## **CHAPTER 10**

# THICK CYLINDERS

#### Summary

The hoop and radial stresses at any point in the wall cross-section of a thick cylinder at radius r are given by the Lamé equations:

hoop stress 
$$\sigma_H = A + \frac{B}{r^2}$$
  
radial stress  $\sigma_r = A - \frac{B}{r^2}$ 

With internal and external pressures  $P_1$  and  $P_2$  and internal and external radii  $R_1$  and  $R_2$  respectively, the *longitudinal stress* in a cylinder with closed ends is

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \text{Lamé constant } A$$

Changes in dimensions of the cylinder may then be determined from the following strain formulae:

circumferential or hoop strain = diametral strain

$$= \frac{\sigma_H}{E} - v \frac{\sigma_r}{E} - v \frac{\sigma_L}{E}$$
  
longitudinal strain 
$$= \frac{\sigma_L}{E} - v \frac{\sigma_r}{E} - v \frac{\sigma_H}{E}$$

For *compound tubes* the resultant hoop stress is the algebraic sum of the hoop stresses resulting from shrinkage and the hoop stresses resulting from internal and external pressures.

For force and shrink fits of cylinders made of different materials, the total interference or shrinkage allowance (on radius) is

$$[\varepsilon_{H_o} - \varepsilon_{H_i}]r$$

where  $\varepsilon_{H_a}$  and  $\varepsilon_{H_i}$  are the hoop strains existing in the outer and inner cylinders respectively at the common radius r. For cylinders of the same material this equation reduces to

$$\frac{r}{E} \left[ \sigma_{H_o} - \sigma_{H_i} \right]$$

For a hub or sleeve shrunk on a solid shaft the shaft is subjected to constant hoop and radial stresses, each equal to the pressure set up at the junction. The hub or sleeve is then treated as a thick cylinder subjected to this internal pressure.

If the internal and external radii of the cylinder are  $R_1$  and  $R_2$  respectively and it is wound with wire until its external radius becomes  $R_3$ , the radial and hoop stresses in the wire at any radius r between the radii  $R_2$  and  $R_3$  are found from:

radial stress = 
$$\left(\frac{r^2 - R_1^2}{2r^2}\right) T \log_e \left(\frac{R_3^2 - R_1^2}{r^2 - R_1^2}\right)$$
  
hoop stress =  $T \left\{ 1 - \left(\frac{r^2 + R_1^2}{2r^2}\right) \log_e \left(\frac{R_3^2 - R_1^2}{r^2 - R_1^2}\right) \right\}$ 

where T is the constant tension stress in the wire.

The hoop and radial stresses in the cylinder can then be determined by considering the cylinder to be subjected to an external pressure equal to the value of the radial stress above when  $r = R_2$ .

When an additional internal pressure is applied the final stresses will be the algebraic sum of those resulting from the internal pressure and those resulting from the wire winding.

Plastic yielding of thick cylinders

For initial yield, the internal pressure  $P_1$  is given by:

$$P_1 = \frac{\sigma_y}{2R_2^2} \left[ R_2^2 - R_1^2 \right]$$

For yielding to a radius  $R_p$ ,

$$P_{1} = \sigma_{y} \left[ \log_{e} \frac{R_{1}}{R_{p}} - \frac{1}{2R_{2}} \left( R_{2}^{2} - R_{p}^{2} \right) \right]$$

and for complete collapse,

$$P_1 = \sigma_y \left[ \log_e \frac{R_1}{R_2} \right]$$

# 10.1. Difference in treatment between thin and thick cylinders – basic assumptions

The theoretical treatment of thin cylinders assumes that the hoop stress is constant across the thickness of the cylinder wall (Fig. 10.1), and also that there is no pressure gradient across the wall. Neither of these assumptions can be used for thick cylinders for which the variation of hoop and radial stresses is shown in Fig. 10.2, their values being given by the *Lamé* equations:

$$\sigma_H = A + \frac{B}{r^2}$$
 and  $\sigma_r = A - \frac{B}{r^2}$ 

Development of the theory for thick cylinders is concerned with sections remote from the



Fig. 10.1. Thin cylinder subjected to internal pressure.



Fig. 10.2. Thick cylinder subjected to internal pressure.

ends since distribution of the stresses around the joints makes analysis at the ends particularly complex. For central sections the applied pressure system which is normally applied to thick cylinders is symmetrical, and all points on an annular element of the cylinder wall will be displaced by the same amount, this amount depending on the radius of the element. Consequently there can be no shearing stress set up on transverse planes and stresses on such planes are therefore principal stresses (see page 331). Similarly, since the radial shape of the cylinder is maintained there are no shears on radial or tangential planes, and again stresses on such planes are principal stresses. Thus, consideration of any element in the wall of a thick cylinder involves, in general, consideration of a mutually prependicular, tri-axial, principal stress system, the three stresses being termed **radial**, **hoop** (tangential or circumferential) and **longitudinal** (axial) stresses.

## 10.2. Development of the Lamé theory

Consider the thick cylinder shown in Fig. 10.3. The stresses acting on an element of unit length at radius r are as shown in Fig. 10.4, the radial stress increasing from  $\sigma$ , to  $\sigma$ ,  $+ d\sigma$ , over the element thickness dr (all stresses are assumed tensile),

For radial equilibrium of the element:

$$(\sigma_r + d\sigma_r)(r + dr)d\theta \times 1 - \sigma_r \times rd\theta \times 1 = 2\sigma_H \times dr \times 1 \times \sin\frac{d\theta}{2}$$

10



Fig. 10.3.



Fig. 10.4.

For small angles:

$$\sin \frac{d\theta}{2} = \frac{d\theta}{2}$$
 radian

Therefore, neglecting second-order small quantities,

$$rd\sigma_{r} + \sigma_{r}dr = \sigma_{H}dr$$

$$\sigma_{r} + r\frac{d\sigma_{r}}{dr} = \sigma_{H}$$

$$\sigma_{H} - \sigma_{r} = r\frac{d\sigma_{r}}{dr}$$
(10.1)

or

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Assuming now that plane sections remain plane, i.e. the longitudinal strain  $\varepsilon_L$  is constant across the wall of the cylinder,

then

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$$\varepsilon_L = \frac{1}{E} [\sigma_L - \nu \sigma_r - \nu \sigma_H]$$
$$= \frac{1}{E} [\sigma_L - \nu (\sigma_r + \sigma_H)] = \text{constant}$$

It is also assumed that the longitudinal stress  $\sigma_L$  is constant across the cylinder walls at points remote from the ends.

$$\sigma_r + \sigma_H = \text{constant} = 2A \text{ (say)} \tag{10.2}$$

§10.3

Substituting in (10.1) for  $\sigma_H$ ,

$$2A - \sigma_r - \sigma_r = r \frac{d\sigma_r}{dr}$$

Multiplying through by r and rearranging,

$$2\sigma_{r}r + r^{2}\frac{d\sigma_{r}}{dr} - 2Ar = 0$$

$$\frac{d}{dr}(\sigma_{r}r^{2} - Ar^{2}) = 0$$

$$\sigma_{r}r^{2} - Ar^{2} = \text{constant} = -B \text{ (say)}$$

$$\sigma_{r} = A - \frac{B}{r^{2}}$$
(10.3)

and from eqn. (10.2)

Therefore, integrating,

$$\sigma_H = A + \frac{B}{r^2} \tag{10.4}$$

The above equations yield the radial and hoop stresses at any radius r in terms of constants A and B. For any pressure condition there will always be two known conditions of stress (usually radial stress) which enable the constants to be determined and the required stresses evaluated.

# 10.3. Thick cylinder-internal pressure only

Consider now the thick cylinder shown in Fig. 10.5 subjected to an internal pressure P, the external pressure being zero.



Fig. 10.5. Cylinder cross-section.

The two known conditions of stress which enable the Lamé constants A and B to be determined are:

At 
$$r = R_1$$
,  $\sigma_r = -P$  and at  $r = R_2$ ,  $\sigma_r = 0$ 

N.B. – The internal pressure is considered as a negative radial stress since it will produce a radial compression (i.e. thinning) of the cylinder walls and the normal stress convention takes compression as negative.

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Substituting the above conditions in eqn. (10.3),

$$-P = A - \frac{B}{R_1^2}$$

$$0 = A - \frac{B}{R_2^2}$$

$$A = \frac{PR_1^2}{(R_2^2 - R_1^2)} \text{ and } B = \frac{PR_1^2R_2^2}{(R_2^2 - R_1^2)}$$

radial stress  $\sigma_r = A - \frac{B}{r^2}$ 

$$= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ 1 - \frac{R_2^2}{r^2} \right]$$
$$= \frac{PR_1^2}{(R_2^2 - R_1^2)} \left[ \frac{r^2 - R_2^2}{r^2} \right] = -P \left[ \frac{(R_2/r)^2 - 1}{k^2 - 1} \right]$$
(10.5)

where k is the diameter ratio  $D_2/D_1 = R_2/R_1$ 

hoop

and

stress 
$$\sigma_{H} = \frac{PR_{1}^{2}}{(R_{2}^{2} - R_{1}^{2})} \left[ 1 + \frac{R_{2}^{2}}{r^{2}} \right]$$
  
$$= \frac{PR_{1}^{2}}{(R_{2}^{2} - R_{1}^{2})} \left[ \frac{r^{2} + R_{2}^{2}}{r^{2}} \right] = P \left[ \frac{(R_{2}/r)^{2} + 1}{k^{2} - 1} \right]$$
(10.6)

These equations yield the stress distributions indicated in Fig. 10.2 with maximum values of both  $\sigma_r$  and  $\sigma_H$  at the inside radius.

# 10.4. Longitudinal stress

Consider now the cross-section of a thick cylinder with closed ends subjected to an internal pressure  $P_1$  and an external pressure  $P_2$  (Fig. 10.6).



Fig. 10.6. Cylinder longitudinal section.

