CHAPTER 11

STRAIN ENERGY

Summary

The energy stored within a material when work has been done on it is termed the *strain energy* or *resilience*,

i.e. $$\text{strain energy} = \text{work done}$$

In general there are four types of loading which can be applied to a material:

1. *Direct load (tension or compression)*

$$\text{Strain energy } U = \int \frac{P^2\,ds}{2AE} \quad \text{or} \quad \frac{P^2L}{2AE}$$

$$= \frac{\sigma^2 AL}{2E} = \frac{\sigma^2}{2E} \times \text{volume of bar}$$

2. *Shear load*

$$\text{Strain energy } U = \int \frac{Q^2\,ds}{2AG} \quad \text{or} \quad \frac{Q^2L}{2AG}$$

$$= \frac{\tau^2}{2G} \times AL = \frac{\tau^2}{2G} \times \text{volume of bar}$$

3. *Bending*

$$\text{Strain energy } U = \int \frac{M^2\,ds}{2EI} \quad \text{or} \quad \frac{M^2L}{2EI} \quad \text{if } M \text{ is constant}$$

4. *Torsion*

$$\text{Strain energy } U = \int \frac{T^2\,ds}{2GJ} \quad \text{or} \quad \frac{T^2L}{2GJ} \quad \text{if } T \text{ is constant}$$

From 1 above, the strain energy or resilience when the tensile stress reaches the proof stress σ_p, i.e. the *proof resilience*, is

$$\frac{\sigma_p^2}{2E} \times \text{volume of bar}$$

and the *modulus of resilience* is

$$\frac{\sigma_p^2}{2E}$$

The *strain energy per unit volume* of a *three-dimensional principal stress system* is

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

The *volumetric or "dilatational" strain energy per unit volume* is then

$$\frac{(1-2\nu)}{6E}[(\sigma_1+\sigma_2+\sigma_3)^2]$$

and the *shear*, or *"distortional", strain energy per unit volume* is

$$\frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

The *maximum instantaneous stress* in a uniform bar caused by a weight W falling through a distance h on to the bar is given by

$$\sigma=\frac{W}{A}\pm\sqrt{\left[\left(\frac{W}{A}\right)^2+\frac{2WEh}{AL}\right]}$$

The *instantaneous extension* is then given by

$$\delta=\frac{\sigma L}{E}$$

If this is small compared to the height h, then

$$\sigma=\sqrt{\left(\frac{2WEh}{AL}\right)}$$

For *any shock-loaded system* the instantaneous deflection is given by

$$\delta=\delta_s\left[1\pm\sqrt{\left(1+\frac{2h}{\delta_s}\right)}\right]$$

where δ_s is the deflection under an equal static load.

Castigliano's first theorem for deflection states that:

If the total strain energy expressed in terms of the external loads is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load (see Examples 11.5 and 11.6):

i.e. Deflection in direction of $W=\frac{\partial U}{\partial W}=\delta$

In applications where bending provides practically all of the strain energy,

$$\delta=\frac{\partial}{\partial W}\int\frac{M^2ds}{2EI}=\int\frac{M}{EI}\frac{\partial M}{\partial W}ds$$

This is sometimes written in the form

$$\delta=\int\frac{Mm}{EI}ds$$

where $m=\frac{\partial M}{\partial W}$ = the bending moment resulting from a unit load only in the place of W. This method of solution is then termed the *unit load method.*

Castigliano's theorem also applies to ***angular movements***:

If the total strain energy expressed in terms of the external moments be partially differentiated with respect to one of the moments, the result is the angular deflection in radians of the point of application of that moment and in its direction

$$\theta = \int \frac{M}{EI}\frac{\partial M}{\partial M_i}\, ds$$

where M_i is the actual or imaginary moment at the point where θ is required.

Deflections due to shear

Beam loading	Shear deflection	
	Rectangular-section beam	I-section beam
Cantilever–concentrated end load W'	$\frac{6WL}{5AG}$	$\frac{WL}{AG}$
Cantilever–u.d.l.	$\frac{3wL^2}{5AG}$	$\frac{wL^2}{2AG} = \frac{WL}{2AG}$
Simply supported beam – central concentrated load W	$\frac{3WL}{10AG}$	$\frac{WL}{4AG}$
Simply supported beam – concentrated load dividing span into lengths a and b	$\frac{6Wab}{5AGL}$	
Simply supported beam–u.d.l.	$\frac{3wL^2}{20AG}$	$\frac{wL^2}{8AG} = \frac{WL}{8AG}$

Introduction

Energy is normally defined as the *capacity to do work* and it may exist in any of many forms, e.g. mechanical (potential or kinetic), thermal, nuclear, chemical, etc. The potential energy of a body is the form of energy which is stored by virtue of the work which has previously been done on that body, e.g. in lifting it to some height above a datum. Strain energy is a particular form of potential energy which is stored within materials which have been subjected to strain, i.e. to some change in dimension. The material is then capable of doing work, equivalent to the amount of strain energy stored, when it returns to its original unstrained dimension.

Strain energy is therefore defined as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

i.e. strain energy U = work done

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load–extension graph of Fig. 11.1.

$$U = \tfrac{1}{2}P\delta$$

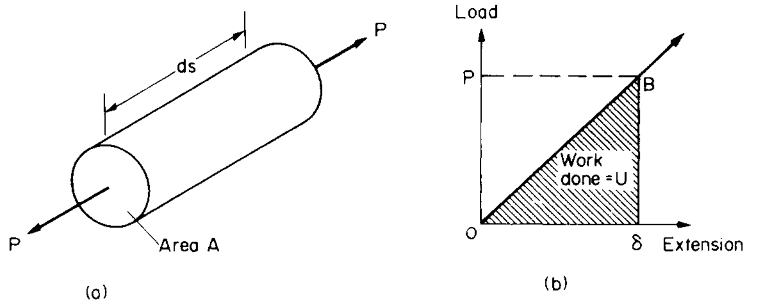

Fig. 11.1. Work done by a gradually applied load.

The strain energy per unit volume is often referred to as the *resilience*. The value of the resilience at the yield point or at the proof stress for non-ferrous materials is then termed the *proof resilience*.

The unshaded area above the line *OB* of Fig. 11.1 is called the *complementary energy*, a quantity which is utilised in some advanced energy methods of solution and is not considered within the terms of reference of this text.†

11.1. Strain energy – tension or compression

(a) *Neglecting the weight of the bar*

Consider a small element of a bar, length *ds*, shown in Fig. 11.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

i.e. $$\text{strain energy } U = \tfrac{1}{2}P\delta$$

Now $$\text{Young's modulus } E = \frac{\text{stress}}{\text{strain}} = \frac{P}{A} \times \frac{ds}{\delta}$$

$\therefore$ $$\delta = \frac{P\,ds}{AE}$$

$\therefore$ $$\text{for the bar element } U = \frac{P^2\,ds}{2AE}$$

$\therefore$ $$\text{total strain energy for a bar of length } L = \int_0^L \frac{P^2\,ds}{2AE}$$

Thus, assuming that the area of the bar remains constant along the length,

$$U = \frac{P^2 L}{2AE} \qquad (11.1)$$

† See H. Ford and J. M. Alexander, *Advanced Mechanics of Materials* (Longmans, London, 1963).

or, in terms of the stress $\sigma\,(=P/A)$,

$$U = \frac{\sigma^2 AL}{2E} = \frac{\boldsymbol{\sigma}^2}{\mathbf{2E}} \times \textbf{volume of bar} \tag{11.2}$$

i.e. strain energy, or resilience, *per unit volume* of a bar subjected to direct load, tensile or compressive

$$= \frac{\boldsymbol{\sigma}^2}{\mathbf{2E}} \tag{11.3}$$

or, alternatively,

$$= \frac{1}{2}\sigma \times \frac{\sigma}{E} = \tfrac{1}{2}\sigma \times \varepsilon$$

i.e. **resilience = ½ stress × strain**

(b) Including the weight of the bar

Consider now a bar of length L mounted vertically, as shown in Fig. 11.2. At any section AB the total load on the section will be the external load P together with the weight of the bar material below AB.

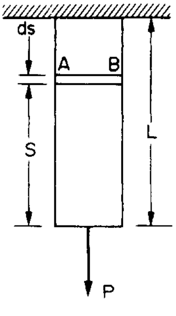

Fig. 11.2. Direct load – tension or compression.

Assuming a uniform cross-section of area A with density ρ,

$$\text{load on section } AB = P \pm \rho g A s$$

the positive sign being used when P is tensile and the negative sign when P is compressive. Thus, for a tensile force P the extension of the element ds is given by the definition of Young's modulus E to be

$$\delta = \frac{\sigma ds}{E}$$

$$= \frac{(P + \delta g A s)}{AE}\, ds$$

$$\therefore \quad \text{work done} = \tfrac{1}{2} \times \text{load} \times \text{extension}$$

$$= \tfrac{1}{2}(P + \rho g A s)\frac{(P + \rho g A s)}{AE}\,ds$$

$$= \frac{P^2}{2AE}\,ds + \frac{P\rho g}{E}\,s\,ds + \frac{(\rho g)^2 A}{2E}\,s^2\,ds$$

$\therefore$ total strain energy or work done

$$= \int_0^L \frac{P^2}{2AE}\,ds + \int_0^L \frac{P\rho g}{E}\,s\,ds + \int_0^L \frac{(\rho g)^2 A}{2E}\,s^2\,ds$$

$$= \frac{P^2 L}{2AE} + \frac{P\rho g L^2}{2E} + \frac{(\rho g)^2 A L^3}{6E} \qquad (11.4)$$

The last two terms are therefore the modifying terms to eqn. (11.1) to account for the body-weight effect of the bar.

11.2. Strain energy – shear

Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle γ (the shear strain) and a shear deflection δ, as shown in Fig. 11.3.

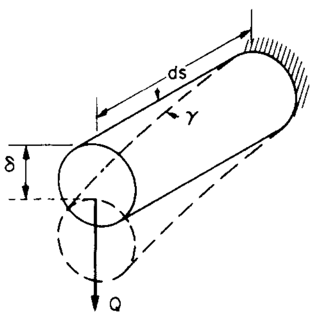

Fig. 11.3. Shear.

$$\text{Strain energy } U = \text{work done} = \tfrac{1}{2}Q\delta = \tfrac{1}{2}Q\gamma\,ds$$

Now

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$$

$$\therefore \quad \gamma = \frac{Q}{AG}$$

$$\therefore \quad \text{shear strain energy} = \tfrac{1}{2}Q \times \frac{Q}{AG} \times ds = \frac{Q^2}{2AG}\,ds$$

∴ total strain energy resulting from shear

$$= \int_0^L \frac{Q^2 ds}{2AG} = \frac{Q^2 L}{2AG} \tag{11.5}$$

or, in terms of the shear stress $\tau = (Q/A)$,

$$U = \frac{\tau^2 AL}{2G} = \frac{\tau^2}{2G} \times \textbf{volume of bar} \tag{11.6}$$

11.3. Strain energy – bending

Let the element now be subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle $d\theta$ at the centre (Fig. 11.4). The beam will also have moved through an angle $d\theta$.

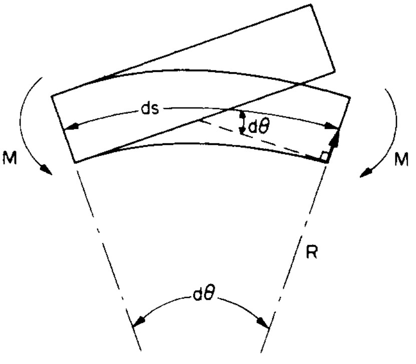

Fig. 11.4. Bending.

