CHAPTER 8

TORSION

Summary

For a *solid or hollow shaft* of uniform circular cross-section throughout its length, the theory of pure torsion states that

$$\frac{T}{J}=\frac{\tau}{R}=\frac{G\theta}{L}$$

where T is the applied external torque, constant over length L;

J is the polar second moment of area of shaft cross-section

$= \dfrac{\pi D^4}{32}$ for a solid shaft and $\dfrac{\pi(D^4-d^4)}{32}$ for a hollow shaft;

D is the outside diameter; R is the outside radius;

d is the inside diameter;

τ is the shear stress at radius R and is the maximum value for both solid and hollow shafts;

G is the modulus of rigidity (shear modulus); and

θ is the angle of twist in *radians* on a length L.

For *very thin-walled hollow shafts*

$J = 2\pi r^3 t$, where r is the mean radius of the shaft wall and t is the thickness.

Shear stress and shear strain are related to the angle of twist thus:

$$\tau=\frac{G\theta}{L}R=G\gamma$$

Strain energy in torsion is given by

$$U=\frac{T^2L}{2GJ}=\frac{GJ\theta^2}{2L}\left(=\frac{\tau^2}{4G}\times\text{volume for } \textit{solid shafts}\right)$$

For a circular shaft subjected to *combined bending and torsion* the *equivalent bending moment* is

$$M_e=\tfrac{1}{2}[M+\sqrt{(M^2+T^2)}]$$

and the *equivalent torque* is

$$T_e=\tfrac{1}{2}\sqrt{(M^2+T^2)}$$

where M and T are the applied bending moment and torque respectively.
The *power transmitted* by a shaft carrying torque T at ω rad/s $= T\omega$.

8.1. Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear (Fig. 8.1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory to describe the behaviour of shafts subjected to torque it is necessary to make the following basic assumptions:

(1) The material is homogeneous, i.e. of uniform elastic properties throughout.
(2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.
(3) The stress does not exceed the elastic limit or limit of proportionality.
(4) Circular sections remain circular.
(5) Cross-sections remain plane. (This is certainly not the case with the torsion of non-circular sections.)
(6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.

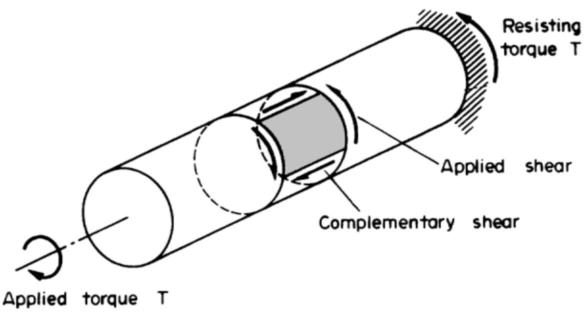

Fig. 8.1. Shear system set up on an element in the surface of a shaft subjected to torsion.

Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

(a) *Angle of twist*

Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed (Fig. 8.2). Under the action of this torque a radial line at the free end of the shaft twists through an angle θ, point A moves to B, and AB subtends an angle γ at the fixed end. This is then the angle of distortion of the shaft, i.e. *the shear strain.*

Since angle in radians = arc ÷ radius

$$\text{arc } AB = R\theta = L\gamma$$

$$\therefore \quad \gamma = R\theta/L \tag{8.1}$$

From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

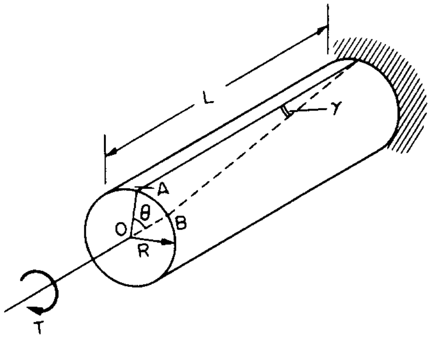

Fig. 8.2.

$$\therefore \qquad \gamma = \frac{\tau}{G} \tag{8.2}$$

where τ is the shear stress set up at radius R.

Therefore equating eqns. (8.1) and (8.2),

$$\frac{R\theta}{L} = \frac{\tau}{G}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right) \tag{8.3}$$

where τ' is the shear stress at any other radius r.

(*b*) *Stresses*

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness dr as shown in Fig. 8.3 each subjected to a shear stress τ'.

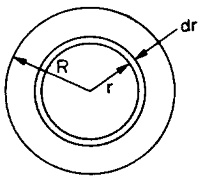

Fig. 8.3. Shaft cross-section.

The force set up on each element

$$= \text{stress} \times \text{area}$$

$$= \tau' \times 2\pi r\, dr \text{ (approximately)}$$

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

$$= (\tau' \times 2\pi r\, dr) \times r$$

$$= 2\pi\tau' r^2\, dr$$

The total torque on the section T will then be the sum of all such contributions across the section,

i.e.

$$T = \int_0^R 2\pi\tau' r^2\, dr$$

Now the shear stress τ' will vary with the radius r and must therefore be replaced in terms of r before the integral is evaluated.