For horizontal equilibrium:

$$P_1 \times \pi R_1^2 - P_2 \times \pi R_2^2 = \sigma_L \times \pi (R_2^2 - R_1^2)$$



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where  $\sigma_L$  is the longitudinal stress set up in the cylinder walls,

longitudinal stress 
$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{R_2^2 - R_1^2}$$
 (10.7)

i.e. a constant.

It can be shown that the constant has the same value as the constant A of the Lame equations. This can be verified for the "internal pressure only" case of §10.3 by substituting  $P_2 = 0$  in eqn. (10.7) above.

For combined internal and external pressures, the relationship  $\sigma_L = A$  also applies.

# 10.5. Maximum shear stress

It has been stated in §10.1 that the stresses on an element at any point in the cylinder wall are principal stresses.

It follows, therefore, that the maximum shear stress at any point will be given by eqn. (13.12) as

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}$$

i.e. half the difference between the greatest and least principal stresses.

Therefore, in the case of the thick cylinder, normally,

$$\tau_{\max} = \frac{\sigma_H - \sigma_H}{2}$$

since  $\sigma_H$  is normally tensile, whilst  $\sigma_r$  is compressive and both exceed  $\sigma_L$  in magnitude.

$$\tau_{\max} = \frac{1}{2} \left[ \left( A + \frac{B}{r^2} \right) - \left( A - \frac{B}{r^2} \right) \right]$$
  
$$\tau_{\max} = \frac{B}{r^2}$$
(10.8)

The greatest value of  $\tau_{max}$  thus normally occurs at the inside radius where  $r = R_1$ .

# 10.6. Change of cylinder dimensions

#### (a) Change of diameter

It has been shown in §9.3 that the diametral strain on a cylinder equals the hoop or circumferential strain.

Therefore change of diameter = diametral strain × original diameter

= circumferential strain × original diameter

With the principal stress system of hoop, radial and longitudinal stresses, all assumed tensile, the circumferential strain is given by

$$\varepsilon_{H} = \frac{1}{E} \left[ \sigma_{H} - v \sigma_{r} - v \sigma_{L} \right]$$

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222

Thus the change of diameter at any radius r of the cylinder is given by

$$\Delta D = \frac{2r}{E} \left[ \sigma_H - \nu \sigma_r - \nu \sigma_L \right] \tag{10.9}$$

§10.7

# (b) Change of length

Similarly, the change of length of the cylinder is given by

$$\Delta L = \frac{L}{E} [\sigma_L - \sigma v_r - v \sigma_H]$$
(10.10)

# 10.7. Comparison with thin cylinder theory

In order to determine the limits of D/t ratio within which it is safe to use the simple thin cylinder theory, it is necessary to compare the values of stress given by both thin and thick cylinder theory for given pressures and D/t values. Since the maximum hoop stress is normally the limiting factor, it is this stress which will be considered. From thin cylinder theory:

$$\sigma_{H} = P \frac{D}{2t}$$
$$\frac{\sigma_{H}}{P} = \frac{K}{2} \text{ where } K = D/t$$

For thick cylinders, from eqn. (10.6),

$$\sigma_{H} = \frac{PR_{1}^{2}}{(R_{2}^{2} - R_{1}^{2})} \left[ 1 + \frac{R_{2}^{2}}{r^{2}} \right]$$
$$\sigma_{H_{\text{max}}} = P \frac{(R_{1}^{2} + R_{2}^{2})}{(R_{2}^{2} - R_{1}^{2})} \quad \text{at } r = R_{1}$$
(10.11)

i.e.

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Now, substituting for  $R_2 = R_1 + t$  and  $D = 2R_1$ ,

$$\sigma_{H_{\max}} = \left[\frac{\frac{1}{2}D^{2} + t(D+t)}{t(D+t)}\right]P$$
$$= \left[\frac{D^{2}}{2t^{2}(D/t+1)} + 1\right]P$$
$$\frac{\sigma_{H_{\max}}}{P} = \frac{K^{2}}{2(K+1)} + 1$$
(10.12)

i.e.

Thus for various D/t ratios the stress values from the two theories may be plotted and compared; this is shown in Fig. 10.7.

Also indicated in Fig. 10.7 is the percentage error involved in using the thin cylinder theory. It will be seen that the error will be held within 5% if D/t ratios in excess of 15 are used.



Fig. 10.7. Comparison of thin and thick cylinder theories for various diameter/thickness ratios.

However, if D is taken as the *mean* diameter for calculation of the thin cylinder values instead of the inside diameter as used here, the percentage error reduces from 5% to approximately 0.25% at D/t = 15.

#### 10.8. Graphical treatment - Lamé line

The Lamé equations when plotted on stress and  $1/r^2$  axes produce straight lines, as shown in Fig. 10.8.



Fig. 10.8. Graphical representation of Lamé equations - Lamé line.

Both lines have exactly the same intercept A and the same magnitude of slope B, the only difference being the sign of their slopes. The two are therefore combined by plotting hoop stress values to the left of the  $\sigma$  axis (again against  $1/r^2$ ) instead of to the right to give the single line shown in Fig. 10.9. In most questions one value of  $\sigma_r$  and one value of  $\sigma_H$ , or alternatively two values of  $\sigma_r$ , are given. In both cases the single line can then be drawn.

When a thick cylinder is subjected to external pressure only, the radial stress at the inside radius is zero and the graph becomes the straight line shown in Fig. 10.10.



Fig. 10.9. Lamé line solution for cylinder with internal and external pressures.



Fig. 10.10. Lamé line solution for cylinder subjected to external pressure only.

**N.B.** – From §10.4 the value of the longitudinal stress  $\sigma_L$  is given by the intercept A on the  $\sigma$  axis.

It is not sufficient simply to read off stress values from the axes since this can introduce appreciable errors. Accurate values must be obtained from proportions of the figure using similar triangles.

#### 10.9. Compound cylinders

From the sketch of the stress distributions in Fig. 10.2 it is evident that there is a large variation in hoop stress across the wall of a cylinder subjected to internal pressure. The material of the cylinder is not therefore used to its best advantage. To obtain a more uniform hoop stress distribution, cylinders are often built up by shrinking one tube on to the outside of another. When the outer tube contracts on cooling the inner tube is brought into a state of

compression. The outer tube will conversely be brought into a state of tension. If this compound cylinder is now subjected to internal pressure the resultant hoop stresses will be the algebraic sum of those resulting from internal pressure and those resulting from shrinkage as drawn in Fig. 10.11; thus a much smaller total fluctuation of hoop stress is obtained. A similar effect is obtained if a cylinder is wound with wire or steel tape under tension (see §10.19).



Fig. 10.11. Compound cylinders - combined internal pressure and shrinkage effects.

# (a) Same materials

The method of solution for compound cylinders constructed from similar materials is to break the problem down into three separate effects:

- (a) shrinkage pressure only on the inside cylinder;
- (b) shrinkage pressure only on the outside cylinder;
- (c) internal pressure only on the complete cylinder (Fig. 10.12).



Fig. 10.12. Method of solution for compound cylinders.

For each of the resulting load conditions there are two known values of radial stress which enable the Lamé constants to be determined in each case.

i.e. condition (a) shrinkage-internal cylinder: At  $r = R_1$ ,  $\sigma_r = 0$ At  $r = R_c$ ,  $\sigma_r = -p$  (compressive since it tends to reduce the wall thickness) condition (b) shrinkage-external cylinder: At  $r = R_2$ ,  $\sigma_r = 0$ At  $r = R_c$ ,  $\sigma_r = -p$ condition (c) internal pressure-compound cylinder: At  $r = R_2$ ,  $\sigma_r = 0$ At  $r = R_1$ ,  $\sigma_r = -P_1$  Mechanics of Materials

Thus for each condition the hoop and radial stresses at any radius can be evaluated and the principle of superposition applied, i.e. the various stresses are then combined algebraically to produce the stresses in the compound cylinder subjected to both shrinkage and internal pressure. In practice this means that the compound cylinder is able to withstand greater internal pressures before failure occurs or, alternatively, that a thinner compound cylinder (with the associated reduction in material cost) may be used to withstand the same internal pressure as the single thick cylinder it replaces.



Fig. 10.13. Distribution of hoop and radial stresses through the walls of a compound cylinder.

(b) Different materials

(See §10.14.)

## 10.10. Compound cylinders - graphical treatment

The graphical, or Lamé line, procedure introduced in §10.8 can be used for solution of compound cylinder problems. The vertical lines representing the boundaries of the cylinder walls may be drawn at their appropriate  $1/r^2$  values, and the solution for condition (c) of Fig. 10.12 may be carried out as before, producing a single line across both cylinder sections (Fig. 10.14a).

The graphical representation of the effect of shrinkage does not produce a single line, however, and the effect on each cylinder must therefore be determined by projection of known lines on the radial side of the graph to the respective cylinder on the hoop stress side, i.e. conditions (a) and (b) of Fig. 10.12 must be treated separately as indeed they are in the analytical approach. The resulting graph will then appear as in Fig. 10.14b.

The total effect of combined shrinkage and internal pressure is then given, as before, by the algebraic combination of the separate effects, i.e. the graphs must be added together, taking due account of sign to produce the graph of Fig. 10.14c. In practice this is the only graph which need be constructed, all effects being considered on the single set of axes. Again, all values should be calculated from proportions of the figure, i.e. by the use of similar triangles.

## 10.11. Shrinkage or interference allowance

In the design of compound cylinders it is important to relate the difference in diameter of the mating cylinders to the stresses this will produce. This difference in diameter at the





Fig. 10.14. Graphical (Lamé line) solution for compound cylinders.

"common" surface is normally termed the *shrinkage* or *interference allowance* whether the compound cylinder is formed by a shrinking or a force fit procedure respectively. Normally, however, the shrinking process is used, the outer cylinder being heated until it will freely slide over the inner cylinder thus exerting the required junction or shrinkage pressure on cooling.

Consider, therefore, the compound cylinder shown in Fig. 10.15, the material of the two cylinders not necessarily being the same.

