CHAPTER 15

THEORIES OF ELASTIC FAILURE

Summary

TABLE 15.1

Theory	Value in tension test at failure	Value in complex stress system	Criterion for failure
Maximum principal stress (*Rankine*)	σ_y	σ_1	$\sigma_1 = \sigma_y$
Maximum shear stress (*Guest–Tresca*)	$\frac{1}{2}\sigma_y$	$\frac{1}{2}(\sigma_1 - \sigma_3)$	$\sigma_1 - \sigma_3 = \sigma_y$
Maximum principal strain (*Saint-Venant*)	$\frac{\sigma_y}{E}$	$\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E}$	$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma_y$
Total strain energy per unit volume (*Haigh*)	$\frac{\sigma_y^2}{2E}$	$\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$	$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$
Shear strain energy per unit volume Distortion energy theory (*Maxwell–Huber–von Mises*)	$\frac{\sigma_y^2}{6G}$	$\frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$	$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \sigma_y^2$
Modified shear stress Internal friction theory (*Mohr*)			$\frac{\sigma_1}{\sigma_{y_t}} + \frac{\sigma_2}{\sigma_{y_c}} = 1$

Introduction

When dealing with the design of structures or components the physical properties of the constituent materials are usually found from the results of laboratory experiments which have only subjected the materials to the simplest stress conditions. The most usual test is the simple tensile test in which the value of the stress at yield or at fracture (whichever occurs first) is easily determined. The strengths of materials under complex stress systems are not generally known except in a few particular cases. In practice it is these complicated systems of stress which are more often encountered, and therefore it is necessary to have some basis for determining allowable working stresses so that failure will not occur. Thus the function of the theories of elastic failure is to predict from the behaviour of materials in a simple tensile test when elastic failure will occur under *any* condition of applied stress.

A number of theoretical criteria have been proposed each seeking to obtain adequate correlation between estimated component life and that actually achieved under service load conditions for both brittle and ductile material applications. The five main theories are:

(a) Maximum principal stress theory (Rankine).
(b) Maximum shear stress theory (Guest–Tresca).
(c) Maximum principal strain (Saint-Venant).
(d) Total strain energy per unit volume (Haigh).
(e) Shear strain energy per unit volume (Maxwell–Huber–von Mises).

In each case the value of the selected critical property implied in the title of the theory is determined for both the simple tension test and a three-dimensional complex stress system. These values are then equated to produce the so-called *criterion for failure* listed in the last column of Table 15.1.

In Table 15.1 σ_y is the stress at the yield point in the simple tension test, and σ_1, σ_2 and σ_3 are the three principal stresses in the three-dimensional complex stress system in order of magnitude. Thus in the case of the maximum shear stress theory $\sigma_1 - \sigma_3$ is the greatest numerical difference between two principal stresses taking into account signs and the fact that one principal stress may be zero.

Each of the first five theories listed in Table 15.1 will be introduced in detail in the following text, as will a sixth theory, (f) **Mohr's modified shear stress theory**. Whereas the previous theories (a) to (e) assume equal material strength in tension and compression, the Mohr's modified theory attempts to take into account the additional strength of brittle materials in compression.

15.1. Maximum principal stress theory

This theory assumes that when the maximum principal stress in the complex stress system reaches the elastic limit stress in simple tension, failure occurs. The criterion of failure is thus

$$\sigma_1 = \sigma_y$$

It should be noted, however, that failure could also occur in compression if the least principal stress σ_3 were compressive and its value reached the value of the yield stress in compression for the material concerned before the value of σ_{y_t} was reached in tension. An additional criterion is therefore

$$\sigma_3 = \sigma_y \quad \text{(compressive)}$$

Whilst the theory can be shown to hold fairly well for brittle materials, there is considerable experimental evidence that the theory should not be applied for ductile materials. For example, even in the case of the pure tension test itself, failure for ductile materials takes place not because of the direct stresses applied but in shear on planes at 45° to the specimen axis. Also, truly homogeneous materials can withstand very high hydrostatic pressures without failing, thus indicating that maximum direct stresses alone do not constitute a valid failure criteria for all loading conditions.

15.2. Maximum shear stress theory

This theory states that failure can be assumed to occur when the maximum shear stress in the complex stress system becomes equal to that at the yield point in the simple tensile test.

Since the maximum shear stress is half the greatest difference between two principal stresses the criterion of failure becomes

$$\tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}(\sigma_y - 0)$$

i.e.

$$\boldsymbol{\sigma_1 - \sigma_3 = \sigma_y} \qquad (15.1)$$

the value of σ_3 being algebraically the smallest value, i.e. taking account of sign *and the fact that one stress may be zero.* This produces fairly accurate correlation with experimental results particularly for ductile materials, and is often used for ductile materials in machine design. The criterion is often referred to as the "Tresca" theory and is one of the widely used laws of plasticity.

15.3. Maximum principal strain theory

This theory assumes that failure occurs when the maximum strain in the complex stress system equals that at the yield point in the tensile test,

i.e.

$$\frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} - \nu\frac{\sigma_3}{E} = \frac{\sigma_y}{E}$$

$$\boldsymbol{\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma_y} \qquad (15.2)$$

This theory is contradicted by the results obtained from tests on flat plates subjected to two mutually perpendicular tensions. The Poisson's ratio effect of each tension reduces the strain in the perpendicular direction so that according to this theory failure should occur at a higher load. This is not always the case. The theory holds reasonably well for cast iron but is not generally used in design procedures these days.

15.4. Maximum total strain energy per unit volume theory

The theory assumes that failure occurs when the total strain energy in the complex stress system is equal to that at the yield point in the tensile test.

