CHAPTER 14

COMPLEX STRAIN AND THE ELASTIC CONSTANTS

Summary

The *relationships between the elastic constants* are

$$E = 2G(1+\nu) \quad \text{and} \quad E = 3K(1-2\nu)$$

Poisson's ratio ν being defined as the ratio of lateral strain to longitudinal strain and bulk modulus K as the ratio of volumetric stress to volumetric strain.

The *strain in the x direction* in a material subjected to three mutually perpendicular stresses in the x, y and z directions is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

Similar equations apply for ε_y and ε_z.

Thus the *principal strain* in a given direction can be found in terms of the principal stresses, since

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E} = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

For a *two-dimensional* stress system (i.e. $\sigma_3 = 0$), *principal stresses* can be found from known principal strains, since

$$\sigma_1 = \frac{(\varepsilon_1 + \nu\varepsilon_2)}{(1-\nu^2)}E \quad \text{and} \quad \sigma_2 = \frac{(\varepsilon_2 + \nu\varepsilon_1)}{(1-\nu^2)}E$$

When the linear strains in two perpendicular directions are known, together with the associated shear strain, or when three linear strains are known, the principal strains are easily determined by the use of *Mohr's strain circle*.

14.1. Linear strain for tri-axial stress state

Consider an element subjected to three mutually perpendicular tensile stresses σ_x, σ_y and σ_z as shown in Fig. 14.1.

If σ_y and σ_z were not present the strain in the x direction would, from the basic definition of Young's modulus E, be

$$\varepsilon_x = \frac{\sigma_x}{E}$$

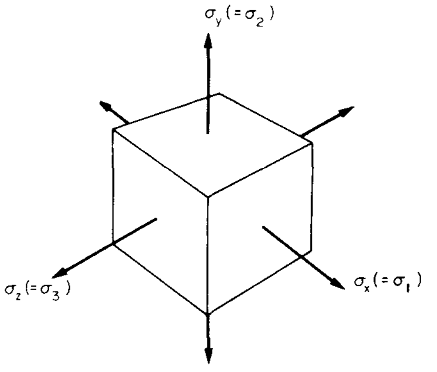


Fig. 14.1.

The effects of σ_y and σ_z in the x direction are given by the definition of Poisson's ratio ν to be

$$-\nu\frac{\sigma_y}{E} \quad \text{and} \quad -\nu\frac{\sigma_z}{E} \text{ respectively}$$

the negative sign indicating that if σ_y and σ_z are positive, i.e. tensile, then they tend to reduce the strain in the x direction.

Thus the total linear strain in the x direction is given by

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

i.e.

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) \qquad (14.1)$$

Similarly the strains in the y and z directions would be

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$

The three equations being known as the "generalised Hookes Law" from which the simple uniaxial form of §1.5 is obtained (when two of the three stresses are reduced to zero).

14.2. Principal strains in terms of stresses

In the absence of shear stresses on the faces of the element shown in Fig. 14.1 the stresses σ_x, σ_y and σ_z are in fact principal stresses. Thus the principal strain in a given direction is obtained from the principal stresses as

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

or
$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3) \tag{14.2}$$

or
$$\varepsilon_3 = \frac{1}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

14.3. Principal stresses in terms of strains – two-dimensional stress system

For a two-dimensional stress system, i.e. $\sigma_3 = 0$, the above equations reduce to

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

and
$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

with
$$\varepsilon_3 = \frac{1}{E}(-\nu\sigma_1 - \nu\sigma_2)$$

$\therefore$
$$E\varepsilon_1 = \sigma_1 - \nu\sigma_2$$
$$E\varepsilon_2 = \sigma_2 - \nu\sigma_1$$

Solving these equations simultaneously yields the following values for the principal stresses:

$$\sigma_1 = \frac{E}{(1-\nu^2)}(\varepsilon_1 + \nu\varepsilon_2)$$

and
$$\sigma_2 = \frac{E}{(1-\nu^2)}(\varepsilon_2 + \nu\varepsilon_1) \tag{14.3}$$

14.4. Bulk modulus K

It has been shown previously that Young's modulus E and the shear modulus G are defined as the ratio of stress to strain under direct load and shear respectively. Bulk modulus is similarly defined as a ratio of stress to strain under uniform pressure conditions. Thus if a material is subjected to a uniform pressure (or volumetric stress) σ in all directions then

$$\text{bulk modulus} = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

i.e.
$$K = \frac{\sigma}{\varepsilon_v} \tag{14.4}$$

the volumetric strain being defined below.

14.5. Volumetric strain

Consider a rectangular block of sides x, y and z subjected to a system of equal direct stresses σ on each face. Let the sides be changed in length by δx, δy and δz respectively under stress (Fig. 14.2).

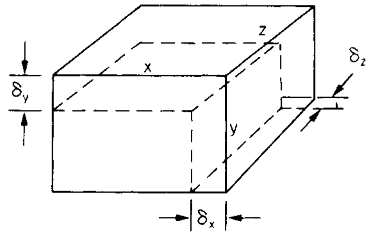


Fig. 14.2. Rectangular element subjected to uniform compressive stress on all faces producing decrease in size shown.

The volumetric strain is defined as follows:

$$\text{volumetric strain} = \frac{\text{change in volume}}{\text{original volume}} = \frac{\delta V}{V} = \frac{\delta V}{xyz}$$

The change in volume can best be found by calculating the volume of the strips to be cut off the original size of block to reduce it to the dotted block shown in Fig. 14.2.

Then

$$\delta V = \underbrace{xy\delta z}_{\text{strip at back}} + \underbrace{y(z-\delta z)\delta x}_{\text{strip at side}} + \underbrace{(x-\delta x)(z-\delta z)\delta y}_{\text{strip on top}}$$

and neglecting the products of small quantities

$$\delta V = xy\delta z + yz\delta x + xz\delta y$$

$\therefore$
$$\text{volumetric strain} = \frac{(xy\delta z + yz\delta x + xz\delta y)}{xyz} = \varepsilon_v$$

$\therefore$
$$\varepsilon_v = \frac{\delta z}{z} + \frac{\delta x}{x} + \frac{\delta y}{y} = \varepsilon_z + \varepsilon_x + \varepsilon_y \tag{14.5}$$

i.e. **volumetric strain = sum of the three mutually perpendicular linear strains**

14.6. Volumetric strain for unequal stresses

It has been shown above that the volumetric strain is the sum of the three perpendicular linear strains

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

Substituting for the strains in terms of stresses as given by eqn. (14.1),

$$\varepsilon_v = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z) + \frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z) + \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$

$$\boldsymbol{\varepsilon_v = \frac{1}{E}(\sigma_x + \sigma_y + \sigma_z)(1 - 2\nu)} \tag{14.6}$$

It will be shown later that the following relationship applies between the elastic constants E, ν and K,

$$E = 3K(1-2\nu)$$

Thus the volumetric strain may be written in terms of the bulk modulus as follows:

$$\varepsilon_v = \frac{(\sigma_x + \sigma_y + \sigma_z)}{3K} \tag{14.7}$$

This equation applies to solid bodies only and cannot be used for the determination of internal volume (or capacity) changes of hollow vessels. It may be used, however, for changes in cylinder wall volume.

14.7. Change in volume of circular bar

A simple application of eqn. (14.6) is to the determination of volume changes of circular bars under direct load.

Consider, therefore, a circular bar subjected to a direct stress σ applied axially as shown in Fig. 14.3.

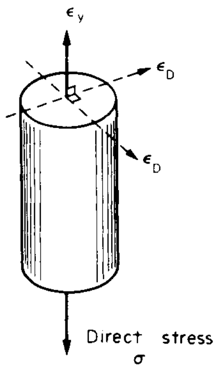


Fig. 14.3. Circular bar subjected to direct axial stress σ.

Here

$$\sigma_y = \sigma, \quad \sigma_x = 0 \quad \text{and} \quad \sigma_z = 0$$

Therefore from eqn. (14.6)

$$\varepsilon_v = \frac{\sigma}{E}(1-2\nu) = \frac{\delta V}{V}$$

$$\therefore \qquad \text{change of volume} = \delta V = \frac{\sigma V}{E}(1-2\nu) \tag{14.8}$$

This formula could have been obtained from eqn. (14.5) with

$$\varepsilon_y = \frac{\sigma}{E} \quad \text{and} \quad \varepsilon_x = \varepsilon_z = \varepsilon_D = -\nu\frac{\sigma}{E}$$

then

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\delta V}{V}$$

$$\therefore \qquad \delta V = V\left(\frac{\sigma}{E} - 2\nu\frac{\sigma}{E}\right) = \frac{\sigma V}{E}(1-2\nu)$$

14.8. Effect of lateral restraint

(a) Restraint in one direction only

Consider a body subjected to a two-dimensional stress system with a rigid lateral restraint provided in the y direction as shown in Fig. 14.4. Whilst the material is free to contract laterally in the x direction the "Poisson's ratio" extension along the y axis is totally prevented.

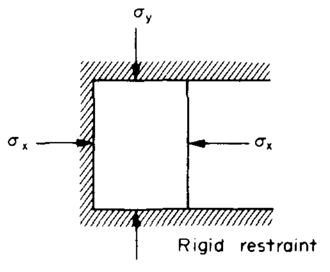


Fig. 14.4. Material subjected to lateral restraint in the y direction.

Therefore strain in the y direction with σ_x and σ_y both compressive, i.e. negative,

$$= \varepsilon_y = -\frac{1}{E}(\sigma_y - \nu\sigma_x) = 0$$

$$\therefore \qquad \sigma_y = \nu\sigma_x$$

Thus strain in the x direction

$$= \varepsilon_x = -\frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$= -\frac{1}{E}(\sigma_x - \nu^2\sigma_x)$$

$$= -\frac{\boldsymbol{\sigma_x}}{\boldsymbol{E}}(\mathbf{1} - \boldsymbol{\nu}^2) \qquad (14.9)$$

Thus the introduction of a lateral restraint affects the stiffness and hence the load-carrying capacity of the material by producing an **effective change of Young's modulus** from

$$E \quad \text{to} \quad E/(1-\nu^2)$$

(b) Restraint in two directions

Consider now a material subjected to a three-dimensional stress system σ_x, σ_y and σ_z with restraint provided in both the y and z directions. In this case,

$$\varepsilon_y = -\frac{1}{E}(\sigma_y - \nu\sigma_x - \nu\sigma_z) = 0 \qquad (1)$$

and

$$\varepsilon_z = -\frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y) = 0 \qquad (2)$$

From (1),

$$\sigma_y = \nu\sigma_x + \nu\sigma_z$$

$$\therefore \qquad \sigma_z = (\sigma_y - \nu\sigma_x)\frac{1}{\nu} \qquad (3)$$

Substituting in (2),

$$\frac{1}{\nu}(\sigma_y - \nu\sigma_x) - \nu\sigma_x - \nu\sigma_y = 0$$

$$\therefore \qquad \sigma_y - \nu\sigma_x - \nu^2\sigma_x - \nu^2\sigma_y = 0$$

$$\sigma_y(1-\nu^2) = \sigma_x(\nu+\nu^2)$$

$$\sigma_y = \sigma_x\frac{\nu(1+\nu)}{(1-\nu^2)}$$

$$= \frac{\sigma_x\nu}{(1-\nu)}$$

and from (3),

$$\sigma_z = \frac{1}{\nu}\left[\frac{\nu\sigma_x}{(1-\nu)} - \nu\sigma_x\right]$$

$$= \sigma_x\left[\frac{1-(1-\nu)}{(1-\nu)}\right] = \frac{\nu\sigma_x}{(1-\nu)}$$

$$\therefore \qquad \text{strain in } x \text{ direction} = -\frac{\sigma_x}{E} + \nu\frac{\sigma_y}{E} + \nu\frac{\sigma_z}{E}$$

$$= -\frac{\sigma_x}{E}\left[1 - \frac{\nu^2}{(1-\nu)} - \frac{\nu^2}{(1-\nu)}\right]$$

$$= -\frac{\sigma_x}{E}\left[1 - \frac{2\nu^2}{(1-\nu)}\right] \qquad (14.10)$$

Again Young's modulus E is effectively changed, this time to

$$E\bigg/\left[1 - \frac{2\nu^2}{(1-\nu)}\right]$$

14.9. Relationship between the elastic constants E, G, K and ν

(a) E, G and ν

Consider a cube of material subjected to the action of the shear and complementary shear forces shown in Fig. 14.5 producing the strained shape indicated.

Assuming that the strains are small the angle ACB may be taken as 45°.

Therefore strain on diagonal OA

$$= \frac{BC}{OA} \simeq \frac{AC\cos 45^\circ}{a\sqrt{2}} = \frac{AC}{a\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{AC}{2a}$$

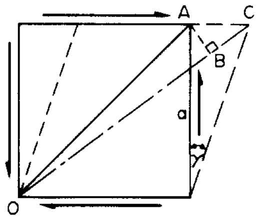


Fig. 14.5. Element subjected to shear and associated complementary shear.