From eqn. (8.3)

$$\tau' = \frac{G\theta}{L} r$$

$$\therefore \qquad T = \int_0^R 2\pi \frac{G\theta}{L} r^3\, dr$$

$$= \frac{G\theta}{L} \int_0^R 2\pi r^3\, dr$$

The integral $\int_0^R 2\pi r^3\, dr$ is called the *polar second moment of area J*, and may be evaluated as a standard form for solid and hollow shafts as shown in §8.2 below.

$$\therefore \qquad T = \frac{G\theta}{L} J$$

or

$$\frac{T}{J} = \frac{G\theta}{L} \tag{8.4}$$

Combining eqns. (8.3) and (8.4) produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \tag{8.5}$$

8.2. Polar second moment of area

As stated above the polar second moment of area J is defined as

$$J = \int_0^R 2\pi r^3\, dr$$

For a solid shaft,

$$J = 2\pi\left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32} \tag{8.6}$$

For a hollow shaft of internal radius r,

$$J = 2\pi\int_r^R r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_r^R$$

$$= \frac{\pi}{2}(R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32}(D^4 - d^4) \tag{8.7}$$

For thin-walled hollow shafts the values of D and d may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

Now

$$J = \int_0^R 2\pi r^3\,dr = \Sigma(2\pi r\,dr)r^2$$

$$= \Sigma\,Ar^2$$

where $A(= 2\pi r\,dr)$ is the area of each small element of Fig. 8.3, i.e. J is the sum of the Ar^2 terms for all elements.

If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness $t = dr$, then

$$J = Ar^2 = (2\pi r\,t)r^2$$

$$= 2\pi r^3 t \text{ (approximately)} \tag{8.8}$$

8.3. Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Fig. 8.1, their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L}R$$

Now from the definition of the shear or rigidity modulus G,

$$\tau = G\gamma$$

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L}R = G\gamma \tag{8.9}$$

or, in terms of some internal radius r,

$$\tau' = \frac{G\theta}{L} r = G\gamma \tag{8.10}$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius (Fig. 8.4). The applied shear stresses in the plane of the cross-section are accompanied by complementary stresses of equal value on longitudinal planes as indicated in Figs. 8.1 and 8.4. The significance of these longitudinal shears to material failure is discussed further in §8.10.

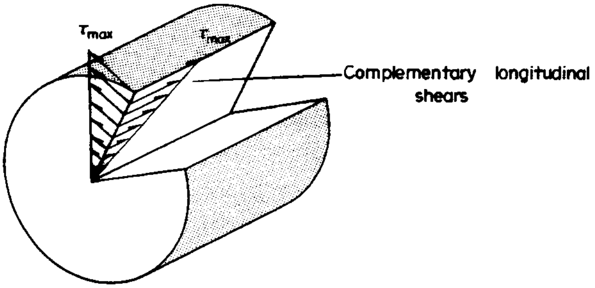

Fig. 8.4. Complementary longitudinal shear stress in a shaft subjected to torsion.

8.4. Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\therefore \qquad \tau = \frac{TR}{J}$$

With R the outside radius of the shaft the above equation yields the greatest value possible for τ (Fig. 8.4),

i.e. $$\tau_{max} = \frac{TR}{J}$$

$$\therefore \qquad \tau_{max} = \frac{T}{Z} \tag{8.11}$$

where $Z = J/R$ is termed the *polar section modulus*. It will be seen from the preceding section that:

for solid shafts, $$Z = \frac{\pi D^3}{16} \tag{8.12}$$

and for hollow shafts, $$Z = \frac{\pi(D^4 - d^4)}{16D} \tag{8.13}$$

8.5. Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L} = \frac{T}{GJ}$$

The quantity GJ is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ = \frac{T}{\theta/L} \tag{8.14}$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

8.6. Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. It is clear, therefore, that if there is to be some limit set on the maximum allowable working stress in the shaft material then only the outer surface of the shaft will reach this limit. The material within the shaft will work at a lower stress and, particularly near the centre, will not contribute as much to the torque-carrying capacity of the shaft. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

The relevant formulae for hollow shafts have been introduced in §8.2 and will not be repeated here. As an example of the increased torque-to-weight ratio possible with hollow shafts, however, it should be noted for a hollow shaft with an inside diameter half the outside diameter that the maximum stress increases by 6% over that for a solid shaft of the same outside diameter whilst the weight reduction achieved is approximately 25%.

8.7. Torsion of thin-walled tubes

The torsion of thin-walled tubes of circular and non-circular cross-section is treated fully in *Mechanics of Materials 2*.†

8.8. Composite shafts – series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed *series-connected* (Fig. 8.5) (see Example 8.3). In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion theory to each in turn; the composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torques in each shaft, e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \tag{8.15}$$

† E. J. Hearn, *Mechanics of Materials 2*, 3rd edition (Butterworth-Heinemann, Oxford, 1997).

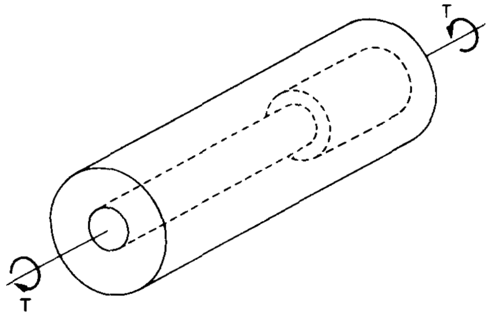

Fig. 8.5. "Series-connected" shaft – common torque.