From the work of §14.17 the criterion of failure is thus

$$\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = \frac{\sigma_y^2}{2E}$$

i.e.

$$\boldsymbol{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2} \qquad (15.3)$$

The theory gives fairly good results for ductile materials but is seldom used in preference to the theory below.

15.5. Maximum shear strain energy per unit volume (or distortion energy) theory

Section 14.17 again indicates how the strain energy of a stressed component can be divided into volumetric strain energy and shear strain energy components, the former being associated with volume change and no distortion, the latter producing distortion of the stressed elements. This theory states that failure occurs when the maximum shear strain energy component in the complex stress system is equal to that at the yield point in the tensile test,

i.e. $$\frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=\frac{\sigma_y^2}{6G} \quad \text{(eqn. (14.23a))}$$

or $$\frac{1}{6G}[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)=\frac{\sigma_y^2}{6G}$$

$$\therefore \quad (\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_y^2 \tag{15.4}$$

This theory has received considerable verification in practice and is widely regarded as the most reliable basis for design, particularly when dealing with ductile materials. It is often referred to as the "von Mises" or "Maxwell" criteria and is probably the best theory of the five. It is also sometimes referred to as the **distortion energy** or **maximum octahedral shear stress theory**.

In the above theories it has been assumed that the properties of the material in tension and compression are similar. It is well known, however, that certain materials, notably concrete, cast iron, soils, etc., exhibit vastly different properties depending on the nature of the applied stress. For brittle materials this has been explained by Griffith,† who has introduced the principle of surface energy at microscopic cracks and shown that an existing crack will propagate rapidly if the available elastic strain energy release is greater than the surface energy of the crack.‡ In this way Griffith indicates the greater seriousness of tensile stresses compared with compressive ones with respect to failure, particularly in fatigue environments. A further theory has been introduced by Mohr to predict failure of materials whose strengths are considerably different in tension and shear; this is introduced below.

15.6. Mohr's modified shear stress theory for brittle materials (sometimes referred to as the internal friction theory)

Brittle materials in general show little ability to deform plastically and hence will usually fracture at, or very near to, the elastic limit. Any of the so-called "yield criteria" introduced above, therefore, will normally imply fracture of a brittle material. It has been stated previously, however, that brittle materials are usually considerably stronger in compression than in tension and to allow for this Mohr has proposed a construction based on his stress circle in the application of the maximum shear stress theory. In Fig. 15.1 the circle on diameter OA is that for pure tension, the circle on diameter OB that for pure compression and the circle centre O and diameter CD is that for pure shear. Each of these types of test can be performed to failure relatively easily in the laboratory. An envelope to these curves, shown dotted, then represents the failure envelope according to the Mohr theory. A failure condition is then indicated when the stress circle for a particular complex stress condition is found to cut the envelope.

† A. A. Griffith, The phenomena of rupture and flow of solids, *Phil. Trans. Royal Soc.*, London, 1920.
‡ J. F. Knott, *Fundamentals of Fracture Mechanics* (Butterworths, London), 1973.

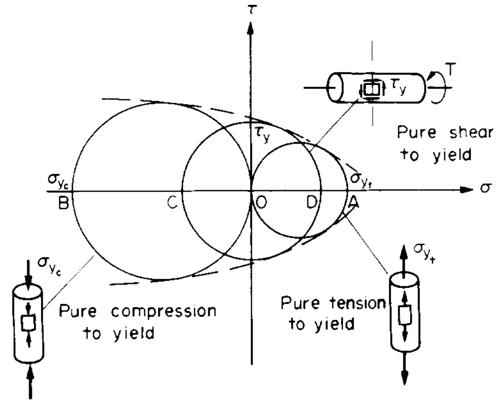

Fig. 15.1. Mohr theory on σ–τ axes.

As a close approximation to this procedure Mohr suggests that only the pure tension and pure compression failure circles need be drawn with OA and OB equal to the yield or fracture strengths of the brittle material. Common tangents to these circles may then be used as the failure envelope as shown in Fig. 15.2. Circles drawn tangent to this envelope then represent the condition of failure at the point of tangency.

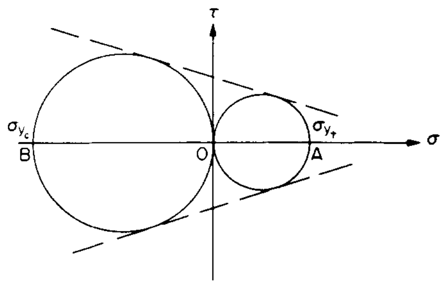

Fig. 15.2. Simplified Mohr theory on σ–τ axes.

In order to develop a theoretical expression for the failure criterion, consider a general stress circle with principal stresses of σ_1 and σ_2. It is then possible to develop an expression relating σ_1, σ_2, the principal stresses, and $\sigma_{y_t}, \sigma_{y_c}$, the yield strengths of the brittle material in tension and compression respectively.

From the geometry of Fig. 15.3,

$$\frac{KL}{KM} = \frac{JL}{MH}$$

Now, in terms of the stresses,

$$KL = \tfrac{1}{2}(\sigma_1 + \sigma_2) - \sigma_1 + \tfrac{1}{2}\sigma_{y_t} = \tfrac{1}{2}(\sigma_{y_t} - \sigma_1 + \sigma_2)$$

$$KM = \tfrac{1}{2}\sigma_{y_t} + \tfrac{1}{2}\sigma_{y_c} = \tfrac{1}{2}(\sigma_{y_t} + \sigma_{y_c})$$

$$JL = \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}\sigma_{y_t} = \tfrac{1}{2}(\sigma_1 + \sigma_2 - \sigma_{y_t})$$

$$MH = \tfrac{1}{2}\sigma_{y_c} - \tfrac{1}{2}\sigma_{y_t} = \tfrac{1}{2}(\sigma_{y_c} - \sigma_{y_t})$$

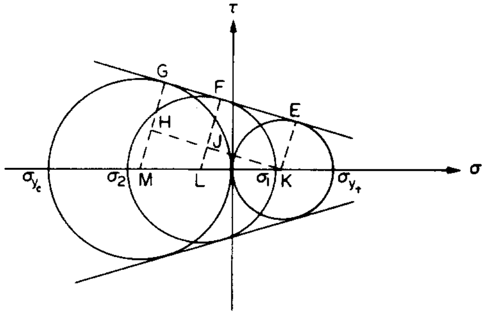

Fig. 15.3.