Strain energy = work done = $\frac{1}{2}$ × moment × angle turned through (in radians)

$$= \tfrac{1}{2} M\, d\theta$$

But $\qquad ds = R\, d\theta \quad$ and $\quad \dfrac{M}{I} = \dfrac{E}{R}$

$$\therefore \qquad d\theta = \frac{ds}{R} = \frac{M}{EI} ds$$

$$\therefore \qquad \text{strain energy} = \tfrac{1}{2} M \times \frac{M}{EI} ds = \frac{M^2 ds}{2EI}$$

Total strain energy resulting from bending,

$$U = \int_0^L \frac{M^2 ds}{2EI} \tag{11.7}$$

If the bending moment is constant this reduces to

$$U = \frac{M^2 L}{2EI}$$

11.4. Strain energy – torsion

The element is now considered subjected to a torque T as shown in Fig. 11.5, producing an angle of twist $d\theta$ radians.

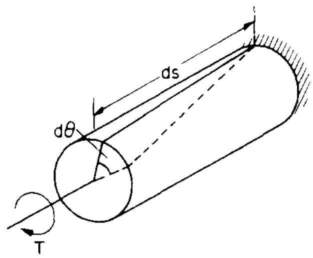

Fig. 11.5. Torsion.

$$\text{Strain energy} = \text{work done} = \tfrac{1}{2} T\, d\theta$$

But, from the simple torsion theory,

$$\frac{T}{J} = \frac{G\,d\theta}{ds} \quad \text{and} \quad d\theta = \frac{T\,ds}{GJ}$$

∴ total strain energy resulting from torsion,

$$U = \int_0^L \frac{T^2\,ds}{2GJ} = \frac{T^2 L}{2GJ} \tag{11.8}$$

since in most practical applications T is constant.

For a hollow circular shaft eqn. (11.8) still applies

i.e. $$\text{Strain energy } U = \frac{T^2 L}{2GJ}$$

Now, from the simple bending theory

$$\frac{T}{J} = \frac{\tau}{r} = \frac{\tau_{max}}{R}$$

where R is the outer radius of the shaft and

$$J = \frac{\pi}{2}(R^4 - r^4)$$

∴ $$T = \frac{\pi}{2R}\tau_{max}(R^4 - r^4)$$

Substituting in the strain energy equation (11.8) we have:

$$U = \frac{\left[\frac{\pi\tau_{max}}{2R}(R^4 - r^4)\right]^2 L}{2G\frac{\pi}{2}(R^4 - r^4)}$$

$$= \frac{\tau_{max}^2}{4G}\frac{\pi(R^4 - r^4)L}{R^2}$$

$$= \frac{\tau_{max}^2}{4G}\frac{[R^2 + r^2]}{R^2} \times \text{volume of shaft}$$

or

$$\textbf{Strain energy/unit volume} = \frac{\tau_{max}^2}{4G}\frac{[R^2 + r^2]}{R^2} \qquad (11.8a)$$

It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

i.e.

$$\text{strain energy } U = \frac{(\text{applied "load"})^2 \times L}{2 \times \text{product of two related constants}}$$

the constants being related to the type of loading considered. In bending, for example, the relevant constants which appear in the bending theory are E and I, whilst for torsion G and J are more applicable. Thus the above standard equations for strain energy should easily be remembered.

11.5. Strain energy of a three-dimensional principal stress system

The reader is referred to §14.17 for the derivation of the following expression for the strain energy of a system of three principal stresses:

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \textbf{per unit volume}$$

It is then shown in §14.17 that this total strain energy can be conveniently considered as made up of two parts:

(a) the *volumetric* or *dilatational* strain energy;
(b) the *shear* or *distortional* strain energy.

11.6. Volumetric or dilatational strain energy

This is the strain energy associated with a mean or hydrostatic stress of $\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \bar{\sigma}$ acting equally in all three mutually perpendicular directions giving rise to no distortion, merely a change in volume.

Then from eqn. (14.22),

$$\textbf{volumetric strain energy} = \frac{(1 - 2\nu)}{6E}[(\sigma_1 + \sigma_2 + \sigma_3)^2] \quad \textbf{per unit volume}$$

11.7. Shear or distortional strain energy

In order to consider the general principal stress case it has been shown necessary, in §14.6, to add to the mean stress $\bar{\sigma}$ in the three perpendicular directions, certain so-called deviatoric stress values to return the stress system to values of σ_1, σ_2 and σ_3. These *deviatoric stresses* are then associated directly with change of shape, i.e. distortion, without change in volume and the strain energy associated with this mechanism is shown to be given by

$$\textbf{shear strain energy} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad \text{per unit volume}$$

$$= \frac{1}{6G}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad \textbf{per unit volume}$$

This equation is used as the basis of the Maxwell–von Mises theory of elastic failure which is discussed fully in Chapter 15.

11.8. Suddenly applied loads

If a load P is applied gradually to a bar to produce an extension δ the load–extension graph will be as shown in Fig. 11.1 and repeated in Fig. 11.6, the work done being given by $U = \frac{1}{2}P\delta$.

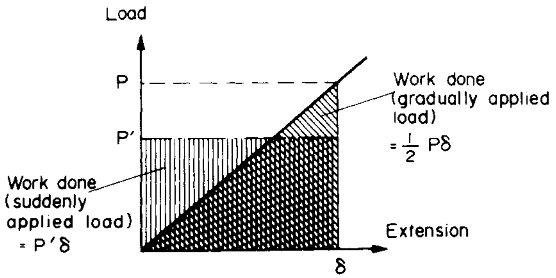

Fig. 11.6. Work done by a suddenly applied load.

If now a load P' is suddenly applied (i.e. applied with an instantaneous value, not gradually increasing from zero to P') to produce the same extension δ, the graph will now appear as a horizontal straight line with a work done or strain energy $= P'\delta$.

The bar will be strained by an equal amount δ in both cases and the energy stored must therefore be equal,

i.e. $$P'\delta = \tfrac{1}{2}P\delta$$

or $$P' = \tfrac{1}{2}P$$

Thus the suddenly applied load which is required to produce a certain value of instantaneous strain is half the equivalent value of static load required to perform the same function. It is then clear that vice versa *a load P which is suddenly applied will produce twice the effect of the same load statically applied.* Great care must be exercised, therefore, in the design of, for example, machine parts to exclude the possibility of sudden applications of load since associated stress levels are likely to be doubled.

11.9. Impact loads – axial load application

Consider now the bar shown vertically in Fig. 11.7 with a rigid collar firmly attached at the end. The load W is free to slide vertically and is suspended by some means at a distance h above the collar. When the load is dropped it will produce a maximum instantaneous extension δ of the bar, and will therefore have done work (neglecting the mass of the bar and collar)

$$= \text{force} \times \text{distance} = W(h+\delta)$$

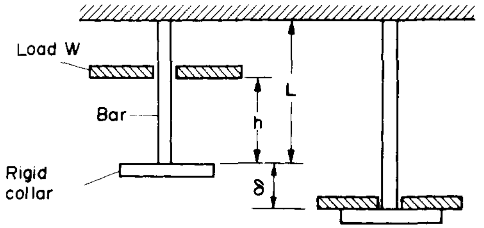

Fig. 11.7. Impact load – axial application.

This work will be stored as strain energy and is given by eqn. (11.2):

$$U = \frac{\sigma^2 AL}{2E}$$

where σ is the instantaneous stress set up.

$$\therefore \qquad \frac{\sigma^2 AL}{2E} = W(h+\delta) \tag{11.9}$$

If the extension δ is small compared with h it may be ignored and then, approximately,

$$\sigma^2 = 2WEh/AL$$

i.e.

$$\sigma = \sqrt{\left(\frac{2WEh}{AL}\right)} \tag{11.10}$$

If, however, δ is not small compared with h it must be expressed in terms of σ, thus

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma L}{\delta} \quad \text{and} \quad \delta = \frac{\sigma L}{E}$$

Therefore substituting in eqn. (11.9)

$$\frac{\sigma^2 AL}{2E} = Wh + \frac{W\sigma L}{E}$$

$$\therefore \qquad \frac{\sigma^2 AL}{2E} - \sigma\frac{WL}{E} - Wh = 0$$

$$\sigma^2 - \frac{2W}{A}\sigma - \frac{2WEh}{AL} = 0$$

Solving by "the quadratic formula" and ignoring the negative sign,

$$\sigma = \tfrac{1}{2}\left\{\frac{2W}{A} + \sqrt{\left[\left(\frac{2W}{A}\right)^2 + 4\left(\frac{2WEh}{AL}\right)\right]}\right.$$

i.e.

$$\sigma = \frac{W}{A} + \sqrt{\left[\left(\frac{W}{A}\right)^2 + \frac{2WEh}{AL}\right]} \tag{11.11}$$

This is the *accurate* equation for the *maximum* stress set up, and should always be used if there is any doubt regarding the relative magnitudes of δ and h.

Instantaneous extensions can then be found from

$$\delta = \frac{\sigma L}{E}$$

If the load is not dropped but *suddenly applied* from effectively zero height, $h = 0$, and eqn. (11.11) reduces to

$$\sigma = \frac{W}{A} + \frac{W}{A} = \frac{2W}{A}$$

This verifies the work of §11.8 and confirms that stresses resulting from suddenly applied loads are twice those resulting from statically applied loads of the same magnitude. Inspection of eqn. (11.11) shows that stresses resulting from impact loads of similar magnitude will be even higher than this and any design work in applications where impact loading is at all possible should always include a safety factor well in excess of two.

11.10. Impact loads – bending applications

Consider the beam shown in Fig. 11.8 subjected to a shock load W falling through a height h and producing an instantaneous deflection δ.

$$\text{Work done by falling load} = W(h + \delta)$$

In these cases it is often convenient to introduce an *equivalent static load* W_E defined as that load which, when gradually applied, produces the same deflection as the shock load

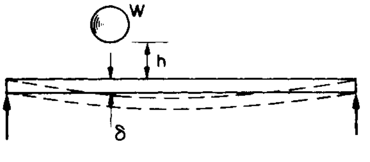

Fig. 11.8. Impact load – bending application.

which it replaces, then

$$\text{work done by equivalent static load} = \tfrac{1}{2}W_E\delta$$

$$W(h+\delta) = \tfrac{1}{2}W_E\delta \qquad (11.12)$$

Thus if δ is obtained in terms of W_E using the standard deflection equations of Chapter 5 for the support conditions in question, the above equation becomes a quadratic equation in one unknown W_E. Hence W_E can be determined and the required stresses or deflections can be found on the equivalent beam system using the usual methods for static loading, i.e. the dynamic load case has been reduced to the equivalent static load condition.

Alternatively, if W produces a deflection δ_s when applied statically then, by proportion,

$$\frac{W_E}{\delta} = \frac{W}{\delta_s} \quad \text{or} \quad W_E = \frac{\delta}{\delta_s}W$$

Substituting in eqn. (11.12)

$$W(h+\delta) = \tfrac{1}{2}W \times \frac{\delta}{\delta_s} \times \delta$$

$$\therefore \quad \delta^2 - 2\delta_s\delta - 2\delta_s h = 0$$

$$\therefore \quad \delta = \delta_s \pm \sqrt{(\delta_s + 2\delta_s h)}$$

$$\delta = \delta_s\left[1 \pm \left(1 + \frac{2h}{\delta_s}\right)^{\frac{1}{2}}\right] \qquad (11.13)$$

The instantaneous deflection of any shock-loaded system is thus obtained from a knowledge of the static deflection produced by an equal load. Stresses are then calculated as before.

11.11. Castigliano's first theorem for deflection

Castigliano's first theorem states that:

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

i.e. if U is the total strain energy, the deflection in the direction of load $W = \partial U/\partial W$.