Let the pressure set up at the junction of two cylinders owing to the force or shrink fit be p. Let the hoop stresses set up at the junction on the inner and outer tubes resulting from the pressure p be  $\sigma_{H_i}$  (compressive) and  $\sigma_{H_a}$  (tensile) respectively.

Then, if

 $\delta_o$  = radial shift of outer cylinder

 $\delta_i$  = radial shift of inner cylinder (as shown in Fig. 10.15)

and



Fig. 10.15. Interference or shrinkage allowance for compound cylinders – total interference =  $\delta_o + \delta_i$ .

since circumferential strain = diametral strain circumferential strain at radius r on outer cylinder =  $\frac{2\delta_o}{2r} = \frac{\delta_o}{r} = \varepsilon_{H_o}$ 

circumferential strain at radius r on inner cylinder  $=\frac{2\delta_i}{2r}=\frac{\delta_i}{r}=-\varepsilon_{H_i}$ 

(negative since it is a decrease in diameter).

Total interference or shrinkage =  $\delta_0 + \delta_i$ 

$$= r\varepsilon_{H_o} + r(-\varepsilon_{H_i})$$
$$= (\varepsilon_{H_o} - \varepsilon_{H_i})r$$

Now assuming open ends, i.e.  $\sigma_L = 0$ ,

$$\varepsilon_{H_o} = \frac{\sigma_{H_o}}{E_1} - \frac{v_1}{E_1}(-p) \quad \text{since } \sigma_{r_o} = -p$$

$$\varepsilon_{H_i} = \frac{\sigma_{H_i}}{E_2} - \frac{v_2}{E_2}(-p)$$
 since  $\sigma_{r_i} = -p$ 

where  $E_1$  and  $v_1$ ,  $E_2$  and  $v_2$  are the elastic modulus and Poisson's ratio of the two tubes respectively.

Therefore total interference or shrinkage allowance (based on radius)

$$= \left[\frac{1}{E_{1}}(\sigma_{H_{o}} + v_{1}p) - \frac{1}{E_{2}}(\sigma_{H_{i}} + v_{2}p)\right]r$$
(10.13)

where r is the initial nominal radius of the mating surfaces.

N.B.  $\sigma_{H_i}$  being compressive, will change the negative sign to a positive one when its value is substituted. Shrinkage allowances based on diameter will be twice this value, i.e. replacing radius r by diameter d.

Generally, however, the tubes are of the same material.

$$E_1 = E_2 = E$$
 and  $v_1 = v_2 = v$   
Shrinkage allowance  $= \frac{r}{E} (\sigma_{H_0} - \sigma_{H_i})$  (10.14)

and

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Thick Cylinders

229

The values of  $\sigma_{H_o}$  and  $\sigma_{H_i}$  may be determined graphically or analytically in terms of the shrinkage pressure p which can then be evaluated for any known shrinkage or interference allowance. Other stress values throughout the cylinder can then be determined as described previously.

#### 10.12. Hub on solid shaft

The Lamé equations give

$$\sigma_H = A + \frac{B}{r^2}$$
 and  $\sigma_r = A - \frac{B}{r^2}$ 

Since  $\sigma_H$  and  $\sigma_r$  cannot be infinite when r = 0, i.e. at the centre of the solid shaft, it follows that B must be zero since this is the only solution which can yield finite values for the stresses.

From the above equations, therefore, it follows that  $\sigma_H = \sigma_r = A$  for all values of r.

Now at the outer surface of the shaft

$$\sigma_r = -p$$
 the shrinkage pressure.

Therefore the hoop and radial stresses throughout a solid shaft are everywhere equal to the shrinkage or interference pressure and both are compressive. The maximum shear stress  $=\frac{1}{2}(\sigma_1 - \sigma_2)$  is thus zero throughout the shaft.

# 10.13. Force fits

It has been stated that compound cylinders may be formed by *shrinking* or by *force-fit* techniques. In the latter case the interference allowance is small enough to allow the outer cylinder to be pressed over the inner cylinder with a large axial force.

If the interference pressure set up at the common surface is p, the normal force N between the mating cylinders is then

$$N = p \times 2\pi r L$$

where L is the axial length of the contact surfaces.

The friction force F between the cylinders which has to be overcome by the applied force is thus

$$F = \mu N$$

where  $\mu$  is the coefficient of friction between the contact surfaces.

$$F = \mu(p \times 2\pi rL)$$

$$= 2\pi \mu nrL \qquad (10.15)$$

With a knowledge of the magnitude of the applied force required the value of p may be determined.

Alternatively, for a known interference between the cylinders the procedure of 10.11 may be carried out to determine the value of p which will be produced and hence the force F which will be required to carry out the press-fit operation.

#### 10.14. Compound cylinder-different materials

The value of the shrinkage or interference allowance for compound cylinders constructed from cylinders of different materials is given by eqn. (10.13). The value of the shrinkage pressure set up owing to a known amount of interference can then be calculated as with the standard compound cylinder treatment, each component cylinder being considered separately subject to the shrinkage pressure.

Having constructed the compound cylinder, however, the treatment is different for the analysis of stresses owing to applied internal and/or external pressures. Previously the compound cylinder has been treated as a single thick cylinder and, e.g., a single Lamé line drawn across both cylinder walls for solution. In the case of cylinders of different materials, however, each component cylinder must be considered separately as with the shrinkage effects. Thus, for a known internal pressure  $P_1$  which sets up a common junction pressure p, the Lamé line solution takes the form shown in Fig. 10.16.

Fig. 10.16. Graphical solution for compound tubes of different materials.

For a full solution of problems of this type it is often necessary to make use of the equality of *diametral* strains at the common junction surface, i.e. to realise that for the cylinders to maintain contact with each other the diametral strains must be equal at the common surface.

Now diametral strain = circumferential strain

$$=\frac{1}{E}\left[\sigma_{H}-v\sigma_{r}-v\sigma_{L}\right]$$

Therefore at the common surface, ignoring longitudinal strains and stresses,

$$\frac{1}{E} \left[ \sigma_{H_o} - v_o \sigma_r \right] = \frac{1}{E_i} \left[ \sigma_{H_i} - v_i \sigma_r \right]$$
(10.16)



where	$E_o$ and $v_o =$ Young's modulus and Poisson's ratio of outer cylinder,
	$E_i$ and $v_i$ = Young's modulus and Poisson's ratio of inner cylinder,
	$\sigma_r = -p = radial$ stress at common surface,

and

 $\sigma_{H_o}$  and  $\sigma_{H_i} =$  (as before) the hoop stresses at the common surface for the outer and inner cylinders respectively.

# 10.15. Uniform heating of compound cylinders of different materials

When an initially unstressed compound cylinder constructed from two tubes of the same material is heated uniformly, all parts of the cylinder will expand at the same rate, this rate depending on the value of the coefficient of expansion for the cylinder material.

If the two tubes are of different materials, however, each will attempt to expand at different rates and *differential thermal stresses* will be set up as described in §2.3. The method of treatment for such compound cylinders is therefore similar to that used for compound bars in the section noted.

Consider, therefore, two tubes of different material as shown in Fig. 10.17. Here it is convenient, for simplicity of treatment, to take as an example steel and brass for the two materials since the coefficients of expansion for these materials are known, the value for brass being greater than that for steel. Thus if the inner tube is of brass, as the temperature rises the brass will attempt to expand at a faster rate than the outer steel tube, the "free" expansions being indicated in Fig. 10.17a. In practice, however, when the tubes are joined as a compound cylinder, the steel will restrict the expansion of the brass and, conversely, the brass will force the steel to expand beyond its "free" expansion position. As a result a compromise situation is reached as shown in Fig. 10.17b, both tubes being effectively compressed radially (i.e. on their thickness) through the amounts shown. An effective increase  $p_t$  in "shrinkage" pressure is thus introduced.



Fig. 10.17. Uniform heating of compound cylinders constructed from tubes of different materials - in this case, steel and brass.

 $p_t$  is the radial pressure introduced at the common interface by virtue of the differential thermal expansions.

Therefore, as for the compound bar treatment of §2.3:

compression of steel + compression of brass = difference in "free" lengths

$$\varepsilon_s d + \varepsilon_B d = (\alpha_B - \alpha_s) t d$$
$$= (\alpha_B - \alpha_s) (T_2 - T_1) d \qquad (10.17)$$

§10.15

where d is the initial nominal diameter of the mating surfaces,  $\alpha_B$  and  $\alpha_s$  are the coefficients of linear expansion for the brass and steel respectively,  $t = T_2 - T_1$  is the temperature change, and  $\varepsilon_B$  and  $\varepsilon_s$  are the diametral strains in the two materials.

Alternatively, using a treatment similar to that used in the derivation of the thick cylinder "shrinkage fit" expressions

$$\Delta d = (\varepsilon_{H_a} - \varepsilon_{H_a})d = (\alpha_B - \alpha_s)td$$

 $\varepsilon_{H_i}$  being compressive, then producing an identical expression to that obtained above.

Now since diametral strain = circumferential or hoop strain

$$\varepsilon_{s} = \frac{1}{E_{s}} [\sigma_{t_{s}} - v_{s}\sigma_{r}]$$
$$\varepsilon_{B} = \frac{1}{E_{B}} [\sigma_{t_{B}} - v_{B}\sigma_{r}]$$

 $\sigma_{t_s}$  and  $\sigma_{t_B}$  being the hoop stresses set up at the common interface surfaces in the steel and brass respectively due to the differential thermal expansion and  $\sigma_r$  the effective increase in radial stress at the common junction surface caused by the same effect, i.e.  $\sigma_r = -p_t$ .

However, any radial pressure at the common interface will produce hoop tension in the outer cylinder but hoop compression in the inner cylinder. The expression for  $\varepsilon_B$  obtained above will thus always be negative when the appropriate stress values have been inserted. Since eqn. (10.17) deals with **magnitudes** of displacements only, it follows that a negative sign must be introduced to the value of  $\varepsilon_B$  before it can be substituted into eqn. (10.17).