But $AC = a\gamma$, where γ = angle of distortion or shear strain.

$\therefore$ $$\text{strain on diagonal} = \frac{a\gamma}{2a} = \frac{\gamma}{2}$$

Now $$\frac{\text{shear stress } \tau}{\text{shear strain } \gamma} = G$$

$\therefore$ $$\gamma = \frac{\tau}{G}$$

$\therefore$ $$\text{strain on diagonal} = \frac{\tau}{2G} \qquad (1)$$

From § 13.2 the shear stress system can be replaced by a system of direct stresses at 45°, as shown in Fig. 14.6. One set will be compressive, the other tensile, and both will be equal in value to the applied shear stresses.

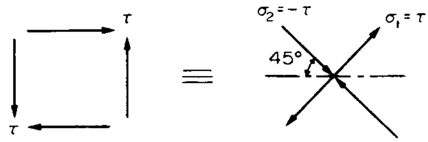


Fig. 14.6. Direct stresses due to shear.

Thus, from the direct stress system which applies along the diagonals:

$$\text{strain on diagonal} = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E}$$

$$= \frac{\tau}{E} - \nu\frac{(-\tau)}{E}$$

$$= \frac{\tau}{E}(1+\nu) \qquad (2)$$

Combining (1) and (2),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

$$\mathbf{E = 2G(1+\nu)} \qquad (14.11)$$

(*b*) *E, K and ν*

Consider a cube subjected to three equal stresses σ as in Fig. 14.7 (i.e. volumetric stress $= \sigma$).

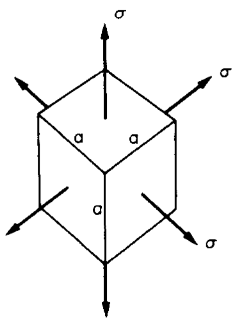


Fig. 14.7. Cubical element subjected to uniform stress σ on all faces ("volumetric" or "hydrostatic" stress).

$$\text{Total strain along one edge} = \frac{\sigma}{E} - \nu\frac{\sigma}{E} - \nu\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1-2\nu)$$

But

$$\text{volumetric strain} = 3 \times \text{linear strain} \qquad \text{(see eqn. 14.5)}$$

$$= \frac{3\sigma}{E}(1-2\nu) \qquad (3)$$

By definition:

$$\text{bulk modulus } K = \frac{\text{volumetric stress}}{\text{volumetric strain}}$$

$$\text{volumetric strain} = \frac{\sigma}{K} \qquad (4)$$

Equating (3) and (4),

$$\frac{\sigma}{K} = \frac{3\sigma}{E}(1-2\nu)$$

$$\therefore \qquad \mathbf{E = 3K(1-2\nu)} \qquad (14.12)$$

(*c*) *G, K and ν*

Equations (14.11) and (14.12) can now be combined to give the final relationship as follows: From eqn. (14.11),

$$\nu = \frac{E}{2G} - 1$$

and from eqn. (14.12),

$$\nu = \frac{1}{2} - \frac{E}{6K}$$

Therefore, equating,

$$\frac{E}{2G} - 1 = \frac{1}{2} - \frac{E}{6K}$$

$$E\left[\frac{1}{2G} + \frac{1}{6K}\right] = \frac{3}{2}$$

$$\therefore \quad E\left[\frac{6K+2G}{12KG}\right] = \frac{3}{2}$$

i.e.

$$E = \frac{9KG}{(3K+G)} \tag{14.13}$$

14.10. Strains on an oblique plane

(a) Linear strain

Consider a rectangular block of material $OLMN$ as shown in the xy plane (Fig. 14.8). The strains along Ox and Oy are ε_x and ε_y, and γ_{xy} is the shearing strain.

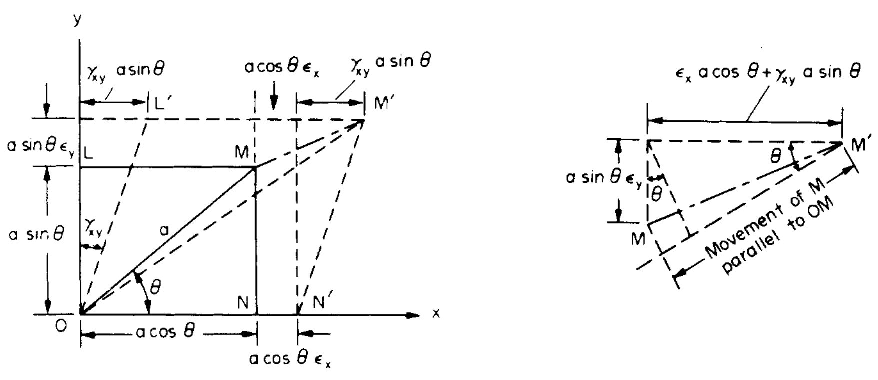


Fig. 14.8. Strains on an inclined plane.

Let the diagonal OM be of length a; then $ON = a\cos\theta$ and $OL = a\sin\theta$, and the increases in length of these sides under strain are $\varepsilon_x a\cos\theta$ and $\varepsilon_y a\sin\theta$ (i.e. strain × original length).

If M moves to M', the movement of M parallel to the x axis is

$$\varepsilon_x a\cos\theta + \gamma_{xy} a\sin\theta$$

and the movement parallel to the y axis is

$$\varepsilon_y a\sin\theta$$

Thus the movement of M parallel to OM, which since the strains are small is practically coincident with MM', is

$$(\varepsilon_x a \cos\theta + \gamma_{xy} a \sin\theta)\cos\theta + (\varepsilon_y a \sin\theta)\sin\theta$$

Then

$$\text{strain along } OM = \frac{\text{extension}}{\text{original length}}$$

$$= (\varepsilon_x \cos\theta + \gamma_{xy}\sin\theta)\cos\theta + (\varepsilon_y \sin\theta)\sin\theta$$

$\therefore$

$$\varepsilon_\theta = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy}\sin\theta\cos\theta$$

$\therefore$

$$\boldsymbol{\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta} \qquad (14.14)$$

This is identical in form with the equation defining the direct stress on any inclined plane θ with ε_x and ε_y replacing σ_x and σ_y and $\frac{1}{2}\gamma_{xy}$ replacing τ_{xy}, i.e. **the shear stress is replaced by HALF the shear strain.**

(b) Shear strain

To determine the shear strain in the direction OM consider the displacement of point P at the foot of the perpendicular from N to OM (Fig. 14.9).

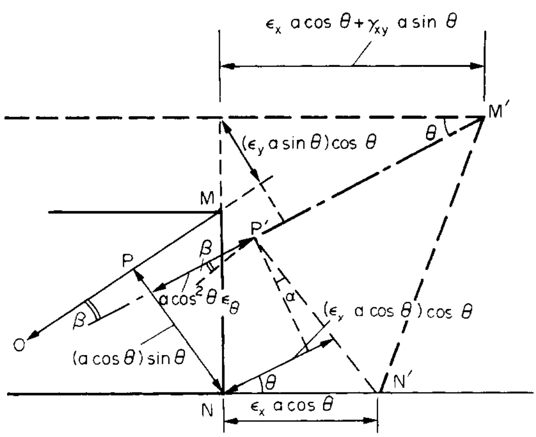


Fig. 14.9. Enlarged view of part of Fig. 14.8.

In the strained condition this point moves to P'.

Since $\quad$ strain along $OM = \varepsilon_\theta$

$$\text{extension of } OM = OM \cdot \varepsilon_\theta$$

$\therefore$

$$\text{extension of } OP = OP \cdot \varepsilon_\theta$$

But

$$OP = (a\cos\theta)\cos\theta$$

$\therefore$

$$\text{extension of } OP = a\cos^2\theta\,\varepsilon_\theta$$

During straining the line *PN* rotates counterclockwise through a small angle α.

$$\alpha = \frac{(\varepsilon_x a \cos\theta)\cos\theta - a\cos^2\theta\,\varepsilon_\theta}{a\cos\theta\sin\theta}$$

$$= (\varepsilon_x - \varepsilon_\theta)\cot\theta$$

The line *OM* also rotates, but clockwise, through a small angle

$$\beta = \frac{(\varepsilon_x a\cos\theta + \gamma_{xy} a\sin\theta)\sin\theta - (\varepsilon_y a\sin\theta)\cos\theta}{a}$$

Thus the required shear strain γ_θ in the direction *OM*, i.e. the amount by which the angle *OPN* changes, is given by

$$\gamma_\theta = \alpha + \beta = (\varepsilon_x - \varepsilon_\theta)\cot\theta + (\varepsilon_x\cos\theta + \gamma_{xy}\sin\theta)\sin\theta - \varepsilon_y\sin\theta\cos\theta$$

Substituting for ε_θ from eqn. (14.14) gives

$$\gamma_\theta = 2(\varepsilon_x - \varepsilon_y)\cos\theta\sin\theta - \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\therefore \quad \tfrac{1}{2}\gamma_\theta = \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta - \tfrac{1}{2}\gamma_{xy}\cos 2\theta$$

which again is similar in form to the expression for the shear stress τ on any inclined plane θ.

For consistency of sign convention, however (see § 14.11 below), because *OM'* moves clockwise with respect to *OM* it is considered to be a negative shear strain, i.e.

$$\tfrac{1}{2}\gamma_\theta = -\left[\tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta - \tfrac{1}{2}\gamma_{xy}\cos 2\theta\right] \qquad (14.15)$$

14.11. Principal strain – Mohr's strain circle

Since the equations for stress and strain on oblique planes are identical in form, as noted above, it is evident that Mohr's stress circle construction can be used equally well to represent strain conditions using the horizontal axis for linear strains and the vertical axis for *half* the shear strain. It should be noted, however, that angles given by Mohr's stress circle refer to the directions of the planes on which the stresses act and not to the direction of the stresses themselves. The directions of the stresses and hence the associated strains are therefore normal (i.e. at 90°) to the directions of the planes. Since angles are doubled in Mohr's circle construction it follows therefore that for true similarity of working a relative rotation of the axes of 2 × 90 = 180° must be introduced. This is achieved by plotting positive shear strains vertically *downwards* on the strain circle construction as shown in Fig. 14.10.

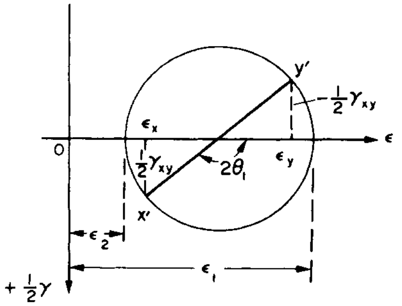


Fig. 14.10. Mohr's strain circle.

The **sign convention** adopted for strains is as follows:

Linear strains: extension positive
compression negative.

Shear strains:

The convention for shear strains is a little more difficult. The first subscript in the symbol γ_{xy} usually denotes the shear strain associated with that direction, i.e. with Ox. Similarly, γ_{yx} is the shear strain associated with Oy. If, under strain, the line associated with the first subscript moves counterclockwise with respect to the other line, the shearing strain is said to be positive, and if it moves clockwise it is said to be negative. It will then be seen that positive shear strains are associated with planes carrying positive shear stresses and negative shear strains with planes carrying negative shear stresses.

Thus, $$\gamma_{xy} = -\gamma_{yx}$$

Mohr's circle for strains ε_x, ε_y and shear strain γ_{xy} (positive referred to x direction) is therefore constructed as for the stress circle with $\frac{1}{2}\gamma_{xy}$ replacing τ_{xy} and the axis of shear reversed, as shown in Fig. 14.10.

The maximum principal strain is then ε_1 at an angle θ_1 to ε_x in the same angular direction as that in Mohr's circle (Fig. 14.11).

Again, **angles are doubled on Mohr's circle.**

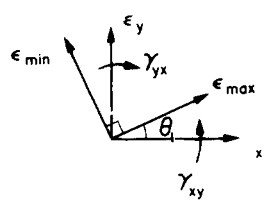


Fig. 14.11. Strain system at a point, including the principal strains and their inclination.

Strain conditions at any angle α to ε_x are found as in the stress circle by marking off an angle 2α from the point representing the x direction, i.e. x'. The coordinates of the point on the circle thus obtained are the strains required.

Alternatively, the principal strains may be determined *analytically* from eqn. (14.14),

i.e. $$\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta$$

As for the derivation of the principal stress equations on page 331, the principal strains, i.e. the maximum and minimum values of strain, occur at values of θ obtained by equating $d\varepsilon_\theta/d\theta$ to zero.

The procedure is identical to that of page 331 for the stress case and will not be repeated here. The values obtained are

$$\varepsilon_1 \text{ or } \varepsilon_2 = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \tfrac{1}{2}\sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}{}^2]} \qquad (14.16)$$

i.e. once again identical in form to the principal stress equation with ε replacing σ and $\frac{1}{2}\gamma$ replacing τ.

