CHAPTER 13

COMPLEX STRESSES

Summary

The normal stress σ and shear stress τ on oblique planes resulting from direct loading are

$$\sigma = \sigma_y \sin^2\theta \quad \text{and} \quad \tau = \tfrac{1}{2}\sigma_y \sin 2\theta$$

The stresses on oblique planes owing to a complex stress system are:

$$\text{normal stress} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\text{shear stress} = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

The *principal stresses* (i.e. the maximum and minimum direct stresses) are then

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

$$\sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

and these occur on planes at an angle θ to the plane on which σ_x acts, given by either

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)} \quad \text{or} \quad \tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}$$

where $\sigma_p = \sigma_1$, or σ_2, the planes being termed *principal planes.* The principal planes are always at 90° to each other, and the *planes of maximum shear* are then located at 45° to them.

The *maximum shear stress* is

$$\tau_{\max} = \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} = \tfrac{1}{2}(\sigma_1 - \sigma_2)$$

In problems where the principal stress in the third dimension σ_3 either is known or can be assumed to be zero, the true maximum shear stress is then

$\tfrac{1}{2}$(greatest principal stress − least principal stress)

Normal stress on plane of maximum shear $= \tfrac{1}{2}(\sigma_x + \sigma_y)$

Shear stress on plane of maximum direct stress (principal plane) = 0

Most problems can be solved graphically by *Mohr's stress circle.* All questions which are capable of solution by this method have been solved both analytically and graphically.

13.1. Stresses on oblique planes

Consider the general case, shown in Fig. 13.1, of a bar under direct load F giving rise to stress σ_y vertically.

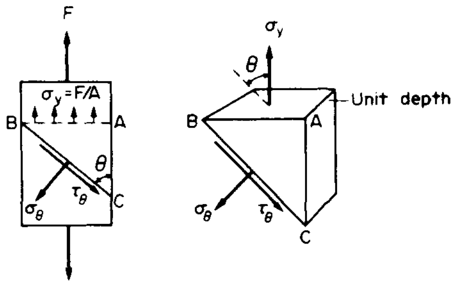

Fig. 13.1. Bar subjected to direct stress, showing stresses acting on any inclined plane.

Let the block be of *unit depth*; then considering the equilibrium of *forces* on the triangular portion *ABC*:

resolving forces perpendicular to *BC*,

$$\sigma_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \sin\theta$$

But $AB = BC \sin\theta$,

$$\therefore \qquad \boldsymbol{\sigma_\theta = \sigma_y \sin^2\theta} \qquad (13.1)$$

Now resolving forces parallel to *BC*,

$$\tau_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \cos\theta$$

Again $AB = BC \sin\theta$,

$$\therefore \qquad \tau_\theta = \sigma_y \sin\theta\cos\theta$$

$$= \tfrac{1}{2}\boldsymbol{\sigma_y \sin 2\theta} \qquad (13.2)$$

The stresses on the inclined plane, therefore, are not simply the resolutions of σ_y perpendicular and tangential to that plane. The direct stress σ_θ has a maximum value of σ_y when $\theta = 90°$ whilst the shear stress τ_θ has a maximum value of $\tfrac{1}{2}\sigma_y$ when $\theta = 45°$.

Thus any material whose yield stress in shear is less than half that in tension or compression will yield initially in shear under the action of direct tensile or compressive forces.

This is evidenced by the typical "cup and cone" type failure in tension tests of ductile specimens such as low carbon steel where failure occurs initially on planes at 45° to the specimen axis. Similar effects occur in compression tests on, for example, timber where failure is again due to the development of critical shear stresses on 45° planes.

13.2. Material subjected to pure shear

Consider the element shown in Fig. 13.2 to which shear stresses have been applied to the sides *AB* and *DC*. *Complementary shear stresses* of equal value but of opposite effect are then set up on sides *AD* and *BC* in order to prevent rotation of the element. Since the applied and complementary shears are of equal value on the x and y planes, they are both given the symbol τ_{xy}.

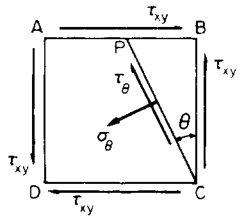

Fig. 13.2. Stresses on an element subjected to pure shear.

Consider now the equilibrium of portion *PBC*.

Resolving normal to *PC assuming unit depth,*

$$\sigma_\theta \times PC = \tau_{xy} \times BC \sin\theta + \tau_{xy} \times PB \cos\theta$$

$$= \tau_{xy} \times PC \cos\theta \sin\theta + \tau_{xy} \times PC \sin\theta \cos\theta$$

$$\therefore \quad \boldsymbol{\sigma_\theta = \tau_{xy} \sin 2\theta} \quad (13.3)$$

The maximum value of σ_θ is τ_{xy} when $\theta = 45^\circ$.

Similarly, resolving forces parallel to *PC*,

$$\tau_\theta \times PC = \tau_{xy} \times PB \sin\theta - \tau_{xy} BC \cos\theta$$

$$= \tau_{xy} \times PC \sin^2\theta - \tau_{xy} \times PC \cos^2\theta$$

$$\therefore \quad \boldsymbol{\tau_\theta = -\tau_{xy} \cos 2\theta} \quad (13.4)$$

The negative sign means that the sense of τ_θ is opposite to that assumed in Fig. 13.2.

The maximum value of τ_θ is τ_{xy} when $\theta = 0^\circ$ or 90° and it has a value of zero when $\theta = 45^\circ$, i.e. on the planes of maximum direct stress.

Further consideration of eqns. (13.3) and (13.4) shows that the system of pure shear stresses produces an equivalent direct stress system as shown in Fig. 13.3, one set compressive and one tensile, each at 45° to the original shear directions, and equal in magnitude to the applied shear.

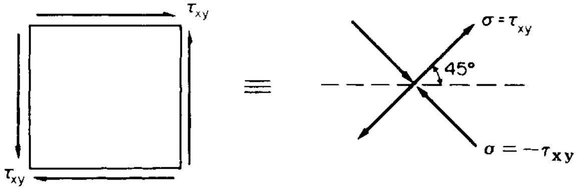

Fig. 13.3. Direct stresses due to shear.

This has great significance in the measurement of shear stresses or torques on shafts using strain gauges where the gauges are arranged to record the direct strains at 45° *to the shaft axis.*

Practical evidence of the theory is also provided by the failure of brittle materials in shear. A shaft of a brittle material subjected to torsion will fail under direct stress on planes at 45° to the shaft axis. (This can be demonstrated easily by twisting a piece of blackboard chalk in one's hands; see Fig. 8.8a on page 185.) Tearing of a wet cloth when it is being wrung out is also attributed to the direct stresses introduced by the applied torsion.

13.3. Material subjected to two mutually perpendicular direct stresses

Consider the rectangular element of *unit depth* shown in Fig. 13.4 subjected to a system of two direct stresses, both tensile, at right angles, σ_x and σ_y.

For equilibrium of the portion ABC, resolving perpendicular to AC,

$$\sigma_\theta \times AC \times 1 = \sigma_x \times BC \times 1 \times \cos\theta + \sigma_y \times AB \times 1 \times \sin\theta$$

$$= \sigma_x \times AC\cos^2\theta + \sigma_y \times AC\sin^2\theta$$

$$\therefore \quad \sigma_\theta = \tfrac{1}{2}\sigma_x(1+\cos 2\theta) + \tfrac{1}{2}\sigma_y(1-\cos 2\theta)$$

i.e.
$$\boldsymbol{\sigma_\theta = \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta} \qquad (13.5)$$

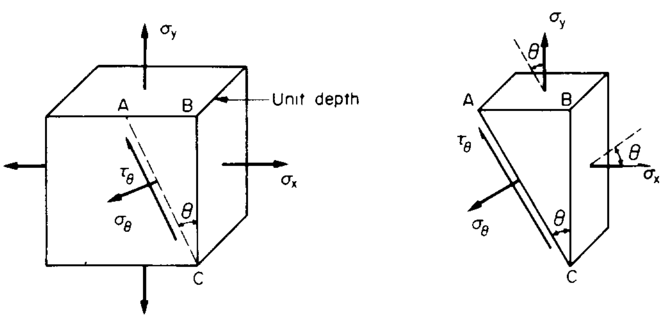

Fig. 13.4. Element from a material subjected to two mutually perpendicular direct stresses.

Resolving parallel to AC:

$$\tau_\theta \times AC \times 1 = \sigma_x \times BC \times 1 \times \sin\theta - \sigma_y \times AB \times 1 \times \cos\theta$$

$$\tau_\theta = \sigma_x \cos\theta\sin\theta - \sigma_y\cos\theta\sin\theta$$

$$\therefore \quad \boldsymbol{\tau_\theta = \tfrac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta} \qquad (13.6)$$

The maximum direct stress will equal σ_x or σ_y, whichever is the greater, when $\theta = 0$ or $90°$.

The maximum shear stress *in the plane of the applied stresses* (see §13.8) occurs when $\theta = 45°$,

i.e.
$$\boldsymbol{\tau_{max} = \tfrac{1}{2}(\sigma_x-\sigma_y)} \qquad (13.7)$$

13.4. Material subjected to combined direct and shear stresses

Consider the complex stress system shown in Fig. 13.5 acting on an element of material.

The stresses σ_x and σ_y may be compressive or tensile and may be the result of direct forces or bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear forces or torsion.

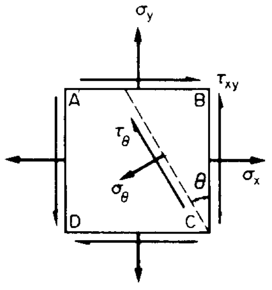

Fig. 13.5. Two-dimensional complex stress system.

The diagram thus represents a complete stress system for any condition of applied load in two dimensions and represents an addition of the stress systems previously considered in §§13.2 and 13.3.

The formulae obtained in these sections may therefore be combined to give

$$\boldsymbol{\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta} \qquad (13.8)$$

and

$$\boldsymbol{\tau_\theta = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta} \qquad (13.9)$$

The *maximum and minimum stresses* which occur on any plane in the material can now be determined as follows:

For σ_θ to be a maximum or minimum $\dfrac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$\therefore$

$$\frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

or

$$\boldsymbol{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}} \qquad (13.10)$$

$\therefore$ from Fig. 13.6

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

$$\cos 2\theta = \frac{(\sigma_x - \sigma_y)}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

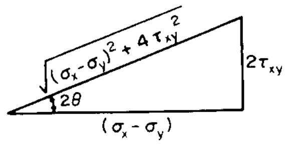

Fig. 13.6.