In some applications it is convenient to ensure that the angles of twist in each shaft are equal, i.e. $\theta_1 = \theta_2$, so that for similar materials in each shaft

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

or

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \qquad (8.16)$$

8.9. Composite shafts – parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be *connected in parallel* (Fig. 8.6).

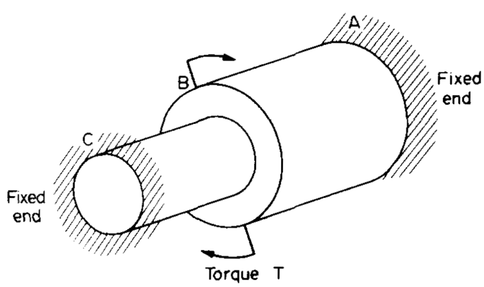

Fig. 8.6. "Parallel-connected" shaft – shared torque.

For parallel connection,

$$\text{total torque } T = T_1 + T_2 \qquad (8.17)$$

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \qquad (8.18)$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \tag{8.19}$$

Thus two equations are obtained in terms of the torques in each part of the composite shaft and these torques can therefore be determined.

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

8.10. Principal stresses

It will be shown in §13.2 that a state of pure shear as produced by the torsion of shafts is equivalent to a system of biaxial direct stresses, one stress tensile, one compressive, of equal value and at 45° to the shaft axis as shown in Fig. 8.7; these are then the principal stresses.

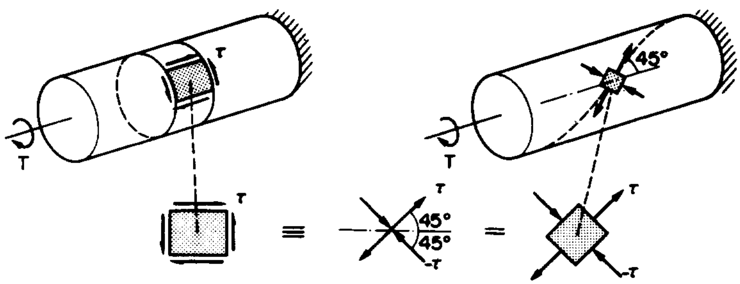

Fig. 8.7. Shear and principal stresses in a shaft subjected to torsion.

Thus shafts which are constructed from brittle materials which are notably weaker under direct stress than in shear (cast-iron, for example) will fail by cracking along a helix inclined at 45° to the shaft axis. This can be demonstrated very simply by twisting a piece of chalk to failure (Fig. 8.8a). Ductile materials, however, which are weaker in shear, fail on the shear planes at right angles to the shaft axis (Fig. 8.8b). In some cases, e.g. timber, failure occurs by cracking along the shear planes parallel to the shaft axis owing to the nature of the material with fibres generally parallel to the axis producing a weakness in shear longitudinally rather than transversely. The complementary shears of Fig. 8.4 then assume greater significance.

8.11. Strain energy in torsion

It will be shown in §11.4 that the strain energy stored in a solid circular bar or shaft subjected to a torque T is given by the alternative expressions

$$U = \tfrac{1}{2}T\theta = \frac{T^2 L}{2GJ} = \frac{GJ\theta^2}{2L} = \frac{\tau^2}{4G} \times \text{volume} \tag{8.20}$$

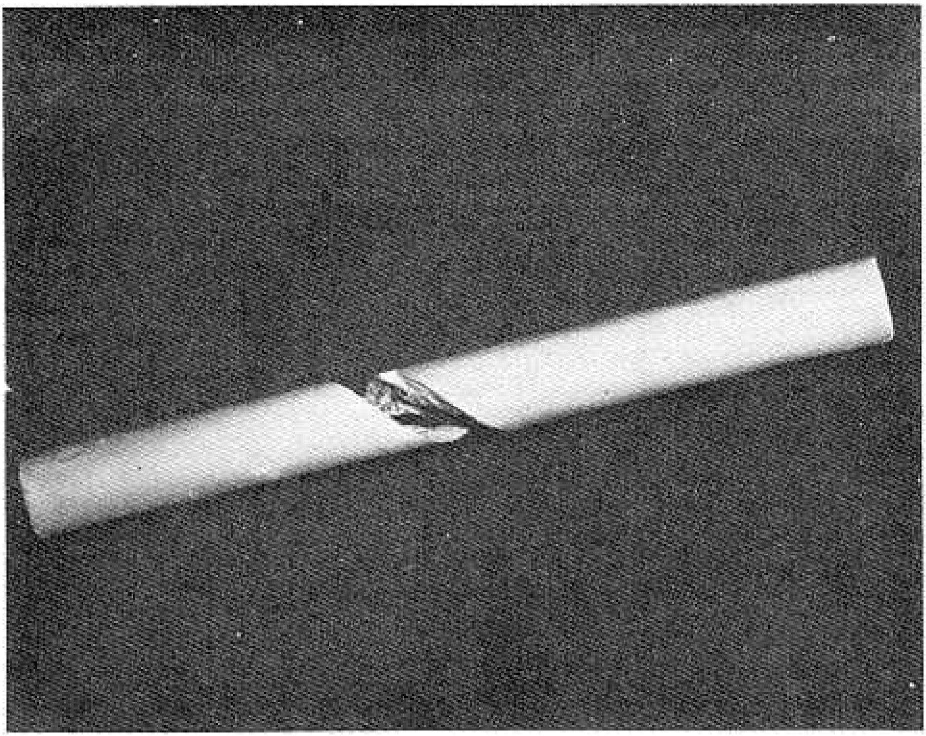

Fig. 8.8a. Typical failure of a brittle material (chalk) in torsion. Failure occurs on a 45° helix owing to the action of the direct tensile stresses produced at 45° by the applied torque.