Substituting for  $\varepsilon_s$  and  $\varepsilon_b$  in eqn. (10.17) with  $\sigma_r = -p_t$ 

$$\frac{1}{E_{s}}[\sigma_{t_{s}} + v_{s}p_{t}] - \frac{1}{E_{B}}[\sigma_{t_{B}} + v_{B}p_{t}] = [a_{B} - a_{s}][T_{2} - T_{1}]$$
(10.18)

The values of  $\sigma_{t_s}$  and  $\sigma_{t_g}$  are found in terms of the radial stress  $p_t$  at the junction surfaces by calculation or by graphical means as shown in Fig. 10.18.



Substitution in eqn. (10.18) then yields the value of the "unknown"  $p_t$  and hence the other resulting stresses.

10.16. Failure theories - yield criteria

For thick cylinder design the Tresca (maximum shear stress) criterion is normally used for *ductile materials* (see Chapter 15), i.e. the maximum shear stress in the cylinder wall is equated to the maximum shear stress at yield in simple tension,

$$\tau_{\rm max} = \sigma_v/2$$

Now the maximum shear stress is at the inside radius (§10.5) and is given by

$$\tau_{\max} = \frac{\sigma_H - \sigma_r}{2}$$

Therefore, for cylinder failure

$$\frac{\sigma_y}{2} = \frac{\sigma_H - \sigma_r}{2}$$
$$\sigma_y = \sigma_H - \sigma_r$$

i.e.

Here,  $\sigma_H$  and  $\sigma_r$  are the hoop and radial stresses at the inside radius and  $\sigma_y$  is the allowable yield stress of the material taking into account any safety factors which may be introduced by the company concerned.

For brittle materials such as cast iron the Rankine (maximum principal stress) theory is used. In this case failure is deemed to occur when

$$\sigma_y = \sigma_{H_{\max}}$$

# 10.17. Plastic yielding-"auto-frettage"

It has been shown that the most highly stressed part of a thick cylinder is at the inside radius. It follows, therefore, that if the internal pressure is increased sufficiently, yielding of the cylinder material will take place at this position. Fortunately the condition is not too serious at this stage since there remains a considerable bulk of elastic material surrounding the yielded area which contains the resulting strains within reasonable limits. As the pressure is increased further, however, plastic penetration takes place deeper and deeper into the cylinder wall and eventually the whole cylinder will yield.

If the pressure is such that plastic penetration occurs only partly into the cylinder wall, on release of that pressure the elastic outer zone attempts to return to its original dimensions but is prevented from doing so by the permanent deformation or "set" of the yielded material. The result is that the elastic material is held in a state of residual tension whilst the inside is brought into residual compression. This process is termed *auto-frettage* and it has the same effect as shrinking one tube over another without the necessary complications of the shrinking procedure, i.e. on subsequent loading cycles the cylinder is able to withstand a higher internal pressure since the compressive residual stress at the inside surface has to be overcome before this region begins to experience tensile stresses. For this reason gun barrels and other pressure vessels are often pre-stressed in this way prior to service.

A full theoretical treatment of the auto-frettage process is introduced in Chapter 18 together with associated plastic collapse theory.

#### 10.18. Wire-wound thick cylinders

Consider a thick cylinder with inner and outer radii  $R_1$  and  $R_2$  respectively, wound with wire under tension until its external radius becomes  $R_3$ . The resulting hoop and radial stresses developed in the cylinder will depend upon the way in which the tension T in the wire varies. The simplest case occurs when the tension in the wire is held constant throughout the winding process, and the solution for this condition will be introduced here. Solution for more complicated tension conditions will be found in more advanced texts and are not deemed appropriate for this volume. The method of solution, however, is similar.

#### (a) Stresses in the wire

Let the combined tube and wire be considered as a thick cylinder. The tension in the wire produces an "effective" external pressure on the tube and hence a compressive hoop stress.

Now for a thick cylinder subjected to an *external* pressure P the hoop and radial stresses are given by

$$\sigma_{H} = -\frac{PR_{2}^{2}}{(R_{2}^{2} - R_{1}^{2})} \left[ \frac{r^{2} + R_{1}^{2}}{r^{2}} \right]$$

and 
$$\sigma_r = -\frac{PR_2^2}{(R_2^2 - R_1^2)} \left[ \frac{r^2 - R_1^2}{r^2} \right]$$

i.e. 
$$\sigma_H = \sigma_r \left[ \frac{r^2 + R_1^2}{r^2 - R_1^2} \right]$$

If the initial tensile stress in the wire is T the final tensile hoop stress in the winding at any radius r is less than T by an amount equal to the compressive hoop stress set up by the effective "external" pressure caused by the winding,

i.e. final hoop stress in the winding at radius 
$$r = T - \sigma_r \left[ \frac{(r^2 + R_1^2)}{(r^2 - R_1^2)} \right]$$
 (10.19)

Using the same analysis outlined in §10.2,

$$\sigma_{H} = \sigma_{r} + r \frac{d\sigma_{r}}{dr}$$

$$\sigma_{r} + r \frac{d\sigma_{r}}{dr} = T - \sigma_{r} \left[ \frac{(r^{2} + R_{1}^{2})}{(r^{2} - R_{1}^{2})} \right]$$

$$r \frac{d\sigma_{r}}{dr} = T - \sigma_{r} \left[ \frac{r^{2} + R_{1}^{2} - r^{2} + R_{1}^{2}}{(r^{2} - R_{1}^{2})} \right]$$

$$= T - \left[ \frac{2R_{1}^{2}\sigma_{r}}{r^{2} - R_{1}^{2}} \right]$$

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Multiplying through by  $\frac{r}{(r^2 - R_1^2)}$  and rearranging,

$$\frac{r^{2}}{(r^{2} - R_{1}^{2})} \frac{d\sigma_{r}}{dr} + 2 \frac{R_{1}^{2}r}{(r^{2} - R_{1}^{2})^{2}} \sigma_{r} = \frac{Tr}{(r^{2} - R_{1}^{2})}$$
$$\frac{d}{dr} \left[ \frac{r^{2}}{(r^{2} - R_{1}^{2})} \sigma_{r} \right] = \frac{Tr}{(r^{2} - R_{1}^{2})}$$
$$\frac{r^{2}}{(r^{2} - R_{1}^{2})} \sigma_{r} = \frac{T}{2} \log_{e}(r^{2} - R_{1}^{2}) + A \qquad (10.20)$$

But  $\sigma_r = 0$  when  $r = R_3$ ,

$$0 = \frac{T}{2}\log_e(R_3^2 - R_1^2) + A$$
$$A = -\frac{T}{2}\log_e(R_3^2 - R_1^2)$$

Therefore substituting in eqn. (10.20),

$$\frac{r^{2}}{(r^{2} - R_{1}^{2})}\sigma_{r} = \frac{T}{2}\log_{e}\frac{(r^{2} - R_{1}^{2})}{(R_{3}^{2} - R_{1}^{2})}$$

$$\sigma_{r} = \frac{(r^{2} - R_{1}^{2})}{2r^{2}}T\log_{e}\frac{(r^{2} - R_{1}^{2})}{(R_{3}^{2} - R_{1}^{2})}$$

$$= -\frac{(r^{2} - R_{1}^{2})}{2r^{2}}T\log_{e}\frac{(R_{3}^{2} - R_{1}^{2})}{(r^{2} - R_{1}^{2})}$$
(10.21)

From eqn. (10.19),

$$\sigma_{H} = T - \sigma_{r} \left[ \frac{r^2 + R_1^2}{r^2 - R_1^2} \right]$$

Therefore since the sign of  $\sigma_r$  has been taken into account in setting up eqn. (10.19)

$$\sigma_{H} = T \left[ 1 - \frac{(r^{2} + R_{1}^{2})}{2r^{2}} \log_{e} \frac{(R_{3}^{2} - R_{1}^{2})}{(r^{2} - R_{1}^{2})} \right]$$
(10.22)

Thus eqns. (10.21) and (10.22) give the stresses in the wire winding for all radii between  $R_2$  and  $R_3$ .

# (b) Stresses in the tube

The stresses in the tube due to wire winding may be found from the normal thick cylinder expressions when it is considered subject to an external pressure  $P_2$  at radius  $R_2$ . The value of  $P_2$  is that obtained from eqn. (10.21) with  $r = R_2$ .

If an additional internal pressure is applied to the wire-wound cylinder it may be treated as a single thick cylinder and the resulting stresses combined algebraically with those due to winding to obtain the resultant effect.

# Examples

# Example 10.1 (B)

A thick cylinder of 100 mm internal radius and 150 mm external radius is subjected to an internal pressure of  $60 \text{ MN/m}^2$  and an external pressure of  $30 \text{ MN/m}^2$ . Determine the hoop and radial stresses at the inside and outside of the cylinder together with the longitudinal stress if the cylinder is assumed to have closed ends.

Solution (a): analytical



Fig. 10.19.