Therefore substituting in eqn. (13.8), the maximum and minimum direct stresses are given by

$$\sigma_1 \quad \text{or} \quad \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\frac{(\sigma_x - \sigma_y)(\sigma_x - \sigma_y)}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}} + \frac{\tau_{xy} \times 2\tau_{xy}}{\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}}$$

$$= \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \qquad (13.11)$$

These are then termed the *principal stresses* of the system.

The solution of eqn. (13.10) yields two values of 2θ separated by 180°, i.e. two values of θ separated by 90°. Thus the two principal stresses occur on mutually perpendicular planes termed *principal planes*, and substitution for θ from eqn. (13.10) into the shear stress expression eqn. (13.9) will show that $\tau_\theta = 0$ on the principal planes.

The complex stress system of Fig. 13.5 can now be reduced to the equivalent system of principal stresses shown in Fig. 13.7.

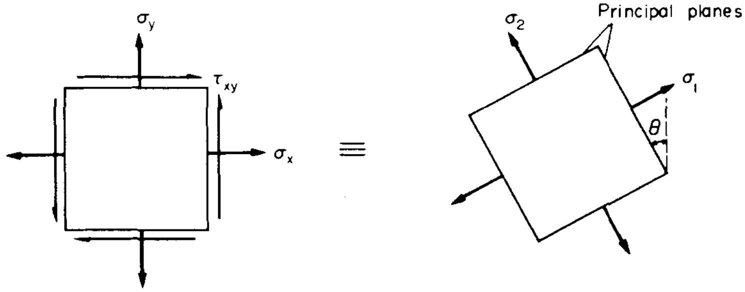

Fig. 13.7. Principal planes and stresses.

From eqn. (13.7) the maximum shear stress present in the system is given by

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) \qquad (13.12)$$

$$= \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \qquad (13.13)$$

and this occurs **on planes at 45° to the principal planes.**

This result could have been obtained using a similar procedure to that used for determining the principal stresses, i.e. by differentiating expression (13.9), equating to zero and substituting the resulting expression for θ.

13.5. Principal plane inclination in terms of the associated principal stress

It has been stated in the previous section that expression (13.10), namely

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of θ, i.e. the inclination of the two principal planes on which the principal stresses σ_1 and σ_2 act. It is uncertain, however, which stress acts on which plane unless eqn. (13.8) is used, substituting *one* value of θ obtained from eqn. (13.10) and observing which one of the two principal stresses is obtained. The following alternative solution is therefore to be preferred.

Consider once again the equilibrium of a triangular block of material of unit depth (Fig. 13.8); this time AC is a principal plane on which a principal stress σ_p acts, and the shear stress is zero (from the property of principal planes).

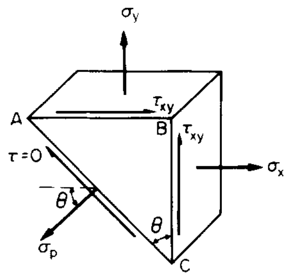

Fig. 13.8.

Resolving forces horizontally,

$$(\sigma_x \times BC \times 1) + (\tau_{xy} \times AB \times 1) = (\sigma_p \times AC \times 1)\cos\theta$$

$$\sigma_x + \tau_{xy}\tan\theta = \sigma_p$$

$$\therefore \qquad \mathbf{\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}} \qquad (13.14)$$

Thus we have an equation for the inclination of the principal planes *in terms of the principal stress*. If, therefore, the principal stresses are determined and substituted in the above equation, each will give the corresponding angle of the plane on which it acts and there can then be no confusion.

The above formula has been derived with two tensile direct stresses and a shear stress system, as shown in the figure; should any of these be reversed in action, then the appropriate minus sign must be inserted in the equation.

13.6. Graphical solution – Mohr's stress circle

Consider the complex stress system of Fig. 13.5 (p. 330). As stated previously this represents a complete stress system for any condition of applied load in two dimensions.

In order to find graphically the direct stress σ_θ and shear stress τ_θ on any plane inclined at θ to the plane on which σ_x acts, proceed as follows:

(1) Label the block $ABCD$.
(2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate) (Fig. 13.9).
(3) Plot the stresses acting on two *adjacent* faces, e.g. AB and BC, using the following sign conventions:

direct stresses: tensile, positive; compressive, negative;
shear stresses: tending to turn block clockwise, positive; tending to turn block counterclockwise, negative.

This gives two points on the graph which may then be labelled $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

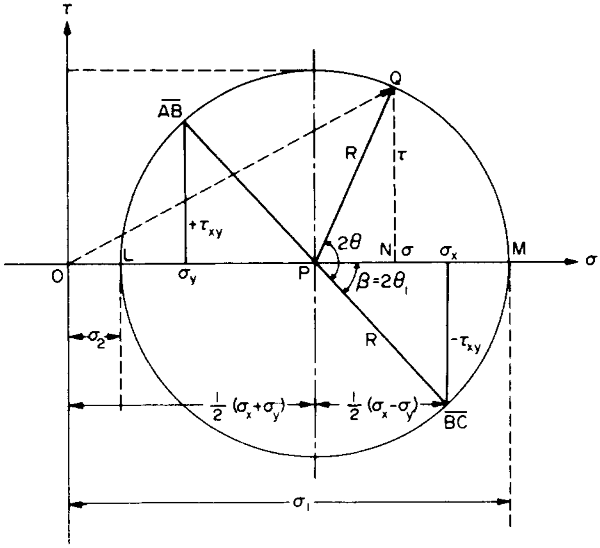

Fig. 13.9. Mohr's stress circle.

(4) Join $\overline{AB}$ and $\overline{BC}$.

(5) The point P where this line cuts the σ axis is then the centre of Mohr's circle, and the line is the diameter; therefore the circle can now be drawn.

Every point on the circumference of the circle then represents a state of stress on some plane through C.

Proof

Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with $\overline{BC}$, and drop a perpendicular from Q to meet the σ axis at N.

Coordinates of Q:

$$ON = OP + PN = \tfrac{1}{2}(\sigma_x + \sigma_y) + R\cos(2\theta - \beta)$$

$$= \tfrac{1}{2}(\sigma_x + \sigma_y) + R\cos 2\theta\cos\beta + R\sin 2\theta\sin\beta$$

But $$R\cos\beta = \tfrac{1}{2}(\sigma_x - \sigma_y) \quad \text{and} \quad R\sin\beta = \tau_{xy}$$

$\therefore$ $$ON = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

On inspection this is seen to be eqn. (13.8) for the direct stress σ_θ on the plane inclined at θ to BC in Fig. 13.5.

Similarly,

$$QN = R\sin(2\theta - \beta)$$

$$= R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$$

$$= \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Again, on inspection this is seen to be eqn. (13.9) for the shear stress τ_θ on the plane inclined at θ to BC.

Thus the coordinates of Q are the normal and shear stresses on a plane inclined at θ to BC in the original stress system.

N.B. – Single angle $\overline{BCPQ}$ is 2θ on Mohr's circle and not θ, it is evident that *angles are doubled on Mohr's circle.* This is the only difference, however, as they are measured in the same direction and from the same plane in both figures (in this case counterclockwise from $\overline{BC}$).

Further points to note are:

(1) The direct stress is a maximum when Q is at M, i.e. OM is the length representing the maximum principal stress σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from BC. Similarly, OL is the other principal stress.

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; *therefore the centre of the circle will always lie on the σ axis midway between σ_x and σ_y.*

(3) From the above point the direct stress on the plane of maximum shear must be midway between σ_x and σ_y, i.e. $\frac{1}{2}(\sigma_x + \sigma_y)$.

(4) The shear stress on the principal planes is zero.

(5) Since the resultant of two stresses at 90° can be found from the parallelogram of vectors as the diagonal, as shown in Fig. 13.10, the resultant stress on the plane at θ to BC is given by OQ on Mohr's circle.

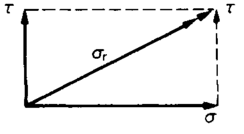

Fig. 13.10. Resultant stress (σ_r) on any plane.

The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction. It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

With the growing availability and power of programmable calculators and microcomputers it may be that the practical use of Mohr's circle for the analytical determination of stress (and strain – see Chapter 14) values will become limited. It will remain, however, a highly effective medium for the teaching and understanding of complex stress systems.

A free-hand sketch of the Mohr circle construction, for example, provides a convenient mechanism for the derivation (by simple geometric relationships) of the principal stress equations (13.11) or of the equations for the shear and normal stresses on any inclined plane in terms of the principal stresses as shown in Fig. 13.11.

13.7. Alternative representations of stress distributions at a point

The way in which the stress at a point varies with the angle at which a plane is taken through the point may be better understood with the aid of the following alternative graphical representations.

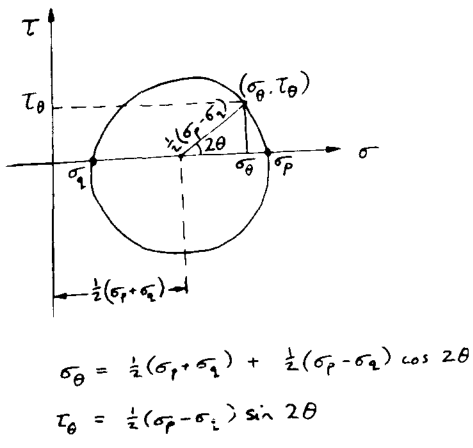

Fig. 13.11. Free-hand sketch of Mohr's stress circle.

Equations (13.8) and (13.9) give the values of the direct stress σ_θ and shear stress τ_θ on any plane inclined at an angle θ to the plane on which the direct stress σ_x acts within a two-dimensional complex stress system, viz:

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

(*a*) *Uniaxial stresses*

For the special case of a single uniaxial stress σ_x as in simple tension or on the surface of a beam in bending, $\sigma_y = \tau_{xy} = 0$ and the equations (13.8) and (13.9) reduce to

$$\sigma_\theta = \tfrac{1}{2}\sigma_x(1 + \cos 2\theta) = \sigma_x \cos^2\theta.$$

N.B. If the single stress were selected as σ_y then the relationship would have reduced to that of eqn. (13.1), i.e. $\sigma_\theta = \sigma_y \sin^2\theta.$

Similarly: $\tau_\theta = \tfrac{1}{2}\sigma_x \sin 2\theta.$

Plotting these equations on simple Cartesian axes produces the stress distribution diagrams of Fig. 13.12, both sinusoidal in shape with shear stress "shifted" by 45° from the normal stress.

Principal stresses σ_p and σ_q occur, as expected, at 90° intervals and the amplitude of the normal stress curve is given by the difference between the principal stress values. It should also be noted that shear stress is proportional to the derivative of the normal stress with respect to θ, i.e. τ_θ is a maximum where $d\sigma_\theta/d\theta$ is a maximum and τ_θ is zero where $d\sigma_\theta/d\theta$ is zero, etc.