Similarly, $$\tfrac{1}{2}\gamma_{max} = \pm\tfrac{1}{2}\sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}{}^2]} \qquad (14.17)$$

14.12. Mohr's strain circle – alternative derivation from the general stress equations

The direct stress on any plane within a material inclined at an angle θ to the xy axes is given by eqn. (13.8) as:

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\therefore \quad \sigma_{\theta+90} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos(2\theta + 180^\circ) + \tau_{xy}\sin(2\theta + 180^\circ)$$

$$= \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta$$

Also, from eqn. (13.9),

$$\tau_\theta = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta \qquad (1)$$

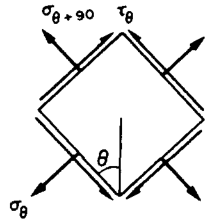


Fig. 14.12.

Now for the two-dimensional stress system shown in Fig. 14.12,

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_{\theta+90})$$

$$= \frac{1}{E}\{[\tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta]$$

$$- \nu[\tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta]\}$$

$$= \frac{1}{E}[\tfrac{1}{2}(1 - \nu)(\sigma_x + \sigma_y) + \tfrac{1}{2}(1 + \nu)(\sigma_x - \sigma_y)\cos 2\theta + (1 + \nu)\tau_{xy}\sin 2\theta]$$

But $$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

and $$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

from which $$\sigma_y = \frac{E}{(1 - \nu^2)}[\varepsilon_y + \nu\varepsilon_x]$$

and $$\sigma_x = \frac{E}{(1 - \nu^2)}[\varepsilon_x + \nu\varepsilon_y]$$

$$\therefore \quad \tfrac{1}{2}(\sigma_x + \sigma_y) = \frac{E(1 + \nu)}{2(1 - \nu^2)}(\varepsilon_y + \varepsilon_x)$$

and $$\tfrac{1}{2}(\sigma_x - \sigma_y) = \frac{E(1-\nu)}{2(1-\nu^2)}(\varepsilon_x - \varepsilon_y)$$

$$\therefore \quad \varepsilon_\theta = \frac{1}{E}\left[\frac{E(1-\nu)(1+\nu)(\varepsilon_y + \varepsilon_x)}{2(1-\nu^2)} + \frac{E(1+\nu)(1-\nu)(\varepsilon_x - \varepsilon_y)}{2(1-\nu^2)}\cos 2\theta + (1+\nu)\tau_{xy}\sin 2\theta\right]$$

$$= \tfrac{1}{2}(\varepsilon_y + \varepsilon_x) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tau_{xy}\frac{(1+\nu)}{E}\sin 2\theta$$

Now $$\frac{\tau}{\gamma} = G \quad \therefore \quad \tau = G\gamma \quad \text{and} \quad E = 2G(1+\nu)$$

$$\therefore \quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\therefore \quad \boldsymbol{\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta} \qquad (14.14)$$

Similarly, substituting for $\tfrac{1}{2}(\sigma_x - \sigma_y)$ and τ_{xy} in (1),

$$\tau_\theta = \frac{E(\varepsilon_x - \varepsilon_y)(1-\nu)}{2(1-\nu^2)}\sin 2\theta - \frac{E}{2(1+\nu)}\gamma_{xy}\cos 2\theta$$

But $$\tau_\theta = \frac{E}{2(1+\nu)}\gamma_\theta$$

$$\therefore \quad \frac{E}{2(1+\nu)}\gamma_\theta = \frac{E(\varepsilon_x - \varepsilon_y)}{2(1+\nu)}\sin 2\theta - \frac{E}{2(1+\nu)}\gamma_{xy}\cos 2\theta$$

$$\gamma_\theta = (\varepsilon_x - \varepsilon_y)\sin 2\theta - \gamma_{xy}\cos 2\theta$$

$$\therefore \quad \tfrac{1}{2}\gamma_\theta = \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta - \tfrac{1}{2}\gamma_{xy}\cos 2\theta$$

Again, for consistency of sign convention, since OM will move clockwise under strain, the above shear strain must be considered negative,

i.e. $$\boldsymbol{\tfrac{1}{2}\gamma_\theta = -\left[\tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta - \tfrac{1}{2}\gamma_{xy}\cos 2\theta\right]} \qquad (14.15)$$

Equations (14.14) and (14.15) are similar in form to eqns. (13.8) and (13.9) which are the basis of Mohr's circle solution for stresses provided that $\tfrac{1}{2}\gamma_{xy}$ is used in place of τ_{xy} and linear stresses σ are replaced by linear strains ε. **These equations will therefore provide a graphical solution known as Mohr's strain circle if axes of ε and $\tfrac{1}{2}\gamma$ are used.**

14.13. Relationship between Mohr's stress and strain circles

Consider now a material subjected to the two-dimensional principal stress system shown in Fig. 14.13a. The stress and strain circles are then as shown in Fig. 14.13(b) and (c).

For Mohr's stress circle (Fig. 14.13b),

$$OA \times \text{stress scale} = \frac{(\sigma_1 + \sigma_2)}{2}$$

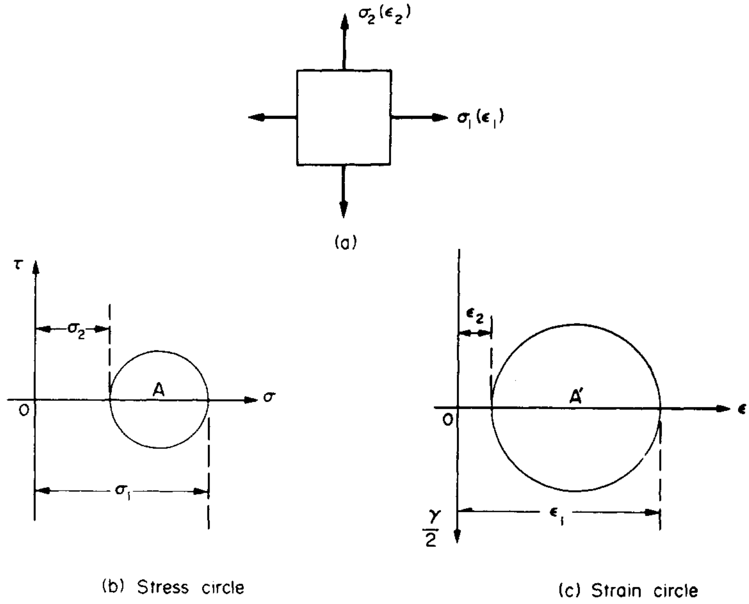


Fig. 14.13.

$$\therefore \qquad OA = \frac{(\sigma_1 + \sigma_2)}{2 \times \text{stress scale}} \tag{1}$$

and $\qquad$ radius of stress circle × stress scale $= \frac{1}{2}(\sigma_1 - \sigma_2)$ $\qquad$ (2)

For Mohr's strain circle (Fig. 14.13c),

$$OA' \times \text{strain scale} = \frac{(\varepsilon_1 + \varepsilon_2)}{2}$$

But

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2)$$

and

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1)$$

$$\therefore \qquad \varepsilon_1 + \varepsilon_2 = \frac{1}{E}[(\sigma_1 + \sigma_2) - \nu(\sigma_1 + \sigma_2)]$$

$$= \frac{1}{E}(\sigma_1 + \sigma_2)(1 - \nu)$$

$$\therefore \qquad OA' = \frac{(\sigma_1 + \sigma_2)(1 - \nu)}{2E \times \text{strain scale}} \tag{3}$$

Thus, in order that the circles shall be concentric (Fig. 14.14),

$$OA = OA'$$

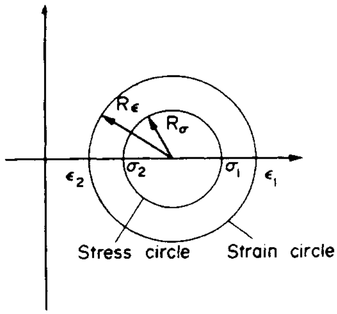


Fig. 14.14. Combined stress and strain circles.

Therefore from (1) and (3)

$$\frac{(\sigma_1+\sigma_2)}{2\times \text{stress scale}} = \frac{(\sigma_1+\sigma_2)(1-\nu)}{2E\times \text{strain scale}}$$

$$\therefore \qquad \textbf{stress scale} = \frac{E}{(1-\nu)} \times \textbf{strain scale} \qquad (14.18)$$

Now radius of strain circle × strain scale

$$= \tfrac{1}{2}(\varepsilon_1-\varepsilon_2)$$

$$= \frac{1}{2E}[(\sigma_1-\nu\sigma_2)-(\sigma_2-\nu\sigma_1)]$$

$$= \frac{1}{2E}(\sigma_1-\sigma_2)(1+\nu) \qquad (4)$$

$$\therefore \qquad \frac{\text{radius of stress circle}\times\text{stress scale}}{\text{radius of strain circle}\times\text{strain scale}} = \frac{\tfrac{1}{2}(\sigma_1-\sigma_2)}{\frac{1}{2E}(\sigma_1-\sigma_2)(1+\nu)}$$

$$= \frac{E}{(1+\nu)}$$

$$\frac{\text{radius of stress circle}}{\text{radius of strain circle}} = \frac{E}{(1+\nu)}\times\frac{\text{strain scale}}{\text{stress scale}}$$

i.e.

$$\frac{R_\sigma}{R_\varepsilon} = \frac{E}{(1+\nu)}\times\frac{(1-\nu)}{E}$$

$$= \frac{(1-\nu)}{(1+\nu)} \qquad (14.19)$$

In other words, provided suitable scales are chosen so that

$$\textbf{stress scale} = \frac{E}{(1-\nu)}\times \textbf{strain scale}$$

the stress and strain circles will have the same centre. If the radius of one circle is known the radius of the other circle can then be determined from the relationship

$$\textbf{radius of stress circle} = \frac{(1-\nu)}{(1+\nu)} \times \textbf{radius of strain circle}$$

Other relationships for the stress and strain circles are shown in Fig. 14.15.

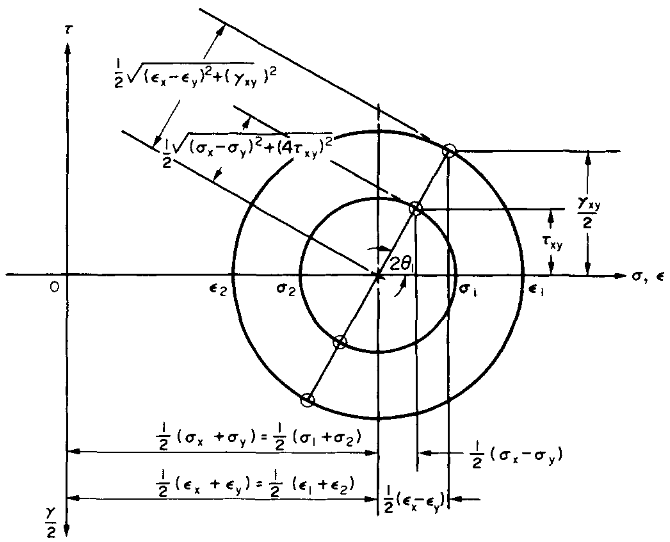


Fig. 14.15. Other relationships for Mohr's stress and strain circles.

14.14. Construction of strain circle from three known strains (McClintock method)–rosette analysis

In order to measure principal strains on the surface of engineering components the normal experimental technique involves the bonding of a strain gauge rosette at the point under consideration. This gives the values of strain in three known directions and enables Mohr's strain circle to be constructed as follows.

Consider the three-strain system shown in Fig. 14.16, the known directions of strain being at angles α_a, α_b and α_c to a principal strain direction (this being one of the primary requirements of such readings). The construction sequence is then:

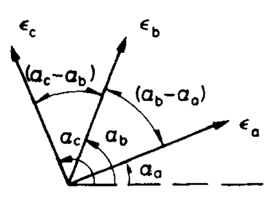


Fig. 14.16. System of three known strains.

(1) On a horizontal line *da* mark off the known strains ε_a, ε_b and ε_c to the same scale to give points *a*, *b* and *c* (see Fig. 14.18).

(2) From *a*, *b* and *c* draw perpendiculars to the line *da*.

(3) From a convenient point *X* on the perpendicular through *b* mark off lines corresponding to the known strain directions of ε_a and ε_c to intersect (projecting back if necessary) perpendiculars through *c* and *a* at *C* and *A*.

Note that these directions must be identical relative to *Xb* as they are relative to ε_b in Fig. 14.16,

i.e. *XC* is $\alpha_c - \alpha_b$ counterclockwise from *Xb* and *XA* is $\alpha_b - \alpha_a$ clockwise from *Xb*

(4) Construct perpendicular bisectors of the lines *XA* and *XC* to meet at the point *Y*, which is then the centre of Mohr's strain circle (Fig. 14.18).

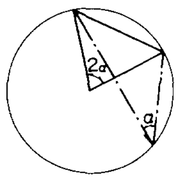


Fig. 14.17. Useful relationship for development of Mohr's strain circle (see Fig. 14.18).

(5) With centre *Y* and radius *YA* or *YC* draw the strain circle to cut *Xb* in the point *B*.

(6) The vertical shear strain axis can now be drawn through the zero of the strain scale *da*; the horizontal linear strain axis passes through *Y*.