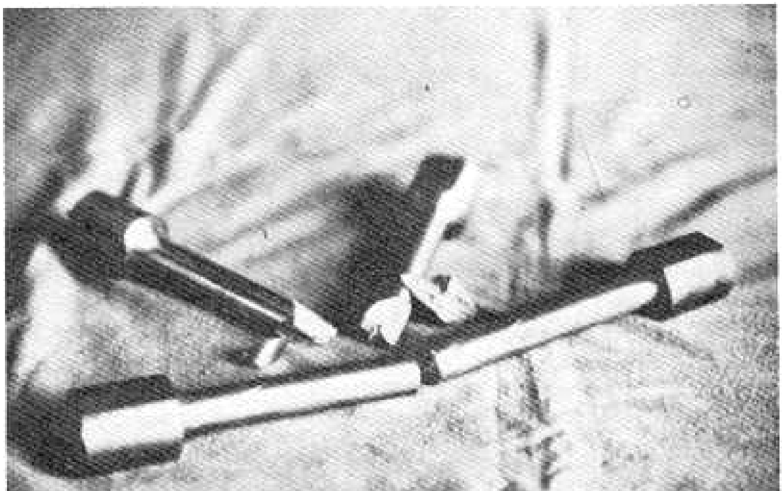

Fig. 8.8b. (Foreground) Failure of a ductile steel in torsion on a plane perpendicular to the specimen longitudinal axis. Scribed lines on the surface of the specimen which were parallel to the longitudinal axis before torque application indicate the degree of twist applied to the specimen. (Background) Equivalent failure of a more brittle, higher carbon steel in torsion. Failure again occurs on 45° planes but in this case, as often occurs in practice, a clean fracture into two pieces did not take place.

8.12. Variation of data along shaft length – torsion of tapered shafts

This section illustrates the procedure which may be adopted when any of the quantities normally used in the torsion equations vary along the length of the shaft. Provided the variation is known in terms of x, the distance along the shaft, then a solution can be obtained.

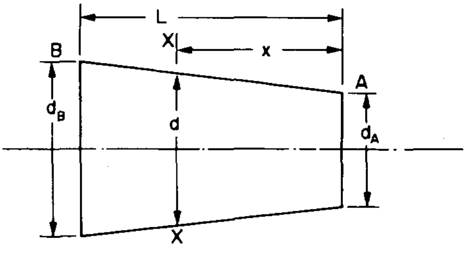

Fig. 8.9. Torsion of a tapered shaft.

Consider, therefore, the tapered shaft shown in Fig. 8.9 with its diameter changing linearly from d_A to d_B over a length L. The diameter at any section x from end A is then given by

$$d = d_A + (d_B - d_A)\frac{x}{L}$$

Provided that the angle of the taper is not too great, the simple torsion theory may be applied to an element at section XX in order to determine the angle of twist of the shaft, i.e. for the element shown,

$$\frac{G d\theta}{dx} = \frac{T}{J_{XX}}$$

Therefore the total angle of twist of the shaft is given by

$$\theta = \int_0^L \frac{T}{GJ_{XX}}\,dx$$

Now $$J_{XX} = \frac{\pi d^4}{32} = \frac{\pi}{32}\left[d_A + (d_B - d_A)\frac{x}{L}\right]^4$$

Substituting and integrating,

$$\theta = \frac{32TL}{3\pi G}\left[\frac{1}{d_A^3} - \frac{1}{d_B^3}\right]\left[\frac{1}{d_B} - \frac{1}{d_A}\right] = \frac{32TL}{3\pi G}\left[\frac{d_A^2 + d_A d_B + d_B^2}{d_A^3 d_B^3}\right]$$

When $d_A = d_B = d$ this reduces to $\theta = \dfrac{32TL}{\pi G d^4}$ the standard result for a parallel shaft.

8.13. Power transmitted by shafts

If a shaft carries a torque T Newton metres and rotates at ω rad/s it will do work at the rate of

$$T\omega \text{ Nm/s (or joule/s).}$$

Now the rate at which a system works is defined as its power, the basic unit of power being the Watt (1 Watt = 1 Nm/s).
Thus, the power transmitted by the shaft:

$$= T\omega \text{ Watts.}$$

Since the Watt is a very small unit of power in engineering terms use is normally made of S.I. multiples, i.e. kilowatts (kW) or megawatts (MW).

8.14. Combined stress systems – combined bending and torsion

In most practical transmission situations shafts which carry torque are also subjected to bending, if only by virtue of the self-weight of the gears they carry. Many other practical applications occur where bending and torsion arise simultaneously so that this type of loading represents one of the major sources of complex stress situations.