Alternatively, plotting the same equations on polar graph paper, as in Fig. 13.13, gives an even more readily understood pictorial representation of the stress distributions showing a peak value of direct stress in the direction of application of the applied stress σ_x falling to zero

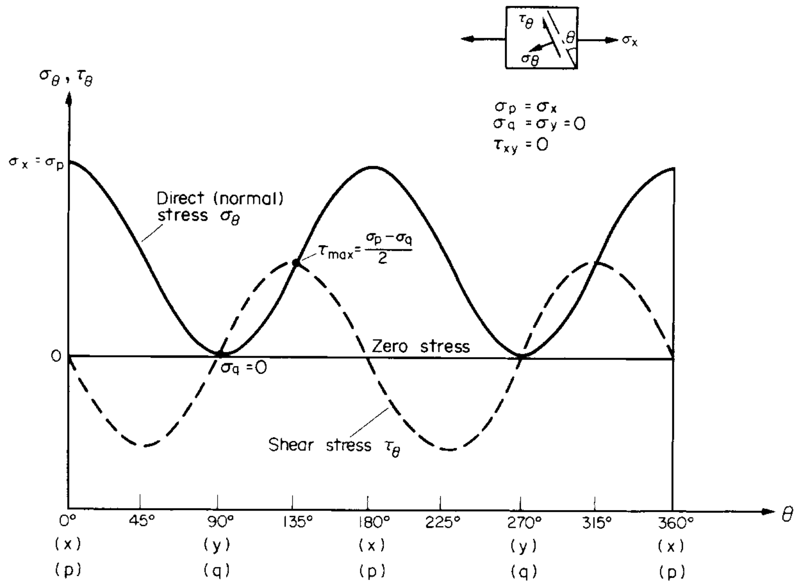

Fig. 13.12. Cartesian plot of stress distribution at a point under uniaxial applied stress.

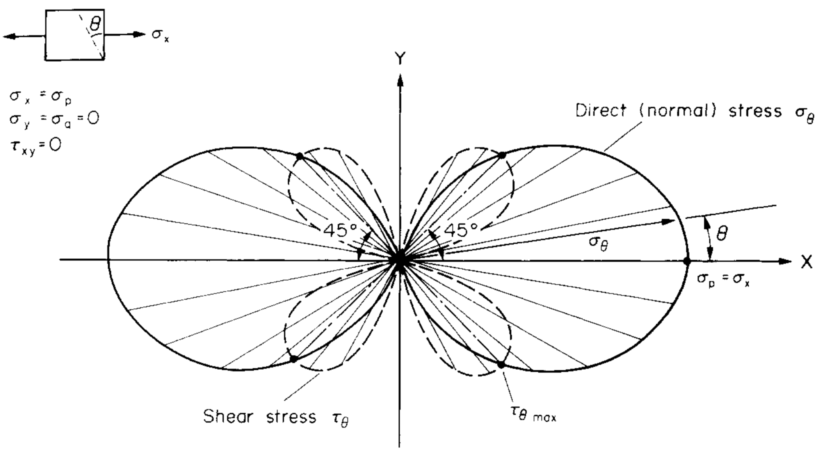

Fig. 13.13. Polar plot of stress distribution at a point under uniaxial applied stress.

in directions at right angles and maximum shearing stresses on planes at 45° with zero shear on the x and y (principal) axes.

(*b*) *Biaxial stresses*

In almost all modes of loading on structural members or engineering components the stresses produced are a maximum at the free (outside) surface. This is particularly evident for the cases of pure bending or torsion as shown by the stress diagrams of Figs. 4.4 and 8.4, respectively, but is also true for other more complex combined loading situations with the major exception of direct bearing loads where maximum stress conditions can be sub-surface. Additionally, at free surfaces the stress normal to the surface is always zero so that the most severe stress condition often reduces, at worst, to a two-dimensional plane stress system within the surface of the component. It should be evident, therefore that the biaxial stress system is of considerable importance to practical design considerations.

The Cartesian plot of a typical bi-axial stress state is shown in Fig. 13.14 whilst Fig. 13.15 shows the polar plot of stresses resulting from the bi-axial stress system present on the surface of a thin cylindrical pressure vessel for which $\sigma_p = \sigma_H$ and $\sigma_q = \sigma_L = \frac{1}{2}\sigma_H$ with $\tau_{xy} = 0$.

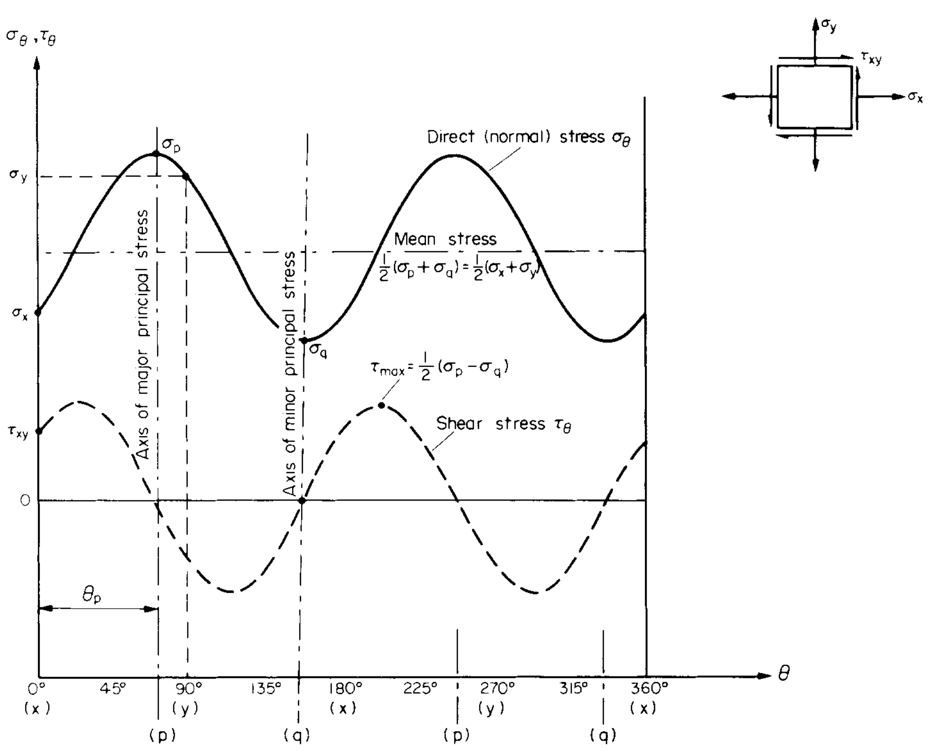

Fig. 13.14. Cartesian plot of stress distribution at a point under a typical biaxial applied stress system.

It should be noted that the whole of the information conveyed on these alternative representations is also available from the relevant Mohr circle which, additionally, is more amenable to quantitative analysis. They do not, therefore, replace Mohr's circle but are included merely to provide alternative pictorial representations which may aid a clearer understanding of the general problem of stress distribution at a point. The equivalent diagrams for strain are given in §14.16.

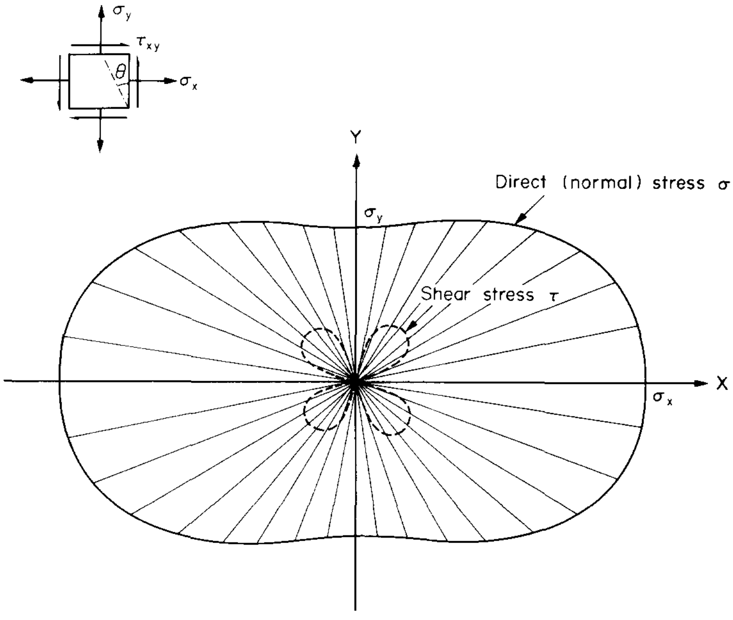

Fig. 13.15. Polar plot of stress distribution under typical biaxial applied stress system.

13.8. Three-dimensional stresses – graphical representation

Figure 13.16 shows the general *three-dimensional* state of stress at any point in a body, i.e. the body will be subjected to three mutually perpendicular direct stresses and three shear stresses.

Figure 13.17 shows a *principal element* at the same point, i.e. one in general rotated relative to the first until the stresses on the faces are principal stresses with no associated shear.

Figure 13.18 then represents true views on the various faces of the principal element, and for each two-dimensional stress condition so obtained a Mohr circle may be drawn. These

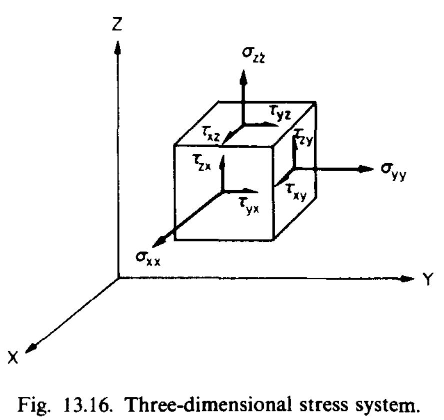

Fig. 13.16. Three-dimensional stress system.

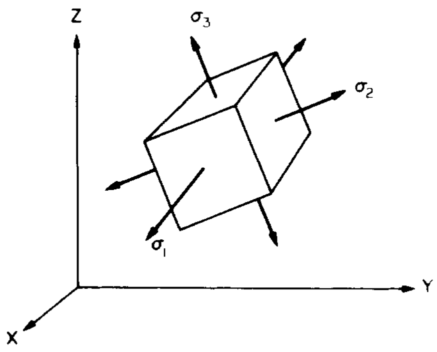

Fig. 13.17. Principal element.

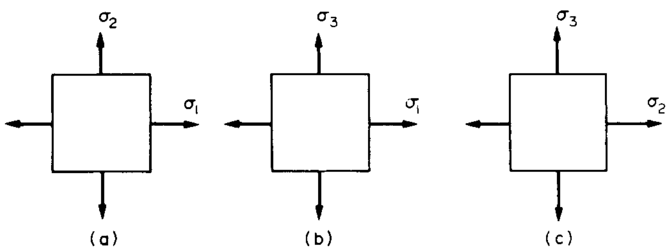

Fig. 13.18. True views on the various faces of the principal element.

can then be combined to produce the complete three-dimensional Mohr circle representation shown in Fig. 13.19.