In order to prove the theorem, consider the beam or structure shown in Fig. 11.9 with forces P_A, P_B, P_C, etc., acting at points A, B, C, etc.

If a, b, c, etc., are the deflections in the direction of the loads then the total strain energy of the system is equal to the work done.

$$U = \tfrac{1}{2}P_A a + \tfrac{1}{2}P_B b + \tfrac{1}{2}P_C c + \ldots \qquad (11.14)$$

N.B. *Limitations of theory.* The above simplified approach to impact loading suffers severe limitations. For example, the distribution of stress and strain under impact conditions will not strictly be the same as under static loading, and perfect elasticity of the bar will not be exhibited. These and other effects are discussed by Roark and Young in their advanced treatment of dynamic stresses: *Formulas for Stress & Strain*, 5th edition (McGraw Hill), Chapter 15.

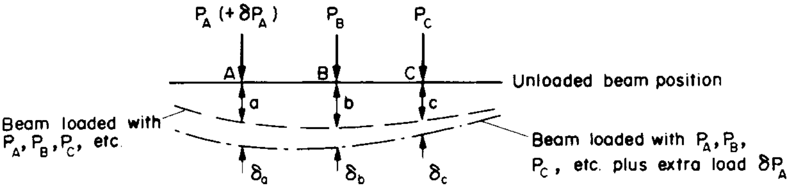

Fig. 11.9. Any beam or structure subjected to a system of applied concentrated loads P_A, P_B, $P_C \dots P_N$, etc.

If one of the loads, P_A, is now increased by an amount δP_A the changes in deflections will be δa, δb and δc, etc., as shown in Fig. 11.9.

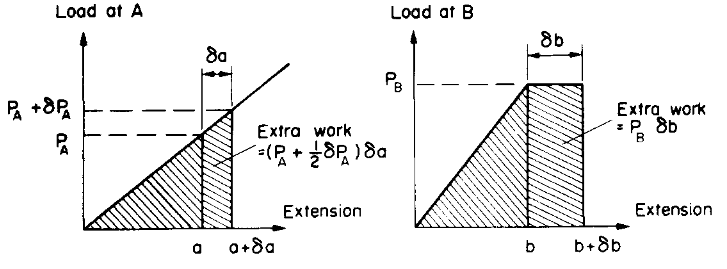

Fig. 11.10. Load–extension curves for positions A and B.

Extra work done at A (see Fig. 11.10)

$$= (P_A + \tfrac{1}{2}\delta P_A)\delta a$$

Extra work done at B, C, etc. (see Fig. 11.10)

$$= P_B\,\delta b,\; P_C\,\delta c, \text{ etc.}$$

Increase in strain energy

$$= \text{total extra work done}$$

$$\therefore \qquad \delta U = P_A\,\delta a + \tfrac{1}{2}\delta P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \dots$$

and neglecting the product of small quantities

$$\delta U = P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \dots \qquad (11.15)$$

But if the loads $P_A + \delta P_A$, P_B, P_C, etc., were applied gradually from zero the total strain energy would be

$$U + \delta U = \sum \tfrac{1}{2} \times \text{load} \times \text{extension}$$

$$U + \delta U = \tfrac{1}{2}(P_A + \delta P_A)(a + \delta a) + \tfrac{1}{2}P_B(b + \delta b) + \tfrac{1}{2}P_C(c + \delta c) + \dots$$

$$= \tfrac{1}{2}P_A\,a + \tfrac{1}{2}P_A\,\delta a + \tfrac{1}{2}\,\delta P_A\,a + \tfrac{1}{2}\delta P_A\,\delta a + \tfrac{1}{2}P_B\,b + \tfrac{1}{2}P_B\,\delta b + \tfrac{1}{2}P_C\,c + \tfrac{1}{2}P_C\,\delta c + \dots$$

Neglecting the square of small quantities ($\tfrac{1}{2}\delta P_A \delta a$) and subtracting eqn. (11.14),

$$\delta U = \tfrac{1}{2}\delta P_A\,a + \tfrac{1}{2}P_A\,\delta a + \tfrac{1}{2}P_B\,\delta b + \tfrac{1}{2}P_C\,\delta c + \dots$$

or

$$2\delta U = \delta P_A\,a + P_A\,\delta a + P_B\,\delta b + P_C\,\delta c + \dots$$

Subtracting eqn. (11.15),

$$\delta U = \delta P_A\, a \quad \therefore \quad \frac{\delta U}{\delta P_A} = a$$

or, in the limit,

$$\frac{\partial U}{\partial P_A} = a$$

i.e. the partial differential of the strain energy U with respect to P_A gives the deflection under and in the direction of P_A. Similarly,

$$\frac{\partial U}{\partial P_B} = b \quad \text{and} \quad \frac{\partial U}{\partial P_C} = c, \text{ etc.}$$

In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

$$\therefore \qquad U = \int \frac{M^2}{2EI}\, ds$$

$$\frac{\partial U}{\partial P} = \frac{\partial U}{\partial M} \times \frac{\partial M}{\partial P} = \int \frac{2M}{2EI}\, ds \times \frac{\partial M}{\partial P}$$

i.e.

$$\delta = \frac{\partial U}{\partial P} = \int \frac{M}{EI} \frac{\partial M}{\partial P}\, ds \qquad (11.16)$$

which is the usual form of Castigliano's first theorem. The integral is evaluated as it stands to give the deflection under an existing load P, the value of the bending moment M at some general section having been determined in terms of P. If no general expression for M in terms of P can be obtained to cover the whole beam then the beam, and hence the integral limits, can be divided into any number of convenient parts and the results added. In cases where the deflection is required at a point or in a direction in which there is no load applied, an imaginary load P is introduced in the required direction, the integral obtained in terms of P and then evaluated with P equal to zero.

The above procedures are illustrated in worked examples at the end of this chapter.

11.12. "Unit-load" method

It has been shown in §11.11 that in applications where bending provides practically all of the total strain energy of a system

$$\delta = \int \frac{M}{EI} \frac{\partial M}{\partial W}\, ds$$

Now W is an applied concentrated load and M will therefore include terms of the form Wx, where x is some distance from W to the point where the bending moment (B.M.) is required plus terms associated with the other loads. The latter will reduce to zero when partially differentiated with respect to W since they do not include W.

Now

$$\frac{\partial}{\partial W}(Wx) = x = 1 \times x$$

i.e. the partial differential of the B.M. term containing W is identical to the result achieved if W is replaced by unity in the B.M. expression. Using this information the Castigliano expression can be simplified to remove the partial differentiation procedure, thus

$$\delta = \int \frac{Mm}{EI}\, ds \tag{11.17}$$

where m is the B.M. resulting from a *unit load only* applied at the point of application of W and in the direction in which the deflection is required. The value of M remains the same as in the standard Castigliano procedure and is therefore the B.M. due to the *applied load system, **including W***.

This so-called "unit load" method is particularly powerful for cases where deflections are required at points where no external load is applied or in directions different from those of the applied loads. The method mentioned previously of introducing imaginary loads P and then subsequently assuming P is zero often gives rise to confusion. It is much easier to simply apply a unit load at the point, and in the direction, in which deflection is required regardless of whether external loads are applied there or not (see Example 11.6).

11.13. Application of Castigliano's theorem to angular movements

Castigliano's theorem can also be applied to angular rotations under the action of bending moments or torques. For the bending application the theorem becomes:

> *If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction,*

i.e.

$$\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M_i}\, ds \tag{11.18}$$

where M_i is the imaginary or applied moment at the point where θ is required.

Alternatively the "unit-load" procedure can again be used, this time replacing the applied or imaginary moment at the point where θ is required by a "unit moment". Castigliano's expression for slope or angular rotation then becomes

$$\theta = \int \frac{Mm}{EI} \cdot ds$$

where M is the bending moment at a general point due to the applied loads or moments and m is the bending moment at the same point due to the unit moment at the point where θ is required and in the required direction. See Example 11.8 for a simple application of this procedure.

11.14. Shear deflection

(a) Cantilever carrying a concentrated end load

In the majority of beam-loading applications the deflections due to bending are all that need be considered. For very short, deep beams, however, a secondary deflection, that due to shear, must also be considered. This may be determined using the strain energy formulae derived earlier in this chapter.

For bending, $$U_B = \int_0^L \frac{M^2\,ds}{2EI}$$

For shear, $$U_S = \int_0^L \frac{Q^2\,ds}{2AG} = \frac{\tau^2}{2G} \times \text{volume}$$

Consider, therefore, the cantilever, of solid rectangular section, shown in Fig. 11.11.

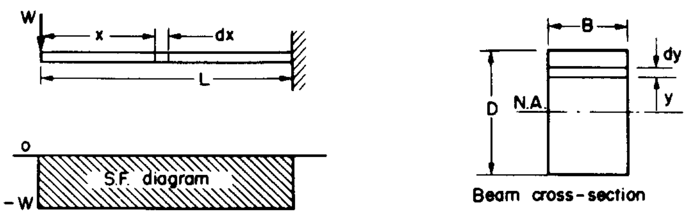

Fig. 11.11.

For the element of length dx

$$U_S = \int \frac{\tau^2}{2G} \times B\,dy\,dx$$

But $$\tau = \frac{QA\bar{y}}{Ib} \quad \text{(see §7.1)}$$

$$= Q \times \frac{B\left(\dfrac{D}{2} - y\right)}{IB} \quad \left[\frac{\left(\dfrac{D}{2} - y\right)}{2} + y\right]$$

$$= \frac{Q}{2I}\left(\frac{D^2}{4} - y^2\right)$$

∴ $$U_S = \frac{1}{2G}\int\left\{\frac{Q}{2I}\left(\frac{D^2}{4} - y^2\right)\right\}^2 B\,dx\,dy$$

$$= \frac{Bdx}{2G}\int_{-D/2}^{D/2}\left\{\frac{Q}{2I}\left(\frac{D^2}{4} - y^2\right)\right\}^2 dy$$

$$= \frac{Q^2B}{8GI^2}\,dx\left(\frac{D^5}{30}\right)$$

To obtain the total strain energy we must now integrate this along the length of the cantilever. In this case Q is constant and equal to W and the integration is simple.

$$U_s = \int_0^L \frac{W^2 B}{8GI^2}\frac{D^5}{30}\,dx$$

$$= \frac{W^2 B}{8GI^2}\frac{D^5}{30}L = \frac{W^2 BLD^5}{240G}\left(\frac{12}{BD^3}\right)^2$$

$$= \frac{3W^2 L}{5AG}$$

where $A = BD$.

Therefore deflection due to shear

$$\delta_s = \frac{\partial U_s}{\partial W} = \frac{\mathbf{6}WL}{\mathbf{5}AG} \qquad (11.19)$$

Similarly, since $M = -Wx$

$$U_B = \int_0^L \frac{(-Wx)^2}{2EI}\,ds = \frac{W^2 L^3}{6EI}$$

Therefore deflection due to bending

$$\delta_B = \frac{\partial U}{\partial W} = \frac{WL^3}{\mathbf{3}EI} \qquad (11.20)$$

Comparison of eqns. (11.19) and (11.20) then yields the relationship between the shear and bending deflections. For very short beams, where the length equals the depth, the shear deflection is almost twice that due to bending. For longer beams, however, the bending deflection is very much greater than that due to shear and the latter can usually be neglected, e.g. for $L = 10D$ the deflection due to shear is less than 1 % of that due to bending.

(b) Cantilever carrying uniformly distributed load

Consider now the same cantilever but carrying a uniformly distributed load over its complete length as shown in Fig. 11.12.

The shear force at any distance x from the free end

$$Q = wx$$

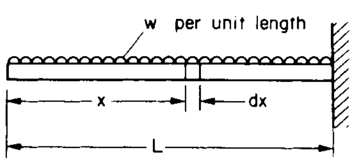

Fig. 11.12.