Substituting,

$$\frac{\sigma_{y_t} - \sigma_1 + \sigma_2}{\sigma_{y_t} + \sigma_{y_c}} = \frac{\sigma_1 + \sigma_2 - \sigma_{y_t}}{\sigma_{y_c} - \sigma_{y_t}}$$

Cross-multiplying and simplifying this reduces to

$$\frac{\sigma_1}{\sigma_{y_t}} + \frac{\sigma_2}{\sigma_{y_c}} = 1 \tag{15.5}$$

which is then the Mohr's modified shear stress criterion for brittle materials.

15.7. Graphical representation of failure theories for two-dimensional stress systems (one principal stress zero)

Having obtained the equations for the elastic failure criteria above in the general three-dimensional stress state it is relatively simple to obtain the corresponding equations when one of the principal stresses is zero.

Each theory may be represented graphically as described below, the diagrams often being termed *yield loci*.

(a) *Maximum principal stress theory*

For simplicity of treatment, ignore for the moment the normal convention for the principal stresses, i.e. $\sigma_1 > \sigma_2 > \sigma_3$ and consider the two-dimensional stress state shown in Fig. 15.4

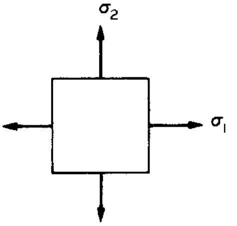

Fig. 15.4. Two-dimensional stress state ($\sigma_3 = 0$).

where σ_3 is zero and σ_2 may be tensile or compressive as appropriate, i.e. σ_2 may have a value less than σ_3 for the purpose of this development.

The maximum principal stress theory then states that failure will occur when σ_1 or $\sigma_2 = \sigma_{y_t}$ or σ_{y_c}. Assuming $\sigma_{y_t} = \sigma_{y_c} = \sigma_y$, these conditions are represented graphically on σ_1, σ_2 coordinates as shown in Fig. 15.5. If the point with coordinates (σ_1, σ_2) representing any complex two-dimensional stress system falls outside the square, then failure will occur according to the theory.

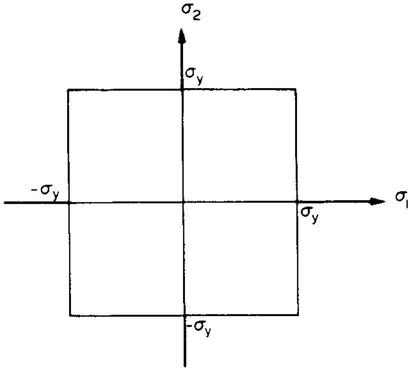

Fig. 15.5. Maximum principal stress failure envelope (locus).

(*b*) *Maximum shear stress theory*

For *like stresses*, i.e. σ_1 and σ_2, both tensile or both compressive (first and third quadrants), the maximum shear stress criterion is

$$\tfrac{1}{2}(\sigma_1 - 0) = \tfrac{1}{2}\sigma_y \quad \text{or} \quad \tfrac{1}{2}(\sigma_2 - 0) = \tfrac{1}{2}\sigma_y$$

i.e.

$$\sigma_1 = \sigma_y \quad \text{or} \quad \sigma_2 = \sigma_y$$

thus producing the same result as the previous theory in the first and third quadrants.

For *unlike stresses* the criterion becomes

$$\tfrac{1}{2}(\sigma_1 - \sigma_2) = \tfrac{1}{2}\sigma_y$$

since consideration of the third stress as zero will not produce as large a shear as that when σ_2 is negative. Thus for the second and fourth quadrants,

$$\frac{\sigma_1}{\sigma_y} - \frac{\sigma_2}{\sigma_y} = 1 \quad \left(\text{or} \quad \frac{\sigma_2}{\sigma_y} - \frac{\sigma_1}{\sigma_y} = 1\right)$$

These are straight lines and produce the failure envelope of Fig. 15.6. Again, any point outside the failure envelope represents a condition of potential failure.

(*c*) *Maximum principal strain theory*

For yielding in tension the theory states that

$$\sigma_1 - \nu\sigma_2 = \sigma_y$$

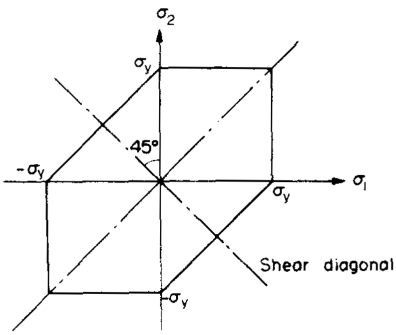

Fig. 15.6. Maximum shear stress failure envelope.

and for compressive yield, with σ_2 compressive,

$$\sigma_2 - \nu\sigma_1 = \sigma_y$$

Since this theory does not find general acceptance in any engineering field it is sufficient to note here, without proof, that the above equations produce the rhomboid failure envelope shown in Fig. 15.7.