The internal and external pressures both have the effect of decreasing the thickness of the cylinder; the radial stresses at both the inside and outside radii are thus compressive, i.e. negative (Fig. 10.19).

and at r = 0.1 m,  $\sigma_r = -60 \text{ MN/m}^2$ and  $\sigma_r = -30 \text{ MN/m}^2$ 

Therefore, from eqn. (10.3), with stress units of  $MN/m^2$ ,

$$-60 = A - 100B$$
 (1)

and

$$30 = A - 44.5B$$
 (2)

Subtracting (2) from (1),

$$-30 = -55.5B$$
$$B = 0.54$$
$$A = -60 + 100 \times 0.54$$
$$A = -6$$

Therefore, from (1),

Therefore, at r = 0.1 m, from eqn. (10.4),

$$\sigma_{H} = A + \frac{B}{r^{2}} = -6 + 0.54 \times 100$$
$$= 48 \,\mathrm{MN/m^{2}}$$

Thick Cylinders

and at  $r = 0.15 \, \text{m}$ ,

$$= 18 \, MN/m^2$$

 $\sigma_{H} = -6 + 0.54 \times 44.5 = -6 + 24$ 

From eqn. (10.7) the longitudinal stress is given by

$$\sigma_L = \frac{P_1 R_1^2 - P_2 R_2^2}{(R_2^2 - R_1^2)} = \frac{(60 \times 0.1^2 - 30 \times 0.15^2)}{(0.15^2 - 0.1^2)}$$
$$= \frac{10^2 (60 - 30 \times 2.25)}{1.25 \times 10^2} = -6 \,\mathrm{MN/m^2} \quad \text{i.e. compressive}$$

## Solution (b): graphical

The graphical solution is shown in Fig. 10.20, where the boundaries of the cylinder are given by

$$\frac{1}{r^2} = 100 \text{ for the inner radius where } r = 0.1 \text{ m}$$
$$\frac{1}{r^2} = 44.5 \text{ for the outer radius where } r = 0.15 \text{ m}$$



Fig. 10.20.

The two conditions which enable the Lamé line to be drawn are the same as those used above for the analytical solution,

i.e.

$$\sigma_r = -60 \text{ MN/m}^2 \text{ at } r = 0.1 \text{ m}$$
  
 $\sigma_r = -30 \text{ MN/m}^2 \text{ at } r = 0.15 \text{ m}$ 

The hoop stresses at these radii are then given by points P and Q on the graph. For complete accuracy these values should be calculated by proportions of the graph thus:

by similar triangles PAS and BAC

$$\frac{PS}{100 + 44.5} = \frac{CB}{100 - 44.5}$$
$$\frac{PL + LS}{144.5} = \frac{30}{55.5}$$

i.e. hoop stress at radius r = 0.1 m

$$= PL = \frac{30 \times 144.5}{55.5} - LS$$
$$= 78 - 30 = 48 \text{ MN/m}^2$$

Similarly, the hoop stress at radius r = 0.15 m is QM and given by the similar triangles QAT and BAC,

i.e. 
$$\frac{QM + MT}{44.5 + 44.5} = \frac{30}{55.5}$$
$$QM = \frac{30 \times 89}{55.5} - 30$$
$$= 48 - 30 = 18 \text{ MN/m}^2$$

The longitudinal stress  $\sigma_L$  = the intercept on the  $\sigma$  axis (which is negative)

= DO = OE - DE = 30 - DE

·.	DE = 24		
•	DE = 24 $\sigma_{1} = 30 - 24 - 6$	$MN/m^2$	compressive

# Example 10.2 (B)

An external pressure of 10 MN/m<sup>2</sup> is applied to a thick cylinder of internal diameter 160 mm and external diameter 320 mm. If the maximum hoop stress permitted on the inside wall of the cylinder is limited to 30 MN/m<sup>2</sup>, what maximum internal pressure can be applied assuming the cylinder has closed ends? What will be the change in outside diameter when this pressure is applied?  $E = 207 \text{ GN/m}^2$ , v = 0.29.

Solution (a): analytical

The conditions for the cylinder are:

When  $r = 0.08 \text{ m}, \sigma_r = -p$  and  $\frac{1}{r^2} = 156$ 

. <sup>.</sup> .

when r = 0.16 m,  $\sigma_r = -10 \text{ MN/m}^2$  and  $\frac{1}{r^2} = 39$ 

and when r = 0.08 m,  $\sigma_H = 30 \text{ MN/m}^2$ 

since the maximum hoop stress occurs at the inside surface of the cylinder.

Using the latter two conditions in eqns. (10.3) and (10.4) with units of  $MN/m^2$ ,

$$-10 = A - 39B$$
 (1)  
30 = A + 156B (2)

Subtracting (1) from (2),

Substituting in (1),

 $40 = 195B \qquad \therefore \ B = 0.205$  $A = -10 + (39 \times 0.205)$  $= -10 + 8 \qquad \therefore \ A = -2$ 

Therefore, at r = 0.08, from eqn. (10.3),

$$\sigma_r = -p = A - 156B$$
  
=  $-2 - 156 \times 0.205$   
=  $-2 - 32 = -34 \text{ MN/m}^2$ 

i.e. the allowable internal pressure is  $34 \text{ MN/m}^2$ .

From eqn. (10.9) the change in diameter is given by

$$\Delta D = \frac{2r_0}{E} \left( \sigma_H - v \sigma_r - v \sigma_L \right)$$

Now at the outside surface

. .

$$\sigma_{r} = -10 \text{ MN/m}^{2} \text{ and } \sigma_{H} = A + \frac{B}{r^{2}}$$

$$= -2 + (39 \times 0.205)$$

$$= -2 + 8 = 6 \text{ MN/m}^{2}$$

$$\sigma_{L} = \frac{P_{1}R_{1}^{2} - P_{2}R_{2}^{2}}{(R_{2}^{2} - R_{1}^{2})} = \frac{(34 \times 0.08^{2} - 10 \times 0.16^{2})}{(0.16^{2} - 0.08^{2})}$$

$$= \frac{(34 \times 0.64 - 10 \times 2.56)}{(2.56 - 0.64)} = \frac{21.8 - 25.6}{1.92}$$

$$= -\frac{3.8}{1.92} = 1.98 \text{ MN/m}^{2} \text{ compressive}$$

$$\Delta D = \frac{0.32}{207 \times 10^{9}} [6 - 0.29(-10) - 0.29(-1.98)] 10^{6}$$

$$= \frac{0.32}{207 \times 10^{3}} (6 + 2.9 + 0.575)$$

$$= 14.7 \ \mu\text{m}$$

Solution (b): graphical



Fig. 10.21.

The graphical solution is shown in Fig. 10.21. The boundaries of the cylinder are as follows:

for 
$$r = 0.08$$
 m,  $\frac{1}{r^2} = 156$   
and for  $r = 0.16$  m,  $\frac{1}{r^2} = 39$ 

The two fixed points on the graph which enable the line to be drawn are, therefore,

 $\sigma_r = -10 \text{ MN/m}^2$  at r = 0.16 and  $\sigma_H = 30 \text{ MN/m}^2$  at r = 0.08 m

The allowable internal pressure is then given by the value of  $\sigma_r$  at  $r = 0.08 \text{ m} \left(\frac{1}{r^2} = 156\right)$ , i.e. 34 MN/m<sup>2</sup>.

Similarly, the hoop stress at the outside surface is given by the value of  $\sigma_H$  at  $\frac{1}{r^2} = 39$ , i.e.  $6 \text{ MN/m}^2$ , and the longitudinal stress by the intercept on the  $\sigma$  axis, i.e.  $2 \text{ MN/m}^2$  compressive.

N.B.-In practice all these values should be calculated by proportions.

# Example 10.3 (B)

(a) In an experiment on a thick cylinder of 100 mm external diameter and 50 mm internal diameter the hoop and longitudinal strains as measured by strain gauges applied to the outer

surface of the cylinder were  $240 \times 10^{-6}$  and  $60 \times 10^{-6}$ , respectively, for an internal pressure of 90 MN/m<sup>2</sup>, the external pressure being zero.

Determine the actual hoop and longitudinal stresses present in the cylinder if  $E = 208 \text{ GN/m}^2$  and v = 0.29. Compare the hoop stress value so obtained with the theoretical value given by the Lame equations.

(b) Assuming that the above strain readings were obtained for a thick cylinder of 100 mm external diameter but unkonwn internal diameter calculate this internal diameter.

# Solution

(a)

$$\varepsilon_H = \frac{1}{E} (\sigma_H - v\sigma_L) \text{ and } \varepsilon_L = \frac{1}{E} (\sigma_L - v\sigma_H)$$

since  $\sigma_r = 0$  at the outside surface of the cylinder for zero external pressure.

$$\therefore \qquad 240 \times 10^{-6} \times 208 \times 10^9 = \sigma_H - 0.29 \sigma_L = 50 \times 10^6 \tag{1}$$

$$60 \times 10^{-6} \times 208 \times 10^9 = \sigma_L - 0.29 \sigma_H = 12.5 \times 10^6 \tag{2}$$

(1) × 0.29  
(2) 
$$0.29\sigma_H - 0.084\sigma_L = 14.5 \times 10^6$$
  
 $\sigma_L - 0.29\sigma_H = 12.5 \times 10^6$ 

(3) + (2) 
$$0.916\sigma_L = 27 \times 10^6$$

$$\therefore \qquad \sigma_L = 29.5 \text{ MN/m}^2$$
Substituting in (2) 
$$0.29\sigma_H = 29.5 - 12.5 = 17 \times 10^6$$

 $\sigma_{H} = 58.7 \text{ MN/m}^{2}$ 

The theoretical values of  $\sigma_H$  for an internal pressure of 90 MN/m<sup>2</sup> may be obtained from Fig. 10.22, the boundaries of the cylinder being given by r = 0.05 and r = 0.025,

i.e.  $\frac{1}{r^2} = 400$  and 1600 respectively i.e.  $\sigma_H = 60 \text{ MN/m}^2$  theoretically



Fig. 10.22.

(3)

Mechanics of Materials

(b) From part (a)	$\sigma_H = 58.7 \text{ MN/m}^2$	at $r = 0.05$
and 	$\sigma_r = 0$ 58.7 = A + 400B	at $r = 0.05$
	0 = A - 400B	
Adding:	$58.7 = 2A$ $\therefore$ $A =$	29.35
and since	$A = 400B$ $\therefore$ B	R = 0.0734

Therefore for the internal radius  $R_1$  where  $\sigma_r = 90 \text{ MN/m}^2$ 

$$-90 = 29.35 - \frac{0.0734}{R_1^2}$$

$$R_1^2 = \frac{0.0734}{119.35} = 0.000615$$

$$= 6.15 \times 10^{-4}$$

$$R_1 = 2.48 \times 10^{-2} \text{ m} = 24.8 \text{ mm}$$

... ..