Therefore shear deflection over the length of the small element dx

$$= \frac{6}{5}\frac{(wx)}{AG}\,dx \qquad \text{from (11.19)}$$

Therefore total shear deflection

$$\delta_S = \int_0^L \frac{6}{5}\frac{wx\,dx}{AG} = \frac{3wL^2}{5AG} \tag{11.21}$$

(*c*) *Simply supported beam carrying central concentrated load*

In this case it is convenient to treat the beam as two cantilevers each of length equal to half the beam span and each carrying an end load half that of the central beam load (Fig. 11.13). The required central deflection due to shear will equal that of the end of each cantilever, i.e. from eqn. (11.19), with $W = W/2$ and $L = L/2$,

$$\delta_s = \frac{6}{5AG}\left(\frac{W}{2}\times\frac{L}{2}\right) = \frac{3WL}{10AG} \tag{11.22}$$

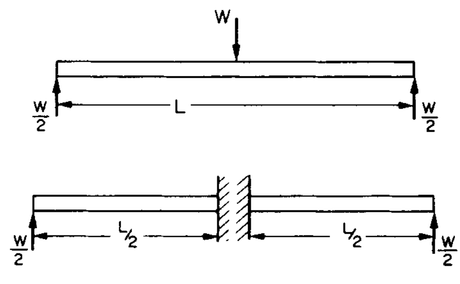

Fig. 11.13. Shear deflection of simply supported beam carrying central concentrated load–equivalent loading diagram.

(*d*) *Simply supported beam carrying a concentrated load in any position*

If the load divides the beam span into lengths a and b the reactions at each end will be Wa/L and Wb/L. The equivalent cantilever system is then shown in Fig. 11.14 and the shear

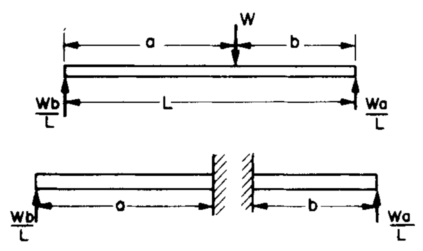

Fig. 11.14. Equivalent loading for offset concentrated load.

deflection under the load is equal to the end deflection of either cantilever and given by eqn. (11.19),

$$\delta_s = \frac{6}{5AG}\left(\frac{Wa}{L}\right)b \quad \text{or} \quad \delta_s = \frac{6}{5AG}\left(\frac{Wb}{L}\right)a$$

$$\therefore \qquad \boldsymbol{\delta_s = \frac{6Wab}{5AGL}} \qquad (11.23)$$

(e) Simply supported beam carrying uniformly distributed load

Using a similar treatment to that described above, the equivalent cantilever system is shown in Fig. 11.15, i.e. each cantilever now carries an end load of $wL/2$ in one direction and a uniformly distributed load w over its complete length $L/2$ in the opposite direction.

From eqns. (11.19) and (11.20)

$$\delta_s = \frac{6}{5AG}\left(\frac{wL}{2} \times \frac{L}{2}\right) - \frac{3}{5AG}\,w\left(\frac{L}{2}\right)^2$$

$$\boldsymbol{\delta_s = \frac{3wL^2}{20AG}} \qquad (11.24)$$

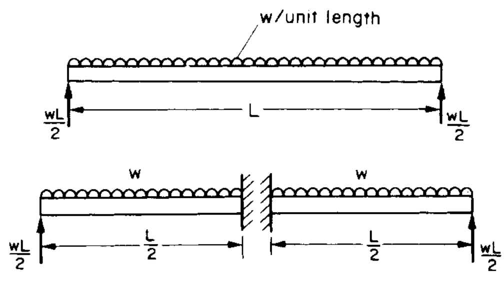

Fig. 11.15. Equivalent loading for uniformly loaded beam.

(f) I-section beams

If the shear force is assumed to be uniformly distributed over the web area A, a similar treatment to that described above yields the following approximate results:

cantilever with concentrated end load W	$\delta_s = \boldsymbol{\frac{WL}{AG}}$
cantilever with uniformly distributed load w	$\delta_s = \frac{wL^2}{2AG} = \boldsymbol{\frac{WL}{2AG}}$
simply supported beam with concentrated end load W	$\delta_s = \boldsymbol{\frac{WL}{4AG}}$
simply supported beam with uniformly distributed load w	$\delta_s = \frac{wL^2}{8AG} = \boldsymbol{\frac{WL}{8AG}}$

In the above expressions the effect of the flanges has been neglected and it therefore follows that the same formulae would apply for rectangular sections if it were assumed that the shear stress is evenly distributed across the section. The result of WL/AG for the cantilever carrying a concentrated end load is then directly comparable to that obtained in eqn. (11.19) taking full account of the variation of shear across the section, i.e. 6/5 (WL/AG). Since the shear strain $\gamma = \delta/L$ it follows that both the deflection and associated shear strain is underestimated by 20% if the shear is assumed to be uniform.

(g) *Shear deflections at points other than loading points*

In the case of simply supported beams, deflections at points other than loading positions are found by simple proportion, deflections increasing linearly from zero at the supports (Fig. 11.16). For cantilevers, however, if the load is not at the free end, the above remains true between the load and the support but between the load and the free end the beam remains horizontal, i.e. there is no shear deflection. This, of course, must not be confused with deflections due to bending when there will always be some deflection of the end of a cantilever whatever the position of loading.

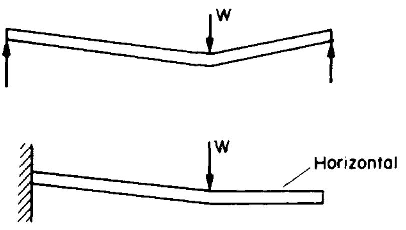

Fig. 11.16. Shear deflections of simply supported beams and cantilevers. *These must not be confused with bending deflections.*

Examples

Example 11.1

Determine the diameter of an aluminium shaft which is designed to store the same amount of strain energy per unit volume as a 50 mm diameter steel shaft of the same length. Both shafts are subjected to equal compressive axial loads.

What will be the ratio of the stresses set up in the two shafts?

$E_{steel} = 200\,GN/m^2$; $E_{aluminium} = 67\,GN/m^2$.

Solution

$$\text{Strain energy per unit volume} = \frac{\sigma^2}{2E}$$

Since the strain energy/unit volume in the two shafts is equal,

then $$\frac{\sigma_A^2}{2E_A} = \frac{\sigma_S^2}{2E_S}$$

$\therefore$ $$\frac{\sigma_A^2}{\sigma_S^2} = \frac{E_A}{E_S} = \frac{67}{200} = \tfrac{1}{3} \text{ (approximately)} \qquad (1)$$

$\therefore$ $$3\sigma_A^2 = \sigma_S^2 \qquad (2)$$

Now $$\sigma = \frac{P}{\text{area}} \quad \text{where } P \text{ is the applied load}$$

Therefore from (1) $$\left[\frac{P}{\frac{\pi}{4}D_A^2}\right]^2 \times \left[\frac{\frac{\pi}{4}D_S^2}{P}\right]^2 = \frac{1}{3}$$

$\therefore$ $$\frac{D_S^4}{D_A^4} = \frac{1}{3}$$

$\therefore$ $$D_A^4 = 3 \times D_S^4 = 3 \times (50)^4$$
$$= 3 \times 625 \times 10^4$$

$\therefore$ $$D_A = \sqrt[4]{(1875 \times 10^4)} = \mathbf{65.8\,mm}$$

The required diameter of the aluminium shaft is 65.8 mm.

From (2) $$3\sigma_A^2 = \sigma_S^2$$

$\therefore$ $$\frac{\sigma_S}{\sigma_A} = \sqrt{3}$$

Example 11.2

Two shafts are of the same material, length and weight. One is solid and 100 mm diameter, the other is hollow. If the hollow shaft is to store 25% more energy than the solid shaft when transmitting torque, what must be its internal and external diameters?

Assume the same maximum shear stress applies to both shafts.

Solution

Let A be the solid shaft and B the hollow shaft. If they are the same weight and the same material their volume must be equal.

$\therefore$ $$\frac{\pi}{4}D_A^2 \times L = \frac{\pi}{4}[D_B^2 - d_B^2]\,L$$

$\therefore$ $$D_A^2 = D_B^2 - d_B^2 = \frac{100^2}{10^6}\,m^2 = \mathbf{10 \times 10^{-3}\,m^2} \qquad (1)$$

Now for the same maximum shear stress

$$\tau = \frac{Tr}{J} = \frac{TD}{2J}$$

i.e.
$$\frac{T_A D_A}{J_A} = \frac{T_B D_B}{J_B}$$

$$\therefore \quad \frac{T_A}{T_B} = \frac{D_B J_A}{D_A J_B} \qquad (2)$$

But the strain energy of $B = 1.25 \times$ strain energy of A.

$$\therefore \quad \text{since } U = \frac{T^2 L}{2GJ}$$

then
$$\frac{T_B^2 L}{2GJ_B} = 1.25 \frac{T_A^2 L}{2GJ_A} \quad \text{or} \quad \frac{T_A^2}{T_B^2} = \frac{J_A}{1.25 J_B}$$

Therefore substituting from (2),

$$\frac{D_B^2}{D_A^2} = \frac{J_B}{1.25 J_A}$$

$$\therefore \quad \frac{D_B^2}{D_A^2} = \frac{\frac{\pi}{32}[D_B^4 - d_B^4]}{1.25 \frac{\pi}{32} D_A^4} = \frac{D_B^4 - d_B^4}{1.25 D_A^4}$$

$$D_B^2 = \frac{D_B^4 - d_B^4}{1.25 D_A^2}$$

$$= \frac{D_B^4 - (D_B^2 - 10 \times 10^{-3})^2}{1.25 \times 10 \times 10^{-3}}$$

$$12.5 \times 10^{-3} D_B^2 = D_B^4 - D_B^4 + 20 \times 10^{-3} D_B^2 - 100 \times 10^{-6}$$

$$\therefore \quad 7.5 \times 10^{-3} \times D_B^2 = 100 \times 10^{-6}$$

$$\therefore \quad D_B^2 = \frac{100 \times 10^{-6}}{7.5 \times 10^{-3}} = 13.3 \times 10^{-3}$$

$$\mathbf{D_B = 115.47\,mm}$$

$$d_B^2 = D_B^2 - D_A^2 = \frac{13.3}{10^3} - \frac{10}{10^3} = \frac{3.3}{10^3}$$

$$\therefore \quad \mathbf{d_B = 57.74\,mm}$$

The internal and external diameters of the hollow tube are therefore 57.7 mm and 115.5 mm respectively.

Example 11.3

(a) What will be the instantaneous stress and èlongation of a 25 mm diameter bar, 2.6 m long, suspended vertically, if a mass of 10 kg falls through a height of 300 mm on to a collar which is rigidly attached to the bottom end of the bar?

Take $g = 10\,\text{m/s}^2$.

(b) When used horizontally as a simply supported beam, a concentrated force of 1 kN applied at the centre of the support span produces a static deflection of 5 mm. The same load will produce a maximum bending stress of 158 MN/m^2.

Determine the magnitude of the instantaneous stress produced when a mass of 10 kg is allowed to fall through a height of 12 mm on to the beam at mid-span.

What will be the instantaneous deflection?