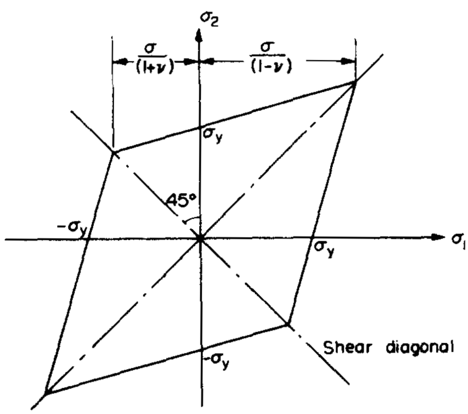

Fig. 15.7. Maximum principal strain failure envelope.

(d) *Maximum strain energy per unit volume theory*

With $\sigma_3 = 0$ this failure criterion reduces to

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma_y^2$$

i.e.

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - 2\nu\left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

This is the equation of an ellipse with major and minor semi-axes

$$\frac{\sigma_y}{\sqrt{(1-\nu)}} \quad \text{and} \quad \frac{\sigma_y}{\sqrt{(1+\nu)}}$$

respectively, each at 45° to the coordinate axes as shown in Fig. 15.8.

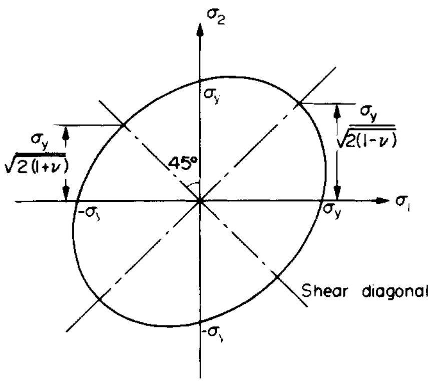

Fig. 15.8. Failure envelope for maximum strain energy per unit volume theory.

(*e*) *Maximum shear strain energy per unit volume theory*

With $\sigma_3 = 0$ the criteria of failure for this theory reduces to

$$\tfrac{1}{2}[(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2] = \sigma_y^2$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$$

$$\left(\frac{\sigma_1}{\sigma_y}\right)^2 + \left(\frac{\sigma_2}{\sigma_y}\right)^2 - \left(\frac{\sigma_1}{\sigma_y}\right)\left(\frac{\sigma_2}{\sigma_y}\right) = 1$$

again an ellipse with semi-axes $\sqrt{(2)}\sigma_y$ and $\sqrt{(\frac{2}{3})}\sigma_y$ at 45° to the coordinate axes as shown in Fig. 15.9. The ellipse will circumscribe the maximum shear stress hexagon.

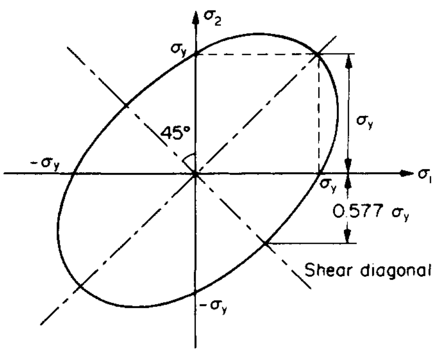

Fig. 15.9. Failure envelope for maximum shear strain energy per unit volume theory.

(*f*) *Mohr's modified shear stress theory* ($\sigma_{y_c} > \sigma_{y_t}$)

For the original formulation of the theory based on the results of pure tension, pure compression and pure shear tests the Mohr failure envelope is as indicated in Fig. 15.10. In its simplified form, however, based on just the pure tension and pure compression results, the failure envelope becomes that of Fig. 15.11.

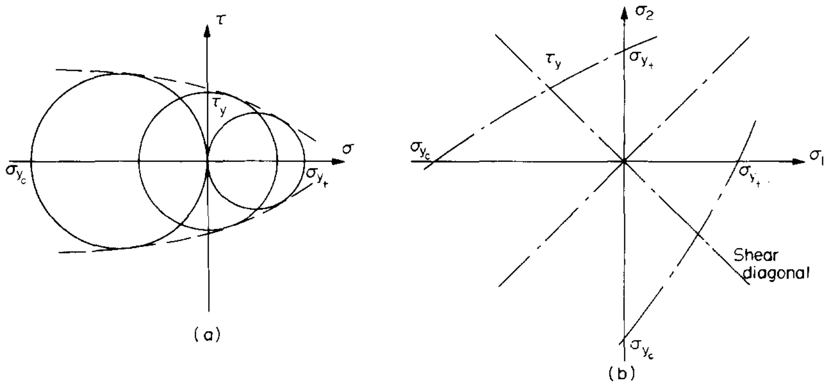

Fig. 15.10. (a) Mohr theory on σ–τ axes. (b) Mohr theory failure envelope on σ_1–σ_2 axes.

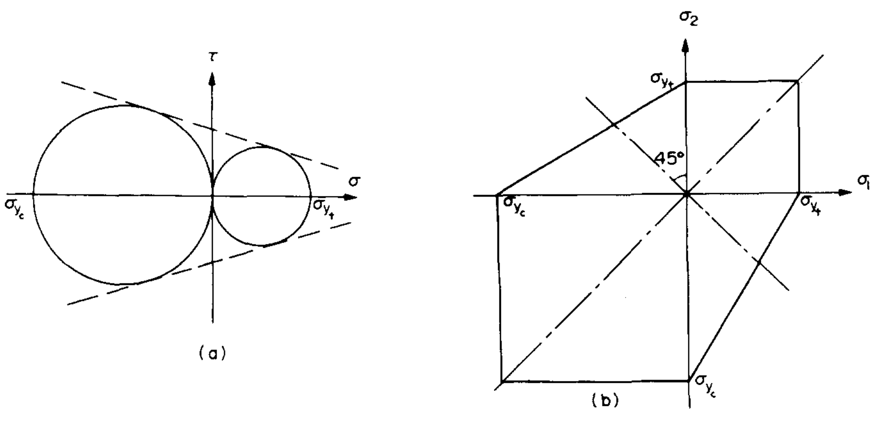

Fig. 15.11. (a) Simplified Mohr theory on σ–τ axes. (b) Failure envelope for simplified Mohr theory.