The large circle between points σ_1 and σ_3 represents stresses on all planes through the point in question containing the σ_2 axis. Likewise the small circle between σ_2 and σ_3 represents

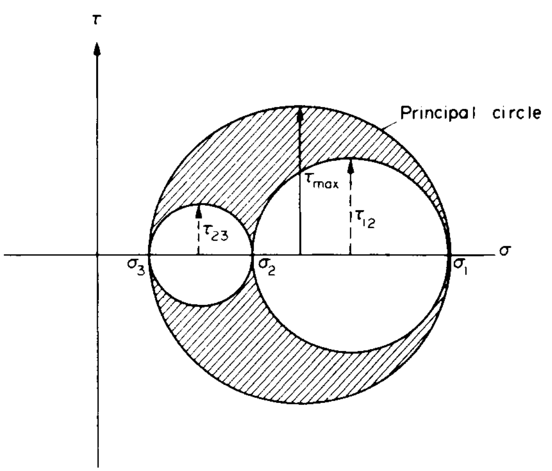

Fig. 13.19. Mohr circle representation of three-dimensional stress state showing the principal circle, the radius of which is equal to the greatest shear stress present in the system.

stresses on all planes containing the σ_1 axis and the circle between σ_1 and σ_2 all planes containing the σ_3 axis.

There are, of course, an infinite number of planes passing through the point which do not contain any of the three principal axes, but it can be shown that all such planes are represented by the shaded area between the circles. The procedure involved in the location of a particular point in the shaded area which corresponds to any given plane is covered in *Mechanics of Materials 2*.† In practice, however, it is often the maximum direct and shear stresses which will govern the elastic failure of materials. These are determined from the larger of the three circles which is thus termed the *principal circle* (τ_{max} = radius).

It is perhaps evident now that in many two-dimensional cases the maximum (greatest) shear stress value will be missed by not considering $\sigma_3 = 0$ and constructing the principal circle.

Consider the stress state shown in Fig. 13.20(a). If the principal stresses σ_1, σ_2 and σ_3 all have non-zero values the system will be termed "three-dimensional"; if one of the principal stresses is zero the system is said to be "two-dimensional" and with two principal stresses zero a "uniaxial" stress condition is obtained. In all cases, however, it is necessary to consider all three principal stress values in the determination of the maximum shear stress since out-of-plane shear stresses will be dependent on all three values and one will be a maximum – see Fig. 13.20(b), (c) and (d).

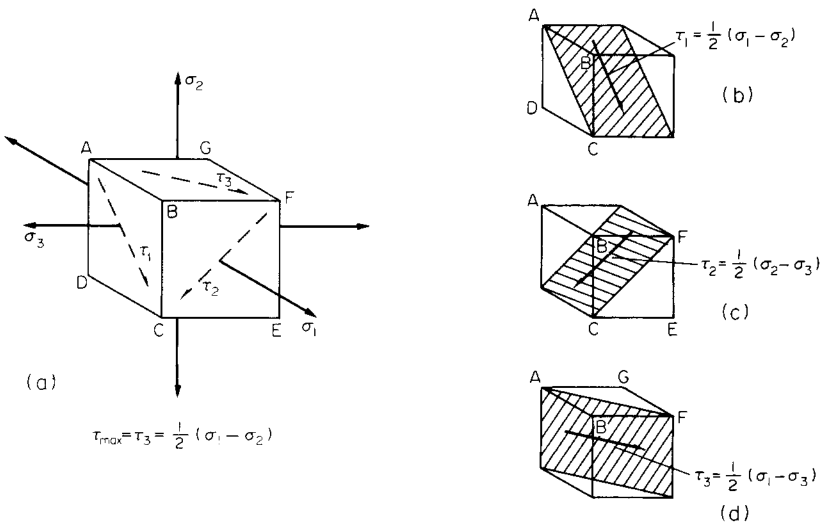

Fig. 13.20. Maximum shear stresses in a three-dimensional stress system.

Examples of the crucial effect of consideration of the third (zero) principal stress value in apparently "two-dimensional" stress states are given below:

(a) Thin cylinder.

An element in the surface of a thin cylinder subjected to internal pressure p will have principal stresses:

$$\sigma_1 = \sigma_H = pd/2t$$

$$\sigma_2 = \sigma_L = pd/4t$$

†E. J. Hearn, *Mechanics of Materials 2*, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

with the third, radial, stress σ_r assumed to be zero – see Fig. 13.21(a).

A two-dimensional Mohr circle representation of the stresses in the element will give Fig. 13.21(b) with a maximum shear stress:

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2)$$
$$= \tfrac{1}{2}\left(\frac{pd}{2t} - \frac{pd}{4t}\right) = \frac{pd}{8t}$$

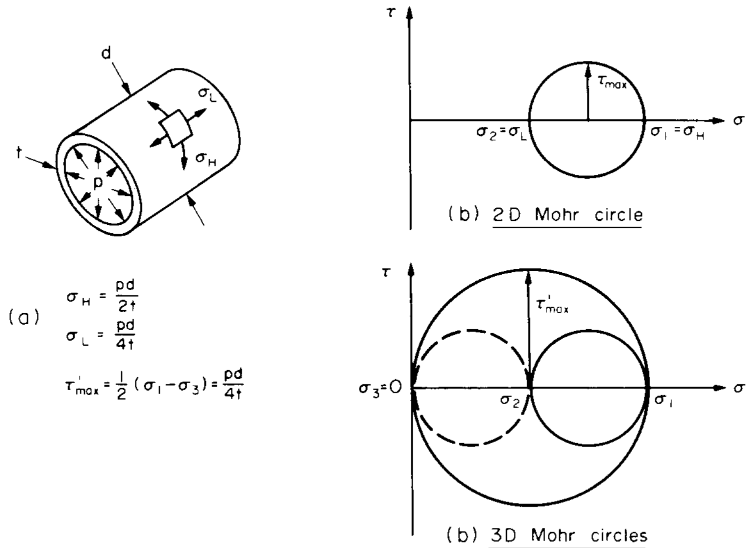

Fig. 13.21. Maximum shear stresses in a pressurised thin cylinder.

A three-dimensional Mohr circle construction, however, is shown in Fig. 13.21(c), the zero value of σ_3 producing a much larger principal circle and a maximum shear stress:

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}\left(\frac{pd}{2t} - 0\right) = \frac{pd}{4t}$$

i.e. **twice** the value obtained from the two-dimensional circle.

(b) Sphere

Consider now an element in the surface of a sphere subjected to internal pressure p as shown in Fig. 13.22(a). Principal stresses on the element will then be $\sigma_1 = \sigma_2 = \dfrac{pd}{4t}$ with $\sigma_r = \sigma_3 = 0$ normal to the surface.

The two-dimensional Mohr circle is shown in Fig. 13.22(b), in this case reducing to a point since σ_1 and σ_2 are equal. The maximum shear stress, which always equals the radius of Mohr's, circle is thus zero and would seem to imply that, although the material of the vessel may well be ductile and susceptible to shear failure, no shear failure could ensue. However,

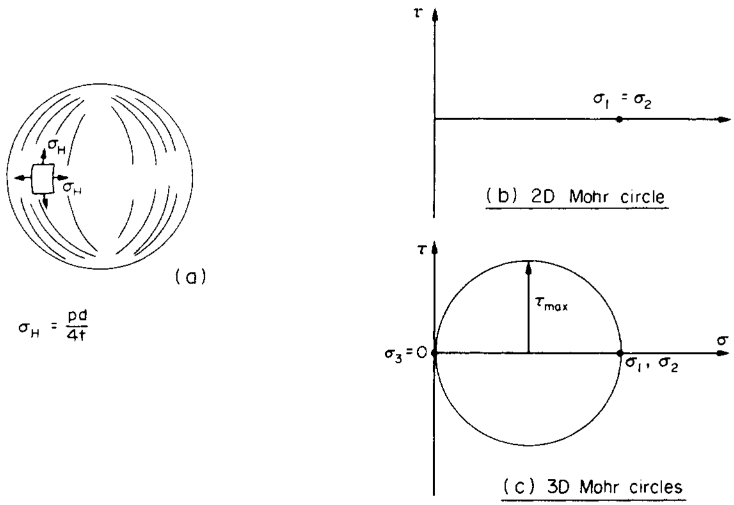

Fig. 13.22. Maximum shear stresses in a pressurised thin sphere.

this is far from the truth as will be evident when the full three-dimensional representation is drawn as in Fig. 13.22(c) with the third, zero, principal stress taken into account.

A maximum shear stress is now produced within the $\sigma_1\sigma_3$ plane of value:

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = \boldsymbol{pd/8t}$$

The greatest value of τ can be obtained *analytically* by using the statement

$$\tau_{max} = \tfrac{1}{2}(\text{greatest principal stress} - \text{least principal stress})$$

and considering separately the principal stress conditions as illustrated in Fig. 13.18.

Examples

Example 13.1 (A)

A circular bar 40 mm diameter carries an axial tensile load of 100 kN. What is the value of the shear stress on the planes on which the normal stress has a value of 50 MN/m² tensile?

Solution

Tensile stress $\qquad \sigma_y = \dfrac{F}{A} = \dfrac{100 \times 10^3}{\pi \times (0.02)^2} = 79.6\ \text{MN/m}^2$

Now the normal stress on an oblique plane is given by eqn. (13.1):

$$\sigma_\theta = \sigma_y \sin^2\theta$$

$$50 \times 10^6 = 79.6 \times 10^6 \sin^2\theta$$

$$\theta = 52^\circ\,28'$$

The shear stress on the oblique plane is then given by eqn. (13.2):

$$\begin{aligned}\tau_\theta &= \tfrac{1}{2}\sigma_y \sin 2\theta \\ &= \tfrac{1}{2} \times 79.6 \times 10^6 \times \sin 104^\circ\, 56' \\ &= 38.6 \times 10^6\end{aligned}$$

The required shear stress is 38.6 MN/m².

Example 13.2 (A/B)

Under certain loading conditions the stresses in the walls of a cylinder are as follows:

(a) 80 MN/m² tensile;
(b) 30 MN/m² tensile at right angles to (a);
(c) shear stresses of 60 MN/m² on the planes on which the stresses (a) and (b) act; the shear couple acting on planes carrying the 30 MN/m² stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged?

Solution

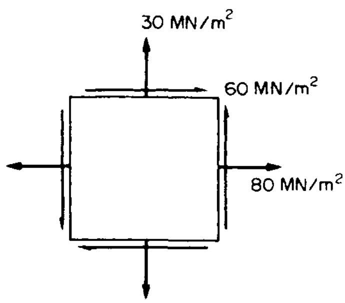

Fig. 13.23.

