.2 = a + b(0.6931 - 2.3026)$$

$$200 = a - 1.6095\ b \qquad (1)$$

also
$$100 = a + b\log_e 0.3 = a + b(1.0986 - 2.3026)$$

$$100 = a - 1.204\ b \qquad (2)$$

(2) − (1),
$$100 = -0.4055\ b$$

$$b = -246.5 = -247$$

Also
$$\frac{E\alpha}{2(1-\nu)} = \frac{207 \times 10^9 \times 11 \times 10^{-6}}{2(1-0.29)}$$

$$= 1.6 \times 10^6$$

Therefore substituting in the given expression for radial stress,

$$\sigma_r = A - \frac{B}{r^2} - 1.6 \times 10^6 T$$

At $r = 0.3$, $\sigma_r = 0$ and $T = 100$

$$0 = A - \frac{B}{0.09} - 1.6 \times 10^6 \times 100 \qquad (3)$$

At $r = 0.2$, $\sigma_r = 0$ and $T = 200$

$$0 = A - \frac{B}{0.04} - 1.6 \times 10^6 \times 200 \qquad (4)$$

(4) − (3), $$0 = B(11.1 - 25) - 1.6 \times 10^8$$

$$B = -11.5 \times 10^6$$

and from (4),

$$A = 25B + 3.2 \times 10^8$$

$$= (-2.88 + 3.2)10^8 = 0.32 \times 10^8$$

substituting in the given expression for hoop stress,

$$\sigma_H = 0.32 \times 10^8 - \frac{11.5 \times 10^6}{r^2} - 1.6 \times 10^6 T + 1.6 \times 10^6 \times 247$$

At $r = 0.2$, $\sigma_H = (0.32 - 2.88 - 3.2 + 3.96)10^8 = \mathbf{-180\ MN/m^2}$

At $r = 0.3$, $\sigma_H = (0.32 - 1.28 - 1.6 + 3.96)10^8 = \mathbf{+140\ MN/m^2}$

The maximum tensile circumferential stress therefore occurs at the outside radius and has a value of 140 MN/m^2. The maximum compressive stress is 180 MN/m^2 at the inside radius.

Problems

Unless otherwise stated take the following material properties for steel:

$$\rho = 7470\ kg/m^3; \quad \nu = 0.3; \quad E = 207\ GN/m^2$$

4.1 (B). Determine equations for the hoop and radial stresses set up in a solid rotating disc of radius R commencing with the following relationships:

$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu)\frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu)\frac{\rho\omega^2 r^2}{8}$$

Hence determine the maximum stress and the stress at the outside of a 250 mm diameter disc which rotates at 12000 rev/min. [76, 32.3 MN/m^2.]

4.2 (B). Determine from first principles the hoop stress at the inside and outside radius of a thin steel disc of 300 mm diameter having a central hole of 100 mm diameter, if the disc is made to rotate at 5000 rev/min. What will be the position and magnitude of the maximum radial stress?

[38.9, 12.3 MN/m^2; 87 mm rad; 8.4 MN/m^2.]

4.3 (B). Show that the tensile hoop stress set up in a thin rotating ring or cylinder is given by

$$\sigma_H = \rho\omega^2 r^2$$

Hence determine the maximum angular velocity at which the disc can be rotated if the hoop stress is limited to 20 MN/m^2. The ring has a mean diameter of 260 mm. [3800 rev/min.]

4.4 (B). A solid steel disc 300 mm diameter and of small constant thickness has a steel ring of outer diameter 450 mm and the same thickness shrunk onto it. If the interference pressure is reduced to zero at a rotational speed of 3000 rev/min, calculate

(a) the radial pressure at the interface when stationary;
(b) the difference in diameters of the mating surfaces of the disc and ring before assembly.

The radial and circumferential stresses at radius r in a ring or disc rotating at ω rad/s are obtained from the following relationships:

$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu)\frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu)\frac{\rho\omega^2 r^2}{8} \qquad [8.55\ \text{MN/m}^2, 0.045\ \text{mm.}]$$

4.5 (B). A steel rotor disc of uniform thickness 50 mm has an outer rim of diameter 800 mm and a central hole of diameter 150 mm. There are 200 blades each of weight 2 N at an effective radius of 420 mm pitched evenly around the periphery. Determine the rotational speed at which yielding first occurs according to the maximum shear stress criterion.

Yield stress in simple tension = 750 MN/m^2.

The basic equations for radial and hoop stresses given in Example 4.4 may be used without proof.

[7300 rev/min.]

4.6 (B). A rod of constant cross-section and of length $2a$ rotates about its centre in its own plane so that each end of the rod describes a circle of radius a. Find the maximum stress in the rod as a function of the peripheral speed V. [$\frac{1}{2}(\rho\omega^2 a^2)$.]