15.8. Graphical solution of two-dimensional theory of failure problems

The graphical representations of the failure theories, or yield loci, may be combined onto a single set of σ_1 and σ_2 coordinate axes as shown in Fig. 15.12. Inside any particular locus or failure envelope elastic conditions prevail whilst points outside the loci suggest that yielding or fracture will occur. It will be noted that in most cases the maximum shear stress criterion is the most conservative of the theories. The combined diagram is particularly useful since it allows experimental points to be plotted to give an immediate assessment of failure

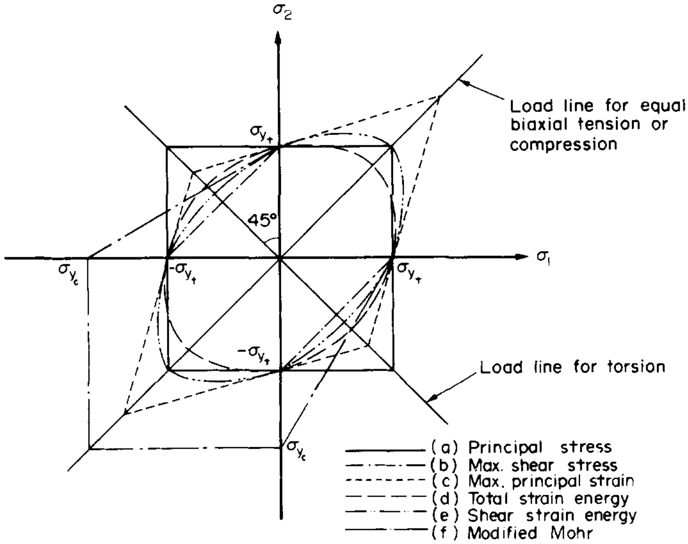

Fig. 15.12. Combined yield loci for the various failure theories.

probability according to the various theories. In the case of equal biaxial tension or compression for example $\sigma_1/\sigma_2 = 1$ and a so-called *load line* may be drawn through the origin with a slope of unity to represent this loading case. This line cuts the yield loci in the order of theories *d*; (*a*, *b*, *e*, *f*); and *c*. In the case of pure torsion, however, $\sigma_1 = \tau$ and $\sigma_2 = -\tau$, i.e. $\sigma_1/\sigma_2 = -1$. This load line will therefore have a slope of -1 and the order of yield according to the various theories is now changed considerably to (*b*; *e*, *f*, *d*, *c*, *a*). The load line procedure may be used to produce rapid solutions of failure problems as shown in Example 15.2.

15.9. Graphical representation of the failure theories for three-dimensional stress systems

15.9.1. Ductile materials

(a) Maximum shear strain energy or distortion energy (von Mises) theory

It has been stated earlier that the failure of most ductile materials is most accurately governed by the distortion energy criterion which states that, at failure,

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2 = \text{constant}$$

In the special case where $\sigma_3 = 0$, this has been shown to give a yield locus which is an ellipse symmetrical about the shear diagonal. For a three-dimensional stress system the above equation defines the surface of a regular prism having a circular cross-section, i.e. a cylinder with its central axis along the line $\sigma_1 = \sigma_2 = \sigma_3$. The axis thus passes through the origin of the principal stress coordinate system shown in Fig. 15.13 and is inclined at equal angles to each

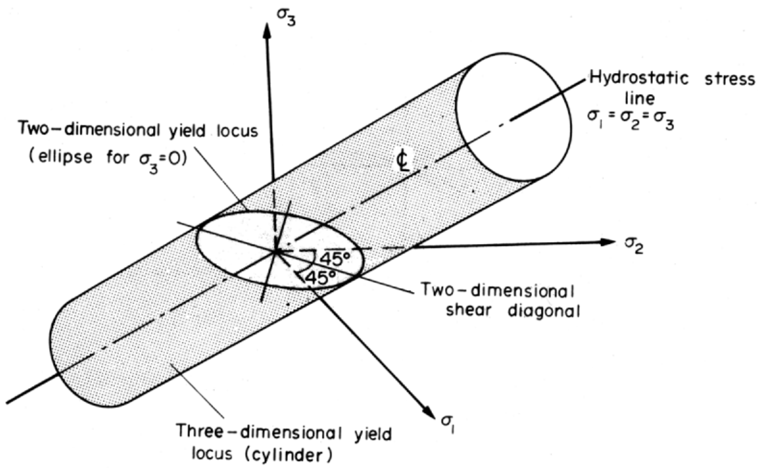

Fig. 15.13. Three-dimensional yield locus for Maxwell–von Mises distortion energy (shear strain energy per unit volume) theory.

axis. It will be observed that when $\sigma_3 = 0$ the failure condition reverts to the ellipse mentioned above, i.e. that produced by intersection of the (σ_1, σ_2) plane with the inclined cylinder.

The yield locus for the von Mises theory in a three-dimensional stress system is thus the *surface* of the inclined cylinder. Points within the cylinder represent safe conditions, points outside indicate failure conditions. It should be noted that the cylinder axis extends indefinitely along the $\sigma_1 = \sigma_2 = \sigma_3$ line, this being termed the ***hydrostatic stress line***. It can be shown that hydrostatic stress alone cannot cause yielding and it is presumed that all other stress conditions which fall within the cylindrical boundary may be considered equally safe.

(*b*) *Maximum shear stress (Tresca) theory*

With a few exceptions, e.g. aluminium alloys and certain steels, the yielding of most ductile materials is adequately governed by the Tresca maximum shear stress condition, and because of its relative simplicity it is often used in preference to the von Mises theory. For the Tresca theory the three-dimensional yield locus can be shown to be a regular prism with hexagonal cross-section (Fig. 15.14). The central axis of this figure is again on the line $\sigma_1 = \sigma_2 = \sigma_3$ (the hydrostatic stress line) and again extends to infinity.