The principal stresses are given by the formula

$$\begin{aligned}\sigma_1 \text{ and } \sigma_2 &= \tfrac{1}{2}(\sigma_x+\sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]} \\ &= \tfrac{1}{2}(80+30) \pm \tfrac{1}{2}\sqrt{[(80-30)^2+(4\times 60^2)]} \\ &= 55 \pm 5\sqrt{(25+144)} \\ &= 55 \pm 65\end{aligned}$$

$\therefore$ $\quad \sigma_1 = \mathbf{120\ MN/m^2}$

and $\quad \sigma_2 = \mathbf{-10\ MN/m^2}$ (i.e. compressive)

The planes on which these stresses act can be determined from eqn. (13.14),

i.e. $$\tan\theta_1 = \frac{\sigma_p - \sigma_x}{\tau_{xy}}$$

$\therefore$ $$\tan\theta_1 = \frac{120-80}{60} = 0.6667$$

$\therefore$ $$\theta_1 = \mathbf{33^\circ\,41'}$$

Also $$\tan\theta_2 = = \frac{-10-80}{60} = 1.50$$

$\therefore$ $$\theta_2 = \mathbf{-56^\circ\,19'} \quad \textbf{or} \quad \mathbf{123^\circ\,41'}$$

N.B. – The resulting angles are at 90° to each other as expected.

If the loading is now changed so that the 80 MN/m² stress becomes compressive:

$$\sigma_1 = \tfrac{1}{2}(-80+30) + \tfrac{1}{2}\sqrt{[(-80-30)^2 + (4\times 60^2)]}$$
$$= -25 + 5\sqrt{(121+144)}$$
$$= -25 + 81.5 = \mathbf{56.5\ MN/m^2}$$

and $$\sigma_2 = -25 - 81.5 = \mathbf{-106.5\ MN/m^2}$$

Then $$\tan\theta_1 = \frac{56.5-(-80)}{60} = 2.28$$

$\therefore$ $$\theta_1 = \mathbf{66^\circ\,19'}$$

and $$\theta_2 = 66^\circ\,19' + 90 = \mathbf{156^\circ\,19'}$$

Mohr's circle solutions

In the first part of the question the stress system and associated Mohr's circle are as drawn in Fig. 13.24.

By measurement: $\sigma_1 = \mathbf{120\ MN/m^2}$ **tensile**

$\sigma_2 = \mathbf{10\ MN/m^2}$ **compressive**

and $\theta_1 = \mathbf{34^\circ}$ **counterclockwise from *BC***

$\theta_2 = \mathbf{124^\circ}$ **counterclockwise from *BC***

When the 80 MN/m² stress is reversed, the stress system is that in Fig. 13.25, giving Mohr's circle as drawn.

The required values are then:

$\sigma_1 = \mathbf{56.5\ MN/m^2}$ **tensile**

$\sigma_2 = \mathbf{106.5\ MN/m^2}$ **compressive**

$\theta_1 = \mathbf{66^\circ\,15'}$ **counterclockwise to *BC***

and $\theta_2 = \mathbf{156^\circ\,15'}$ **counterclockwise to *BC***

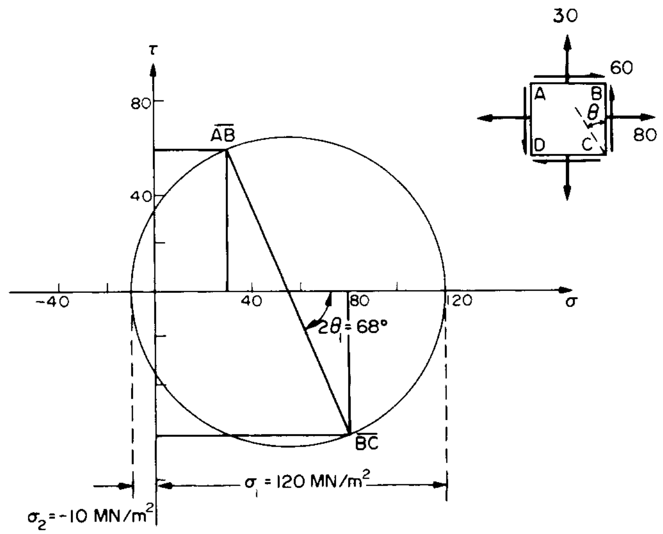

Fig. 13.24.

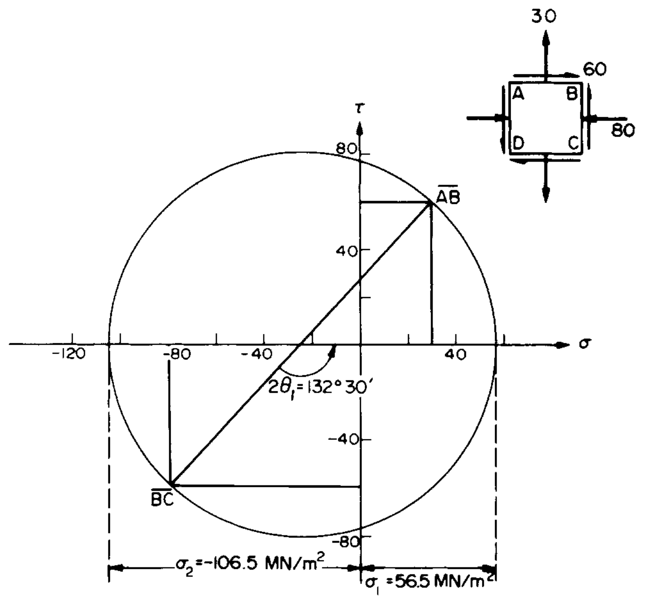

Fig. 13.25.

Example 13.3 (B)

A material is subjected to two mutually perpendicular direct stresses of 80 MN/m^2 tensile and 50 MN/m^2 compressive, together with a shear stress of 30 MN/m^2. The shear couple acting on planes carrying the 80 MN/m^2 stress is clockwise in effect. Calculate

(a) the magnitude and nature of the principal stresses;
(b) the magnitude of the maximum shear stresses in the plane of the given stress system;
(c) the direction of the planes on which these stresses act.

Confirm your answer by means of a Mohr's stress circle diagram, and from the diagram determine the magnitude of the normal stress on a plane inclined at 20° counterclockwise to the plane on which the 50 MN/m² stress acts.

Solution

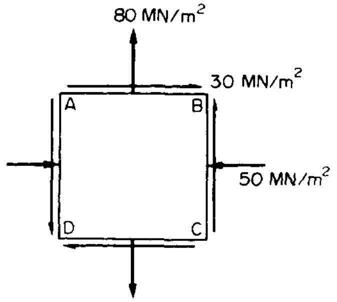

Fig. 13.26.

(a) To find the principal stresses:

$$\sigma_1 \quad \text{and} \quad \sigma_2 = \tfrac{1}{2}(\sigma_x+\sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]}$$
$$= \tfrac{1}{2}(-50+80) \pm \tfrac{1}{2}\sqrt{[(-50-80)^2+(4\times 900)]}$$
$$= 5[3 \pm \sqrt{(169+36)}] = 5[3 \pm 14.31]$$

$$\therefore \qquad \sigma_1 = 86.55\ \text{MN/m}^2$$
$$\sigma_2 = -56.55\ \text{MN/m}^2$$

The principal stresses are

86.55 MN/m² tensile and **56.55 MN/m² compressive**

(b) To find the maximum shear stress:

$$\tau_{max} = \frac{\sigma_1-\sigma_2}{2} = \frac{86.55-(-56.55)}{2} = \frac{143.1}{2} = 71.6\ \text{MN/m}^2$$

Maximum shear stress = **71.6 MN/m²**

(c) To find the directions of the principal planes:

$$\tan\theta_1 = \frac{\sigma_p-\sigma_x}{\tau_{xy}} = \frac{86.55-(-50)}{30}$$
$$= \frac{136.55}{30} = 4.552$$

$$\therefore \qquad \theta_1 = \mathbf{77^\circ\,36'}$$

$$\therefore \qquad \theta_2 = 77^\circ\,36' + 90^\circ = \mathbf{167^\circ\,36'}$$

The principal planes are inclined at 77° 36′ to the plane on which the 50 MN/m^2 stress acts. The maximum shear planes are at 45° to the principal planes.

Mohr's circle solution

The stress system shown in Fig. 13.26 gives the Mohr's circle in Fig. 13.27.

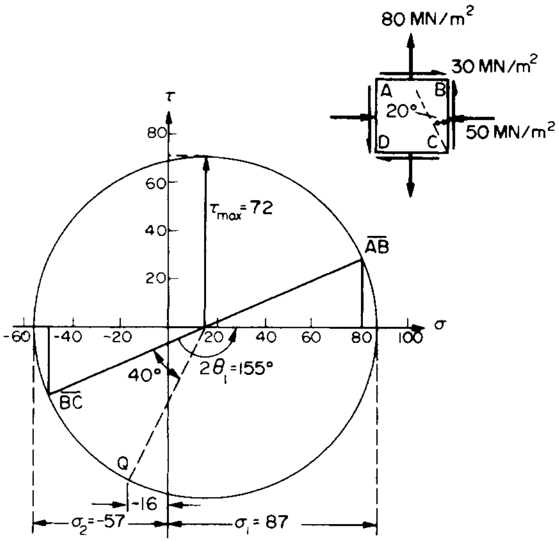

Fig. 13.27.

By measurement

$$\sigma_1 = \mathbf{87\,MN/m^2\ tensile}$$

$$\sigma_2 = \mathbf{57\,MN/m^2\ compressive}$$

$$\tau_{max} = \mathbf{72\,MN/m^2}$$

and $$\theta_1 = \frac{155^\circ}{2} = \mathbf{77^\circ\,30'}$$

The direct or normal stress on a plane inclined at 20° counterclockwise to *BC* is obtained by measuring from $\overline{BC}$ on the Mohr's circle through 2 × 20° = 40° in the same direction.

This gives $$\sigma = \mathbf{16\,MN/m^2\ compressive}$$

Example 13.4 (B)

At a given section a shaft is subjected to a bending stress of 20 MN/m^2 and a shear stress of 40 MN/m^2. Determine:

(a) the principal stresses;
(b) the directions of the principal planes;
(c) the maximum shear stress and the planes on which this acts;
(d) the tensile stress which, acting alone, would produce the same maximum shear stress;
(e) the shear stress which, acting alone, would produce the same maximum tensile principal stress.

Solution

(a) The bending stress is a direct stress and can be treated as acting on the x axis, so that $\sigma_x = 20\,MN/m^2$; since no other direct stresses are given, $\sigma_y = 0$.