4.7 (B). A turbine blade is to be designed for constant tensile stress σ under the action of centrifugal force by varying the area A of the blade section. Consider the equilibrium of an element and show that the condition is

$$A = A_h e^{[-\rho\omega^2(r^2 - r_h^2)]/(2\sigma)}$$

where A_h and r_h are the cross-sectional area and radius at the hub (i.e. base of the blade).

4.8 (B). A steel turbine rotor of 800 mm outside diameter and 200 mm inside diameter is 50 mm thick. The rotor carries 100 blades each 200 mm long and of mass 0.5 kg. The rotor runs at 3000 rev/min. Assuming the shaft to be rigid, calculate the expansion of the *inner* bore of the disc due to rotation and hence the initial shrinkage allowance necessary. [0.14 mm.]

4.9 (B). A steel disc of 750 mm diameter is shrunk onto a steel shaft of 80 mm diameter. The interference on the diameter is 0.05 mm.

(a) Find the maximum tangential stress in the disc at standstill.

(b) Find the speed in rev/min at which the contact pressure is zero.

(c) What is the maximum tangential stress at the speed found in (b)? [65 MN/m^2; 3725; 65 MN/m^2.]

4.10 (B). A flat steel turbine disc of 600 mm outside diameter and 120 mm inside diameter rotates at 3000 rev/min at which speed the blades and shrouding cause a tensile rim loading of 5 MN/m^2. The maximum stress at this speed is to be 120 MN/m^2. Find the maximum shrinkage allowance on the diameter when the disc is put on the shaft. [0.097 mm.]

4.11 (B). Find the maximum permissible speed of rotation for a steel disc of outer and inner radii 150 mm and 70 mm respectively if the outer radius is not to increase in magnitude due to centrifugal force by more than 0.03 mm. [7900 rev/min.]

4.12 (B). The radial and hoop stresses at any radius r for a disc of uniform thickness rotating at an angular speed ω rad/s are given respectively by

$$\sigma_r = A - \frac{B}{r^2} - (3 + \nu)\frac{\rho\omega^2 r^2}{8}$$

$$\sigma_H = A + \frac{B}{r^2} - (1 + 3\nu)\frac{\rho\omega^2 r^2}{8}$$

where A and B are constants, ν is Poisson's ratio and ρ is the density of the material. Determine the greatest values of the radial and hoop stresses for a disc in which the outer and inner radii are 300 mm and 150 mm respectively.

Take $\omega = 150$ rad/s, $\nu = 0.304$ and $\rho = 7470$ kg/m^3. [U.L.] [1.56, 13.2 MN/m^2.]

4.13 (B). Derive an expression for the tangential stress set up when a thin hoop, made from material of density ρ kg/m^3, rotates about its polar axis with a tangential velocity of v m/s.

What will be the greatest value of the mean radius of such a hoop, made from flat mild-steel bar, if the maximum allowable tensile stress is 45 MN/m^2 and the hoop rotates at 300 rev/min?

Density of steel = 7470 kg/m^3. [2.47 m.]

4.14 (C). Determine the hoop stresses at the inside and outside surfaces of a long thick cylinder inside radius = 75 mm, outside radius = 225 mm, which is rotated at 4000 rev/min.

Take $\nu = 0.3$ and $\rho = 7470$ kg/m^3. [57.9, 11.9 MN/m^2.]

4.15 (C). Calculate the maximum principal stress and maximum shear stress set up in a thin disc when rotating at 12 000 rev/min. The disc is of 300 mm outside diameter and 75 mm inside diameter.

Take $\nu = 0.3$ and $\rho = 7470$ kg/m^3. [221, 110.5 MN/m^2.]

4.16 (B). A thin-walled cylindrical shell made of material of density ρ has a mean radius r and rotates at a constant angular velocity of ω rad/s. Assuming the formula for centrifugal force, establish a formula for the circumferential (hoop) stress induced in the cylindrical shell due to rotation about the longitudinal axis of the cylinder and, if necessary, adjust the derived expression to give the stress in MN/m^2.

A drum rotor is to be used for a speed of 3000 rev/min. The material is steel with an elastic limit stress of 248 MN/m^2 and a density of 7.8 Mg/m^3. Determine the mean diameter allowable if a factor of safety of 2.5 on the elastic limit stress is desired. Calculate also the expansion of this diameter (in millimetres) when the shell is rotating.

For steel, $E = 207$ GN/m^2. [I.Mech.E.] [0.718 m; 0.344 mm.]

4.17 (B). A forged steel drum, 0.524 m outside diameter and 19 mm wall thickness, has to be mounted in a machine and spun about its longitudinal axis. The centrifugal (hoop) stress induced in the cylindrical shell is not to exceed 83 MN/m^2. Determine the maximum speed (in rev/min) at which the drum can be rotated.