Points representing stress conditions plotted on the principal stress coordinate axes indicate safe conditions if they lie within the surface of the hexagonal cylinder. The two-dimensional yield locus of Fig. 15.6 is obtained as before by the intersection of the σ_1, σ_2 plane $(\sigma_3 = 0)$ with this surface.

15.9.2. Brittle materials

Failure of brittle materials has been shown previously to be governed by the maximum principal tensile stress present in the three-dimensional stress system. This is thought to be

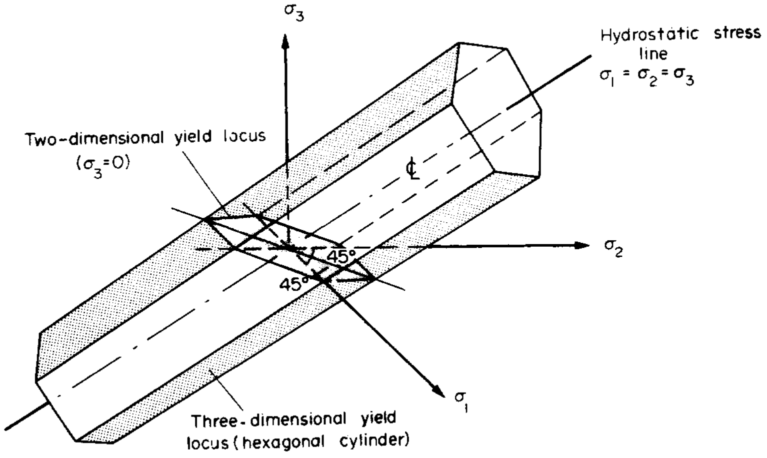

Fig. 15.14. Three-dimensional yield locus for Tresca (maximum shear stress) theory.

due to the microscopic cracks, flaws or discontinuities which are present in most brittle materials and which act as local stress raisers. These stress raisers, or *stress concentrations*, have a much greater adverse effect in tension and hence produce the characteristic weaker behaviour of brittle materials in tension than in compression.

Thus if the greatest tensile principal stress exceeds the yield stress then failure occurs, and such a simple condition does not require a graphical representation.

15.10. Limitations of the failure theories

It is important to remember that the theories introduced above are those of *elastic* failure, i.e. they relate to the "failure" which is assumed to occur under elastic loading conditions at an equivalent stage to that of yielding in a simple tensile test. If it is anticipated that loading conditions are such that the component may fail in service in a way which cannot easily be related to standard simple loading tests (e.g. under fatigue, creep, buckling, impact loading, etc.) then the above "classical" elastic failure theories should not be applied. A good example of this is the brittle fracture failure of steel under low temperature or very high strain rate (impact) conditions compared with simple ductile failure under normal ambient conditions. If any doubt exists about the relevance of the failure theories then, ideally, specially designed tests should be carried out on the component with loading conditions as near as possible to those expected in service. If, however, elastic failure can be assumed to be relevant it is necessary to consider which of the theories is the most appropriate for the material in question and for the service loading condition expected.

In most cases the Von Mises "distortion energy" theory is considered to be the most reliable and relevant theory with the following exceptions:

(a) For brittle materials the maximum principal stress or Mohr "internal friction" theories are most suitable. (It must be noted, however, that the former is definitely unsafe for ductile materials.) Some authorities also recommend the Mohr theory for extension of the theories to ductile *fracture* consideration as opposed to ductile yielding as assumed in the elastic theories.

(b) All theories produce similar results in loading situations where one principal stress is large compared to another. This can be readily appreciated from the graphical representations if a load-line is drawn with a very small positive or negative slope.

(c) The greatest discrepancy between the theories is found in the second and fourth quadrants of the graphical representations where the principal stresses are of opposite sign but numerically equal.

(d) For bi-axial stress conditions, the Mohr modified theory is often preferred, provided that reliable test data are available for tension, compression and torsion.

(e) In most general bi-axial and tri-axial stress conditions the Tresca maximum shear stress theory is the most conservative (i.e. the safest) theory and this, together with its easily applied and simple formula, probably explains its widespread use in industry.

(f) The St. Venant maximum principal strain and Haigh total strain energy per unit volume theories are now rarely, if ever, used in general engineering practice.

15.11. Effect of stress concentrations

Whilst stress concentrations have their most significant effect under fatigue loading conditions and impact situations, nevertheless, there are also some important considerations for static loading applications, namely:

(a) In the presence of ductile yielding, stress concentrations are relatively unimportant since the yielding which will occur at the concentration, e.g. the tip of a notch, will merely redistribute the stresses and not necessarily lead to failure. If, however, there is only marginal ductility, or in the presence of low temperatures, then stress concentrations become more significant as the likelihood of brittle failure increases. It is wise, therefore, to keep stress concentration factors as low as possible.

(b) For brittle materials like cast iron, internal stress concentrations arise within the material due to the presence of, e.g., flaws, impurities or graphite flakes. These produce stress increases at least as large as those given by surface stress concentrations which, therefore, may have little or no effect on failure. A cast iron bar with a small transverse hole, for example, may not fracture at the hole when a tensile load is applied!

15.12. Safety factors

When using elastic design procedures incorporating any of the failure theories introduced in this chapter it is normal to incorporate safety factors to take account of various imponderables which arise when one attempts to forecast accurately service loads or operating conditions or to make allowance for variations in material properties or behaviour from those assumed by the acceptance of "standard" values. "Ideal" application of the theories, i.e. a rigorous mathematical analysis, is thus rarely possible and the following factors indicate in a little more detail the likely sources of inaccuracy:

1. Whilst design may have been based up nominally static loading, changing service conditions or misuse by operators can often lead to dynamic, fluctuating or impact loading situations which will produce significant increases in maximum stress levels.