# Internal diameter = 49.6 mm

For a graphical solution of part (b), see Fig. 10.23, where the known points which enable the Lame line to be drawn are, as above:

$$\sigma_H = 58.7$$
 at  $\frac{1}{r^2} = 400$  and  $\sigma_r = 0$  at  $\frac{1}{r^2} = 400$ 

It is thus possible to determine the value of  $1/R_1^2$  which will produce  $\sigma_r = -90 \text{ MN/m}^2$ .



Fig. 10.23.

Let the required value of 
$$\frac{1}{R_1^2} = x$$
  
then by proportions  $\frac{90}{x - 400} = \frac{58.7}{800}$ 

 $x - 400 = \frac{90 \times 800}{58.7} = 1225$ x = 1625. <sup>.</sup> .  $R_1 = 24.8 \text{ mm}$ · ·

i.e. required internal diameter = 49.6 mm

# Example 10.4 (B)

A compound cylinder is formed by shrinking a tube of 250 mm internal diameter and 25 mm wall thickness onto another tube of 250 mm external diameter and 25 mm wall thickness, both tubes being made of the same material. The stress set up at the junction owing to shrinkage is  $10 \text{ MN/m}^2$ . The compound tube is then subjected to an internal pressure of  $80 \text{ MN/m}^2$ . Compare the hoop stress distribution now obtained with that of a single cylinder of 300 mm external diameter and 50 mm thickness subjected to the same internal pressure.

### Solution (a): analytical

A solution is obtained as described in §10.9, i.e. by considering the effects of shrinkage and internal pressure separately and combining the results algebraically.

Shrinkage only-outer tube

At r = 0.15,  $\sigma_r = 0$  and at r = 0.125,  $\sigma_r = -10 \text{ MN/m}^2$  $0 = A - \frac{B}{0.15^2} = A - 44.5B$ · . (1)

$$-10 = A - \frac{B}{0.125^2} = A - 64B \tag{2}$$

Subtracting (1) – (2), 10 = 19.5B  $\therefore B = 0.514$ A = 44.5B  $\therefore A = 22.85$ Substituting in (1), · .

hoop stress at 0.15 m radius = 
$$A + 44.5B = 45.7 \text{ MN/m}^2$$

hoop stress at 0.125 m radius =  $A + 64B = 55.75 \text{ MN/m}^2$ 

Shrinkage only-inner tube

· .

At r = 0.10,  $\sigma_r = 0$  and at r = 0.125,  $\sigma_r = -10 \text{ MN/m}^2$ 

$$0 = A - \frac{B}{0.1^2} = A - 100B \tag{3}$$

$$-10 = A - \frac{B}{0.125^2} = A - 64B \tag{4}$$

Subtracting (3) – (4), 10 = -36B  $\therefore B = -0.278$ A = -27.8A = 100BSubstituting in (3),

. ho	op stress at 0.125 m	radius :	= A + 64B =	$= -45.6 \text{ MN}/\text{m}^2$	
ho	oop stress at 0.10 m	radius :	= A + 100B	$= -55.6 \text{ MN}/\text{m}^2$	
Considering inte	rnal pressure only (	on compl	ete cylinder	)	
At $r = 0.15$ , $\sigma_r =$	= 0 and at $r = 0$ .	10, $\sigma_r =$	- 80		
<i>.</i>		0 =	= A - 44.5B		(5)
		- 80 =	= A - 100B		(6)
Subtracting $(5) - (6)$	6), 80 = 5	5.5 <i>B</i>	$\therefore B = 1.4$	4	
From (5),	A = 4	4.5 <i>B</i>	A = 64.	2	
÷.	At $r = 0.15$ m,	$\sigma_H = A$	+44.5B = 1	28.4 MN/m <sup>2</sup>	
	r = 0.125  m,	$\sigma_H = A$	+64B = 1	56.4 MN/m <sup>2</sup>	
	r = 0.1  m,	$\sigma_H = A$	+100B = 2	$208.2 \text{ MN/m}^2$	
The recultent et	increase for combined	ahninka	an and inter		

The resultant stresses for combined shrinkage and internal pressure are then:

outer tube:	r = 0.15	$\sigma_{H} =$	128.4	+	45.7	=	174.1	MN/m <sup>2</sup>
	r = 0.125	$\sigma_{H} =$	156.4	+	55.75	=	212.15	MN/m <sup>2</sup>
inner tube:	r = 0.125	$\sigma_{H} =$	156.4	_	45.6	=	110.8	MN/m <sup>2</sup>
	r = 0.1	$\sigma_{H} =$	208.2	_	55.6	=	152.6	MN/m <sup>2</sup>

Solution (b): graphical

The graphical solution is obtained in the same way by considering the separate effects of shrinkage and internal pressure as shown in Fig. 10.24.



Fig. 10.24.

The final results are illustrated in Fig. 10.25 (values from the graph again being determined by proportion of the figure).



# **Example 10.5 (B)**

A compound tube is made by shrinking one tube of 100 mm internal diameter and 25 mm wall thickness on to another tube of 100 mm external diameter and 25 mm wall thickness. The shrinkage allowance, *based on radius*, is 0.01 mm. If both tubes are of steel (with  $E = 208 \text{ GN/m}^2$ ), calculate the radial pressure set up at the junction owing to shrinkage.

# Solution

Let p be the required shrinkage pressure, then for the inner tube:

At 
$$r = 0.025$$
,  $\sigma_r = 0$  and at  $r = 0.05$ ,  $\sigma_r = -p$   
$$0 = A - \frac{B}{0.025^2} = A - 1600B$$
(1)

$$-p = A - \frac{B}{0.05^2} = A - 400B \tag{2}$$

Subtracting (2) - (1),

$$-p = 1200B$$
  $\therefore B = -p/1200$ 

From (1), 
$$A = 1600B$$
  $\therefore A = -\frac{1600p}{1200} = -\frac{4p}{3}$ 

Therefore at the common radius the hoop stress is given by eqn. (10.4),

$$\sigma_{H_i} = A + B/0.05^2$$
$$= -\frac{4p}{3} + 400\left(-\frac{p}{1200}\right) = -\frac{5p}{3} = -1.67p$$

For the outer tube:

at 
$$r = 0.05$$
,  $\sigma_r = -p$  and at  $r = 0.075$ ,  $\sigma_r = 0$   
 $-p = A - B/0.05^2 = A - 400B$  (3)

$$0 = A - B/0.075^{2} = A - 178B$$
(4)  
Subtracting (4) - (3),  $p = 222B$   $\therefore B = p/222$   
From (4)  $A = 178B$   $\therefore A = \frac{178p}{222}$ 

Therefore at the common radius the hoop stress is given by

$$\sigma_{H_o} = A + \frac{B}{0.05^2}$$
$$= \frac{178p}{222} + \frac{p}{222} \times 400 = \frac{578p}{222} = 2.6p$$

Now from eqn. (10.14) the shrinkage allowance is

$$\frac{r}{E} [\sigma_{H_o} - \sigma_{H_i}]$$

$$0.01 \times 10^{-3} = \frac{50 \times 10^{-3}}{208 \times 10^9} [2.6p - (-1.67p)] 10^6$$

where p has units of  $MN/m^2$ 

$$\therefore \qquad 4.27p = \frac{0.01 \times 208 \times 10^3}{50} = 41.6$$

Hoop and radial stresses in the compound cylinder owing to shrinkage and/or internal pressure can now be determined if desired using the procedure of the previous example.

 $p = 9.74 \text{ MN/m}^2$ 

Once again a graphical solution could have been employed to obtain the values of  $\sigma_{H_o}$  and  $\sigma_{H_i}$  in terms of the unknown pressure p which is set off to some convenient distance on the r = 0.05, i.e.  $1/r^2 = 400$ , line.

# **Example 10.6 (B)**

Two steel rings of radial thickness 30 mm, common radius 70 mm and length 40 mm are shrunk together to form a compound ring. It is found that the axial force required to separate the rings, i.e. to push the inside ring out, is 150 kN. Determine the shrinkage pressure at the mating surfaces and the shrinkage allowance.  $E = 208 \text{ GN/m}^2$ . The coefficient of friction between the junction surfaces of the two rings is 0.15.

# Solution

Let the pressure set up between the rings be  $p \text{ MN/m}^2$ . Then, normal force between rings =  $p \times 2\pi rL = N$ 

$$= p \times 10^6 \times 2\pi \times 70 \times 10^{-3} \times 40 \times 10^{-3}$$

 $= 5600 \pi p$  newtons.

246

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friction force between rings = 
$$\mu N = 0.15 \times 5600 \pi p$$
  
 $0.15 \times 5600 \pi p = 150 \times 10^3$   
 $p = \frac{150 \times 10^3}{0.15 \times 5600 \pi}$   
 $= 57 \text{ MN/m}^2$ 

Now, for the inner tube:

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$$\sigma_r = -57 \text{ at } r = 0.07 \text{ and } \sigma_r = 0 \text{ at } r = 0.04$$
  
 $\therefore \qquad -57 = A - B/0.07^2 = A - 204B \qquad (1)$   
 $0 = A - B/0.04^2 = A - 625B \qquad (2)$ 

Subtracting (2) - (1), 
$$57 = -421B$$
  $\therefore B = -0.135$   
From (2),  $A = 625B$   $\therefore A = -84.5$ 

Therefore at the common radius the hoop stress in the inner tube is given by

$$\sigma_{H_i} = A + \frac{B}{0.07^2} = A + 204B = -112.1 \text{ MN/m}^2$$

For the outer tube:

$$\sigma_r = -57 \text{ at } r = 0.07 \text{ and } \sigma_r = 0 \text{ at } r = 0.1$$
  
 $\therefore \qquad -57 = A - 204B$  (3)  
 $0 = A - 100B$  (4)

Subtracting (4) - (3),
$$57 = 104B$$
 $\therefore B = 0.548$ From (4), $A = 100B$  $\therefore A = 54.8$ 

Therefore at the common radius the hoop stress in the outer tube is given by

$$\sigma_{H_o} = A + \frac{B}{0.07^2} = A + 204B = 166.8 \text{ MN/m}^2$$
shrinkage allowance  $= \frac{r}{E} (\sigma_{H_o} - \sigma_{H_i})$ 
 $= \frac{70 \times 10^{-3}}{208 \times 10^9} [166.8 - (-112.1)] \, 10^6$ 
 $= \frac{70 \times 278.9}{208} \times 10^{-6} \text{ m}$ 
 $= 93.8 \times 10^{-6} = 0.094 \text{ mm}$ 

# **Example 10.7 (B)**

(a) A steel sleeve of 150 mm outside diameter is to be shrunk on to a solid steel shaft of 100 mm diameter. If the shrinkage pressure set up is  $15 \text{ MN/m}^2$ , find the initial difference between the inside diameter of the sleeve and the outside diameter of the shaft.