Solution

(a) From eqn. (11.9)

$$W\left(h+\frac{\sigma L}{E}\right)=\frac{\sigma^2}{2E}\times\text{volume}\quad\text{(Fig. 11.7)}$$

$$\text{volume of bar} = \tfrac{1}{4}\pi\times\frac{25^2}{10^6}\times 2.6 = 12.76\times 10^{-4}$$

Then $$10\times 10\left(0.3+\frac{2.6\sigma}{200\times 10^9}\right)=\frac{\sigma^2\times 12.76\times 10^{-4}}{2\times 200\times 10^9}$$

$\therefore$ $$30+\frac{1.3\sigma}{10^9}=\frac{\sigma^2}{313\times 10^{12}}$$

and $$30\times 313\times 10^{12}+\frac{1.3\sigma}{10^9}\times 313\times 10^{12}=\sigma^2$$

Then $$\sigma^2-406.9\times 10^3\times\sigma-9390\times 10^{12}=0$$

$$\sigma=\frac{406.9\times 10^3\pm\sqrt{(166\times 10^9+37560\times 10^{12})}}{2}$$

$$=\frac{406.9\times 10^3\pm 193.9\times 10^6}{2}$$

$$= \mathbf{97.18\ MN/m^2}$$

If the instantaneous deflection is ignored (the term $\sigma L/E$ omitted) in the above calculation a very small difference in stress is noted in the answer,

i.e. $$W(h)=\frac{\sigma^2\times\text{volume}}{2E}$$

$\therefore$ $$100\times 0.3=\frac{\sigma^2\times 12.76\times 10^{-4}}{2\times 200\times 10^9}$$

$\therefore$ $$\sigma^2=\frac{30\times 400\times 10^9}{12.76\times 10^{-4}}=9404\times 10^{12}$$

$\therefore$ $$\sigma=\mathbf{96.97\ MN/m^2}$$

This suggests that if the deflection δ is small in comparison to h (the distance through which the mass falls) it can, for all practical purposes, be ignored in the above calculation:

$$\text{deflection produced } (\delta) = \frac{\sigma L}{E} = \frac{97.18 \times 2.6 \times 10^6}{200 \times 10^9}$$

i.e. $\quad$ elongation of bar = **1.26 mm**

(b) Consider the loading system shown in Fig. 11.8. Let W_E be the equivalent force that produces the same deflection and stress when gradually applied as that produced by the falling mass.

Then
$$\frac{W_E}{\delta_{max}} = \frac{W_s}{\delta_s}$$

where W_s is a known load, gradually applied to the beam at mid-span, producing deflection δ_s and stress σ_s.

Then
$$\delta_{max} = \frac{W_E \delta_s}{W_s} = \frac{W_E \times 5 \times 10^{-3}}{1 \times 10^3}$$

$\therefore$
$$\delta_{max} = \frac{5}{10^6} W_E$$

Now
$$W(h + \delta_{max}) = \frac{W_E}{2} \delta_{max}$$

$\therefore$
$$100\left[\frac{12}{10^3} + \frac{5 W_E}{10^6}\right] = \frac{W_E}{2} \times \frac{5 W_E}{10^6}$$

$$1.2 + \frac{500 W_E}{10^6} = \frac{2.5 W_E^2}{10^6}$$

$\therefore$
$$W_E^2 - \frac{500 W_E}{2.5} - \frac{1.2 \times 10^6}{2.5} = 0$$

and
$$W_E^2 - 200 W_E - 0.48 \times 10^6 = 0$$

By factors, $\quad W_E = 800\text{ N} \quad \text{or} \quad -600\text{ N}$

$\therefore \quad$ $\boldsymbol{W_E = 800\text{ N}}$

By proportion
$$\frac{\sigma_s}{W_s} = \frac{\sigma_{max}}{W_E}$$

and the maximum stress is given by

$$\sigma_{max} = \frac{\sigma_s}{W_s} \times W_E = \frac{158 \times 10^6 \times 800}{1 \times 10^3} = \mathbf{126.4\ MN/m^2}$$

And since
$$\frac{W_E}{\delta} = \frac{W_s}{\delta_s}$$

the deflection is given by

$$\delta = \frac{W_E}{W_s} \times \delta_s$$

$$= \frac{800 \times 5 \times 10^{-3}}{1 \times 10^3} = 4 \times 10^{-3}$$

$$= \mathbf{4\,mm}$$

Example 11.4

A horizontal steel beam of I-section rests on a rigid support at one end, the other end being supported by a vertical steel rod of 20 mm diameter whose upper end is rigidly held in a support 2.3 m above the end of the beam (Fig. 11.17). The beam is a 200 × 100 mm B.S.B. for which the relevant I-value is $23 \times 10^{-6}\,m^4$ and the distance between its two points of support is 3 m. A load of 2.25 kN falls on the beam at mid-span from a height of 20 mm above the beam.

Determine the maximum stresses set up in the beam and rod, and find the deflection of the beam at mid-span measured from the unloaded position. Assume $E = 200\,GN/m^2$ for both beam and rod.

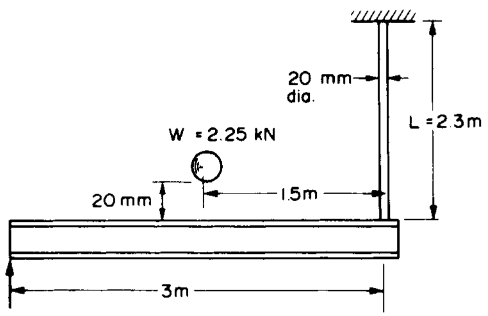

Fig. 11.17.

Solution

Let the shock load cause a deflection δ_B of the beam at the load position and an extension δ_R of the rod. Then if W_E is the equivalent static load which produces the deflection δ_B and P is the maximum tension in the rod,

$$\text{total strain energy} = \frac{P^2 L_R}{2AE} + \frac{1}{2} W_E \delta_B$$

$$= \text{work done by falling mass}$$

Now the mass falls through a distance

$$h+\delta_B+\frac{\delta_R}{2}$$

where $\delta_R/2$ is the effect of the rod extension on the mid-point of the beam. (This assumes that the beam remains straight and rotates about the fixed support position.)

$$\therefore \quad \text{work done by falling mass} = W\left(h+\delta_B+\frac{\delta_R}{2}\right)$$

If $\quad P =$ reaction at one end of beam

then $$P=\frac{W_E}{2}$$

$$\therefore \quad W\left(h+\delta_B+\frac{\delta_R}{2}\right)=\frac{W_E^2 L_B}{8AE}+\frac{W_E\delta_B}{2} \qquad (1)$$

For a centrally loaded beam $$\delta=\frac{WL^3}{48EI}$$

$$\therefore \quad \delta_B=\frac{W_E\times 3^3}{48\times 200\times 10^9\times 23\times 10^{-6}}=\frac{W_E}{8.18\times 10^6} \qquad (2)$$

For an axially loaded rod $$\delta_R=\frac{WL}{AE}$$

$$\therefore \quad \delta_R=\frac{W_E\times 2.3}{\frac{\pi}{4}\times 20^2\times 10^{-6}\times 200\times 10^9}=\frac{W_E}{27.3\times 10^6} \qquad (3)$$

Substituting (2) and (3) in (1),

$$2.25\times 10^3\left[\frac{20}{10^3}+\frac{W_E}{8.18\times 10^6}+\frac{W_E}{54.6\times 10^6}\right]=\frac{W_E^2\times 2.3}{8\,(\frac{\pi}{4}\times 20^2\times 10^{-6})\times 200\times 10^9}+\frac{W_E^2}{2\times 8.18\times 10^6}$$

$$45+\frac{2.25\times 10^3 W_E}{8.18\times 10^6}+\frac{2.25\times 10^3 W_E}{54.6\times 10^6}=\frac{W_E^2\times 2.3}{8\times 314\times 10^{-6}\times 200\times 10^9}+\frac{W_E^2}{16.36\times 10^6}$$

$$45+275\times 10^{-6}\,W_E+41.2\times 10^{-6}\,W_E=4.58\times 10^{-9}\,W_E^2+61.1\times 10^{-9}\,W_E^2$$

$$45+316.2\,W_E\times 10^{-6}=65.68\times 10^{-9}\,W_E^2$$

Then $$W_E^2-\frac{316.2\times 10^{-6}}{65.68\times 10^{-9}}\,W_E-\frac{45}{65.68\times 10^{-9}}=0$$

$$\therefore \quad W_E^2-4.8\times 10^3\,W_E-685\times 10^6=0$$

and $$W_E = \frac{4.8 \times 10^3 \pm \sqrt{(23 \times 10^6 + 2740 \times 10^6)}}{2}$$

$$= \frac{4.8 \times 10^3 \pm \sqrt{(2763 \times 10^6)}}{2}$$

$$= \frac{4.8 \times 10^3 \pm 52.59 \times 10^3}{2}$$

$$= \frac{57.3 \times 10^3}{2}$$

$$= 28.65 \times 10^3 \text{ N}$$

$$\text{Maximum bending moment} = \frac{W_E L}{4}$$

$$= \frac{28.65 \times 10^3 \times 3}{4}$$

$$= 21.5 \times 10^3 \text{ N}$$

$$\text{Then maximum bending stress} = \frac{My}{I}$$

$$= \frac{21.5 \times 10^3 \times 100 \times 10^{-3}}{23 \times 10^{-6}}$$

$$= 93.9 \times 10^6 \text{ N/m}^2$$

$$\text{Maximum stress in rod} = \frac{\frac{1}{2} W_E}{\text{area}}$$

$$= \frac{28.65 \times 10^3}{2 \times \frac{\pi}{4} \times 20^2 \times 10^{-6}}$$

$$= \mathbf{45.9 \times 10^6 \text{ N/m}^2}$$

$$\text{Deflection of beam } \delta_B = \frac{W_E}{8.18 \times 10^6}$$

$$= \frac{28.65 \times 10^3}{8.18 \times 10^6}$$

$$= \mathbf{3.52 \times 10^{-3} \text{ m}}$$

This is the extension at mid-span and neglects the extension of the rod.

$$\text{Extension of rod} = \frac{\sigma L}{E} = \frac{PL}{AE} = \frac{W_E L}{2AE}$$

$$= \frac{28.8 \times 10^3 \times 2.3}{2 \times 314 \times 10^{-6} \times 200 \times 10^9}$$

$$= \mathbf{0.527 \times 10^{-3} \text{ m}}$$

Assuming, as stated earlier, that the beam remains straight and that the beam rotates about the fixed end, then the effect of the rod extension at the mid-span

$$= \frac{\delta_R}{2} = \frac{0.527 \times 10^{-3}}{2} = 0.264 \times 10^{-3}\,\text{m}$$

Then, total deflection at mid-span $= \delta_B + \delta_R/2$

$$= 3.52 \times 10^{-3} + 0.264 \times 10^{-3}$$

$$= \mathbf{3.784 \times 10^{-3}\,m}$$

Example 11.5

Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Fig. 11.18, (b) the deflection at the centre of a simply supported beam carrying a uniformly distributed load.

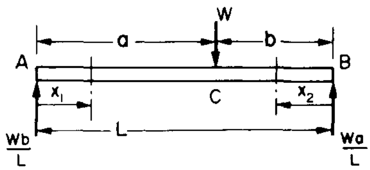

Fig. 11.18.