2. A precise knowledge of the mechanical properties of the material used in the design is seldom available. Standard elastic values found in reference texts assume ideal homogeneous and isotropic materials with equal "strengths" in all directions. This is rarely true in practice and the effect of internal flaws, inclusions or other weaknesses in the material may be quite significant.
3. The method of manufacture or construction of the component can have a significant effect on service life, particularly if residual stresses are introduced by, e.g., welding or straining beyond the elastic limit during the assembly stages.
4. Complex designs often give rise to difficult analysis problems which even after time-consuming and expensive theoretical procedures, at best yield only a reasonable estimate of maximum service stresses.

Despite these problems and the assumptions which are often required to overcome them, it has been shown that elastic design procedures can be made to agree with experimental results within a reasonable margin of error provided that appropriate safety factors are applied.

It has been shown in §1.16 that alternative definitions are used for the safety factor depending upon whether it is based on the tensile strength of the material used or its yield strength, i.e., either

$$\text{safety factor, } n = \frac{\text{tensile strength}}{\text{allowable working stress}}$$

or

$$\text{safety factor, } n = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}}$$

Clearly, *it is important when quoting safety factors to state which definition has been used.*

Safety values vary depending on the type of industry and the area of application of the component being designed. National codes of practice (e.g. British Standards) or other external authority regulations often quote mandatory values to be applied and some companies produce their own guideline values.

Table 15.2 shows the way in which the various factors outlined above contribute to the overall factor of safety for some typical service conditions. These values are based on the yield stress of the materials concerned.

TABLE 15.2. Typical safety factors.

Application	(a) Nature of stress	(b) Nature of load	(c) Type of service	Overall safety factor (a) × (b) × (c)
Steelwork in buildings	1	1	2	2
Pressure vessels	1	1	3	3
Transmission shafts	3	1	2	6
Connecting rods	3	2	1.5	9

It should be noted, however, that the values given in the "type of service" column can be considered to be conservative and severe misuse or overload could increase these (and, hence, the overall factors) by as much as five times.

Recent legislative changes such as "Product Liability" and "Health and Safety at Work" will undoubtedly cause renewed concern that appropriate safety factors are applied, and may lead to the adoption of higher values. Since this could well result in uneconomic utilisation of materials, such a trend would be regrettable and a move to enhanced product testing and service load monitoring is to be preferred.

15.13. Modes of failure

Before concluding this chapter, the first which looks at design procedures to overcome possible failure (in this case elastic overload), it is appropriate to introduce the reader to the many other ways in which components may fail in order that an appreciation is gained of the complexities often facing designers of engineering components. Sub-classification and a certain amount of cross-referencing does make the list appear to be formidably long but even allowing for these it is evident that the designer, together with his supporting materials and stress advisory teams, has an unenviable task if satisfactory performance and reliability of components is to be obtained in the most complex loading situations. The list below is thus a summary of the so-called "*modes (or methods) of failure*"

1. Mechanical overload/under-design
2. Elastic yielding – force and/or temperature induced.
3. Fatigue
   - high cycle
   - low cycle
   - thermal
   - corrosion
   - fretting
   - impact
   - surface
4. Brittle fracture
5. Creep
6. Combined creep and fatigue
7. Ductile rupture
8. Corrosion
   - direct chemical
   - galvanic
   - pitting
   - cavitation
   - stress
   - intergranular
   - crevice
   - erosion
   - hydrogen damage
   - selective leaching
   - biological
   - corrosion fatigue
9. Impact
   - fracture
   - fatigue

deformation
wear
fretting

10. Instability
    buckling
    creep buckling
    torsional instability
11. Wear
    adhesive
    abrasive
    corrosive
    impact
    deformation
    surface fatigue
    fretting
12. Vibration
13. Environmental
    thermal shock
    radiation damage
    lubrication failure
14. Contact
    spalling
    pitting
    galling and seizure
15. Stress rupture
16. Thermal relaxation

Examples

Example 15.1

A material subjected to a simple tension test shows an elastic limit of 240 MN/m^2. Calculate the factor of safety provided if the principal stresses set up in a complex two-dimensional stress system are limited to 140 MN/m^2 tensile and 45 MN/m^2 compressive. The appropriate theories of failure on which your answer should be based are:

(a) the maximum shear stress theory;
(b) the maximum shear strain energy theory.

Solution

(*a*) *Maximum shear stress theory*

This theory states that failure will occur when the maximum shear stress in the material equals the maximum shear stress value at the yield point in a simple tension test, i.e. when

$$\tfrac{1}{2}(\sigma_1 - \sigma_3) = \tfrac{1}{2}\sigma_y$$

or

$$\sigma_1 - \sigma_3 = \sigma_y$$

In this case the system is two-dimensional, i.e. the principal stress in one plane is zero. However, since one of the given principal stresses is a compressive one, it follows that the zero value is that of σ_2 since the negative value of σ_3 associated with the compressive stress will produce a numerically greater value of stress difference $\sigma_1 - \sigma_3$ and hence must be used in the above criterion.

Thus $\sigma_1 = 140\,\text{MN/m}^2$, $\sigma_2 = 0$ and $\sigma_3 = -45\,\text{MN/m}^2$.

Now with a factor of safety applied the design yield point becomes σ_y/n and this must replace σ_y in the yield criterion which then becomes

$$\frac{\sigma_y}{n} = \sigma_1 - \sigma_3$$

$$\therefore \qquad \frac{240}{n} = 140 - (-45) \quad \text{units of MN/m}^2 \text{ throughout}$$

$$n = \frac{240}{185} = \mathbf{1.3}$$

The required factor of safety is 1.3.