(7) Join points *A*, *B* and *C* to *Y*. These radii must then be in the same angular order as the original strain directions. As in Mohr's stress circle, however, angles between them will be double in value, as shown in Fig. 14.18.

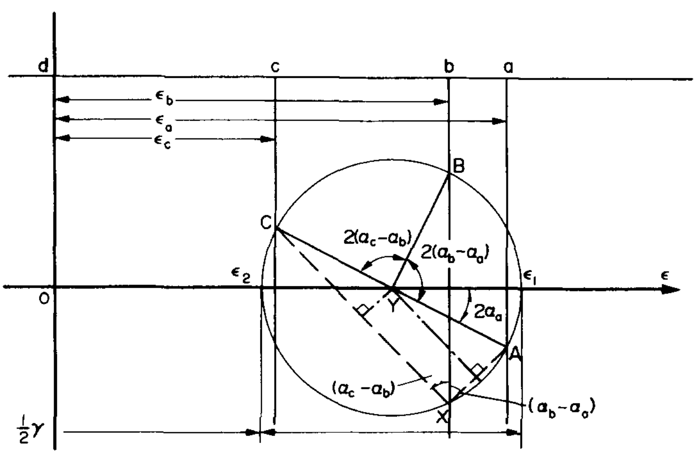


Fig. 14.18. Construction of strain circle from three known strains – McClintock construction. (Strain gauge rosette analysis.)

The principal strains are then ε_1 and ε_2 as indicated. Principal stresses can now be determined either from the relationships

$$\sigma_1 = \frac{E}{(1-\nu^2)}[\varepsilon_1 + \nu\varepsilon_2] \quad \text{and} \quad \sigma_2 = \frac{E}{(1-\nu^2)}[\varepsilon_2 + \nu\varepsilon_1]$$

or by superimposing the stress circle using the relationships established in §14.13.

The above construction applies whatever the values of strain and whatever the angles between the individual gauges of the rosette. The process is simplified, however, if the rosette axes are arranged:

(a) in sequence, in order of ascending or descending strain magnitude,

(b) so that the included angle between axes of maximum and minimum strain is less than 180°.

For example, consider three possible results of readings from the rosette of Fig. 14.16 as shown in Fig. 14.19(i), (ii) and (iii).

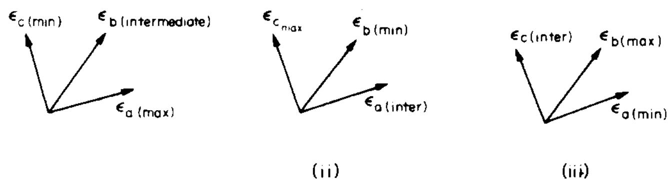


Fig. 14.19. Three possible orders of results from any given strain gauge rosette.

These may be rearranged as suggested above by projecting axes where necessary as shown in Fig. 14.20(i), (ii) and (iii).

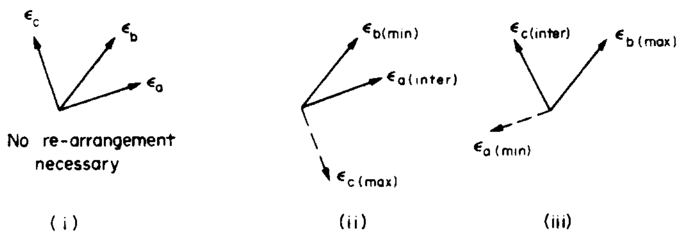
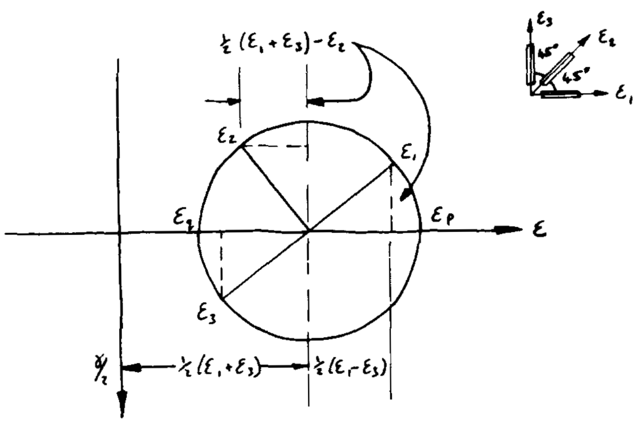


Fig. 14.20. Suitable rearrangement of Fig. 14.19 to facilitate the McClintock construction.

In all the above cases, the most convenient construction still commences with the starting point X on the vertical through the intermediate strain value, and will appear similar in form to the construction of Fig. 14.18.

Mohr's strain circle solution of rosette readings is strongly recommended because of its simplicity, speed and the ease with which principal stresses may be obtained by superimposing Mohr's stress circle. In addition, when one becomes familiar with the construction procedure, there is little opportunity for arithmetical error. As stated in the previous chapter, the advent of cheap but powerful calculators and microcomputers may reduce the effectiveness of Mohr's circle as a quantitative tool. It remains, however, a very powerful medium for the teaching and understanding of complex stress and strain systems and a valuable "aide-memoire" for some of the complex formulae which may be required for solution by other means. For example Fig. 14.21 shows the use of a free-hand sketch of the Mohr circle given by rectangular strain gauge rosette readings to obtain, from simple geometry, the corresponding principal strain equations.



$$\varepsilon_{p,q} = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_3) \pm \sqrt{[\tfrac{1}{2}(\varepsilon_1 - \varepsilon_3)]^2 + [\tfrac{1}{2}(\varepsilon_1 + \varepsilon_3) - \varepsilon_2]^2}$$

$$= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_3) \pm \frac{1}{\sqrt{2}}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2}$$

Fig. 14.21. Free-hand sketch of Mohr's strain circle.

14.15. Analytical determination of principal strains from rosette readings

The values of the principal strains associated with the three strain readings taken from a strain gauge rosette may be found by calculation using eqn. (14.14),

i.e. $$\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta$$

This equation can be applied three times for the three values of θ of the rosette gauges. Thus with three known values of ε_θ for three known values of θ, three simultaneous equations will give the unknown strains ε_x, ε_y and γ_{xy}.

The principal strains can then be determined from eqn. (14.16).

$$\varepsilon_1 \quad \text{or} \quad \varepsilon_2 = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \tfrac{1}{2}\sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}{}^2]}$$

The direction of the principal strain axes are then given by the equivalent strain expression to that derived for stresses [eqn. (13.10)],

i.e. $$\tan 2\theta = \frac{\gamma_{xy}}{(\varepsilon_x - \varepsilon_y)} \qquad (14.20)$$

angles being given relative to the X axis.

The majority of rosette gauges in common use today are either rectangular rosettes with $\theta = 0°$, 45° and 90° or delta rosettes with $\theta = 0°$, 60° and 120° (Fig. 14.22).

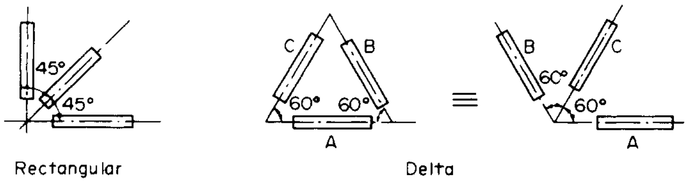


Fig. 14.22. Typical strain gauge rosette configurations.

In each case the calculations are simplified if the X axis is chosen to coincide with $\theta = 0$. Then, for both types of rosette, eqn. (14.14) reduces (for $\theta = 0$) to

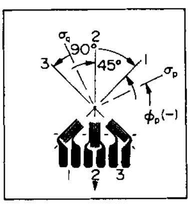
$$\varepsilon_0 = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y) = \varepsilon_x$$

and ε_x is obtained directly from the ε_0 strain gauge reading. Similarly, for the rectangular rosette ε_y is obtained directly from the ε_{90° reading.

If a large number of rosette gauge results have to be analysed, the calculation process may be computerised. In this context the relationship between the rosette readings and resulting principal stresses shown in Table 14.1 for three standard types of strain gauge rosette is recommended.

TABLE 14.1. Principal strains and stresses from strain gauge rosettes*
(Gauge readings = σ_1, σ_2 and σ_3; Principal stresses = σ_p and σ_q.)

Rectangular (45°) Rosette—Arbitrarily oriented with respect to principal axes.



$$\epsilon_{p,q} = \frac{\epsilon_1+\epsilon_3}{2} \pm \frac{1}{\sqrt{2}}\sqrt{(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2}$$

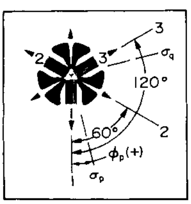
$$\sigma_{p,q} = \frac{E}{2}\left(\frac{\epsilon_1+\epsilon_3}{1-\nu} \pm \frac{\sqrt{2}}{1+\nu}\sqrt{(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2}\right)$$

$$\phi_{p,q} = \frac{1}{2}\tan^{-1}\left(\frac{(\epsilon_2-\epsilon_3)-(\epsilon_1-\epsilon_2)}{\epsilon_1-\epsilon_3}\right) \left(\text{if } \epsilon_1 > \frac{\epsilon_1+\epsilon_3}{2}, \phi_{p,q} = \phi_p\right.$$

$$\text{if } \epsilon_1 < \frac{\epsilon_1+\epsilon_3}{2}, \phi_{p,q} = \phi_q$$

$$\left.\text{if } \epsilon_1 = \frac{\epsilon_1+\epsilon_3}{2}, \phi_p = \pm 45^\circ\right)$$

Delta (equiangular) Rosette—Arbitrarily oriented with respect to principal axes.



$$\epsilon_{p,q} = \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3} \pm \frac{\sqrt{2}}{3}\sqrt{(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_1-\epsilon_3)^2}$$

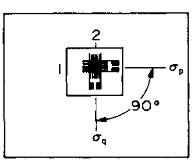
$$\sigma_{p,q} = \frac{E}{3}\left(\frac{\epsilon_1+\epsilon_2+\epsilon_3}{1-\nu} \pm \frac{\sqrt{2}}{1+\nu}\sqrt{(\epsilon_1-\epsilon_2)^2+(\epsilon_2-\epsilon_3)^2+(\epsilon_1-\epsilon_3)^2}\right)$$

$$\phi_{p,q} = \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{3}(\epsilon_2-\epsilon_3)}{(\epsilon_1-\epsilon_2)+(\epsilon_1-\epsilon_3)}\right) \left(\text{if } \epsilon_1 > \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}, \phi_{p,q} = \phi_p\right.$$

$$\text{if } \epsilon_1 < \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}, \phi_{p,q} = \phi_q$$

$$\left.\text{if } \epsilon_1 = \frac{\epsilon_1+\epsilon_2+\epsilon_3}{3}, \phi_p = \pm 45^\circ\right)$$

Tee Rosette—Gage elements must be aligned with principal axes.



$$\epsilon_p = \epsilon_1$$

$$\epsilon_q = \epsilon_2$$

$$\sigma_p = \frac{E}{1-\nu^2}(\epsilon_1+\nu\epsilon_2)$$

$$\sigma_q = \frac{E}{1-\nu^2}(\epsilon_2+\nu\epsilon_1)$$

Rosette Gage-Numbering Considerations

The equations at the left for calculating principal strains and stresses from rosette strain measurements assume that the gage elements are numbered in a particular manner. Improper numbering of the gage elements will lead to ambiguity in the interpretation of $\phi_{p,q}$; and, in the case of the rectangular rosette, can also cause errors in the calculated principal strains and stresses.

Treating the latter situation first, it is always necessary in a rectangular rosette that gage numbers 1 and 3 be assigned to the two mutually perpendicular gages. Any other numbering arrangement will produce incorrect principal strains and stresses.

Ambiguities in the interpretation of $\phi_{p,q}$ for both rectangular and delta rosettes can be eliminated by numbering the gage elements as follows:

In a rectangular rosette, Gage 2 must be 45° away from Gage 1 : and Gage 3 must be 90° away, in the same direction . Similarly, in a delta rosette, Gages 2 and 3 must be 60° and 120° away respectively, in the same direction from Gage 1. By definition, $\phi_{p,q}$ is the angle from the axis of Gage 1 to the nearest principal axis. When $\phi_{p,q}$ is positive, the direction is the same as that of the gage numbering; and, when negative, the opposite.

* Reproduced with permission from Vishay Measurements Ltd wall chart.

14.16. Alternative representations of strain distributions at a point

Alternative forms of representation for the distribution of stress at a point were presented in §13.7; the directly equivalent representations for strain are given below.

The values of the direct strain ε_θ and shear strain γ_θ for any inclined plane θ are given by equations (14.14) and (14.15) as

$$\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \tfrac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\tfrac{1}{2}\gamma_\theta = -[\tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta - \tfrac{1}{2}\gamma_{xy}\cos 2\theta]$$

Plotting these values for the uniaxial stress state on Cartesian axes yields the curves of Fig. 14.23 which can then be compared directly to the equivalent stress distributions of Fig. 13.12. Again the shear curves are "shifted" by 45° from the normal strain curves.