In the case of shafts, bending gives rise to tensile stress on one surface and compressive stress on the opposite surface whilst torsion gives rise to pure shear throughout the shaft. An element on the tensile surface will thus be subjected to the stress system indicated in Fig. 8.10 and eqn. (13.11) or the Mohr circle procedure of §13.6 can be used to obtain the principal stresses present.

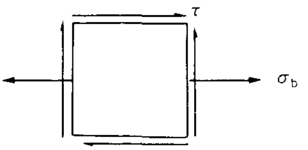

Fig. 8.10. Stress system on the surface of a shaft subjected to torque and bending.

Alternatively, the shaft can be considered to be subjected to *equivalent torques* or *equivalent bending moments* as described below.

8.15. Combined bending and torsion – equivalent bending moment

For shafts subjected to the simultaneous application of a bending moment M and torque T the *principal stresses* set up in the shaft can be shown to be equal to those produced by an *equivalent bending moment*, of a certain value M_e acting alone.

From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment M are given by

$$\sigma = \frac{My_{max}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft $J = 2I$,

$$\therefore \qquad \tau = \frac{TD}{4I}$$

The principal stresses for this system can now be obtained by applying the formula derived in §13.4,

i.e.
$$\sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

and, with $\sigma_y = 0$, the maximum principal stress σ_1 is given by

$$\sigma_1 = \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left[\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2\right]}$$

$$= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

Now if M_e is the bending moment which, acting alone, will produce the same maximum stress, then

$$\sigma_1 = \frac{M_e y_{max}}{I} = \frac{M_e D}{2I}$$

$$\therefore \qquad \frac{M_e D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

i.e. the equivalent bending moment is given by

$$M_e = \tfrac{1}{2}[M + \sqrt{(M^2 + T^2)}] \qquad (8.21)$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.

8.16. Combined bending and torsion – equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment M and a torque T the *maximum shear stress* set up in the shaft may be determined by the application of an *equivalent torque* of value T_e acting alone.

From the preceding section the principal stresses in the shaft are given by

$$\sigma_1 = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M + \sqrt{(M^2 + T^2)}]$$

and
$$\sigma_2 = \frac{1}{2}\left(\frac{D}{2I}\right)[M - \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M - \sqrt{(M^2 + T^2)}]$$

Now the maximum shear stress is given by eqn. (13.12)

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\left(\frac{D}{J}\right)\sqrt{(M^2 + T^2)}$$

But, from the torsion theory, the equivalent torque T_e will set up a maximum shear stress of

$$\tau_{max} = \frac{T_e D}{2J}$$

Thus if these maximum shear stresses are to be equal,

$$T_e = \sqrt{(M^2 + T^2)} \tag{8.22}$$

It must be remembered that the equivalent moment M_e and equivalent torque T_e are merely convenient devices to obtain the maximum principal direct stress or maximum shear stress, respectively, under the combined stress system. They should not be used for other purposes such as the calculation of power transmitted by the shaft; this depends solely on the torque T carried by the shaft (not on T_e).

8.17. Combined bending, torsion and direct thrust

Additional stresses arising from the action of direct thrusts on shafts may be taken into account by adding the direct stress due to the thrust σ_d to that of the direct stress due to bending σ_b taking due account of sign. The complex stress system resulting on any element in the shaft is then as shown in Fig. 8.11 and may be solved to determine the principal stresses using Mohr's stress circle method of solution described in § 13.6.

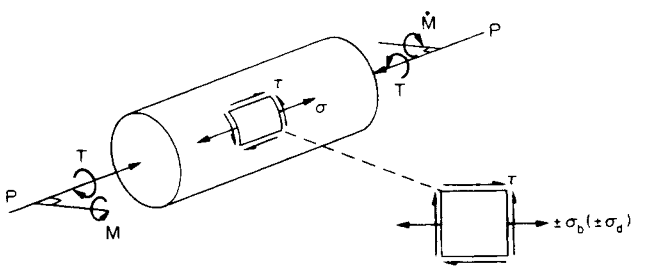

Fig. 8.11. Shaft subjected to combined bending, torque and direct thrust.

This type of problem arises in the service loading condition of marine propeller shafts, the direct thrust being the compressive reaction of the water on the propeller as the craft is pushed forward. This force then exists in combination with the torque carried by the shaft in doing the required work and any bending moments which exist by virtue of the self-weight of the shaft between bearings.

The compressive stress σ_d arising from the propeller reaction is thus superimposed on the bending stresses; on the compressive bending surface it will be additive to σ_b whilst on the "tensile" surface it will effectively reduce the value of σ_b, see Fig. 8.11.

8.18. Combined bending, torque and internal pressure

In the case of pressurised cylinders, direct stresses will be introduced in two perpendicular directions. These have been introduced in Chapters 9 and 10 as the radial and circumferential stresses σ_r and σ_H. If the cylinder also carries a torque then shear stresses will be introduced, their value being calculated from the simple torsion theory of § 8.3. The stress system on an element will thus become that shown in Fig. 8.12.

If bending is present it will generally be on the x axis and will result in a modification to the value of σ_x. If the element is taken on the tensile surface of the cylinder then the bending stress σ_b will add to the value of σ_H, if on the compressive surface it must be subtracted from σ_H.

Once again a solution to such problems can be effected either by application of eqn. (13.11) or by a Mohr circle approach.