Principal stress
$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$
$$= \tfrac{1}{2} \times 20 + \tfrac{1}{2}\sqrt{[20^2 + (4 \times 40^2)]}$$
$$= 10 + 5\sqrt{(68)} = 10 + 5 \times 8.246$$
$$= \mathbf{51.23\,MN/m^2}$$

and
$$\sigma_2 = 10 - 41.23$$
$$= \mathbf{-31.23\,MN/m^2}$$

(b) Then
$$\tan\theta_1 = \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{51.23 - 20}{40} = \frac{31.23}{40} = 0.7808$$

∴
$$\theta_1 = \mathbf{37^\circ\,59'}$$

and
$$\tan\theta_2 = \frac{-31.23 - 20}{40} = \frac{-51.23}{40} = -1.2808$$

∴
$$\theta_2 = \mathbf{-52^\circ\,1'} \quad \textbf{or} \quad \mathbf{127^\circ\,59'}$$

both angles being measured counterclockwise from the plane on which the 20 MN/m^2 stress acts.

(c) Maximum shear stress

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{51.23 - (-31.23)}{2}$$

$$= \frac{82.46}{2} = \mathbf{41.23\,MN/m^2}$$

This acts on planes at 45° to the principal planes,

i.e. **at 82° 59′ or −7° 1′**

(d) Maximum shear stress
$$\tau_{max} = \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Thus if a tensile stress is to act alone to give the same maximum shear stress ($\sigma_x = 0$ and $\tau_{xy} = 0$):

$$\text{maximum shear stress} = \tfrac{1}{2}\sqrt{(\sigma_x^2)} = \tfrac{1}{2}\sigma_x$$

$$41.23 = \tfrac{1}{2}\sigma_x$$

i.e. $$\sigma_x = \mathbf{82.46\ MN/m^2}$$

The required tensile stress is 82.46 MN/m^2.

(e) Principal stress

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Thus if a shear stress is to act alone to give the same principal stress ($\sigma_x = \sigma_y = 0$):

$$\sigma_1 = \tfrac{1}{2}\sqrt{(4\tau_{xy}^2)} = \tau_{xy}$$

$$51.23 = \tau_{xy}$$

The required shear stress is **51.23 MN/m^2**.

Mohr's circle solutions

(a), (b), (c) The stress system and corresponding Mohr's circle are as shown in Fig. 13.28. By measurement:

(a) $\sigma_1 \simeq$ **51 MN/m^2 tensile**

$\sigma_2 \simeq$ **31 MN/m^2 compressive**

(b) $\theta_1 = \dfrac{76^\circ}{2} = \mathbf{38^\circ}$

$\theta_2 = 38^\circ + 90^\circ = \mathbf{128^\circ}$

(c) $\tau_{max} \simeq$ **41 MN/m^2**

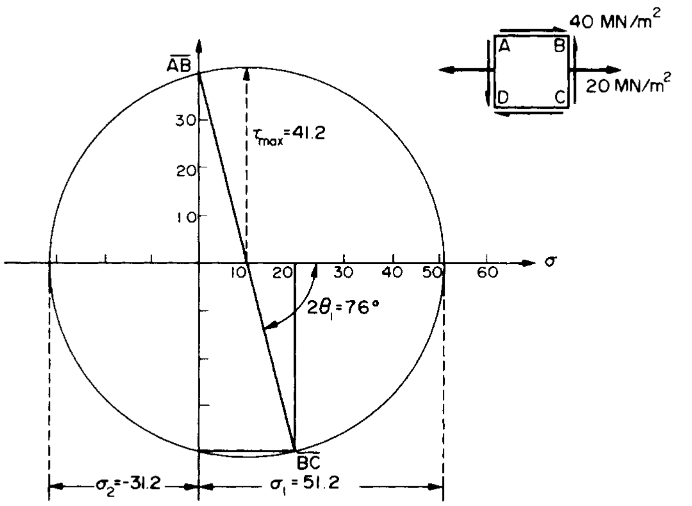

Fig. 13.28.

Angle of maximum shear plane

$$= \frac{166}{2} = \mathbf{83^\circ}$$

(d) If a tensile stress σ_x is to act alone to give the same maximum shear stress, then $\sigma_y = 0$, $\tau_{xy} = 0$ and $\tau_{max} = 41\ \text{MN/m}^2$. The Mohr's circle therefore has a radius of $41\ \text{MN/m}^2$ and passes through the origin (Fig. 13.29).

Hence the required tensile stress is $\mathbf{82\ MN/m^2}$.

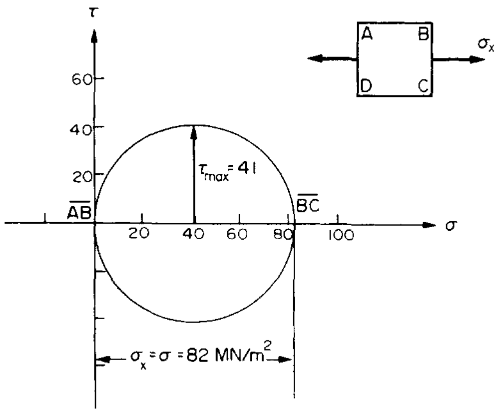

Fig. 13.29.

(e) If a shear stress is to act alone to produce the same principal stress, $\sigma_x = 0$, $\sigma_y = 0$ and $\sigma_1 = 51\ \text{MN/m}^2$. The Mohr's circle thus has its centre at the origin and passes through $\sigma = 51\ \text{MN/m}^2$ (Fig. 13.30).

Hence the required shear stress is $\mathbf{51\ MN/m^2}$.

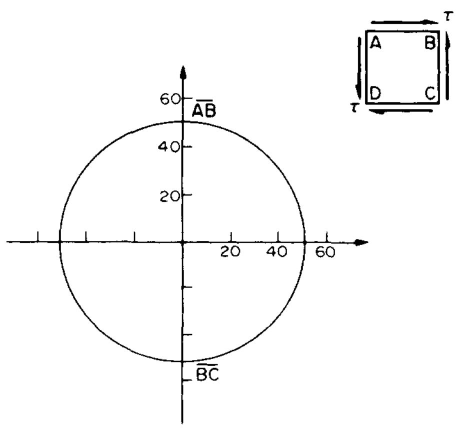

Fig. 13.30.

Example 13.5 (B)

At a point in a piece of elastic material direct stresses of 90 MN/m^2 tensile and 50 MN/m^2 compressive are applied on mutually perpendicular planes. The planes are also subjected to a shear stress. If the greater principal stress is limited to 100 MN/m^2 tensile, determine:

(a) the value of the shear stress;
(b) the other principal stress;
(c) the normal stress on the plane of maximum shear;
(d) the maximum shear stress.

Make a neat sketch showing clearly the positions of the principal planes and planes of maximum shear stress with respect to the planes of the applied stresses.

Solution

(a) Principal stress $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$

This is limited to 100 MN/m^2; therefore shear stress τ_{xy} is given by

$$100 = \tfrac{1}{2}(90 - 50) + \tfrac{1}{2}\sqrt{[(90 + 50)^2 + 4\tau_{xy}^2]}$$

$$\therefore \quad 200 = 40 + 10\sqrt{[14^2 + 0.04\tau_{xy}^2]}$$

$$\therefore \quad \tau_{xy} = \sqrt{\left(\frac{16^2 - 14^2}{0.04}\right)} = \sqrt{\left(\frac{256 - 196}{0.04}\right)} = \frac{\sqrt{60}}{0.2}$$

$$= \mathbf{38.8\ MN/m^2}$$

The required shear stress is 38.8 MN/m^2.

(b) The other principal stress σ_2 is given by

$$\sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

$$= \tfrac{1}{2}[(90 - 50) - 10\sqrt{(14^2 + 60)}] = \frac{40 - 10\sqrt{(256)}}{2}$$

$$= \frac{40 - 160}{2} = \mathbf{-60\ MN/m^2}$$

The other principal stress is 60 MN/m^2 compressive.

(c) The normal stress on the plane of maximum shear

$$= \frac{\sigma_1 + \sigma_2}{2} = \frac{100 - 60}{2}$$

$$= \mathbf{20\ MN/m^2}$$

The required normal stress is 20 MN/m^2 tensile.

(d) The maximum shear stress is given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100 + 60}{2}$$

$$= \mathbf{80\ MN/m^2}$$

The maximum shear stress is 80 MN/m^2.

In order to be able to draw the required sketch (Fig. 13.31) to indicate the relative positions of the planes on which the above stresses act, the angles of the principal planes are required. These are given by

$$\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{100-(-50)}{38.8}$$

$$= \frac{150}{38.8} = 3.87$$

$$\therefore \qquad \theta_1 = \mathbf{75^\circ\, 30'}$$

to the plane on which the $50\,\text{MN/m}^2$ stress acts.

The required sketch is then shown in Fig. 13.31.

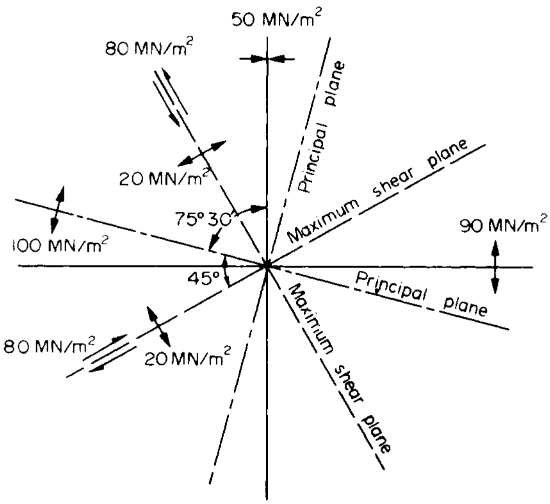

Fig. 13.31. Summary of principal planes and maximum shear planes.

Mohr's circle solution

The stress system is as shown in Fig. 13.32. The centre of the Mohr's circle is positioned midway between the two direct stresses given, and the radius is such that $\sigma_1 = 100\,\text{MN/m}^2$.

By measurement:

$$\tau = \mathbf{39\,MN/m^2}$$

$$\sigma_2 = \mathbf{60\,MN/m^2\ compressive}$$

$$\tau_{max} = \mathbf{80\,MN/m^2}$$

$$\theta_1 = \frac{\mathbf{151}}{\mathbf{2}} = \mathbf{75^\circ\, 30'\ to\ } \boldsymbol{BC}\mathbf{,\ the\ plane\ on\ which\ the\ 50\,MN/m^2\ stress\ acts}$$

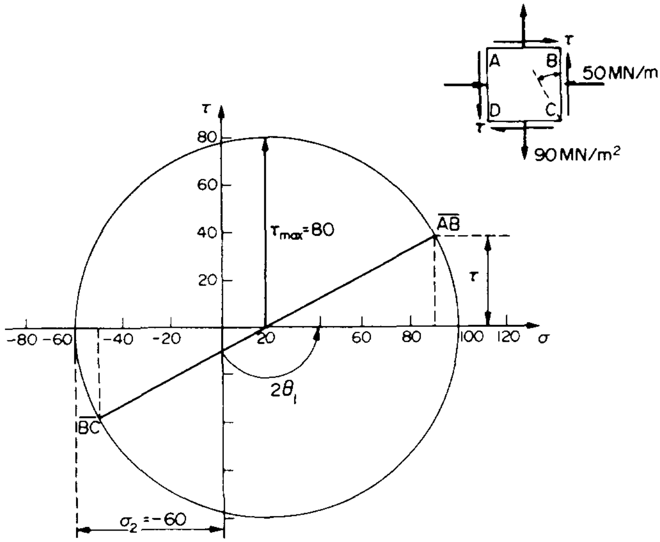

Fig. 13.32.