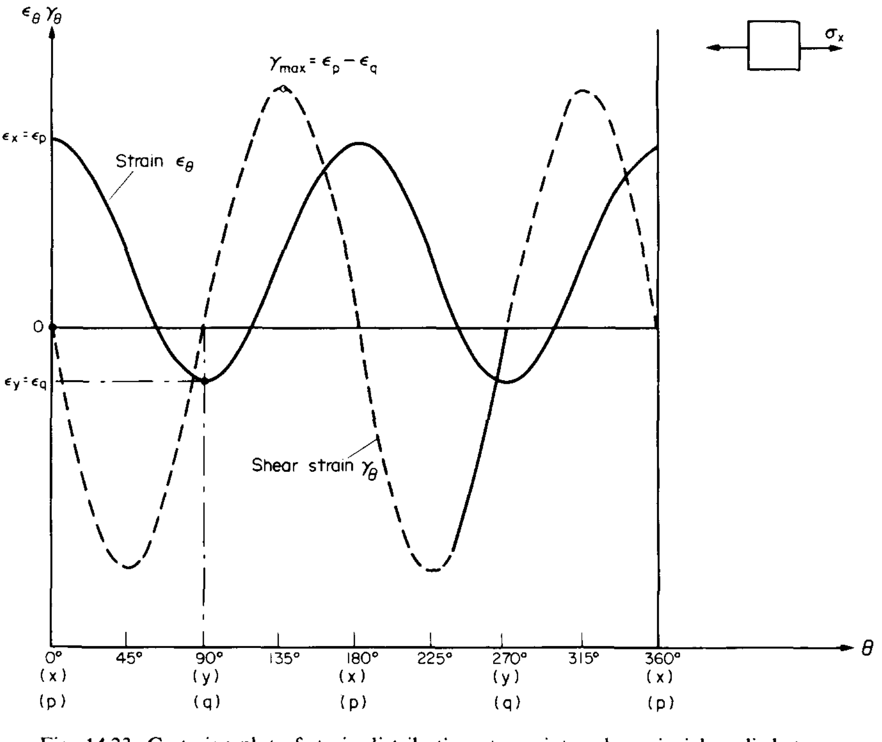


Fig. 14.23. Cartesian plot of strain distribution at a point under uniaxial applied stress.

Comparison with Fig. 13.12 shows that the normal stress and shear stress curves are each in phase with their respective normal strain and shear strain curves. Other relationships between the shear strain and normal strain curves are identical to those listed on page 335 for the normal stress and shear stress distributions.

The alternative polar strain representation for the uniaxial stress system is shown in Fig. 14.24 whilst the Cartesian and polar diagrams for the same biaxial stress systems used for Figs. 13.14 and 13.15 are shown in Figs. 14.25 and 14.26.

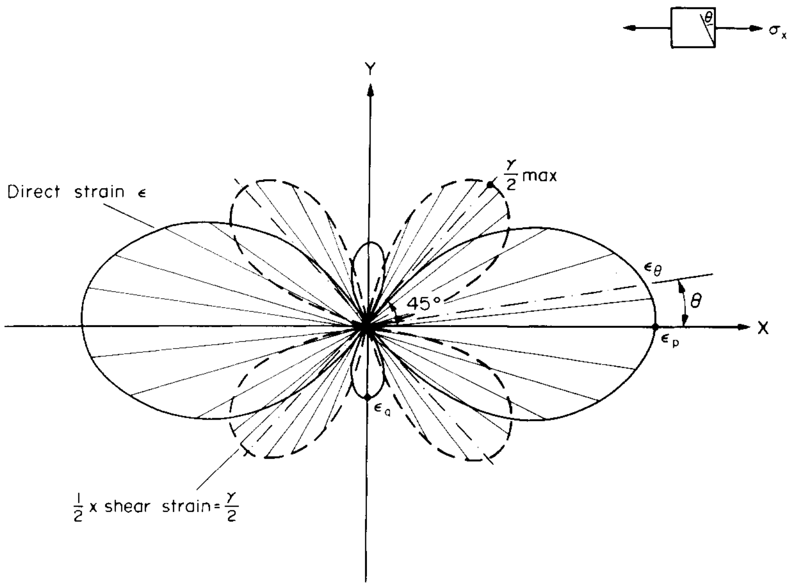


Fig. 14.24. Polar plot of strain distribution at a point under uniaxial applied stress.

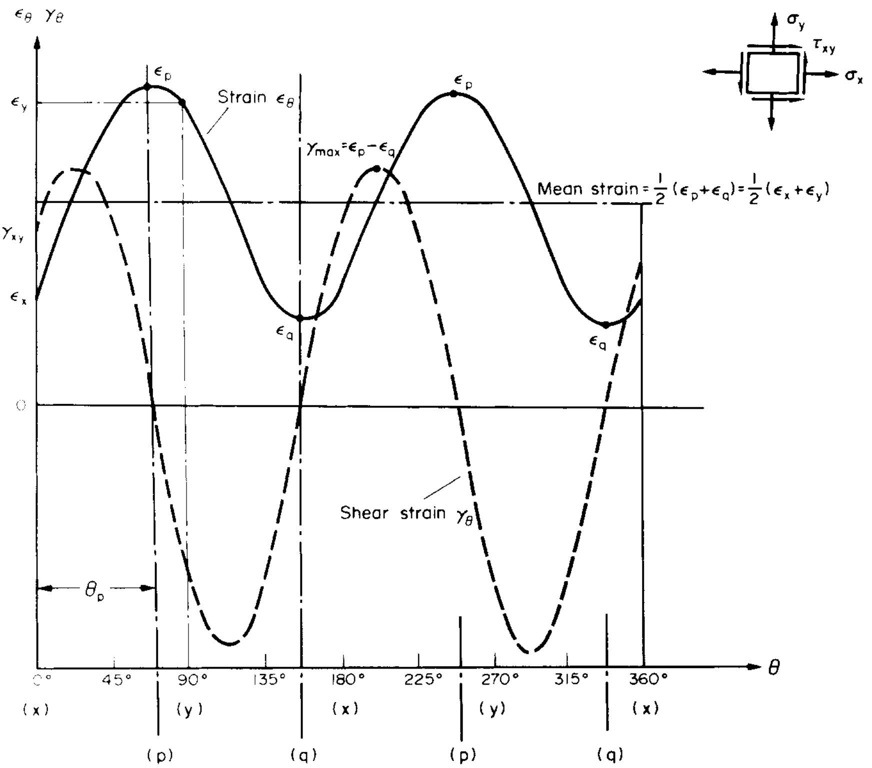


Fig. 14.25. Cartesian plot of strain distribution at a point under a typical biaxial applied stress system.

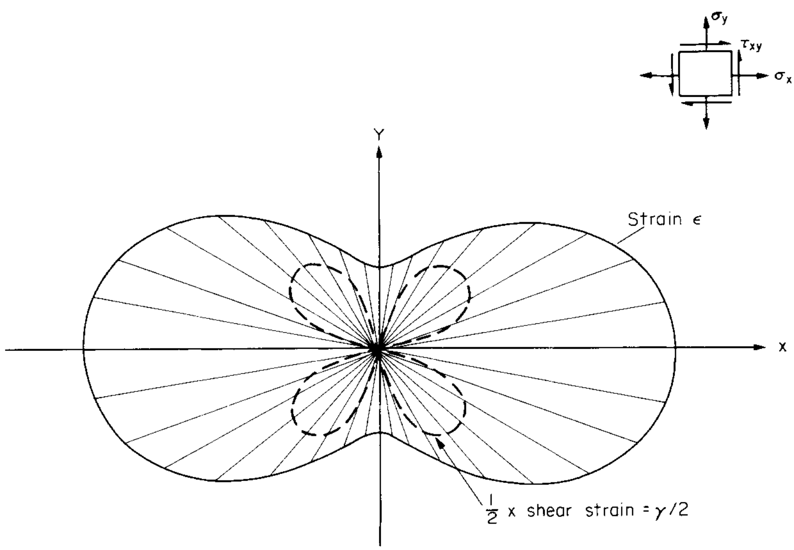


Fig. 14.26. Polar plot of strain distribution at a point under a typical biaxial applied stress system.

14.17. Strain energy of three-dimensional stress system

(a) Total strain energy

Any three-dimensional stress system may be reduced to three principal stresses σ_1, σ_2 and σ_3 acting on a unit cube, the faces of which are principal planes and, therefore, by definition, subjected to zero shear stress. If the corresponding principal strains are ε_1, ε_2 and ε_3, then the total strain energy U_t per unit volume is equal to the total work done by the system and given by the equation

$$U_t = \Sigma \tfrac{1}{2}\sigma\varepsilon$$

since the stresses are applied gradually from zero (see page 258).

$$\therefore \qquad U_t = \tfrac{1}{2}\sigma_1\varepsilon_1 + \tfrac{1}{2}\sigma_2\varepsilon_2 + \tfrac{1}{2}\sigma_3\varepsilon_3$$

Substituting for the principal strains using eqn. (14.2),

$$U_t = \frac{1}{2E}[\sigma_1(\sigma_1 - \nu\sigma_2 - \nu\sigma_3) + \sigma_2(\sigma_2 - \nu\sigma_3 - \nu\sigma_1) + \sigma_3(\sigma_3 - \nu\sigma_2 - \nu\sigma_1)]$$

$$\therefore \qquad \boldsymbol{U_t = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]} \textbf{ per unit volume} \qquad (14.21)$$

(b) Shear (or "distortion") strain energy

As above, consider the three-dimensional stress system reduced to principal stresses σ_1, σ_2 and σ_3 acting on a unit cube as in Fig. 14.27. For convenience the principal stresses may be

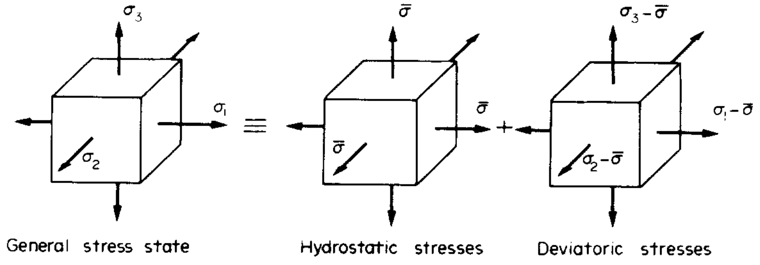


Fig. 14.27. Resolution of general three-dimensional principal stress state into "hydrostatic" and "deviatoric" components.

written in terms of a mean stress $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ and additional shear stress terms,

i.e.
$$\sigma_1 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_1 - \sigma_2) + \tfrac{1}{3}(\sigma_1 - \sigma_3)$$
$$\sigma_2 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_2 - \sigma_1) + \tfrac{1}{3}(\sigma_2 - \sigma_3)$$
$$\sigma_3 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) + \tfrac{1}{3}(\sigma_3 - \sigma_1) + \tfrac{1}{3}(\sigma_3 - \sigma_2)$$

The mean stress term may be considered as a *hydrostatic* tensile stress, equal in all directions, the strains associated with this giving rise to no distortion, i.e. the unit cube under the action of the hydrostatic stress alone would be strained into a cube. The hydrostatic stresses are sometimes referred to as the *spherical* or *dilatational* stresses.

The strain energy associated with the hydrostatic stress is termed the *volumetric strain energy* and is found by substituting

$$\sigma_1 = \sigma_2 = \sigma_3 = \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

into eqn. (14.21),

i.e.
$$\text{volumetric strain energy} = \frac{3}{2E}\left[\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)^2\right](1 - 2\nu)$$

$$\therefore \quad \boldsymbol{U_v = \frac{(1-2\nu)}{6E}[(\sigma_1 + \sigma_2 + \sigma_3)^2]} \textbf{ per unit volume} \qquad (14.22)$$

The remaining terms in the modified principal stress equations are shear stress terms (i.e. functions of principal stress differences in the various planes) and these are the only stresses which give rise to distortion of the stressed element. They are therefore termed *distortional* or *deviatoric* stresses.

Now

total strain energy per unit volume = shear strain energy per unit volume + volumetric strain energy per unit volume

i.e.
$$U_t = U_s + U_v$$

Therefore shear strain energy per unit volume is given by:

$$U_s = U_t - U_v$$

i.e.
$$U_s = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] - \frac{(1-2\nu)}{6E}[(\sigma_1 + \sigma_2 + \sigma_3)^2]$$

This simplifies to

$$U_s = \frac{(1+\nu)}{6E}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

and, since $E = 2G(1+\nu)$,

$$U_s = \frac{1}{12G}[\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \qquad (14.23a)$$

or, alternatively,

$$U_s = \frac{1}{6G}[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)] \qquad (14.23b)$$

It is interesting to note here that even a uniaxial stress condition may be divided into hydrostatic (dilatational) and deviatoric (distortional) terms as shown in Fig. 14.28.