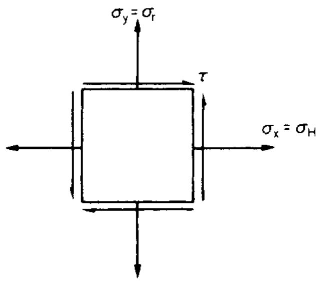

Fig. 8.12. Stress system under combined torque and internal pressure.

Examples

Example 8.1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if $G = 80\ \text{GN/m}^2$.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60 mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution

(a) $$\text{Power} = T\omega \quad \therefore \quad \text{torque } T = \frac{\text{power}}{\omega}$$

$$\therefore \quad T = \frac{75 \times 10^3}{150 \times 2\pi/60} = 4.77\ \text{kN m}$$

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{and} \quad J = \frac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6}\ \text{m}^4$$

$$\therefore \quad \tau_{max} = \frac{TR_{max}}{J} = \frac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = \mathbf{24.3\ MN/m^2}$$

Also from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m}$$

$$= 6.07 \times 10^{-3} \times \frac{360}{2\pi} = \mathbf{0.348 \text{ degrees/m}}$$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}(100^4 - 60^4)10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$$

The torque carried by the modified shaft is then given by

$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = \mathbf{4.15 \times 10^3 \text{ Nm}}$$

Now, weight/metre of original shaft

$$= \frac{\pi}{4}(100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3}\ \rho g$$

where ρ is the density of the shaft material.

Also, weight/metre of modified shaft $= \frac{\pi}{4}(100^2 - 60^2)10^{-6} \times 1 \times \rho g$

$$= 5.027 \times 10^{-3}\ \rho g$$

Power/weight ratio for original shaft $= \dfrac{T\omega}{\text{weight/metre}}$

$$= \frac{4.77 \times 10^3\ \omega}{7.854 \times 10^{-3}\ \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3\ \omega}{5.027 \times 10^{-3}\ \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = \mathbf{36\%}$$

Example 8.2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8° in a length of 4 m.

For the shaft material $G = 80$ GN/m^2.

Solution

The two limiting conditions stated in the question, namely maximum shear stress and angle of twist, will each lead to different values for the required diameter. The larger shaft must then be chosen as the one for which neither condition is exceeded.

Maximum shear stress condition

Since power $= T\omega$ and $\omega = 200 \times \dfrac{2\pi}{60} = 20.94$ rad/s

then $$T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3 \text{ Nm}$$

From the torsion theory

$$J = \frac{TR}{\tau}$$

$$\therefore \quad \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

But $d/D = 0.75$

$$\therefore \quad \frac{\pi}{32} D^4 (1 - 0.75^4) = 20.43 \times 10^{-6} D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$\therefore \quad D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

and $$d = 50.5 \text{ mm}$$

Angle of twist condition

Again from the torsion theory

$$J = \frac{TL}{G\theta}$$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4 (1 - 0.75^4) = 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

and $$d = 56.5 \text{ mm}$$

Thus the dimensions required for the shaft to satisfy both conditions are **outer diameter 75.3 mm; inner diameter 56.5 mm.**

Example 8.3

(a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m^2.

(b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

Solution

(a) This is, in effect, a question on *shafts in series* since each part is subjected to the same torque.

From the torsion theory

$$T = \frac{\tau J}{R}$$

and as the maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts the torque allowable for a known value of shear stress is dependent only on the value of J. This will be least where the internal diameter is greatest since

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$\therefore$ $$\text{least value of } J = \frac{\pi}{32}(50^4 - 38^4)10^{-12} = 0.41 \times 10^{-6}\,\text{m}^4$$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m^2 (at 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3\,\text{Nm}$$

$$\text{Maximum power} = T\omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60}$$

$$= 25.2 \times 10^3 = \mathbf{25.2\,kW}$$

(b) Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, eqn. (8.16) applies,

i.e. $$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\therefore \quad \frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32}(50^4 - 25^4)10^{-12}}{\frac{\pi}{32}(50^4 - 38^4)10^{-12}} = 1.43$$

$$\therefore \quad L_2 = 1.43\, L_1$$

But $$L_1 + L_2 = 510\,\text{mm}$$

$$\therefore \quad L_1(1 + 1.43) = 510$$

$$L_1 = \frac{510}{2.43} = \mathbf{210\,mm}$$

Example 8.4

A circular bar *ABC*, 3 m long, is rigidly fixed at its ends *A* and *C*. The portion *AB* is 1.8 m long and of 50 mm diameter and *BC* is 1.2 m long and of 25 mm diameter. If a twisting moment of 680 N m is applied at *B*, determine the values of the resisting moments at *A* and *C* and the maximum stress in each section of the shaft. What will be the angle of twist of each portion?

For the material of the shaft $G = 80\ \text{GN/m}^2$.

Solution

In this case the two portions of the *shaft* are *in parallel* and the applied torque is shared between them. Let suffix 1 refer to portion *AB* and suffix 2 to portion *BC*.