Solution

(a) For the beam shown in Fig. 11.18

$$\delta = \int_B^A \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \int_A^C \frac{M}{EI}\frac{\partial M}{\partial W}\,ds + \int_C^B \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \frac{1}{EI}\int_0^a \frac{Wbx_1}{L} \times \frac{bx_1}{L} \times dx_1 + \frac{1}{EI}\int_0^b \frac{Wax_2}{L} \times \frac{ax_2}{L} \times dx_2$$

$$= \frac{Wb^2}{L^2EI}\int_0^a x_1^2\,dx_1 + \frac{Wa^2}{L^2EI}\int_0^b x_2^2\,dx_2$$

$$= \frac{Wb^2a^3}{3L^2EI} + \frac{Wa^2b^3}{3L^2EI} = \frac{Wa^2b^2}{3L^2EI}(a+b) = \mathbf{\frac{Wa^2b^2}{3LEI}}$$

(b) For the u.d.l. beam shown in Fig. 11.19a an imaginary load P must be introduced at mid-span; then the mid-span deflection will be

$$\delta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial W}\,ds = 2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

but
$$M_{xx} = \frac{(wL+P)}{2}x - \frac{wx^2}{2} \quad \text{and} \quad \frac{\partial M}{\partial W} = \frac{x}{2}$$

Then
$$\delta = \frac{2}{EI}\int_0^{L/2} \left[\frac{(wL+P)}{2}x - \frac{wx^2}{2}\right]\frac{x}{2}\,dx$$

$$= \frac{1}{2EI}\int_0^{L/2} (wLx^2 - wx^3)\,dx \quad \text{since } P = 0$$

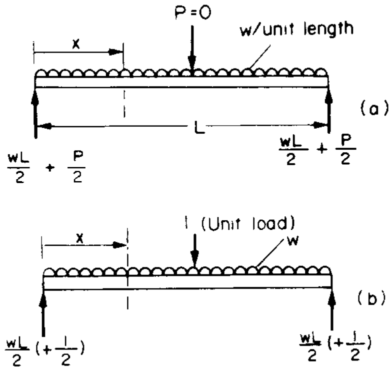

Fig. 11.19.

Alternatively, using a unit load applied vertically at mid-span (Fig. 11.19b),

$$\delta = \int_0^L \frac{Mm}{EI}\,ds = 2\int_0^{L/2} \frac{Mm}{EI}\,ds$$

where
$$M = \frac{wL}{2} - \frac{wx^2}{2} \quad \text{and} \quad m = \frac{x}{2}$$

Then
$$\delta = \frac{2}{EI}\int_0^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

$$= \frac{1}{2EI}\int_0^{L/2} (wLx^2 - wx^3)\,dx$$

as before. Thus, in each case,

$$\delta = \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/2}$$

$$= \frac{wL^4}{2EI}\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{wL^4}{2EI}\left[\frac{8-3}{192}\right] = \mathbf{\frac{5WL^4}{384EI}}$$

Example 11.6

Determine by the methods of unit load and Castigliano's first theorem, (a) the vertical deflection of point A of the bent cantilever shown in Fig. 11.20 when loaded at A with a vertical load of 600 N. (b) What will then be the horizontal movement of A?

The cantilever is constructed from 50 mm diameter bar throughout, with $E = 200\ \text{GN/m}^2$.

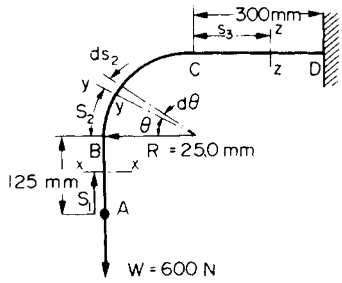

Fig. 11.20.

Solution

The total deflection of A can be considered in three parts, resulting from AB, BC, and CD. Since the question requires solution by two similar methods, they will be worked in parallel.

(a) *For vertical deflection*

Castigliano	Unit load
$\delta = \int \frac{M}{EI}\frac{\partial M}{\partial W} ds$	$\delta = \int \frac{Mm}{EI} ds$ where m = bending moment resulting from a unit load at A.
For AB $M_{xx} = 0$. Hence vertical deflection resulting from $AB = 0$ by both methods.	
For CD $M_{zz} = W(0.25 + s_3)$ $\frac{\partial M}{\partial W} = 0.25 + s_3$ $\delta_{CD} = \int_0^{0.3} \frac{W(0.25+s_3)(0.25+s_3)ds_3}{EI}$	$M_{zz} = W(0.25 + s_3)$ $m = 1(0.25) + s_3)$ $\therefore \delta_{CD} = \int_0^{0.3} \frac{W(0.25+s_3)(0.25+s_3)ds_3}{EI}$

Thus the same equation is achieved by both methods.

Castigliano	Unit load

$\therefore$
$$\delta_{CD} = \frac{W}{EI}\int_0^{0.3} (0.0625 + 0.5\, s_3 + s_3^2)\, ds_3$$
$$= \frac{W}{EI}\left[0.0625\, s_3 + \frac{0.5\, s_3^2}{2} + \frac{s_3^3}{3}\right]_0^{0.3}$$
$$= \frac{W}{EI}[0.01875 + 0.0225 + 0.009]$$
$$= \frac{600}{EI} \times 0.05025 = \frac{\mathbf{30.15}}{\mathbf{EI}}$$

For BC	
$M_{yy} = W(0.25 - 0.25\cos\theta)$	$M_{yy} = W(0.25 - 0.25\cos\theta)$
$\frac{\partial M}{\partial W} = 0.25 - 0.25\cos\theta$	$m = 1(0.25 - 0.25\cos\theta)$
$ds_2 = 0.25\, d\theta$	$ds_2 = 0.25\, d\theta$

Once again the same equation for deflection is obtained

i.e.
$$\delta_{BC} = \int_0^{\pi/2} \frac{W(0.25 - 0.25\cos\theta)}{EI}(0.25 - 0.25\cos\theta)\, 0.25\, d\theta$$
$$= \frac{(0.25)^3 W}{EI}\int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

but
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$\therefore$
$$\delta_{BC} = \frac{(0.25)^3 W}{EI}\int_0^{\pi/2}\left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$$
$$= \frac{(0.25)^3 W}{EI}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$
$$= \frac{(0.25)^3 W}{EI}\left[\frac{\pi}{2} - 2 + \frac{\pi}{4}\right]$$
$$= \frac{(0.25)^3 \times 600}{EI}[\tfrac{3}{4}\pi - 2]$$
$$= \frac{\mathbf{3.34}}{\mathbf{EI}}$$

Total vertical deflection at A

$$= \frac{30.15 + 3.34}{EI} = \frac{33.49 \times 64 \times 10^{12}}{200 \times 10^9 \times \pi \times 50^4} = \mathbf{0.546\ mm}$$

Castigliano	Unit load

Again, working in parallel with Castigliano and unit load methods:–

(b) For the horizontal deflection using Castigliano's method an imaginary load P must be applied horizontally since there is no external load in this direction at A (Fig. 11.21).

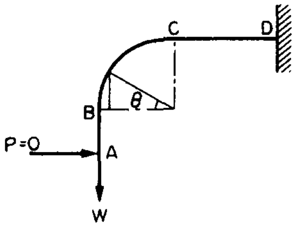

Fig. 11.21.

Then $\delta_H = \int \frac{M}{EI}\frac{\partial M}{\partial P}ds$, with $P = 0$

For AB

$$M_{xx} = P \times s_1 + W \times 0 = Ps_1$$

$$\therefore \quad \frac{\partial M}{\partial P} = s_1$$

$$\therefore \quad \delta_{AB} = \int \frac{Ps_1}{EI} \times s_1\, ds_1$$

but $P = 0$

$$\therefore \quad \delta_{AB} = 0$$

For BC

$$M_{yy} = W(0.25 - 0.25\cos\theta) + P(0.125 + 0.25\sin\theta)$$

$$\frac{\partial M}{\partial P} = 0.125 + 0.25\sin\theta$$

$$ds_2 = 0.25\,d\theta$$

$$\therefore \quad \delta_{BC} = \int_0^{\pi/2} \frac{W}{EI}(0.25 - 0.25\cos\theta) \times (0.125 + 0.25\sin\theta)\,0.25\,d\theta$$

since $P = 0$

For the unit load method a unit load must be applied at A in the direction in which the deflection is required as shown in Fig. 11.22.

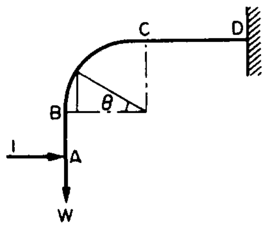

Fig. 11.22.

Then $\delta_H = \int \frac{Mm}{EI} ds$

$$M_{xx} = W \times 0 = 0$$

$$m = 1 \times s_1$$

$$\therefore \quad \delta_{AB} = 0$$

$$M_{yy} = W(0.25 - 0.25\cos\theta)$$

$$m = 1(0.125 + 0.25\sin\theta)$$

$$ds_2 = 0.25\,d\theta$$

$$\therefore \quad \delta_{BC} = \int_0^{\pi/2} \frac{W}{EI}(0.25 - 0.25\cos\theta) \times (0.125 - 0.25\sin\theta)\,0.25\,d\theta$$

Thus, once again, the same equation is obtained. This is always the case and there is little difference in the amount of work involved in the two methods.

$$\therefore \quad \delta_{BC} = \frac{W \times 0.25^3}{EI}\int_0^{\pi/2}(1 - \cos\theta)\,(0.5 + \sin\theta)\,d\theta$$

$$= \frac{0.25^3\,W}{EI}\int_0^{\pi/2}\left(0.5 - \frac{\cos\theta}{2} + \sin\theta - \sin\theta\cos\theta\right)d\theta$$

Castigliano	Unit load

but $\sin\theta\cos\theta = \frac{1}{2}\sin^2\theta$

$$\therefore \quad \delta_{BC} = \frac{0.25^3 W}{EI}\int_0^{\pi/2}\left(\frac{1}{2}-\frac{\cos\theta}{2}+\sin\theta-\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \frac{0.25^3 W}{EI}\left[\frac{\theta}{2}-\frac{\sin\theta}{2}-\cos\theta+\frac{\cos 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{0.25^3 W}{EI}\left[\left(\tfrac{\pi}{4}-\tfrac{1}{2}-\tfrac{1}{4}\right)-\left(-1+\tfrac{1}{4}\right)\right]$$

$$= \frac{0.25^3 \times 600}{EI}\left(\frac{\pi}{4}\right) = \frac{7.36}{EI}$$

For CD, *using unit load method,*

$$M_{zz} = W(0.25+s_3) \qquad m = 1(0.125+0.25) = 0.375$$

$$\delta_{CD} = \frac{1}{EI}\int_0^{0.3} W(0.25+s_3)(0.375)\,ds_3$$

$$= \frac{0.375 W}{EI}\int_0^{0.3}(0.25+s_3)\,ds_3$$

$$= \frac{0.375 W}{EI}\left[0.25 s_3 + \frac{s_3^2}{2}\right]_0^{0.3}$$

$$= \frac{0.375 W}{EI}[0.075+0.045]$$

$$= \frac{0.375\times 600}{EI}\times(0.12) = \frac{27}{EI}$$

Therefore total horizontal deflection

$$= \frac{7.36+27}{EI} = \frac{34.36\times 64\times 10^{12}}{200\times 10^9\times\pi\times 50^4}$$

$$= \mathbf{0.56\,mm}$$

Example 11.7

The frame shown in Fig. 11.23 is constructed from rectangular bar 25 mm wide by 12 mm thick. The end A is constrained by guides to move in a vertical direction and carries a vertical load of 400 N. For the frame material $E = 200\,\text{GN/m}^2$.

Determine (a) the horizontal reaction at the guides, (b) the vertical deflection of A.