Example 13.6 (B)

In a certain material under load a plane *AB* carries a tensile direct stress of 30 MN/m^2 and a shear stress of 20 MN/m^2, while another plane *BC* carries a tensile direct stress of 20 MN/m^2 and a shear stress. If the planes are inclined to one another at 30° and plane *AC* at right angles to plane *AB* carries a direct stress unknown in magnitude and nature, find:

(a) the value of the shear stress on *BC*;
(b) the magnitude and nature of the direct stress on *AC*;
(c) the principal stresses.

Solution

Referring to Fig. 13.33 let the shear stress on *BC* be τ and the direct stress on *AC* be σ_x, assumed tensile. Consider the equilibrium of the elemental wedge *ABC*. Assume this wedge to be of unit depth. A complementary shear stress equal to that on *AB* will be set up on *AC*.

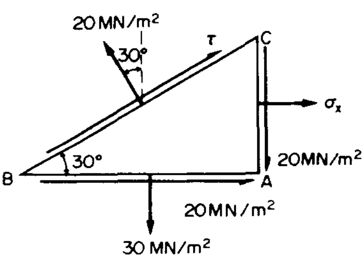

Fig. 13.33.

(a) To find τ, resolve forces vertically:

$$30 \times (AB \times 1) + 20 \times (AC \times 1) = 20 \times (BC \times 1)\cos 30^\circ + \tau \times (BC \times 1)\sin 30^\circ$$

Now $$AB = BC\cos 30 \quad \text{and} \quad AC = BC\sin 30$$

$$\therefore \quad 30 \times BC\cos 30 + 20 \times BC\sin 30 = 20 \times BC\cos 30 + \tau \times BC\sin 30$$

$$30\frac{\sqrt{3}}{2} + 20 \times \frac{1}{2} = 20 \times \frac{\sqrt{3}}{2} + \tau \times \frac{1}{2}$$

$$30\sqrt{3} + 20 = 20\sqrt{3} + \tau$$

$$\therefore \quad \tau = 10\sqrt{3} + 20 = \mathbf{37.32\ MN/m^2}$$

The required shear stress is 37.32 MN/m^2.

(b) To find σ_x, resolve forces horizontally:

$$20 \times (AB \times 1) + \sigma_x \times (AC \times 1) + \tau \times (BC \times 1)\cos 30^\circ = 20 \times (BC \times 1)\sin 30^\circ$$

$$20 \times BC\cos 30^\circ + \sigma_x \times BC\sin 30^\circ + \tau \times BC\cos 30^\circ = 20 \times BC\sin 30^\circ$$

$$20 \times \frac{\sqrt{3}}{2} + \sigma_x \times \frac{1}{2} + \tau \times \frac{\sqrt{3}}{2} = 20 \times \frac{1}{2}$$

$$20\sqrt{3} + \sigma_x + \sqrt{3} \times 37.32 = 10$$

$$\therefore \quad \sigma_x = 10 - \sqrt{3} \times 57.32 = 10 - 99.2$$

$$= \mathbf{-89.2\ MN/m^2,\ i.e.\ compressive}$$

(c) The principal stresses are now given by

$$\sigma_{1,2} = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

$$= \tfrac{1}{2}\{(-89.2 + 30) \pm \sqrt{[(-89.2 - 30)^2 + 4 \times 20^2]}\}$$

$$= 5\{-5.92 \pm \sqrt{[(-11.92)^2 + 16]}\}$$

$$= 5[-5.92 \pm \sqrt{158}] = 5[-5.92 \pm 12.57]$$

$$\therefore \quad \sigma_1 = \mathbf{33.25\ MN/m^2}$$

$$\sigma_2 = \mathbf{-92.45\ MN/m^2}$$

The principal stresses are 33.25 MN/m^2 tensile and 92.45 MN/m^2 compressive.

Example 13.7 (B)

A hollow steel shaft of 100 mm external diameter and 50 mm internal diameter transmits 0.75 MW at 500 rev/min and is also subjected to an axial end thrust of 50 kN. Determine the maximum bending moment which can be safely applied in conjunction with the applied torque and thrust if the maximum compressive principal stress is not to exceed 100 MN/m^2 compressive. What will then be the value of:

(a) the other principal stress;
(b) the maximum shear stress?

Solution

The torque on the shaft may be found from

$$\text{power} = T \times \omega$$

$$\therefore \qquad T = \frac{0.75 \times 10^6 \times 60}{2\pi \times 500} = 14.3 \times 10^3 = 14.3\,\text{kN m}$$

The shear stress in the shaft at the surface is then given by the torsion theory

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{TR}{J} = \frac{14.3 \times 10^3 \times 50 \times 10^{-3} \times 2}{\pi(50^4 - 25^4)10^{-12}}$$

$$= 0.78 \times 10^8$$

$$= 78\,\text{MN/m}^2$$

The direct stress resulting from the end thrust is given by

$$\sigma_d = \frac{\text{load}}{\text{area}} = \frac{-50 \times 10^3}{\pi(50^2 - 25^2)} 10^{-6}$$

$$= -8.5 \times 10^6$$

$$= -8.5\,\text{MN/m}^2$$

The bending moment to be applied will produce a direct stress in the same direction as σ_d. Thus the total stress in the x direction is

$$\sigma_x = \sigma_b + \sigma_d$$

the greatest value of σ_x being obtained where the bending stress is of the same sign as the end thrust or, in other words, compressive. The stress system is therefore as shown in Fig. 13.34.

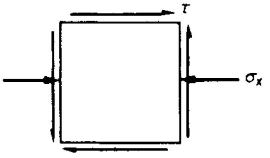

Fig. 13.34.

N.B. $\sigma_y = 0$; there is no stress in the y direction.

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Therefore substituting all stresses in units of MN/m^2,

$$-100 = \tfrac{1}{2}\sigma_x \pm \tfrac{1}{2}\sqrt{(\sigma_x^2 + 4\tau^2)}$$

$$\therefore \qquad -200 - \sigma_x = \pm\sqrt{(\sigma_x^2 + 4\tau^2)}$$

$$\therefore \quad 4 \times 10^4 + 400\sigma_x + \sigma_x^2 = \sigma_x^2 + 4\tau^2$$

$$\therefore \quad 400\sigma_x = 4\tau^2 - 4 \times 10^4$$

$$= 24320 - 40000$$

$$\therefore \quad \sigma_x = -39.2\,\text{MN/m}^2$$

Therefore stress owing to bending

$$\sigma_b = \sigma_x - \sigma_d = -39.2 - (-8.5)$$

$$= -30.7\,\text{MN/m}^2 \quad \text{(i.e. compressive)}$$

But from bending theory

$$\sigma_b = \frac{My}{I}$$

$$\therefore \quad M = \frac{30.7 \times 10^6 \times \pi(50^4 - 25^4)10^{-12}}{50 \times 10^{-3} \times 4}$$

$$= 2830\,\text{N m}$$

$$= 2.83\,\text{kN m}$$

i.e. the bending moment which can be safely applied is **2.83 kN m.**

(a) The other principal stress

$$\sigma_2 = \tfrac{1}{2}\sigma_x + \tfrac{1}{2}\sqrt{(\sigma_x^2 + 4\tau^2)}$$

$$= -19.6 + \tfrac{1}{2}\sqrt{(39.2^2 + 24320)}$$

$$= -19.6 + 80.5$$

$$= \mathbf{60.9\,MN/m^2} \text{ (tensile)}$$

(b) The maximum shear stress is given by

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2)$$

$$= \tfrac{1}{2}(-100 - 60.9)$$

$$= \mathbf{-80.45\,MN/m^2}$$

i.e. the maximum shear stress is $80.45\,\text{MN/m}^2$.

Example 13.8

A beam of symmetrical I-section is simply supported at each end and loaded at the centre of its 3 m span with a concentrated load of 100 kN. The dimensions of the cross-section are: flanges 150 mm wide by 30 mm thick; web 30 mm thick; overall depth 200 mm.

For the transverse section at the point of application of the load, and considering a point at the top of the web where it meets the flange, calculate the magnitude and nature of the principal stresses. Neglect the self-mass of the beam.

Solution

At any section of the beam there will be two sets of stresses acting simultaneously:

(1) bending stresses $$\sigma_b = \frac{My}{I}$$

(2) shear stresses $$\tau = \frac{QA\bar{y}}{Ib}$$

together with their associated complementary shear stresses of the same value (Fig. 13.35a).

The stress system on any element of the beam can therefore be represented as in Fig. 13.36. The stress distribution diagrams are shown in Fig. 13.35b.

Bending stress

$$\sigma_b = \frac{My}{I}$$

$$M = \text{maximum bending moment}$$

$$= \frac{WL}{4} = \frac{100 \times 10^3 \times 3}{4} = 75\,\text{kN m}$$

and $$I = \frac{0.15 \times 0.2^3 - 0.12 \times 0.14^3}{12}\ \text{m}^4$$

$$= 72.56 \times 10^{-6}\,\text{m}^4$$

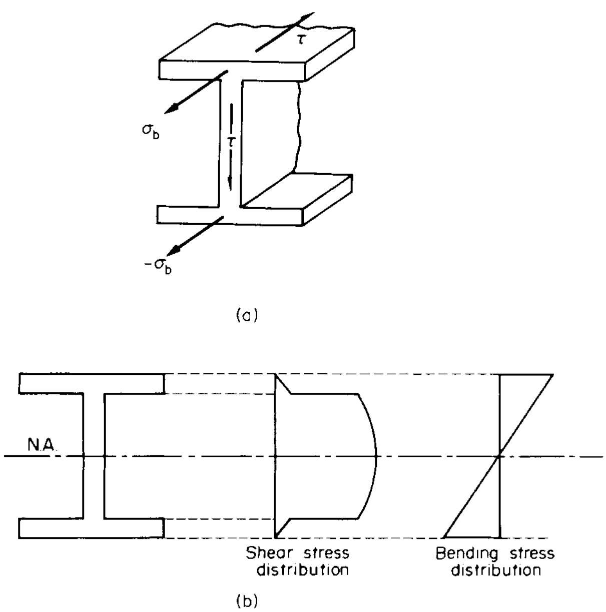

Fig. 13.35.