Since the angles of twist in each portion are equal and *G* is common to both sections,

then $$\frac{T_1 L_1}{J_1} = \frac{T_2 L_2}{J_2}$$

$$\therefore \quad T_1 = \frac{J_1}{J_2} \times \frac{L_2}{L_1} \times T_2 = \frac{\frac{\pi}{32} \times 50^4}{\frac{\pi}{32} \times 25^4} \times \frac{1.2}{1.8} \times T_2$$

$$= \frac{16 \times 1.2}{1.8} T_2 = 10.67 T_2$$

Total torque $= T_1 + T_2 = T_2(10.67 + 1) = 680$

$$\therefore \quad T_2 = \frac{680}{11.67} = \mathbf{58.3\,N\,m}$$

and $$T_1 = \mathbf{621.7\,N\,m}$$

For portion *AB*,

$$\tau_{max} = \frac{T_1 R_1}{J_1} = \frac{621.7 \times 25 \times 10^{-3}}{\frac{\pi}{32} \times 50^4 \times 10^{-12}} = \mathbf{25.33 \times 10^6\ N/m^2}$$

For portion *BC*,

$$\tau_{max} = \frac{T_2 R_2}{J_2} = \frac{58.3 \times 12.5 \times 10^{-3}}{\frac{\pi}{32} \times 25^4 \times 10^{-12}} = \mathbf{19.0 \times 10^6\ N/m^2}$$

$$\text{Angle of twist for each portion} = \frac{T_1 L_1}{J_1 G}$$

$$= \frac{621.7 \times 1.8}{\frac{\pi}{32} \times 50^4 \times 10^{-12} \times 80 \times 10^9} = 0.0228\,\text{rad} = \mathbf{1.3\,degrees}$$

Problems

8.1 (A). A solid steel shaft *A* of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m^2 in the steel.

It is proposed to replace *A* by a hollow shaft *B*, of the same external diameter but with a limiting shearing stress of 75 MN/m^2. Determine the internal diameter of *B* to transmit the same power at the same speed.

[38.6 kW, 33.4 mm.]

8.2 (A). Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20 % and the greatest intensity of shear stress is limited to 75 MN/m^2. The internal diameter of the shaft is to be 80 % of the external diameter. (The mean torque is that derived from the horsepower equation.) [135.2, 108.2 mm.]

8.3 (A). A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the shaft are:

(a) that the shaft must not twist more than 0.02 radian on a length of 10 diameters;
(b) that the working stress must not exceed 60 MN/m^2.

If the modulus of rigidity of steel is 80 GN/m^2 what is

(i) the diameter of the shaft required;
(ii) the actual working stress;
(iii) the angle of twist of the 3 m length? [B.P.] [150 mm; 60 MN/m^2; 0.030 rad.]

8.4 (A). A hollow shaft has to transmit 6 MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m^2 nor the angle of twist 0.3° per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. $G = 80$ GN/m^2. [61.5 mm.]

8.5 (A). A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m^2 and 55 MN/m^2 respectively, and the maximum torque is 1.4 times the mean torque, calculate:

(a) the diameter of the shaft;
(b) the diameter of the bolts. [B.P.] [67.2, 13.8 mm.]

8.6 (A). A hollow low carbon steel shaft is subjected to a torque of 0.25 MN m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m^2 determine the required diameters and the angle of twist in degrees per metre length of shaft.

$G = 80$ GN/m^2. [I.Struct.E.] [264, 88 mm; 0.38°.]

8.7 (A). Describe how you would carry out a torsion test on a low carbon steel specimen and how, from data taken, you would find the modulus of rigidity and yield stress in shear of the steel. Discuss the nature of the torque–twist curve and compare it with the shear stress–shear strain relationship. [U.Birm.]

8.8 (A/B). Opposing axial torques are applied at the ends of a straight bar *ABCD*. Each of the parts *AB*, *BC* and *CD* is 500 mm long and has a hollow circular cross-section, the inside and outside diameters being, respectively, *AB* 25 mm and 60 mm, *BC* 25 mm and 70 mm, *CD* 40 mm and 70 mm. The modulus of rigidity of the material is 80 GN/m^2 throughout. Calculate:

(a) the maximum torque which can be applied if the maximum shear stress is not to exceed 75 MN/m^2;
(b) the maximum torque if the twist of *D* relative to *A* is not to exceed 2°. [E.I.E.] [3.085 kN m, 3.25 kN m.]

8.9 (A/B). A solid steel shaft of 200 mm diameter transmits 5 MW at 500 rev/min. It is proposed to alter the horsepower to 7 MW and the speed to 440 rev/min and to replace the solid shaft by a hollow shaft made of the same type of steel but having only 80 % of the weight of the solid shaft. The length of both shafts is the same and the hollow shaft is to have the same maximum shear stress as the solid shaft. Find:

(a) the ratio between the torque per unit angle of twist per metre for the two shafts;

(b) the external and internal diameters for the hollow shaft. [I.Mech.E.] [2.085; 261, 190 mm.]

8.10 (A/B). A shaft ABC rotates at 600 rev/min and is driven through a coupling at the end A. At B a pulley takes off two-thirds of the power, the remainder being absorbed at C. The part AB is 1.3 m long and of 100 mm diameter; BC is 1.7 m long and of 75 mm diameter. The maximum shear stress set up in BC is 40 MN/m^2. Determine the maximum stress in AB and the power transmitted by it, and calculate the total angle of twist in the length AC. Take $G = 80$ GN/m^2. [I.Mech.E.] [16.9 MN/m^2; 208 kW; 1.61°.]