For steel, the density = 7.8 Mg/m^3. [3630.]

4.18 (B). A cylinder, which can be considered as a thin-walled shell, is made of steel plate 16 mm thick and is 2.14 m internal diameter. The cylinder is subjected to an internal fluid pressure of 0.55 MN/m^2 gauge and, at the same time, rotated about its longitudinal axis at 3000 rev/min. Determine:

(a) the hoop stress induced in the wall of the cylinder due to rotation;
(b) the hoop stress induced in the wall of the cylinder due to the internal pressure;
(c) the factor of safety based on an ultimate stress of the material in simple tension of 456 MN/m^2.

Steel has a density of 7.8 Mg/m^3. [89.5, 36.8 MN/m^2; 3.6]

4.19 (B). The "bursting" speed of a cast-iron flywheel rim, 3 m mean diameter, is 850 rev/min. Neglecting the effects of the spokes and boss, and assuming that the flywheel rim can be considered as a thin rotating hoop, determine the ultimate tensile strength of the cast iron. Cast iron has a density of 7.3 Mg/m^3.

A flywheel rim is to be made of the-same material and is required to rotate at 400 rev/min. Determine the maximum permissible mean diameter using a factor of safety of 8. [U.L.C.I.] [2.25 mm]

4.20 (B). An internal combustion engine has a cast-iron flywheel that can be considered to be a uniform thickness disc of 230 mm outside diameter and 50 mm inside diameter. Given that the ultimate tensile stress and density of cast iron are 200 N/mm^2 and 7180 kg/m^3 respectively, calculate the speed at which the flywheel would burst. Ignore any stress concentration effects and assume Poisson's ratio for cast iron to be 0.25.

[C.E.I.] [254.6 rev/s.]

4.21 (B). A thin steel circular disc of uniform thickness and having a central hole rotates at a uniform speed about an axis through its centre and perpendicular to its plane. The outside diameter of the disc is 500 mm and its speed of rotation is 81 rev/s. If the maximum allowable direct stress in the steel is not to exceed 110 MN/m^2 (11.00 h bar), determine the diameter of the central hole.

For steel, density $\rho = 7800$ kg/m^3 and Poisson's ratio $\nu = 0.3$.

Sketch diagrams showing the circumferential and radial stress distribution across the plane of the disc indicating the peak values and state the radius at which the maximum radial stress occurs. [B.P.] [264 mm.]

4.22 (B). (a) Prove that the differential equation for radial equilibrium in cylindrical coordinates of an element in a uniform thin disc rotating at ω rad/s and subjected to principal direct stresses σ_r and σ_θ is given by the following expression:

$$\sigma_\theta - \sigma_r - r\frac{d\sigma_r}{dr} = \rho\omega^2 r^2$$

(b) A thin solid circular disc of uniform thickness has an outside diameter of 300 mm. Using the maximum shear strain energy per unit volume theory of elastic failure, calculate the rotational speed of the disc to just cause initiation of plastic yielding if the yield stress of the material of the disc is 300 MN/m^2, the density of the material is 7800 kg/m^3 and Poisson's ratio for the material is 0.3. [B.P.] [324 rev/s.]

Thermal gradients

4.23 (C). Determine expressions for the stresses developed in a hollow disc subjected to a temperature gradient of the form $T = Kr$. What are the maximum stresses for such a case if the internal and external diameters of the cylinder are 80 mm and 160 mm respectively;

$\alpha = 12 \times 10^{-6}$ per °C and $E = 206.8$ GN/m^2.

The temperature at the outside radius is -50°C. [-34.5, 27.6 MN/m^2.]

4.24 (C). Calculate the maximum stress in a solid magnesium alloy disc 60 mm diameter when the temperature rise is linear from 60°C at the centre to 90°C at the outside.
$\alpha = 7 \times 10^{-6}$ per °C and $E = 105$ GN/m^2. [7.4 MN/m^2.]

4.25 (C). Calculate the maximum compressive and tensile stresses in a hollow steel disc, 100 mm outer diameter and 20 mm inner diameter when the temperature rise is linear from 100°C at the inner surface to 50°C at the outer surface.
$\alpha = 10 \times 10^{-6}$ per °C and $E = 206.8$ GN/m^2. [−62.9, +40.3 MN/m^2.]

4.26 (C). Calculate the maximum tensile and compressive stresses in a hollow copper *cylinder* 20 mm outer diameter and 10 mm inner diameter when the temperature rise is linear from 0°C at the inner surface to 100°C at the outer surface.
$\alpha = 16 \times 10^{-6}$ per °C and $E = 104$ GN/m^2. [142, −114 MN/m^2.]