Solution

(a) Consider the frame of Fig. 11.23. If A were not constrained in guides it would move in some direction (shown dotted) which would have both horizontal and vertical components. If

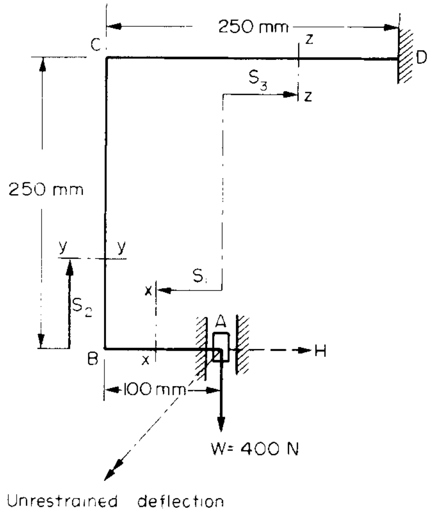

Fig. 11.23.

the horizontal movement is restricted by guides a horizontal reaction H must be set up as shown. Its value is determined by equating the horizontal deflection of A to zero,

i.e.
$$\int \frac{M}{EI}\frac{\partial M}{\partial H}\,ds = 0$$

For AB

$$M_{xx} = Ws_1 \quad \text{and} \quad \frac{\partial M}{\partial H} = 0$$

$\therefore$
$$\delta_{AB} = 0$$

For BC

$$M_{yy} = 0.1\,W - Hs_2 \quad \text{and} \quad \frac{\partial M}{\partial H} = -s_2$$

$\therefore$
$$\delta_{BC} = \int_0^{0.25} \frac{(0.1\,W - Hs_2)}{EI}(-s_2)ds_2$$

$$= \frac{1}{EI}\int_0^{0.25} (-0.1\,Ws_2 + Hs_2^2)ds_2$$

$$= \frac{1}{EI}\left[-\frac{0.1\,Ws_2^2}{2} + \frac{Hs_2^3}{3}\right]_0^{0.25}$$

$$= \frac{1}{EI}\left[-\frac{0.00625\,W}{2} + \frac{0.015625H}{3}\right]$$

$$= \frac{1}{EI \times 10^3}(-3.125\,W + 5.208H)$$

For CD

$$M_{zz} = Ws_3 + 0.25H \quad \text{and} \quad \frac{\partial M}{\partial H} = 0.25$$

$$\therefore \quad \delta_{CD} = \int_{-0.10}^{0.15} \frac{(Ws_3 + 0.25H)}{EI} 0.25 ds_3$$

$$= \frac{1}{EI} \int_{-0.10}^{0.15} (0.25\,Ws_3 + 0.0625H) ds_3$$

$$= \frac{1}{EI} \left[\frac{0.25\,Ws_3^2}{2} + 0.0625Hs_3 \right]_{-0.10}^{0.15}$$

$$= \frac{1}{EI} \left\{ \left[\frac{0.25\,W}{2} \times 0.0225 + 0.0625H \times 0.15 \right] - \left[\frac{0.25\,W}{2} \times 0.01 + 0.0625H(-0.1) \right] \right\}$$

$$= \frac{1}{EI \times 10^3} \{(1.25 \times 2.25\,W + 6.25 \times 1.5H) - (1.25\,W - 6.25H)\}$$

$$= \frac{1}{EI \times 10^3} \{(2.81\,W + 9.375H) - (1.25\,W - 6.25H)\}$$

$$= \frac{1}{EI \times 10^3} (1.56\,W + 15.625H)$$

Now the total horizontal deflection of $A = 0$

$$\therefore \quad -3.125\,W + 5.208H + 1.56\,W + 15.625H = 0$$

$$-1.565\,W + 20.833H = 0$$

$$\therefore \quad H = \frac{1.565 \times 400}{20.833} = \mathbf{30\,N}$$

Since a positive sign has been obtained, H must be in the direction assumed.

(b) For vertical deflection

$$\delta = \int \frac{M}{EI} \frac{\partial M}{\partial W} ds$$

For AB

$$M_{xx} = Ws_1 \quad \text{and} \quad \frac{\partial M}{\partial W} = s_1$$

$$\therefore \quad \delta_{AB} = \int_0^{0.1} \frac{Ws_1 \times s_1}{EI} ds_1$$

$$= \frac{400}{EI}\left[\frac{s_1^3}{3}\right]_0^{0.1}$$

$$= \frac{0.4}{3EI} = \frac{\mathbf{0.133}}{\mathbf{EI}}$$

For BC $\qquad M_{yy} = W \times 0.1 - 30s_2 \quad \text{and} \quad \frac{\partial M}{\partial W} = 0.1$

$$\therefore \qquad \delta_{BC} = \int_0^{0.25} \frac{(0.1W - 30s_2)}{EI} \times 0.1 ds_2$$

$$= \frac{1}{EI}\int_0^{0.25} (0.01 \times 400 - 3s_2) ds_2$$

$$= \frac{1}{EI}\left[4s_2 - \frac{3s_2^2}{2}\right]_0^{0.25}$$

$$= \frac{1}{EI}\left[1 - \frac{3 \times 0.0625}{2}\right]$$

$$= \frac{\mathbf{0.906}}{\mathbf{EI}}$$

For CD

$$M_{zz} = Ws_3 + 0.25H \quad \text{and} \quad \frac{\partial M}{\partial W} = s_3$$

$$\therefore \quad \delta_{CD} = \int_{-0.10}^{+0.15} \frac{(Ws_3 + 0.25H)}{EI} s_3 ds_3$$

$$= \frac{1}{EI}\int_{-0.1}^{+0.15} (Ws_3^2 + 0.25Hs_3) ds_3$$

$$= \frac{1}{EI}\left[\frac{400 \times s_3^3}{3} + \frac{0.25Hs_3^2}{2}\right]_{-0.1}^{0.15}$$

$$= \frac{1}{EI}\left[\frac{400}{3}(3.375 \times 10^{-3} + 1 \times 10^{-3}) + \frac{0.25 \times 30}{2}(22.5 \times 10^{-3} - 10 \times 10^{-3})\right]$$

$$= \frac{1}{EI}\left[\frac{400}{3} \times 4.375 \times 10^{-3} + \frac{0.25 \times 30}{2} \times 12.5 \times 10^{-3}\right]$$

$$= \frac{1}{EI}[0.583 + 0.047]$$

$$= \frac{\mathbf{0.63}}{\mathbf{EI}}$$

Total vertical deflection of A

$$= \frac{1}{EI}(0.133 + 0.906 + 0.63)$$

$$= \frac{1.669}{EI}$$

$$= \frac{1.669 \times 12 \times 10^{12}}{200 \times 10^9 \times 25 \times 12^3} = \mathbf{2.32\ mm}$$

Example 11.8 (B)

Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.

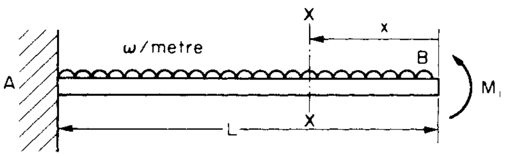

Fig. 11.24.

Solution (a)

Using Castigliano's procedure, apply an imaginary moment M_i in a positive direction at point B where the slope, i.e. rotation, is required.
BM at XX due to applied loading and imaginary couple

$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

from Castigliano's theorem

$$\theta = \int_0^L \frac{M}{EI} \cdot \frac{\partial M}{\partial M_i} \cdot dx$$

$$= \frac{1}{EI} \int_0^L \left(M_i - \frac{wx^2}{2} \right)(1)\, dx$$

which, with $M_i = 0$ in the absence of any applied moment at B, becomes

$$\theta = \frac{-w}{2EI} \int_0^L x^2 . dx = \frac{wL^3}{6EI} \textbf{ radian}$$

The negative sign indicates that rotation of the free end is in the opposite direction to that taken for the imaginary moment, i.e. the beam will slope downwards at B as should have been expected.

Alternative solution (b)

Using the "unit-moment" procedure, apply a unit moment at the point B where rotation is required and since we know that the beam will slope downwards the unit moment can be applied in the appropriate direction as shown.

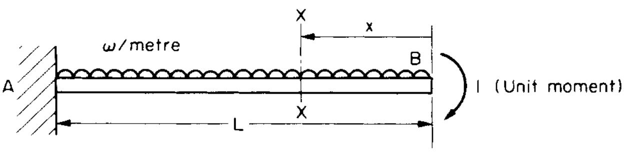

Fig. 11.25.

$$\text{B.M. at } XX \text{ due to applied loading} = M = -\frac{wx^2}{2}$$

$$\text{B.M. at } XX \text{ due to unit moment} = m = -1$$

The required rotation, or slope, is now given by

$$\theta = \int_0^L \frac{Mm}{EI} \cdot dx$$

$$= \frac{1}{EI}\int_0^L \left(-\frac{wx^2}{2}\right)(-1)\,dx.$$

$$= \frac{w}{2EI}\int_0^L x^2\,dx = \frac{wL^3}{6EI} \textbf{ radian.}$$

The answer is thus the same as before and a positive value has been obtained indicating that rotation will occur in the direction of the applied unit moment (i.e. opposite to M_i in the previous solution).

Problems

11.1 (A). Define what is meant by "resilience" or "strain energy". Derive an equation for the strain energy of a uniform bar subjected to a tensile load of P newtons. Hence calculate the strain energy in a 50 mm diameter bar, 4 m long, when carrying an axial tensile pull of 150 kN. $E = 208$ GN/m^2. [110.2 N m.]

11.2 (A). (a) Derive the formula for strain energy resulting from bending of a beam (neglecting shear).

(b) A beam, simply supported at its ends, is of 4 m span and carries, at 3 m from the left-hand support, a load of 20 kN. If I is 120×10^{-6} m^4 and $E = 200$ GN/m^2, find the deflection under the load using the formula derived in part (a). [0.625 mm.]

11.3 (A) Calculate the strain energy stored in a bar of circular cross-section, diameter 0.2 m, length 2 m:
(a) when subjected to a tensile load of 25 kN,
(b) when subjected to a torque of 25 kNm,
(c) when subjected to a uniform bending moment of 25 kNm.
For the bar material $E = 208\ \text{GN/m}^2$, $G = 80\ \text{GN/m}^2$. [0.096, 49.7, 38.2 N m.]

11.4 (A/B). Compare the strain energies of two bars of the same material and length and carrying the same gradually applied compressive load if one is 25 mm diameter throughout and the other is turned down to 20 mm diameter over half its length, the remainder being 25 mm diameter.

If both bars are subjected to pure torsion only, compare the torsional strain energies stored if the shear stress in both bars is limited to 75 MN/m^2. [0.78, 2.22.]

11.5 (A/B). Two shafts, one of steel and the other of phosphor bronze, are of the same length and are subjected to equal torques. If the steel shaft is 25 mm diameter, find the diameter of the phosphor-bronze shaft so that it will store the same amount of energy per unit volume as the steel shaft. Also determine the ratio of the maximum shear stresses induced in the two shafts. Take the modulus of rigidity for phosphor bronze as 50 GN/m^2 and for steel as 80 GN/m^2. [27.04 mm, 1.26.]

11.6 (A/B). Show that the torsional strain energy of a solid circular shaft transmitting power at a constant speed is given by the equation:

$$U = \frac{\tau^2}{4G} \times \text{volume}.$$

Such a shaft is 0.06 m in diameter and has a flywheel of mass 30 kg and radius of gyration 0.25 m situated at a distance of 1.2 m from a bearing. The flywheel is rotating at 200 rev/min when the bearing suddenly seizes. Calculate the maximum shear stress produced in the shaft material and the instantaneous angle of twist under these conditions. Neglect the shaft inertia. For the shaft material $G = 80\ \text{GN/m}^2$. [B.P.] [196.8 MN/m^2, 5.64°.]

11.7 (A/B). A solid shaft carrying a flywheel of mass 100 kg and radius of gyration 0.4 m rotates at a uniform speed of 75 rev/min. During service, a bearing 3 m from the flywheel suddenly seizes producing a fixation of the shaft at this point. Neglecting the inertia of the shaft itself determine the necessary shaft diameter if the instantaneous shear stress produced in the shaft does not exceed 180 MN/m^2. For the shaft material $G = 80\ \text{GN/m}^2$. Assume all kinetic energy of the shaft is taken up as strain energy without any losses. [22.7 mm.]