(b) What percentage error would be involved if the shaft were assumed to be incompressible? For steel,  $E = 208 \text{ GN/m}^2$ ; v = 0.3.

Solution

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(a) Treating the sleeve as a thick cylinder with internal pressure 15 MN/m<sup>2</sup>, at r = 0.05,  $\sigma_r = -15$  MN/m<sup>2</sup> and at r = 0.075,  $\sigma_r = 0$ 

$$-15 = A - B/0.05^2 = A - 400B \tag{1}$$

$$0 = A - B/0.075^2 = A - 178B$$
<sup>(2)</sup>

Subtracting (2) - (1),15 = 222B $\therefore B = 0.0676$ From (2),A = 178B $\therefore A = 12.05$ 

Therefore the hoop stress in the sleeve at r = 0.05 m is given by

$$\sigma_{H_o} = A + B/0.05^2$$
  
=  $A + 400B = 39 \text{ MN/m}^2$ 

The shaft will be subjected to a hoop stress which will be compressive and equal in value to the shrinkage pressure (see §10.12),

i.e. 
$$\sigma_{H_i} = -15 \text{ MN/m}^2$$

Thus the difference in radii or shrinkage allowance

$$= \frac{r}{E} (\sigma_{H_o} - \sigma_{H_i}) = \frac{50 \times 10^{-3}}{208 \times 10^9} [39 + 15] 10^6$$
$$= \frac{50 \times 54}{208} \times 10^{-6} = 13 \times 10^{-6} \text{ m}$$
difference in diameters = **0.026 mm**

(b) If the shaft is assumed incompressible the difference in diameters will equal the necessary change in diameter of the sleeve to fit the shaft. This can be found from the diametral strain, i.e. from eqn. (10.9)

$$\Delta D = \frac{2r}{E} (\sigma_H - v\sigma_r) \text{ assuming } \sigma_L = 0$$
  
change of diameter  $= \frac{100 \times 10^{-3}}{208 \times 10^9} [39 - 0.3(-15)] 10^6$   
 $= \frac{43.5}{208} \times 10^{-4} = 20.9 \times 10^{-6}$   
 $= 0.0209 \text{ mm}$   
percentage error  $= \frac{(0.026 - 0.0209)}{0.026} \times 100 = 19.6\%$ 

# **Example 10.8 (C)**

A thick cylinder of 100 mm external diameter and 50 mm internal diameter is wound with steel wire of 1 mm diameter, initially stressed to  $20 \text{ MN/m}^2$  until the outside diameter is 120 mm. Determine the maximum hoop stress set up in the cylinder if an internal pressure of  $30 \text{ MN/m}^2$  is now applied.

# Solution

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To find the stresses resulting from internal pressure only the cylinder and wire may be treated as a single thick cylinder of 50 mm internal diameter and 120 mm external diameter.

Now 
$$\sigma_r = -30 \text{ MN/m}^2$$
 at  $r = 0.025$  and  $\sigma_r = 0$  at  $r = 0.06$   
 $\therefore -30 = A - B/0.025^2 = A - 1600B$  (1)  
 $0 = A - B/0.06^2 = A - 278B$  (2)  
Subtracting (2) - (1),  $30 = 1322B$   $\therefore B = 0.0227$ 

From (2), A = 278B  $\therefore A = 6.32$ 

hoop stress at 25 mm radius =  $A + 1600B = 42.7 \text{ MN/m}^2$ hoop stress at 50 mm radius =  $A + 400B = 15.4 \text{ MN/m}^2$ 

The external pressure acting on the cylinder owing to wire winding is found from eqn. (10.21),

i.e.  

$$\sigma_r = -\frac{(r^2 - R_1^2)}{2r^2} T \log_e \frac{(R_3^2 - R_1^2)}{(r^2 - R_1^2)} = p$$

$$\therefore \qquad \text{where } r = R_2 = 0.05 \text{ m}, R_1 = 0.025 \quad \text{and } R_3 = 0.06$$

$$p = -\frac{(0.05^2 - 0.025^2)}{2 \times 0.05^2} T \log_e \frac{(0.06^2 - 0.025^2)}{(0.05^2 - 0.025^2)}$$

$$= -\frac{(25 - 6.25)}{50} 20 \log_e \frac{(36 - 6.25)}{(25 - 6.25)} \text{ MN/m}^2$$

$$= -\frac{18.75 \times 20}{50} \log_e \frac{29.75}{18.75}$$

$$= -7.5 \log_e 1.585 = -7.5 \times 0.4606$$

$$= -3.45 \text{ MN/m}^2$$

Therefore for wire winding only the stresses in the tube are found from the conditions

$$\sigma_r = -3.45$$
 at  $r = 0.05$  and  $\sigma_r = 0$  at  $r = 0.025$   
 $-3.45 = A - 400B$   
 $0 = A - 1600B$ 

-3.45 = 1200B  $-B = -2.88 \times 10^{-3}$ 

Subtracting,

$$\therefore A = -4.6$$

hoop stress at 25 mm radius = A + 1600B = -9.2 MN/m<sup>2</sup>

hoop stress at 50 mm radius =  $A + 400B = -5.75 \text{ MN/m}^2$ 

The resultant stresses owing to winding and internal pressure are, therefore:

At 
$$r = 25$$
 mm,  $\sigma_H = 42.7 - 9.2 = 33.5$  MN/m<sup>2</sup>  
At  $r = 50$  mm,  $\sigma_H = 15.4 - 5.75 = 9.65$  MN/m<sup>2</sup>

Thus the maximum hoop stress is  $33.5 \text{ MN/m}^2$ 

# **Example 10.9 (C)**

A thick cylinder of internal and external radii 300 mm and 500 mm respectively is subjected to a gradually increasing internal pressure P. Determine the value of P when:

- (a) the material of the cylinder first commences to yield;
- (b) yielding has progressed to mid-depth of the cylinder wall;
- (c) the cylinder material suffers complete "collapse".

Take  $\sigma_v = 600 \text{ MN}/\text{m}^2$ .

Solution See Chapter 3 from Mechanics of Materials  $2^{\dagger}$ 

From eqn. (3.35) the initial yield pressure

$$= \frac{\sigma_y}{2R_2^2} [R_2^2 - R_1^2] = \frac{600}{2 \times 0.5^2} [0.5^2 - 0.3^2]$$
$$= \frac{600}{2 \times 25} [25 - 9] = 192 \text{ MN/m}^2$$

The pressure required to cause yielding to a depth  $R_p = 40$  mm is given by eqn. (3.36)

$$\sigma_r = \sigma_y \left[ \log_e \frac{R_1}{R_p} - \frac{1}{2R_2^2} (R_2^2 - R_p^2) \right]$$
  
= 600  $\left[ \log_e \frac{0.3}{0.4} - \frac{1}{2 \times 0.5^2} (0.5^2 - 0.4^2) \right]$   
= -600  $\left[ \log_e 1.33 + \left( \frac{25 - 16}{50} \right) \right]$   
= -600(0.2852 + 0.18)  
= -600 × 0.4652 = -280 MN/m<sup>2</sup>

<sup>†</sup>E. J. Hearn, Mechanics of Materials 2, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

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the required internal pressure =  $280 \text{ MN}/\text{m}^2$ 

For complete collapse from eqn. (3.34),

$$p = -\sigma_y \log_e \frac{R_1}{R_2} = \sigma_y \log_e \frac{R_2}{R_1}$$
$$= \sigma_y \log_e \frac{0.5}{0.3}$$
$$= 600 \times \log_e 1.67$$
$$= 600 \times 0.513 = 308 \text{ MN/m}^2$$

## Problems

10.1 (B). A thick cylinder of 150 mm inside diameter and 200 mm outside diameter is subjected to an internal pressure of  $15 \, \text{MN/m^2}$ . Determine the value of the maximum hoop stress set up in the cylinder walls.

[53.4 MN/m<sup>2</sup>.]

**10.2** (B). A cylinder of 100 mm internal radius and 125 mm external radius is subjected to an external pressure of 14 bar  $(1.4 \text{ MN/m}^2)$ . What will be the maximum stress set up in the cylinder?  $[-7.8 \text{ MN/m}^2]$ .

10.3 (B). The cylinder of Problem 10.2 is now subjected to an additional internal pressure of 200 bar  $(20 \text{ MN/m}^2)$ . What will be the value of the maximum stress? [84.7 MN/m<sup>2</sup>.]

10.4 (B). A steel thick cylinder of external diameter 150 mm has two strain gauges fixed externally, one along the longitudinal axis and the other at right angles to read the hoop strain. The cylinder is subjected to an internal pressure of  $75 \text{ MN/m}^2$  and this causes the following strains:

(a) hoop gauge:  $455 \times 10^{-6}$  tensile;

(b) longitudinal gauge:  $124 \times 10^{-6}$  tensile.

Find the internal diameter of the cylinder assuming that Young's modulus for steel is 208 GN/m<sup>2</sup> and Poisson's ratio is 0.283. [B.P.] [96.7 mm.]