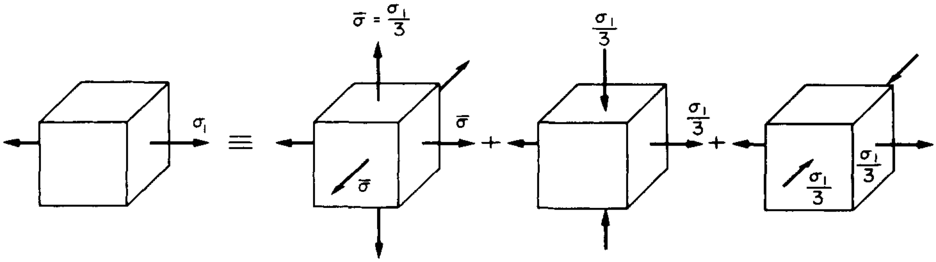


Fig. 14.28. Resolution of uniaxial stress into hydrostatic and deviatoric components.

Examples

Example 14.1

When a bar of 25 mm diameter is subjected to an axial pull of 61 kN the extension on a 50 mm gauge length is 0.1 mm and there is a decrease in diameter of 0.013 mm. Calculate the values of E, ν, G, and K.

Solution

$$\text{Longitudinal stress} = \frac{\text{load}}{\text{area}} = \frac{61\times10^3}{\frac{1}{4}\pi(0.025)^2} = 124.2\ \text{MN/m}^2$$

$$\text{Longitudinal strain} = \frac{\text{extension}}{\text{original length}} = \frac{0.1\times10^3}{10^3\times50} = 2\times10^{-3}$$

$$\text{Young's modulus}\,E = \frac{\text{stress}}{\text{strain}} = \frac{124.2\times10^6}{2\times10^{-3}} = \mathbf{62.1\ GN/m^2}$$

$$\text{Lateral strain} = \frac{\text{change in diameter}}{\text{original diameter}} = \frac{0.013\times10^3}{10^3\times25} = 0.52\times10^{-3}$$

$$\text{Poisson's ratio }(\nu) = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{0.52\times10^{-3}}{2\times10^{-3}} = \mathbf{0.26}$$

Now $$E = 2G(1+\nu) \quad \therefore \quad G = \frac{E}{2(1+\nu)}$$

$$G = \frac{62.1 \times 10^9}{2(1+0.26)} = \mathbf{24.6\ GN/m^2}$$

Also $$E = 3K(1-2\nu) \quad \therefore \quad K = \frac{E}{3(1-2\nu)}$$

$$K = \frac{62.1 \times 10^9}{3 \times 0.48} = \mathbf{43.1\ GN/m^2}$$

Example 14.2

A bar of mild steel 25 mm diameter twists 2 degrees in a length of 250 mm under a torque of 430 N m. The same bar deflects 0.8 mm when simply supported at each end horizontally over a span of 500 mm and loaded at the centre of the span with a vertical load of 1.2 kN. Calculate the values of E, G, K and Poisson's ratio ν for the material.

Solution

$$J = \frac{\pi}{32}D^4 = \frac{\pi}{32}(0.025)^4 = 0.0383 \times 10^{-6}\ \text{m}^4$$

Angle of twist $$\theta = 2 \times \frac{\pi}{180} = 0.0349 \text{ radian}$$

From the simple torsion theory $$\frac{T}{J} = \frac{G\theta}{L} \quad \therefore \quad G = \frac{TL}{J\theta}$$

$$G = \frac{430 \times 250 \times 10^6}{0.0349 \times 10^3 \times 0.0383} = 80.3 \times 10^9\ \text{N/m}^2$$

$$= \mathbf{80.3\ GN/m^2}$$

For a simply supported beam the deflection at mid-span with central load W is

$$\delta = \frac{WL^3}{48EI}$$

Then $$E = \frac{WL^3}{48\delta I} \quad \text{and} \quad I = \frac{\pi}{64}D^4 = \frac{\pi}{64}(0.025)^4 = 0.0192 \times 10^{-6}\ \text{m}^4$$

$$\therefore \quad E = \frac{1.2 \times 10^3 \times (0.5)^3 \times 10^6 \times 10^3}{48 \times 0.0192 \times 0.8} = 203 \times 10^9\ \text{N/m}^2$$

$$= \mathbf{203\ GN/m^2}$$

Now $$E = 2G(1+\nu) \quad \therefore \quad \nu = \frac{E}{2G} - 1$$

$$\therefore \quad \nu = \frac{203}{2 \times 80.3} - 1 = \mathbf{0.268}$$

Also $$E = 3K(1-2\nu) \quad \therefore \quad K = \frac{E}{3(1-2\nu)}$$

$$K = \frac{203 \times 10^9}{3(1-0.536)} = 146 \times 10^9 \text{ N/m}^2$$

$$= \mathbf{146\ GN/m^2}$$

Example 14.3

A rectangular bar of metal 50 mm × 25 mm cross-section and 125 mm long carries a tensile load of 100 kN along its length, a compressive load of 1 MN on its 50 × 125 mm faces and a tensile load of 400 kN on its 25 × 125 mm faces. If $E = 208$ GN/m² and $\nu = 0.3$, find

(a) the change in volume of the bar;
(b) the increase required in the 1 MN load to produce no change in volume.

Solution

(a) $$\sigma_x = \frac{\text{load}}{\text{area}} = \frac{100 \times 10^3 \times 10^6}{50 \times 25} = 80 \text{ MN/m}^2$$

$$\sigma_y = \frac{400 \times 10^3 \times 10^6}{125 \times 25} = 128 \text{ MN/m}^2$$

$$\sigma_z = \frac{-1 \times 10^6 \times 10^6}{125 \times 50} = -160 \text{ MN/m}^2 \quad \text{(Fig. 14.29)}$$

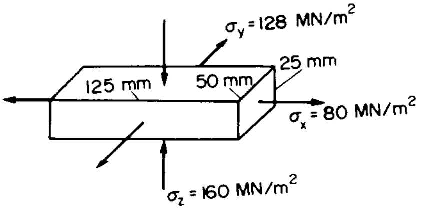


Fig. 14.29.

From §14.6

$$\text{change in volume} = \frac{V}{E}(\sigma_x + \sigma_y + \sigma_z)(1-2\nu)$$

$$= \frac{(125 \times 50 \times 25)}{208 \times 10^9} 10^{-9}[80 + 128 + (-160)]10^6 \times 0.4$$

$$= \frac{125 \times 50 \times 25 \times 48 \times 0.4}{208 \times 10^{12}} \text{m}^3 = \mathbf{14.4\ mm^3}$$

i.e. the bar increases in volume by 14.4 mm³.

(b) If the 1 MN load is to be changed, then σ_z will be changed; therefore the equation for the change in volume becomes

$$\text{change in volume} = 0 = \frac{(125 \times 50 \times 25)}{208 \times 10^9} 10^{-9}(80 + 128 + \sigma_z)10^6 \times 0.4$$

Then
$$0 = 80 + 128 + \sigma_z$$
$$\sigma_z = -208 \text{ MN/m}^2$$

Now
$$\text{load} = \text{stress} \times \text{area}$$

$\therefore$
$$\text{new load required} = -208 \times 10^6 \times 125 \times 50 \times 10^{-6}$$
$$= -1.3 \text{ MN}$$

Therefore the compressive load of 1 MN must be **increased by 0.3 MN** for no change in volume to occur.

Example 14.4

A steel bar ABC is of circular cross-section and transmits an axial tensile force such that the total change in length is 0.6 mm. The total length of the bar is 1.25 m, AB being 750 mm and 20 mm diameter and BC being 500 mm long and 13 mm diameter (Fig. 14.30). Determine for the parts AB and BC the changes in (a) length, and (b) diameter. Assume Poisson's ratio ν for the steel to be 0.3 and Young's modulus E to be 200 GN/m^2.

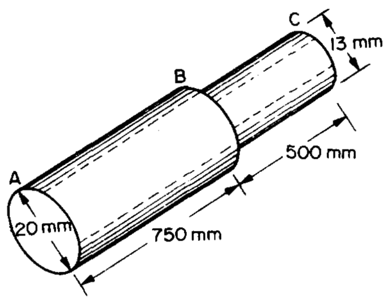


Fig. 14.30.

Solution

(a) Let the tensile force be P newtons.

Then

$$\text{stress in } AB = \frac{\text{load}}{\text{area}} = \frac{P}{\frac{1}{4}\pi(0.02)^2} = \frac{P}{100\pi} \text{ MN/m}^2$$

$$\text{stress in } BC = \frac{P}{\frac{1}{4}\pi(0.013)^2} = \frac{P}{42\pi} \text{ MN/m}^2$$

Then $$\text{strain in } AB = \frac{\text{stress}}{E} = \frac{P \times 10^6}{100\pi \times 200 \times 10^9} = \frac{P}{20\pi} \times 10^{-6}$$

and $$\text{strain in } BC = \frac{P \times 10^6}{42\pi \times 200 \times 10^9} = \frac{P}{8.4\pi} \times 10^{-6}$$

$$\text{change in length of } AB = \frac{P \times 10^{-6}}{20\pi} \times 750 \times 10^{-3} = 11.95P \times 10^{-9}$$

$$\text{change in length of } BC = \frac{P \times 10^{-6}}{8.4\pi} \times 500 \times 10^{-3} = 18.95P \times 10^{-9}$$

$$\text{total change in length} = (11.95P + 18.95P)10^{-9} = 0.6 \times 10^{-3}$$

$\therefore$ $$P(11.95 + 18.95)10^{-9} = 0.6 \times 10^{-3}$$

$\therefore$ $$P = \frac{0.6 \times 10^9}{10^3 \times 30.9} = \mathbf{19.4\ kN}$$

Then $$\text{change in length of } AB = 19.4 \times 10^3 \times 11.95 \times 10^{-9}$$
$$= 0.232 \times 10^{-3}\ \text{m} = \mathbf{0.232\ mm}$$

and $$\text{change in length of } BC = 19.4 \times 10^3 \times 18.95 \times 10^{-9}$$
$$= 0.368 \times 10^{-3} = \mathbf{0.368\ mm}$$

(b) The lateral (in this case "diametral") strain can be found from the definition of Poisson's ratio ν.

$$\nu = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$\therefore$ $$\text{lateral strain} = \text{strain on the diameter } (= \text{diametral strain})$$
$$= \nu \times \text{longitudinal strain}$$

$$\text{Lateral strain on } AB = \frac{\nu P \times 10^{-6}}{20\pi} = \frac{0.3 \times 19.4 \times 10^3}{20\pi \times 10^6}$$
$$= 92.7 \times 10^{-6} (= 92.7\ \mu\varepsilon) \quad \text{compressive}$$

$$\text{Lateral strain on } BC = \frac{\nu P \times 10^{-6}}{8.4 \times \pi} = \frac{0.3 \times 19.4 \times 10^3}{8.4\pi \times 10^6}$$
$$= 220.5 \times 10^{-6}\ (= 220.5\ \mu\varepsilon) \quad \text{compressive}$$

Then, $$\text{change in diameter of } AB = 92.7 \times 10^{-6} \times 20 \times 10^{-3}$$
$$= 1.854 \times 10^{-6} = \mathbf{0.00185\ mm}$$

and $$\text{change in diameter of } BC = 220.5 \times 10^{-6} \times 13 \times 10^{-3}$$
$$= 2.865 \times 10^{-6} = \mathbf{0.00286\ mm}$$

Both these changes are *decreases*.

Example 14.5

At a certain point a material is subjected to the following strains:

$$\varepsilon_x = 400 \times 10^{-6}; \quad \varepsilon_y = 200 \times 10^{-6}; \quad \gamma_{xy} = 350 \times 10^{-6} \text{ radian}$$

Determine the magnitudes of the principal strains, the directions of the principal strain axes and the strain on an axis inclined at 30° clockwise to the x axis.

Solution

Mohr's strain circle is as shown in Fig. 14.31.

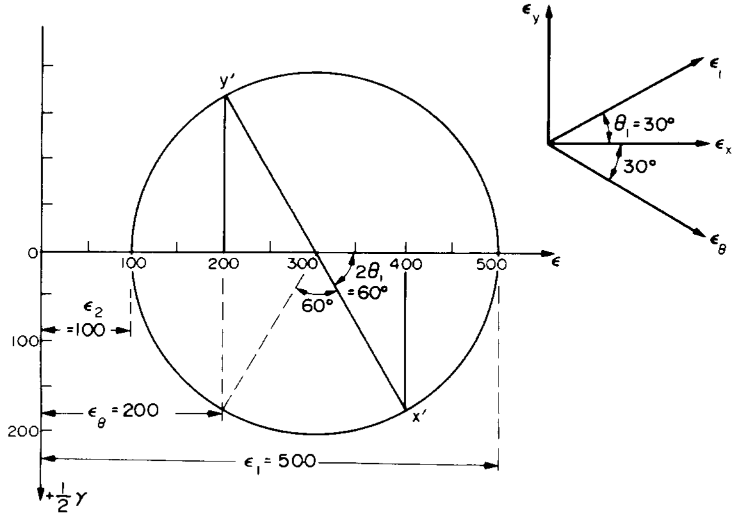


Fig. 14.31.