11.8 (A/B). A multi-bladed turbine disc can be assumed to have a combined mass of 150 kg with an effective radius of gyration of 0.59 m. The disc is rigidly attached to a steel shaft 2.4 m long and, under service conditions, rotates at a speed of 250 rev/min. Determine the diameter of shaft required in order that the maximum shear stress set up in the event of sudden seizure of the shaft shall not exceed 200 MN/m^2. Neglect the inertia of the shaft itself and take the modulus of rigidity G of the shaft material to be 85 GN/m^2. [284 mm.]

11.9 (A/B). Develop from first principles an expression for the instantaneous stress set up in a vertical bar by a weight W falling from a height h on to a stop at the end of the bar. The instantaneous extension x may not be neglected.

A weight of 500 N can slide freely on a vertical steel rod 2.5 m long and 20 mm diameter. The rod is rigidly fixed at its upper end and has a collar at the lower end to prevent the weight from dropping off. The weight is lifted to a distance of 50 mm above the collar and then released. Find the maximum instantaneous stress produced in the rod. $E = 200\ \text{GN/m}^3$. [114 MN/m^2.]

11.10 (A/B). A load of 2 kN falls through 25 mm on to a stop at the end of a vertical bar 4 m long, 600 mm^2 cross-sectional area and rigidly fixed at its other end. Determine the instantaneous stress and elongation of the bar. $E = 200\ \text{GN/m}^2$. [94.7 MN/m^2, 1.9 mm.]

11.11 (A/B). A load of 2.5 kN slides freely on a vertical bar of 12 mm diameter. The bar is fixed at its upper end and provided with a stop at the other end to prevent the load from falling off. When the load is allowed to rest on the stop the bar extends by 0.1 mm. Determine the instantaneous stress set up in the bar if the load is lifted and allowed to drop through 12 mm on to the stop. What will then be the extension of the bar? [365 MN/m^2, 1.65 mm.]

11.12 (A/B). A bar of a certain material, 40 mm diameter and 1.2 m long, has a collar securely fitted to one end. It is suspended vertically with the collar at the lower end and a mass of 2000 kg is gradually lowered on to the collar producing an extension in the bar of 0.25 mm. Find the height from which the load could be dropped on to the collar if the maximum tensile stress in the bar is to be 100 MN/m^2. Take $g = 9.81\ \text{m/s}^2$. The instantaneous extension cannot be neglected. [U.L.] [3.58 mm]

11.13 (A/B). A stepped bar is 2 m long. It is 40 mm diameter for 1.25 m of its length and 25 mm diameter for the remainder. If this bar hangs vertically from a rigid structure and a ring weight of 200 N falls freely from a height of 75 mm on to a stop formed at the lower end of the bar, neglecting all external losses, what would be the maximum instantaneous stress induced in the bar, and the maximum extension? $E = 200\ \text{GN/m}^2$. [99.3 MN/m^2, 0.615 mm.]

11.14 (B). A beam of uniform cross-section, with centroid at mid-depth and length 7 m, is simply supported at its ends and carries a point load of 5 kN at 3 m from one end. If the maximum bending stress is not to exceed 90 MN/m^2 and the beam is 150 mm deep, (i) working from first principles find the deflection under the load, (ii) what load dropped from a height of 75 mm on to the beam at 3 m from one end would produce a stress of 150 MN/m^2 at the point of application of the load? $E = 200\ GN/m^2$. [24 mm; 1.45 kN.]

11.15 (B). A steel beam of length 7 m is built in at both ends. It has a depth of 500 mm and the second moment of area is $300 \times 10^{-6}\ m^4$. Calculate the load which, falling through a height of 75 mm on to the centre of the span, will produce a maximum stress of 150 MN/m^2. What would be the maximum deflection if the load were gradually applied? $E = 200\ GN/m^2$. [B.P.] [7.77 kN, 0.23 mm.]

11.16 (B). When a load of 20 kN is gradually applied at a certain point on a beam it produces a deflection of 13 mm and a maximum bending stress of 75 MN/m^2. From what height can a load of 5 kN fall on to the beam at this point if the maximum bending stress is to be 150 MN/m^2? [U.L.] [78 mm.]

11.17 (B). Show that the vertical and horizontal deflections of the end B of the quadrant shown in Fig. 11.26 are, respectively,

$$\frac{WR^3}{EI}\left[\frac{3\pi}{4} - 2\right] \quad \text{and} \quad \frac{WR^3}{2EI}.$$

What would the values become if W were applied horizontally instead of vertically? $\left[\frac{WR^3}{EI}\left(\frac{\pi}{4}\right);\ \frac{WR^3}{2EI}.\right]$

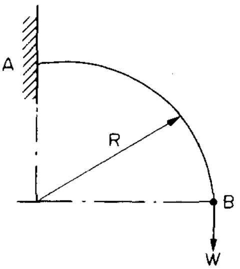

Fig. 11.26.

11.18 (B). A semi-circular frame of flexural rigidity EI is built in at A and carries a vertical load W at B as shown in Fig. 11.27. Calculate the magnitudes of the vertical and horizontal deflections at B and hence the magnitude and direction of the resultant deflection.

$\left[\frac{3\pi}{2}\frac{WR^3}{EI};\ 2\frac{WR^3}{EI};\ 5.12\frac{WR^3}{EI}\text{ at } 23^\circ \text{ to vertical.}\right]$

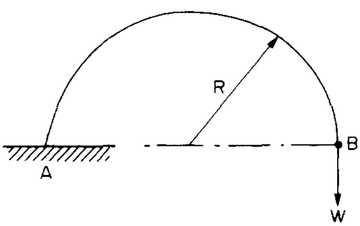

Fig. 11.27.

11.19 (B). A uniform cantilever, length L and flexural rigidity EI carries a vertical load W at mid-span. Calculate the magnitude of the vertical deflection of the free end. $\left[5\frac{WL^3}{48EI}\right]$

11.20 (B). A steel rod, of flexural rigidity EI, forms a cantilever ABC lying in a vertical plane as shown in Fig. 11.28. A horizontal load of P acts at C. Calculate:

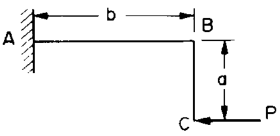

Fig. 11.28.

(a) the horizontal deflection of C;
(b) the vertical deflection of C;
(c) the slope at B.
Consider the strain energy resulting from bending only. [U.E.I.] $\left[\frac{Pa^2}{3EI}[a+3b];\ \frac{Pab^2}{2EI};\ \frac{Pab}{EI}.\right]$

11.21 (B). Derive the formulae for the slope and deflection at the free end of a cantilever when loaded at the end with a concentrated load W. Use a strain energy method for your solution.
A cantilever is constructed from metal strip 25 mm deep throughout its length of 750 mm. Its width, however, varies uniformly from zero at the free end to 50 mm at the support. Determine the deflection of the free end of the cantilever if it carries uniformly distributed load of 300 N/m across its length. $E = 200\ \text{GN/m}^2$. [1.2 mm.]

11.22 (B). Determine the vertical deflection of point A on the bent cantilever shown in Fig. 11.29 when loaded at A with a vertical load of 25 N. The cantilever is built in at B, and EI may be taken as constant throughout and equal to $450\ \text{N m}^2$. [B.P.] [0.98 mm.]

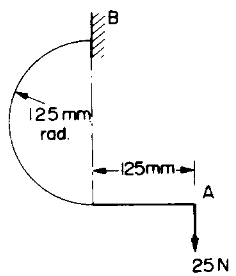

Fig. 11.29.

11.23 (B). What will be the horizontal deflection of A in the bent cantilever of Problem 11.22 when carrying the vertical load of 25 N? [0.56 mm.]

11.24 (B). A steel ring of mean diameter 250 mm has a square section 2.5 mm by 2.5 mm. It is split by a narrow radial saw cut. The saw cut is opened up farther by a tangential separating force of 0.2 N. Calculate the extra separation at the saw cut. $E = 200\ \text{GN}/m^2$. [U.E.I.] [5.65 mm.]

11.25 (B). Calculate the strain energy of the gantry shown in Fig. 11.30 and hence obtain the vertical deflection of the point C. Use the formula for strain energy in bending $U = \int \frac{M^2}{2EI}\,dx$, where M is the bending moment, E is Young's modulus, I is second moment of area of the beam section about axis XX. The beam section is as shown in Fig. 11.30. Bending takes place along AB and BC about the axis XX. $E = 210\ \text{GN/m}^2$. [U.L.C.I.] [53.9 mm.]

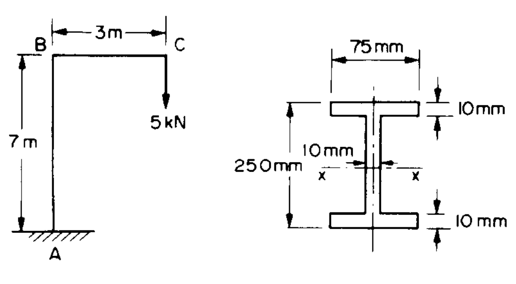

Fig. 11.30.

11.26 (B). A steel ring, of 250 mm diameter, has a width of 50 mm and a radial thickness of 5 mm. It is split to leave a narrow gap 5 mm wide normal to the plane of the ring. Assuming the radial thickness to be small compared with the radius of ring curvature, find the tangential force that must be applied to the edges of the gap to just close it. What will be the maximum stress in the ring under the action of this force? $E = 200\ \text{GN/m}^2$.
[I.Mech.E.] [28.3 N; 34 MN/m^2.]

11.27 (B). Determine, for the cranked member shown in Fig. 11.31:
(a) the magnitude of the force P necessary to produce a vertical movement of P of 25 mm;
(b) the angle, in degrees, by which the tip of the member diverges when the force P is applied.
The member has a uniform width of 50 mm throughout. $E = 200\ \text{GN/m}^2$. [B.P.] [6.58 kN; 4.1°.]

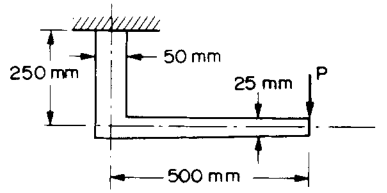

Fig. 11.31.

11.28 (C). A 12 mm diameter steel rod is bent to form a square with sides $2a = 500$ mm long. The ends meet at the mid-point of one side and are separated by equal opposite forces of 75 N applied in a direction perpendicular to the plane of the square as shown in perspective in Fig. 11.32. Calculate the amount by which they will be out of alignment. Consider only strain energy due to bending. $E = 200\ \text{GN/m}^2$. [38.3 mm.]

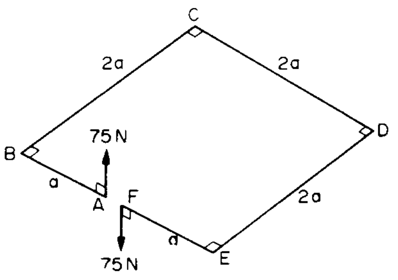

Fig. 11.32

11.29 (B/C). A state of two-dimensional plane stress on an element of material can be represented by the principal stresses σ_1 and σ_2 ($\sigma_1 > \sigma_2$). The strain energy can be expressed in terms of the strain energy per unit volume. Then:
(a) working from first principles show that the strain energy per unit volume is given by the expression

$$\frac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2)$$

for a material which follows Hooke's law where E denotes Young's modulus and ν denotes Poisson's ratio, and
(b) by considering the relations between each of σ_x, σ_y, τ_{xy} respectively and the principal stresses, where x and y are two other mutually perpendicular axes in the same plane, show that the expression

$$\frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2]$$

is identical with the expression given above. [City U.]