4.27 (C). A hollow steel disc has internal and external diameters of 0.2 m and 0.4 m respectively. Determine the circumferential thermal stresses set up at the inner and outer surfaces when the temperature at the outside surface is 100°C. A temperature distribution through the cylinder walls of the form $T = Kr$ may be assumed, i.e. when r = zero, T = zero.

For steel, $E = 207$ GN/m^2 and $\alpha = 11 \times 10^{-6}$ per °C.

What is the significance of (i) the first two terms of the stress eqns. (4.29) and (4.30), (ii) the remaining terms? Hence comment on the relative magnitude of the maximum hoop stresses obtained in a high pressure vessel which is used for (iii) a chemical action which is exothermic, i.e. generating heat, (iv) a chemical reaction which is endothermic, i.e. absorbing heat. [63.2, −50.5 MN/m^2.]

4.28 (C). In the previous problem sketch the thermal hoop and radial stress variation diagrams across the wall thickness of the disc inserting the numerical value of the hoop stresses at the inner, mean and outer radii, and also the maximum radial stress, inserting the radius at which it occurs.

[$\sigma_{\text{mean}} = -2.78$ MN/m^2, $\sigma_{r_{\max}} = 9.65$ MN/m^2.]

4.29 (C). A thin uniform steel disc, 254 mm outside diameter with a central hole 50 mm diameter, rotates at 10000 rev/min. The temperature gradient varies linearly such that the *difference* of temperature between the inner and outer (hotter) edges of the plate is 46°C. For the material of the disc, $E = 205$ GN/m^2, Poisson's ratio = 0.3 and the coefficient of linear expansion = 11×10^{-6} per °C. The density of the material is 7700 kg/m^3.

Calculate the hoop stresses induced at the inner and outer surfaces. [176–12.1 MN/m^2.]

4.30 (C). An unloaded steel cylinder has internal and external diameters of 204 mm and 304 mm respectively. Determine the circumferential thermal stresses at the inner and outer surfaces where the steady temperatures are 200°C and 100°C respectively.

Take $E = 207$ GN/m^2, $\alpha = 11 \times 10^{-6}$ per °C and Poisson's ratio = 0.29.

The temperature distribution through the wall thickness may be regarded as follows:

$$T = a + b\log_e r, \quad \text{where } a \text{ and } b \text{ are constants}$$

With this form of temperature distribution, the radial and circumferential thermal stresses at radius r where the temperature is T are obtained from

$$\sigma_r = A - \frac{B}{r^2} - \frac{E\alpha T}{2(1-\nu)} \quad \text{and} \quad \sigma_H = A + \frac{B}{r^2} - \frac{E\alpha T}{2(1-\nu)} - \frac{E\alpha b}{2(1-\nu)}$$

[−255, 196 MN/m^2.]

4.31 (C). Determine the hoop stresses at the inside and outside surfaces of a long thick cylinder which is rotated at 4000 rev/min. The cylinder has an internal radius of 80 mm and an external radius of 250 mm and is constructed from steel, the relevant properties of which are given above.

How would these values be modified if, under service conditions, the temperatures of the inside and outside surfaces reached maximum levels of 40°C and 90°C respectively?

A linear thermal gradient may be assumed.

For steel $\alpha = 11 \times 10^{-6}$ per °C.

[71.4, 18.9; 164.5, −46.8 MN/m^2.]

4.32 (C). (a) Determine the wall thickness required for a high pressure cylindrical vessel, 800 mm diameter, in order that yielding shall be prevented according to the Tresca criterion of elastic failure when the vessel is subjected to an internal pressure of 450 bar.

(b) Such a vessel is now required to form part of a chemical plant and to contain exothermic reactions which produce a maximum internal temperature of 120°C at a reaction pressure of 450 bar, the outer surface being cooled to an "ambient" temperature of 20°C. In the knowledge that such a thermal gradient condition will introduce additional stresses to those calculated in part (a) the designer proposes to increase the wall thickness by 20% in order that, once again, yielding shall be prevented according to the Tresca theory. Is this a valid proposal?

You may assume that the thermal gradient is of the form $T = a + br^2$ and that the modifying terms to the Lamé expressions to cover thermal gradient conditions are

for radial stress: $$-\frac{\alpha E}{r^2}\int Tr\,dr$$

for hoop stress: $$\frac{\alpha E}{r^2}\int Tr\,dr - \alpha ET.$$

For the material of the vessel, $\sigma_y = 280$ MN/m^2, $\alpha = 12 \times 10^{-6}$ per °C and $E = 208$ GN/m^2.

[52 mm; No-design requires $\sigma_y = 348$ MN/m^2.]