By measurement:

$$\varepsilon_1 = 500 \times 10^{-6} \qquad \varepsilon_2 = 100 \times 10^{-6}$$

$$\theta_1 = \frac{60^\circ}{2} = 30^\circ \qquad \theta_2 = 90^\circ + 30^\circ = 120^\circ$$

$$\varepsilon_{30} = 200 \times 10^{-6}$$

the angles being measured counterclockwise from the direction of ε_x.

Example 14.6

A material is subjected to two mutually perpendicular strains, $\varepsilon_x = 350 \times 10^{-6}$ and $\varepsilon_y = 50 \times 10^{-6}$, together with an unknown shear strain γ_{xy}. If the principal strain in the material is 420×10^{-6}, determine:

(a) the magnitude of the shear strain;
(b) the other principal strain;
(c) the direction of the principal strain axes;
(d) the magnitudes of the principal stresses if $E = 200\ GN/m^2$ and $\nu = 0.3$.

Solution

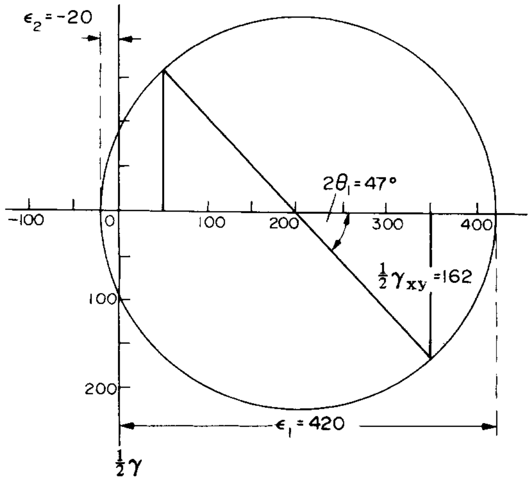


Fig. 14.32.

Mohr's strain circle is as shown in Fig. 14.32. The centre has been positioned half-way between ε_x and ε_y, and the radius is such that the circle passes through the ε axis at 420×10^{-6}. Then, by measurement:

(a) Shear strain $\gamma_{xy} = 2 \times 162 \times 10^{-6} = \mathbf{324 \times 10^{-6}}$ **radian.**
(b) Other principal strain $= \mathbf{-20 \times 10^{-6}}$ **(compressive).**
(c) Direction of principal strain $\varepsilon_1 = \dfrac{47^\circ}{2} = \mathbf{23^\circ 30'}$.

Direction of principal strain $\varepsilon_2 = 90^\circ + 23^\circ 30' = \mathbf{113^\circ 30'}$.

(d) The principal stresses may then be determined from the equations

$$\sigma_1 = \frac{(\varepsilon_1 + \nu\varepsilon_2)}{1-\nu^2}E \quad \text{and} \quad \sigma_2 = \frac{(\varepsilon_2 + \nu\varepsilon_1)}{1-\nu^2}E$$

$$\therefore \qquad \sigma_1 = \frac{[420 + 0.3(-20)]\,10^{-6} \times 200 \times 10^9}{1-(0.3)^2}$$

$$= \frac{414 \times 200 \times 10^3}{0.91} = \mathbf{91\ MN/m^2\ tensile}$$

and
$$\sigma_2 = \frac{(-20 + 0.3 \times 420)10^{-6} \times 200 \times 10^9}{1 - (0.3)^2}$$

$$= \frac{106 \times 200 \times 10^3}{0.91} = \mathbf{23.3\ MN/m^2\ tensile}$$

Thus the principal stresses are 91 MN/m^2 and 23.3 MN/m^2, both tensile.

Example 14.7

The following strain readings were recorded at the angles stated relative to a given horizontal axis:

$$\varepsilon_a = -2.9 \times 10^{-5} \text{ at } 20^\circ$$

$$\varepsilon_b = 3.1 \times 10^{-5} \text{ at } 80^\circ$$

$$\varepsilon_c = -0.5 \times 10^{-5} \text{ at } 140^\circ$$

as shown in Fig. 14.33. Determine the magnitude and direction of the principal stresses. $E = 200$ GN/m^2; $v = 0.3$.

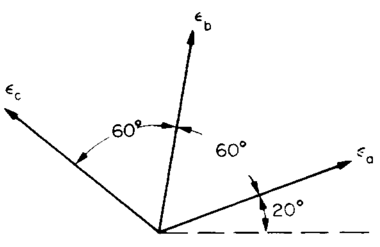


Fig. 14.33.

Solution

Consider now the construction shown in Fig. 14.34 giving the strain circle for the strain values in the question and illustrated in Fig. 14.33.

For a strain scale of 1 cm = 1×10^{-5} strain, in order to superimpose a stress circle concentric with the strain circle, the necessary scale is

$$1 \text{ cm} = \frac{E}{(1-v)} \times 1 \times 10^{-5} = \frac{200 \times 10^9}{0.7} \times 10^{-5}$$

$$= 2.86 \text{ MN/m}^2$$

Also radius of strain circle = 3.5 cm

$\therefore$ radius of stress circle $= 3.5 \times \dfrac{(1-v)}{(1+v)} = \dfrac{3.5 \times 0.7}{1.3}$

$$= 1.886 \text{ cm}$$

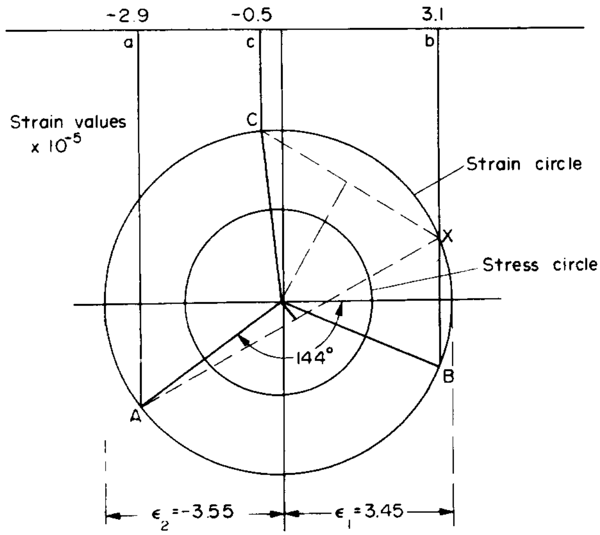


Fig. 14.34.

Superimposing the stress circle of radius 1.886 cm concentric with the strain circle, the principal stresses to a scale 1 cm = 2.86 MN/m^2 are found to be

$$\sigma_1 = 1.8 \times 2.86 \times 10^6 = \mathbf{5.15\ MN/m^2}$$

$$\sigma_2 = -2.0 \times 2.86 \times 10^6 = \mathbf{-5.72\ MN/m^2}$$

The principal strains will be at an angle $\frac{144}{2} = 72°$ and 162° counterclockwise from the direction of ε_a, i.e. **92°** and **182°** counterclockwise from the given horizontal axis. The principal stresses will therefore also be in these directions.

Example 14.8

A rectangular rosette of strain gauges on the surface of a material under stress recorded the following readings of strain:

gauge A	$+450 \times 10^{-6}$
gauge B, at 45° to A	$+200 \times 10^{-6}$
gauge C, at 90° to A	-200×10^{-6}

the angles being counterclockwise from A.

Determine:

(a) the magnitudes of the principal strains,

(b) the directions of the principal strain axes, both by calculation and by Mohr's strain circle.

Solution

If ε_1 and ε_2 are the principal strains and ε_θ is the strain in a direction at θ to the direction of ε_1, then eqn. (14.14) may be rewritten as

$$\varepsilon_\theta = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta$$

since $\gamma_{xy} = 0$ on principal strain axes. Thus if gauge A is at an angle θ to ε_1:

$$450 \times 10^{-6} = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \qquad (1)$$

$$200 \times 10^{-6} = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(90^\circ + 2\theta) \qquad (2)$$

$$-200 \times 10^{-6} = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(180^\circ + 2\theta)$$

$$= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) - \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \qquad (3)$$

Adding (1) and (3),

$$250 \times 10^{-6} = \varepsilon_1 + \varepsilon_2 \qquad (4)$$

Substituting (4) in (2),

$$200 \times 10^{-6} = 125 \times 10^{-6} + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos(90^\circ + 2\theta)$$

$$\therefore \quad 75 \times 10^{-6} = -\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\theta \qquad (5)$$

Substituting (4) in (1),

$$450 \times 10^{-6} = 125 \times 10^{-6} + \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta$$

$$\therefore \quad 325 \times 10^{-6} = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta \qquad (6)$$

Dividing (5) by (6),

$$\frac{75 \times 10^{-6}}{325 \times 10^{-6}} = \frac{-\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\sin 2\theta}{\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 2\theta}$$

$$\therefore \quad \tan 2\theta = -0.231$$

$$\therefore \quad 2\theta = -13^\circ \quad \text{or} \quad 180^\circ - 13^\circ = 167^\circ$$

$$\therefore \quad \theta = \mathbf{83^\circ 30'}$$

Thus gauge A is $83^\circ 30'$ counterclockwise from the direction of ε_1.

Therefore from (6),

$$325 \times 10^{-6} = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)\cos 167^\circ$$

$$\therefore \quad \varepsilon_1 - \varepsilon_2 = \frac{2 \times 325 \times 10^{-6}}{-0.9744} = -667 \times 10^{-6}$$

But from eqn. (4),

$$\varepsilon_1 + \varepsilon_2 = 250 \times 10^{-6}$$

Therefore adding,

$$2\varepsilon_1 = -417 \times 10^{-6} \quad \therefore \quad \varepsilon_1 = \mathbf{-208.5 \times 10^{-6}}$$

and subtracting,

$$2\varepsilon_2 = 917 \times 10^{-6} \quad \therefore \quad \varepsilon_2 = \mathbf{458.5 \times 10^{-6}}$$

Thus the principal strains are -208×10^{-6} and 458.5×10^{-6}, the former being on an axis **83°30′** clockwise from gauge *A*.

Alternatively, these results may be obtained using Mohr's strain circle as shown in Fig. 14.35. The circle has been drawn using the construction procedure of §14.14 and gives principal strains of

ε_1 = **458.5 × 10⁶**, tensile, at **6°30′** counterclockwise from gauge *A*
ε_2 = **208.5 × 10⁶**, compressive, at **83°30′** clockwise from gauge *A*

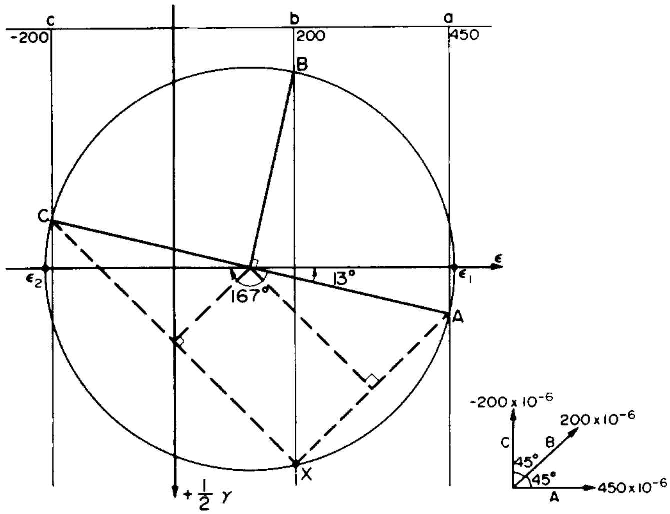


Fig. 14.35.

Problems

14.1 (A). A bar of 40 mm diameter carries a tensile load of 100 kN. Determine the longitudinal extension of a 50 mm gauge length and the contraction of the diameter.
Young's modulus $E = 210\ \text{GN/m}^2$ and Poisson's ratio $\nu = 0.3$. [0.019, 0.0045 mm.]

14.2 (A). Establish the relationship between Young's modulus *E*, the modulus of rigidity *G* and the bulk modulus *K* in the form

$$E = \frac{9KG}{3K + G}$$

14.3 (A). The extension of a 100 mm gauge length of 14.33 mm diameter bar was found to be 0.15 mm when a tensile load of 50 kN was applied. A torsion specimen of the same specification was made with a 19 mm diameter and a 200 mm gauge length. On test it twisted 0.502 degree under the action of a torque of 45 N m. Calculate *E*, *G*, *K* and ν. [206.7, 80.9, 155 GN/m²; 0.278.]

14.4 (A). A rectangular steel bar of 25 mm × 12 mm cross-section deflects 6 mm when simply supported on its 25 mm face over a span of 1.2 m and loaded at the centre with a concentrated load of 126 N. If Poisson's ratio for the material is 0.28 determine the values of (a) the rigidity modulus, and (b) the bulk modulus. [82, 159 GN/m².]

14.5 (A). Calculate the changes in dimensions of a 37 mm × 25 mm rectangular bar when loaded with a tensile load of 600 kN.
Take $E = 210\ \text{GN/m}^2$ and $\nu = 0.3$. [0.034, 0.023 mm.]