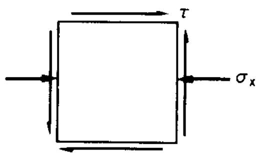

Fig. 13.36.

Therefore at the junction of web and flange

$$\sigma_b = \frac{75 \times 10^3 \times 0.07}{72.56 \times 10^{-6}}$$

$$= 72.35 \times 10^6 = 72.35\,\text{MN/m}^2 \text{ and is compressive}$$

Shear stress

$$\tau = \frac{QA\bar{y}}{Ib}$$

$$= \frac{50 \times 10^3 \times (150 \times 30) \times 85 \times 10^{-9}}{72.56 \times 10^{-4} \times 30 \times 10^{-3}}$$

$$= 8.79\,\text{MN/m}^2$$

The principal stresses are then given by

$$\sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

with $\sigma_x = -\sigma_b$ and $\sigma_y = 0$

$$\therefore \quad \sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(-72.35) \pm \tfrac{1}{2}\sqrt{[(-72.35)^2 + 4 \times 8.79^2]}\,\text{MN/m}^2$$

$$= -36.2 \pm \sqrt{(5544)}$$

$$= -36.2 \pm 74.5$$

$$\therefore \quad \boldsymbol{\sigma_2 = -110.7\,\text{MN/m}^2}$$

$$\boldsymbol{\sigma_1 = +38.3\,\text{MN/m}^2}$$

i.e. the principal stresses are 110.7 MN/m^2 compressive and 38.3 MN/m^2 tensile in the top of the web. At the bottom of the web the stress values obtained would be of the same value but of opposite sign.

Problems

13.1 (A). An axial tensile load of 10 kN is applied to a 12 mm diameter bar. Determine the maximum shearing stress in the bar and the planes on which it acts. Find also the value of the normal stresses on these planes. [44.1 MN/m^2 at 45° and 135°; $\pm 44.2\,\text{MN/m}^2$.]

13.2 (A). A compressive member of a structure is of 25 mm square cross-section and carries a load of 50 kN. Determine, from first principles, the normal, tangential and resultant stresses on a plane inclined at 60° to the axis of the bar. [60, 34.6, 69.3 MN/m^2.]

13.3 (A). A rectangular block of material is subjected to a shear stress of 30 MN/m^2 together with its associated complementary shear stress. Determine the magnitude of the stresses on a plane inclined at 30° to the directions of the applied stresses, which may be taken as horizontal. [26, 15 MN/m^2.]

13.4 (A). A material is subjected to two mutually perpendicular stresses, one 60 MN/m^2 compressive and the other 45 MN/m^2 tensile. Determine the direct, shear and resultant stresses on a plane inclined at 60° to the plane on which the 45 MN/m^2 stress acts. [18.75, 45.5, 49.2 MN/m^2.]

13.5 (A/B). The material of Problem 13.4 is now subjected to an additional shearing stress of 10 MN/m^2. Determine the principal stresses acting on the material and the maximum shear stress. [46, −61, 53.5 MN/m^2.]

13.6 (A/B). At a certain section in a material under stress, direct stresses of 45 MN/m^2 tensile and 75 MN/m^2 tensile act on perpendicular planes together with a shear stress τ acting on these planes. If the maximum stress in the material is limited to 150 MN/m^2 tensile determine the value of τ. [88.7 MN/m^2.]

13.7 (A/B). At a point in a material under stress there is a compressive stress of 200 MN/m^2 and a shear stress of 300 MN/m^2 acting on the same plane. Determine the principal stresses and the directions of the planes on which they act. [216 MN/m^2 at 54.2° to 200 MN/m^2 plane; −416 MN/m^2 at 144.2°.]

13.8 (A/B). At a certain point in a material the following stresses act: a tensile stress of 150 MN/m^2, a compressive stress of 105 MN/m^2 at right angles to the tensile stress and a shear stress clockwise in effect of 30 MN/m^2. Calculate the principal stresses and the directions of the principal planes. [153.5, −108.5 MN/m^2; at 6.7° and 96.7° counterclockwise to 150 MN/m^2 plane.]

13.9 (B). The stresses across two mutually perpendicular planes at a point in an elastic body are 120 MN/m^2 tensile with 45 MN/m^2 clockwise shear, and 30 MN/m^2 tensile with 45 MN/m^2 counterclockwise shear. Find (i) the principal stresses, (ii) the maximum shear stress, and (iii) the normal and tangential stresses on a plane measured at 20° counterclockwise to the plane on which the 30 MN/m^2 stress acts. Draw sketches showing the positions of the stresses found above and the planes on which they act relative to the original stresses. [138.6, 11.4, 63.6, 69.5, −63.4 MN/m^2.]

13.10 (B). At a point in a strained material the stresses acting on planes at right angles to each other are 200 MN/m^2 tensile and 80 MN/m^2 compressive, together with associated shear stresses which may be assumed clockwise in effect on the 80 MN/m^2 planes. If the principal stress is limited to 320 MN/m^2 tensile, calculate:

(a) the magnitude of the shear stresses;
(b) the directions of the principal planes;
(c) the other principal stress;
(d) the maximum shear stress.

[219 MN/m^2, 28.7 and 118.7° counterclockwise to 200 MN/m^2 plane; −200 MN/m^2; 260 MN/m^2.]

13.11 (B). A solid shaft of 125 mm diameter transmits 0.5 MW at 300 rev/min. It is also subjected to a bending moment of 9 kN m and to a tensile end load. If the maximum principal stress is limited to 75 MN/m^2, determine the permissible end thrust. Determine the position of the plane on which the principal stress acts, and draw a diagram showing the position of the plane relative to the torque and the plane of the bending moment. [61.4 kN; 61° to shaft axis.]

13.12 (B). At a certain point in a piece of material there are two planes at right angles to one another on which there are shearing stresses of 150 MN/m^2 together with normal stresses of 300 MN/m^2 tensile on one plane and 150 MN/m^2 tensile on the other plane. If the shear stress on the 150 MN/m^2 planes is taken as clockwise in effect determine for the given point:

(a) the magnitudes of the principal stresses;
(b) the inclinations of the principal planes;
(c) the maximum shear stress and the inclinations of the planes on which it acts;
(d) the maximum strain if $E = 208$ GN/m^2 and Poisson's ratio = 0.29.

[392.7, 57.3 MN/m^2; 31.7°, 121.7°; 167.7 MN/m^2, 76.7°, 166.7°; 1810 $\mu\varepsilon$.]

13.13 (B). A 250 mm diameter solid shaft drives a screw propeller with an output of 7 MW. When the forward speed of the vessel is 35 km/h the speed of revolution of the propeller is 240 rev/min. Find the maximum stress resulting from the torque and the axial compressive stress resulting from the thrust in the shaft; hence find for a point on the surface of the shaft (a) the principal stresses, and (b) the directions of the principal planes relative to the shaft axis. Make a diagram to show clearly the direction of the principal planes and stresses relative to the shaft axis. [U.L.] [90.8, 14.7, 98.4, −83.7 MN/m^2; 47° and 137°.]

13.14 (B). A hollow shaft is 460 mm inside diameter and 25 mm thick. It is subjected to an internal pressure of 2 MN/m^2, a bending moment of 25 kN m and a torque of 40 kN m. Assuming the shaft may be treated as a thin cylinder, make a neat sketch of an element of the shaft, showing the stresses resulting from all three actions. Determine the values of the principal stresses and the maximum shear stress. [21.5, 11.8, 16.6 MN/m^2.]

13.15 (B). In a piece of material a tensile stress σ_1 and a shearing stress τ act on a given plane. Show that the principal stresses are always of opposite sign. If an additional tensile stress σ_2 acts on a plane perpendicular to that of σ_1, find the condition that both principal stresses may be of the same sign. [U.L.] [$\tau = \sqrt{(\sigma_1 \sigma_2)}$.]

13.16 (B). A shaft 100 mm diameter is subjected to a twisting moment of 7 kN m, together with a bending moment of 2 kN m. Find, at the surface of the shaft, (a) the principal stresses, (b) the maximum shear stress. [47.3, -26.9 MN/m^2; 37.1 MN/m^2.]

13.17 (B). A material is subjected to a horizontal tensile stress of 90 MN/m^2 and a vertical tensile stress of 120 MN/m^2, together with shear stresses of 75 MN/m^2, those on the 120 MN/m^2 planes being counterclockwise in effect. Determine:

(a) the principal stresses;
(b) the maximum shear stress;
(c) the shear stress which, acting alone, would produce the same principal stress;
(d) the tensile stress which, acting alone, would produce the same maximum shear stress.

[181.5, 28.5 MN/m^2; 76.5 MN/m^2; 181.5 MN/m^2; 153 MN/m^2.]

13.18 (B). Two planes *AB* and *BC* in an elastic material under load are inclined at 45° to each other. The loading on the material is such that the stresses on these planes are as follows:

On *AB*, 150 MN/m^2 direct stress and 120 MN/m^2 shear.

On *BC*, 80 MN/m^2 shear and a direct stress σ.

Determine the value of the unknown stress σ on *BC* and hence determine the principal stresses which exist in the material. [190, 214, -74 MN/m^2.]

13.19 (B). A beam of I-section, 500 mm deep and 200 mm wide, has flanges 25 mm thick and web 12 mm thick. It carries a concentrated load of 300 kN at the centre of a simply supported span of 3 m. Calculate the principal stresses set up in the beam at the point where the web meets the flange. [83.4, -6.15 MN/m^2.]

13.20 (B). At a certain point on the outside of a shaft which is subjected to a torque and a bending moment the shear stresses are 100 MN/m^2 and the longitudinal direct stress is 60 MN/m^2 tensile. Find, by calculation from first principles or by graphical construction which must be justified:

(a) the maximum and minimum principal stresses;
(b) the maximum shear stress;
(c) the inclination of the principal stresses to the original stresses.

Summarize the answers clearly on a diagram, showing their relative positions to the original stresses.

[E.M.E.U.] [134.4, -74.4 MN/m^2; 104.4 MN/m^2; 35.5°.]

13.21 (B). A short vertical column is firmly fixed at the base and projects a distance of 300 mm from the base. The column is of I-section, 200 mm deep by 100 mm wide, flanges 10 mm thick, web 6 mm thick.

An inclined load of 80 kN acts on the top of the column in the centre of the section and in the plane containing the central line of the web; the line of action is inclined at 30 degrees to the vertical. Determine the position and magnitude of the greatest principal stress at the base of the column.

[U.L.] [48 MN/m^2 at junction of web and flange.]