8.11 (A/B). A composite shaft consists of a steel rod of 75 mm diameter surrounded by a closely fitting brass tube firmly fixed to it. Find the outside diameter of the tube such that when a torque is applied to the composite shaft it will be shared equally by the two materials.

$G_S = 80$ GN/m^2; $G_B = 40$ GN/m^2.

If the torque is 16 kN m, calculate the maximum shearing stress in each material and the angle of twist on a length of 4 m. [U.L.] [98.7 mm; 96.6, 63.5 MN/m^2; 7.38°.]

8.12 (A/B). A circular bar 4 m long with an external radius of 25 mm is solid over half its length and bored to an internal radius of 12 mm over the other half. If a torque of 120 N m is applied at the centre of the shaft, the two ends being fixed, determine the maximum shear stress set up in the surface of the shaft and the work done by the torque in producing this stress. [2.51 MN/m^2; 0.151 N m.]

8.13 (A/B). The shaft of Problem 8.12 is now fixed at one end only and the torque applied at the free end. How will the values of maximum shear stress and work done change? [5.16 MN/m^2; 0.603 N m.]

8.14 (B). Calculate the minimum diameter of a solid shaft which is required to transmit 70 kW at 600 rev/min if the shear stress is not to exceed 75 MN/m^2. If a bending moment of 300 N m is now applied to the shaft find the speed at which the shaft must be driven in order to transmit the same horsepower for the same value of maximum shear stress. [630 rev/min.]

8.15 (B). A solid shaft of 75 mm diameter and 4 m span supports a flywheel of weight 2.5 kN at a point 1.8 m from one support. Determine the maximum direct stress produced in the surface of the shaft when it transmits 35 kW at 200 rev/min. [65.9 MN/m^2.]

8.16 (B). The shaft of Problem 12.15 is now subjected to an axial compressive end load of 80 kN, the other conditions remaining unchanged. What will be the magnitudes of the maximum principal stress in the shaft? [84 MN/m^2.]

8.17 (B). A horizontal shaft of 75 mm diameter projects from a bearing, and in addition to the torque transmitted the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tension test, is 135 MN/m^2 find the safe torque to which the shaft may be subjected using as the criterion (a) the maximum shearing stress, (b) the maximum strain energy per unit volume. Poisson's ratio $\nu = 0.29$. [U.L.] [5.05, 8.3 kN m.]

8.18 (B). A pulley subjected to vertical belt drive develops 10 kW at 240 rev/min, the belt tension ratio being 0.4. The pulley is fixed to the end of a length of overhead shafting which is supported in two self-aligning bearings, the centre line of the pulley overhanging the centre line of the left-hand bearing by 150 mm. If the pulley is of 250 mm diameter and weight 270 N, neglecting the weight of the shafting, find the minimum shaft diameter required if the maximum allowable stress intensity at a point on the top surface of the shaft at the centre line of the left-hand bearing is not to exceed 90 MN/m^2 direct or 40 MN/m^2 shear. [50.5 mm.]

8.19 (B). A hollow steel shaft of 100 mm external diameter and 50 mm internal diameter transmits 0.6 MW at 500 rev/min and is subjected to an end thrust of 45 kN. Find what bending moment may safely be applied if the greater principal stress is not to exceed 90 MN/m^2. What will then be the value of the smaller principal stress? [City U.] [3.6 kN m; −43.1 MN/m^2.]

8.20 (B). A solid circular shaft is subjected to an axial torque T and to a bending moment M. If $M = kT$, determine in terms of k the ratio of the maximum principal stress to the maximum shear stress. Find the power transmitted by a 50 mm diameter shaft, at a speed of 300 rev/min when $k = 0.4$ and the maximum shear stress is 75 MN/m^2. [I.Mech.] [$1 + k/\sqrt{(k^2+1)}$; 57.6 kW.]

8.21 (B). (a) A solid circular steel shaft is subjected to a bending moment of 10 kN m and is required to transmit a maximum power of 550 kW at 420 rev/min. Assuming the shaft to be simply supported at each end and neglecting the shaft weight, determine the ratio of the maximum principal stress to the maximum shear stress induced in the shaft material.

(b) A 300 mm external diameter and 200 mm internal diameter hollow steel shaft operates under the following conditions:

power transmitted = 2280 kW; maximum torque = 1.2 × mean torque; maximum bending moment = 11 kN m; maximum end thrust = 66 kN; maximum principal compressive stress = 40 MN/m^2.

Determine the maximum safe speed of rotation for the shaft. [1.625 : 1; 169 rev/min.]

8.22 (C). A uniform solid shaft of circular cross-section will drive the propeller of a ship. It will therefore necessarily be subject simultaneously to a thrust load and a torque. The magnitude of the thrust can be related to the magnitude of the torque by the simple relationship $N = KT$, where N denotes the magnitude of the thrust, T that of the torque and K is a constant. There will also be some bending moment on the shaft. Assuming that the design requirement is that the maximum shearing stress in the material shall nowhere exceed a certain value, denoted by τ, show that the maximum bending moment that can be allowed is given by the expression

$$\text{bending moment, } M = \left[\left(\frac{\tau\pi^2 r^6}{4T^2} - 1\right)^{1/2} - \frac{Kr}{4}\right]T$$

where r denotes the radius of the shaft cross-section. [City U.]