14.6 (A). A rectangular block of material 125 mm × 100 mm × 75 mm carries loads normal to its faces as follows: 1 MN tensile on the 125 × 100 mm faces; 0.48 MN tensile on the 100 × 75 mm faces; zero load on the 125 × 75 mm faces.

If Poisson's ratio = 0.3 and $E = 200$ GN/m^2, determine the changes in dimensions of the block under load. What is then the change in volume? [0.025, 0.0228, −0.022 mm; 270 mm^3.]

14.7 (A). A rectangular bar consists of two sections, *AB* 25 mm square and 250 mm long and *BC* 12 mm square and 250 mm long. For a tensile load of 20 kN determine:
(a) the change in length of the complete bar;
(b) the changes in dimensions of each portion.
Take $E = 80$ GN/m^2 and Poisson's ratio $\nu = 0.3$. [0.534 mm; 0.434, −0.003, 0.1, −0.0063 mm.]

14.8 (A). A cylindrical brass bar is 50 mm diameter and 250 mm long. Find the change in volume of the bar when an axial compressive load of 150 kN is applied.
Take $E = 100$ GN/m^2 and $\nu = 0.27$. [172.5 mm^3.]

14.9 (A). A certain alloy bar of 32 mm diameter has a gauge length of 100 mm. A tensile load of 25 kN produces an extension of 0.014 mm on the gauge length and a torque of 2.5 kN m produces an angle of twist of 1.63 degrees. Calculate *E*, *G*, *K* and ν. [222, 85.4, 185 GN/m^2, 0.3.]

14.10 (A/B). Derive the relationships which exist between the elastic constants (a) *E*, *G* and ν, and (b) *E*, *K* and ν. Find the change in volume of a steel cube of 150 mm side immersed to a depth of 3 km in sea water.
Take *E* for steel = 210 GN/m^2, $\nu = 0.3$ and the density of sea water = 1025 kg/m^3. [580 mm^3.]

14.11 (B). Two steel bars have the same length and the same cross-sectional area, one being circular in section and the other square. Prove that when axial loads are applied the changes in volume of the bars are equal.

14.12 (B). Determine the percentage change in volume of a bar 50 mm square and 1 m long when subjected to an axial compressive load of 10 kN. Find also the restraining pressure on the sides of the bar required to prevent all lateral expansion.
For the bar material, $E = 210$ GN/m^2 and $\nu = 0.27$. [0.876×10^{-3} %, 1.48 MN/m^2.]

14.13 (B). Derive the formula for longitudinal strain due to axial stress σ_x when all lateral strain is prevented.
A piece of material 100 mm long by 25 mm square is in compression under a load of 60 kN. Determine the change in length of the material if all lateral strain is prevented by the application of a uniform external lateral pressure of a suitable intensity.
For the material, $E = 70$ GN/m^2 and Poisson's ratio $\nu = 0.25$. [0.114 mm.]

14.14 (B). Describe briefly an experiment to find Poisson's ratio for a material.
A steel bar of rectangular cross-section 40 mm wide and 25 mm thick is subjected to an axial tensile load of 100 kN. Determine the changes in dimensions of the sides and hence the percentage decrease in cross-sectional area if $E = 200$ GN/m^2 and Poisson's ratio = 0.3. [-6×10^{-3}, -3.75×10^{-3}, −0.03 %.]

14.15 (B). A material is subjected to the following strain system:

$$\varepsilon_x = 200 \times 10^{-6};\ \varepsilon_y = -56 \times 10^{-6};\quad \gamma_{xy} = 230 \times 10^{-6} \text{ radian}$$

Determine:
(a) the principal strains;
(b) the directions of the principal strain axes;
(c) the linear strain on an axis inclined at 50° counterclockwise to the direction of ε_x.
[244, -100×10^{-6}; 21°; 163×10^{-6}.]

14.16 (B). A material is subjected to two mutually perpendicular linear strains together with a shear strain. Given that this system produces principal strains of 0.0001 compressive and 0.0003 tensile and that one of the linear strains is 0.00025 tensile, determine the magnitudes of the other linear strain and the shear strain.
[-50×10^{-6}, 265×10^{-6}.]

14.17 (C). A 50 mm diameter cylinder is subjected to an axial compressive load of 80 kN. The cylinder is partially enclosed by a well-fitted casing covering almost the whole length, which reduces the lateral expansion by half. Determine the ratio between the axial strain when the casing is fitted and that when it is free to expand in diameter.
Take $\nu = 0.3$. [0.871.]

14.18 (C). A thin cylindrical shell has hemispherical ends and is subjected to an internal pressure. If the radial change of the cylindrical part is to be equal to that of the hemispherical ends, determine the ratio between the thickness necessary in the two parts. Take $\nu = 0.3$. [2.43 : 1.]

14.19 (B). Determine the values of the principal stresses present in the material of Problem 14.16. Describe an experimental technique by which the directions and magnitudes of these stresses could be determined in practice. For the material, take $E = 208$ GN/m^2 and $\nu = 0.3$. [61.6, 2.28 MN/m^2.]

14.20 (B). A rectangular prism of steel is subjected to purely normal stresses on all six faces (i.e. the stresses are principal stresses). One stress is 60 MN/m^2 tensile, and the other two are denoted by σ_x and σ_y and may be either tensile or compressive, their magnitudes being such that there is no strain in the direction of σ_y and that the maximum shearing stress in the material does not exceed 75 MN/m^2 on any plane. Determine the range of values within which σ_x may lie and the corresponding values of σ_y. Make sketches to show the two limiting states of stress, and calculate the strain energy per cubic metre of material in the two limiting conditions. Assume that the stresses are not sufficient to cause elastic failure. For the prism material $E = 208$ GN/m^2; $\nu = 0.286$.

[U.L.] [-90 to 210; -8.6, 77.2 MN/m^2.]

For the following problems on the application of strain gauges additional information may be obtained in §21.2 (*Vol.* 2).

14.21 (A/B). The following strains are recorded by two strain gauges, their axes being at right angles: $\varepsilon_x = 0.00039$; $\varepsilon_y = -0.00012$ (i.e. one tensile and one compressive). Find the values of the stresses σ_x and σ_y acting along these axes if the relevant elastic constants are $E = 208$ GN/m^2 and $\nu = 0.3$. [80.9, -0.69 MN/m^2.]

14.22 (B). Explain how strain gauges can be used to measure shear strain and hence shear stresses in a material.

Find the value of the shear stress present in a shaft subjected to pure torsion if two strain gauges mounted at 45° to the axis of the shaft record the following values of strain: 0.00029; -0.00029. If the shaft is of steel, 75 mm diameter, $G = 80$ GN/m^2 and $\nu = 0.3$, determine the value of the applied torque. [46.4 MN/m^2, 3.84 kN m.]

14.23 (B). The following strains were recorded on a rectangular strain rosette: $\varepsilon_a = 450 \times 10^{-6}$; $\varepsilon_b = 230 \times 10^{-6}$; $\varepsilon_c = 0$.

Determine:

(a) the principal strains and the directions of the principal strain axes;

(b) the principal stresses if $E = 200$ GN/m^2 and $\nu = 0.3$.

[451×10^{-6} at 1° clockwise from A, -1×10^{-6} at 91° clockwise from A; 98, 29.5 MN/m^2.]

14.24 (B). The values of strain given in Problem 14.23 were recorded on a 60° rosette gauge. What are now the values of the principal strains and the principal stresses?

[484×10^{-6}, -27×10^{-6}; 104 MN/m^2, 25.7 MN/m^2.]

14.25 (B). Describe briefly how you would proceed, with the aid of strain gauges, to find the principal stresses present on a material under the action of a complex stress system.

Find, by calculation, the principal stresses present in a material subjected to a complex stress system given that strain readings in directions at 0°, 45° and 90° to a given axis are $+240 \times 10^{-6}$, $+170 \times 10^{-6}$ and $+40 \times 10^{-6}$ respectively.

For the material take $E = 210$ GN/m^2 and $\nu = 0.3$. [59, 25 MN/m^2.]

14.26 (B). Check the calculation of Problem 14.25 by means of Mohr's strain circle.

14.27 (B). A closed-ended steel pressure vessel of diameter 2.5 m and plate thickness 18 mm has electric resistance strain gauges bonded on the outer surface in the circumferential and axial directions. These gauges have a resistance of 200 ohms and a gauge factor of 2.09. When the pressure is raised to 9 MN/m^2 the change of resistance is 1.065 ohms for the circumferential gauge and 0.265 ohm for the axial gauge. Working from first principles calculate the value of Young's modulus and Poisson's ratio. [I.Mech.E.] [0.287, 210 GN/m^2.]

14.28 (B). Briefly describe the mode of operation of electric resistance strain gauges, and a simple circuit for the measurement of a static change in strain.

The torque on a steel shaft of 50 mm diameter which is subjected to pure torsion is measured by a strain gauge bonded on its outer surface at an angle of 45° to the longitudinal axis of the shaft. If the change of the gauge resistance is 0.35 ohm in 200 ohms and the strain gauge factor is 2, determine the torque carried by the shaft. For the shaft material $E = 210$ GN/m^2 and $\nu = 0.3$. [I.Mech.E.] [3.47 kN.]

14.29 (A/B). A steel test bar of diameter 11.3 mm and gauge length 56 mm was found to extend 0.08 mm under a load of 30 kN and to have a contraction on the diameter of 0.00452 mm. A shaft of 80 mm diameter, made of the same quality steel, rotates at 420 rev/min. An electrical resistance strain gauge bonded to the outer surface of the shaft at an angle of 45° to the longitudinal axis gave a recorded resistance change of 0.189 Ω. If the gauge resistance is 100 Ω and the gauge factor is 2.1 determine the maximum power transmitted. [650 kW.]

14.30 (B). A certain equiangular strain gauge rosette is made up of three separate gauges. After it has been installed it is found that one of the gauges has, in error, been taken from an odd batch; its gauge factor is 2.0, that of the other two being 2.2. As the three gauges appear identical it is impossible to say which is the rogue and it is decided to proceed with the test. The following strain readings are obtained using a gauge factor setting on the strain gauge equipment of 2.2:

Gauge direction	0°	60°	120°
Strain $\times 10^{-6}$	+1	−250	+200

Taking into account the various gauge factor values evaluate the greatest possible shear stress value these readings can represent.

For the specimen material $E = 207\ \text{GN/m}^2$ and $\nu = 0.3$. [City U.] [44 MN/m^2.]

14.31 (B). A solid cylindrical shaft is 250 mm long and 50 mm diameter and is made of aluminium alloy. The periphery of the shaft is constrained in such a way as to prevent lateral strain. Calculate the axial force that will compress the shaft by 0.5 mm.

Determine the change in length of the shaft when the lateral constraint is removed but the axial force remains unaltered.

Calculate the required reduction in axial force for the non-constrained shaft if the axial strain is not to exceed 0.2 %

Assume the following values of material constants, $E = 70\ \text{GN/m}^2$; $\nu = 0.3$.

[C.E.I.] [370 kN; 0.673 mm; 95.1 kN.]

14.32 (C). An electric resistance strain gauge rosette is bonded to the surface of a square plate, as shown in Fig. 14.36. The orientation of the rosette is defined by the angle gauge A makes with the X direction. The angle between gauges A and B is 120° and between A and C is 120°. The rosette is supposed to be orientated at 45° to the X direction. To check this orientation the plate is loaded with a uniform tension in the X direction only (i.e. $\sigma_y = 0$), unloaded and then loaded with a uniform tension stress of the same magnitude, in the Ydirection only (i.e. $\sigma_x = 0$), readings being taken from the strain gauges in both loading cases.

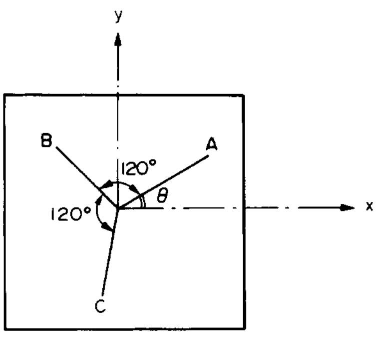


Fig. 14.36.

Denoting the greater principal strain in both loading cases by ε_1, show that if the rosette is correctly orientated, then

(a) the strain shown by gauge A should be

$$\varepsilon_A = \frac{(1-\nu)}{2}\varepsilon_1$$

for both load cases, and

(b) that shown by gauge B should be

$$\varepsilon_B = \frac{(1-\nu)}{2}\varepsilon_1 + \frac{(1+\nu)}{2}\frac{\sqrt{3}}{2}\varepsilon_1 \quad \text{for the } \sigma_x \text{ case}$$

or

$$\varepsilon_B = \frac{(1-\nu)}{2}\varepsilon_1 - \frac{(1+\nu)}{2}\frac{\sqrt{3}}{2}\varepsilon_1 \quad \text{for the } \sigma_y \text{ case}$$

Hence obtain the corresponding expressions for ε_c. [As for B, but reversed.]