**10.5** (B) A compound tube of 300 mm external and 100 mm internal diameter is formed by shrinking one cylinder on to another, the common diameter being 200 mm. If the maximum hoop tensile stress induced in the outer cylinder is 90  $MN/m^2$  find the hoop stresses at the inner and outer diameters of both cylinders and show by means of a sketch how these stresses vary with the radius. [90, 55.35; -92.4, 57.8  $MN/m^2$ .]

10.6 (B). A compound thick cylinder has a bore of 100 mm diameter, a common diameter of 200 mm and an outside diameter of 300 mm. The outer tube is shrunk on to the inner tube, and the radial stress at the common surface owing to shrinkage is  $30 \text{ MN/m}^2$ .

Find the maximum internal pressure the cylinder can receive if the maximum circumferential stress in the outer tube is limited to  $110 \text{ MN/m}^2$ . Determine also the resulting circumferential stress at the outer radius of the inner tube. [B.P.] [79,  $-18 \text{ MN/m}^2$ .]

10.7 (B). Working from first principles find the interference fit per metre of diameter if the radial pressure owing to this at the common surface of a compound tube is 90 MN/m<sup>2</sup>, the inner and outer diameters of the tube being 100 mm and 250 mm respectively and the common diameter being 200 mm. The two tubes are made of the same material, for which  $E = 200 \text{ GN/m}^2$ . If the outside diameter of the inner tube is originally 200 mm, what will be the original inside diameter of the outer tube for the above conditions to apply when compound? [199.44 mm.]

**10.8** (B). A compound cylinder is formed by shrinking a tube of 200 mm outside and 150 mm inside diameter on to one of 150 mm outside and 100 mm inside diameter. Owing to shrinkage the radial stress at the common surface is  $20 \text{ MN/m}^2$ . If this cylinder is now subjected to an internal pressure of  $100 \text{ MN/m}^2$  (1000 bar), what is the magnitude and position of the maximum hoop stress? [164 MN/m<sup>2</sup> at inside of outer cylinder.]

10.9 (B). A thick cylinder has an internal diameter of 75 mm and an external diameter of 125 mm. The ends are closed and it carries an internal pressure of  $60 \text{ MN/m}^2$ . Neglecting end effects, calculate the hoop stress and radial stress at radii of 37.5 mm, 40 mm, 50 mm, 60 mm and 62.5 mm. Plot the values on a diagram to show the variation of these stresses through the cylinder wall. What is the value of the longitudinal stress in the cylinder?

[C.U.] [Hoop: 128, 116, 86.5, 70.2, 67.5 MN/m<sup>2</sup>. Radial: -60, -48.7, -19, -2.9, 0 MN/m<sup>2</sup>; 33.8 MN/m<sup>2</sup>.]

i.e.

10.10 (B). A compound tube is formed by shrinking together two tubes with common radius 150 mm and thickness 25 mm. The shrinkage allowance is to be such that when an internal pressure of  $30 \text{ MN/m}^2$  (300 bar) is applied the final maximum stress in each tube is to be the same. Determine the value of this stress. What must have been the difference in diameters of the tubes before shrinkage?  $E = 210 \text{ GN/m}^2$ . [83.1 MN/m<sup>2</sup>; 0.025 mm.]

10.11 (B). A steel shaft of 75 mm diameter is pressed into a steel hub of 100 mm outside diameter and 200 mm long in such a manner that under an applied torque of 6 kN m relative slip is just avoided. Find the interference fit, assuming a 75 mm common diameter, and the maximum circumferential stress in the hub.  $\mu = 0.3$ .  $E = 210 \text{ GN/m}^2$  [0.0183 mm; 40.4 MN/m<sup>2</sup>.]

**10.12** (B). A steel plug of 75 mm diameter is forced into a steel ring of 125 mm external diameter and 50 mm width. From a reading taken by fixing in a circumferential direction an electric resistance strain gauge on the external surface of the ring, the strain is found to be  $1.49 \times 10^{-4}$ . Assuming  $\mu = 0.2$  for the mating surfaces, find the force required to push the plug out of the ring. Also estimate the greatest hoop stress in the ring.  $E = 210 \text{ GN/m}^2$ . [I.Mech.E.] [65.6 kN; 59 MN/m<sup>2</sup>.]

**10.13** (B). A steel cylindrical plug of 125 mm diameter is forced into a steel sleeve of 200 mm external diameter and 100 mm long. If the greatest circumferential stress in the sleeve is 90 MN/m<sup>2</sup>, find the torque required to turn the sleeve, assuming  $\mu = 0.2$  at the mating surfaces. [U.L.] [19.4 kN m.]

10.14 (B). A solid steel shaft of 0.2 m diameter has a bronze bush of 0.3 m outer diameter shrunk on to it. In order to remove the bush the whole assembly is raised in temperature uniformly. After a rise of  $100^{\circ}$ C the bush can just be moved along the shaft. Neglecting any effect of temperature in the axial direction, calculate the original interface pressure between the bush and the shaft.

For steel: 
$$E = 208 \text{ GN/m}^2$$
,  $v = 0.29$ ,  $\alpha = 12 \times 10^{-6} \text{ per} \,^\circ\text{C}$ .  
For bronze:  $E = 112 \text{ GN/m}^2$ ,  $v = 0.33$ ,  $\alpha = 18 \times 10^{-6} \text{ per} \,^\circ\text{C}$ .

[C.E.I.] [20.2 MN/m<sup>2</sup>.]

10.15 (B). (a) State the Lamé equations for the hoop and radial stresses in a thick cylinder subjected to an internal pressure and show how these may be expressed in graphical form. Hence, show that the hoop stress at the outside surface of such a cylinder subjected to an internal pressure P is given by

$$\sigma_H = \frac{2PR_1^2}{(R_2^2 - R_1^2)}$$

where  $R_1$  and  $R_2$  are the internal and external radii of the cylinder respectively.

(b) A steel tube is shrunk on to another steel tube to form a compound cylinder 60 mm internal diameter, 180 mm external diameter. The initial radial compressive stress at the 120 mm common diameter is  $30 \text{ MN/m}^2$ . Calculate the shrinkage allowance.  $E = 200 \text{ GN/m}^2$ .

(c) If the compound cylinder is now subjected to an internal pressure of  $25 \text{ MN/m}^2$  calculate the resultant hoop stresses at the internal and external surfaces of the compound cylinder. [0.0768 mm; -48.75, +54.25 MN/m<sup>2</sup>.]

10.16 (B). A bronze tube, 60 mm external diameter and 50 mm bore, fits closely inside a steel tube of external diameter 100 mm. When the assembly is at a uniform temperature of  $15^{\circ}$ C the bronze tube is a sliding fit inside the steel tube, that is, the two tubes are free from stress. The assembly is now heated uniformly to a temperature of  $115^{\circ}$ C.

(a) Calculate the radial pressure induced between the mating surfaces and the thermal circumferential stresses, in magnitude and nature, induced at the inside and outside surfaces of each tube.

(b) Sketch the radial and circumferential stress distribution across the combined wall thickness of the assembly when the temperature is 115°C and insert the numerical values. Use the tabulated data given below.

	Young's modulus (E)	Poisson's ratio (v)	Coefficient of linear expansion (a)
Steel	200 GN/m <sup>2</sup>	0.3	$12 \times 10^{-6} / ^{\circ} C$
Bronze	$100  GN/m^2$	0.33	$19 \times 10^{-6} / ^{\circ} C$

10.17 (B) A steel cylinder, 150 mm external diameter and 100 mm internal diameter, just fits over a brass cylinder of external diameter 100 mm and bore 50 mm. The compound cylinder is to withstand an internal pressure of such a

magnitude that the pressure set up between the common junction surfaces is  $30 \text{ MN/m}^2$  when the internal pressure is applied. The external pressure is zero. Determine:

(a) the value of the internal pressure;

(b) the hoop stress induced in the material of both tubes at the inside and outside surfaces.

Lamé's equations for thick cylinders may be assumed without proof, and neglect any longitudinal stress and strain.

For steel,  $E = 207 \text{ GN/m}^2$  (2.07 Mbar) and v = 0.28. For brass,  $E = 100 \text{ GN/m}^2$  (1.00 Mbar) and v = 0.33.

 Sketch the hoop and radial stress distribution diagrams across the combined wall thickness, inserting the peak values.
 [B.P.] [123 MN/m<sup>2</sup>; 125.4, 32.2 MN/m<sup>2</sup>; 78.2, 48.2 MN/m<sup>2</sup>.]

10.18 (C). Assuming the Lamé equations for stresses in a thick cylinder, show that the radial and circumferential stresses in a solid shaft owing to the application of external pressure are equal at all radii.

A solid steel shaft having a diameter of 100 mm has a steel sleeve shrunk on to it. The maximum tensile stress in the sleeve is not to exceed twice the compressive stress in the shaft. Determine (a) the least thickness of the sleeve and (b) the maximum tensile stress in the sleeve after shrinkage if the shrinkage allowance, based on diameter, is 0.015 mm.  $E = 210 \text{ GN/m}^2$ . [I.Mech.E.][36.6 mm; 21 MN/m<sup>2</sup>.]

10.19 (C). A steel tube of internal radius 25 mm and external radius 40 mm is wound with wire of 0.75 mm diameter until the external diameter of the tube and wire is 92 mm. Find the maximum hoop stress set up within the walls of the tube if the wire is wound with a tension of  $15 \text{ MN/m}^2$  and an internal pressure of  $30 \text{ MN/m}^2$  (300 bar) acts within the tube. [49 MN/m<sup>2</sup>.]

**10.20** (C). A thick cylinder of 100 mm internal diameter and 125 mm external diameter is wound with wire until the external diameter is increased by 30 %. If the initial tensile stress in the wire when being wound on the cylinder is  $135 \text{ MN/m}^2$ , calculate the maximum stress set up in the cylinder walls. [-144.5 MN/m<sup>2</sup>.]