(b) Maximum shear strain energy theory

Once again equating the values of the quantity concerned in the tensile test and in the complex stress system,

$$\frac{\sigma_y^2}{6G} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$\sigma_y^2 = \tfrac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

With the three principal stress values used above and with σ_y/n replacing σ_y

$$\left(\frac{240}{n}\right)^2 = \tfrac{1}{2}\{(140 - 0)^2 + [0 - (-45)]^2 + (-45 - 140)^2\}$$

$$\frac{5.76 \times 10^4}{n^2} = \tfrac{1}{2}[1.96 + 0.203 + 3.42]10^4$$

$$n^2 = \frac{2 \times 5.76 \times 10^4}{5.583 \times 10^4} = 2.063$$

$$\therefore \qquad n = \mathbf{1.44}$$

The required factor of safety is now 1.44.

Example 15.2

A steel tube has a mean diameter of 100 mm and a thickness of 3 mm. Calculate the torque which can be transmitted by the tube with a factor of safety of 2.25 if the criterion of failure is (a) maximum shear stress; (b) maximum strain energy; (c) maximum shear strain energy. The elastic limit of the steel in tension is $225\,\text{MN/m}^2$ and Poisson's ratio ν is 0.3.

Solution

From the torsion theory

$$\frac{T}{J} = \frac{\tau}{R} \quad \therefore \quad \tau = \frac{TR}{J}$$

Now mean diameter of tube = 100 mm and thickness = 3 mm.

$$\therefore \qquad J = \pi dt \times r^2 = \frac{\pi d^3 t}{4} \quad \text{(approximately)}$$

$$= \frac{\pi \times 0.1^3 \times 0.003}{4} = 2.36 \times 10^{-6}\,\text{m}^4$$

$$\therefore \qquad \text{shear stress } \tau = \frac{T \times 51.5 \times 10^{-3}}{2.36 \times 10^{-6}} = (2.18 \times 10^4)T\ \text{N/m}^2$$

$$= 21.8T\ \text{kN/m}^2$$

(*a*) *Maximum shear stress*

Torsion introduces pure shear onto elements within the tube material and it has been shown in § 13.2 that pure shear produces an equivalent principal direct stress system, one tensile and one compressive and both equal in value to the applied shear stress,

i.e. $\sigma_1 = \tau, \quad \sigma_3 = -\tau \quad (\text{and } \sigma_2 = 0)$

Thus for the maximum shear stress criterion, taking account of the safety factor,

$$\frac{\sigma_y}{n} = \sigma_1 - \sigma_3 = \tau - (-\tau)$$

$$\therefore \qquad \frac{225 \times 10^6}{2.25} = 2\tau = 2 \times 21.8T \times 10^3$$

$$\therefore \qquad T = \frac{100 \times 10^6}{2 \times 21.8 \times 10^3} = \mathbf{2.3 \times 10^3\,N\,m}$$

The torque which can be safely applied = 2.3 kN m.

(*b*) *Maximum strain energy*

From eqn. (15.3) the relevant criterion of failure is

$$\sigma_y^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$

Taking account of the safety factor

$$\left(\frac{225 \times 10^6}{2.25}\right)^2 = \tau^2 + 0 + (-\tau)^2 - 2 \times 0.3[\tau \times (-\tau)]$$

$$= 2.6\tau^2$$

$$= 2.6(21.8 \times 10^3 T)^2$$

$$\therefore \quad T = \frac{100 \times 10^6}{\sqrt{(2.6)} \times 21.8 \times 10^3} = \mathbf{2.84 \times 10^3\ N\,m}$$

The safe torque is now 2.84 kN m.

(*c*) *Maximum shear strain energy*

From eqn. (15.4) the criterion of failure is

$$\sigma_y^2 = \tfrac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$\therefore \quad \left(\frac{225 \times 10^6}{2.25}\right)^2 = \tfrac{1}{2}\{(\tau - 0)^2 + [0 - (-\tau)]^2 + (-\tau - \tau)^2\}$$

$$= 3\tau^2$$

$$\therefore \quad \tau = \frac{100 \times 10^6}{\sqrt{3}} = 21.8 \times 10^3 T$$

$$\therefore \quad T = \frac{100 \times 10^6}{21.8 \times 10^3 \times \sqrt{3}} = \mathbf{2.65 \times 10^3\ N\,m}$$

The safe torque is now 2.65 kN m.

Example 15.3

A structure is composed of circular members of diameter *d*. At a certain position along one member the loading is found to consist of a shear force of 10 kN together with an axial tensile load of 20 kN. If the elastic limit in tension of the material of the members is 270 MN/m² and there is to be a factor of safety of 4, estimate the magnitude of *d* required according to (a) the maximum principal stress theory, and (b) the maximum shear strain energy per unit volume theory. Poisson's ratio $\nu = 0.283$.

Solution

The stress system at the point concerned is as shown in Fig. 15.15, the principal stress normal to the surface of the member being zero.

Now the direct stress along the axis of the bar is tensile, i.e. positive, and given by

$$\sigma_x = \frac{\text{load}}{\text{area}} = \frac{20}{\pi d^2/4} = \frac{80}{\pi d^2} \quad \text{kN/m}^2$$

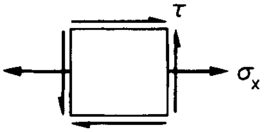

Fig. 15.15.

and the shear stress is

$$\tau = \frac{\text{shear load}}{\text{area}} = \frac{10}{\pi d^2/4} = \frac{40}{\pi d^2} \quad \text{kN/m}^2$$

The principal stresses are given by Mohr's circle construction (πd^2 being a common denominator) or from

$$\sigma_1 \quad \text{and} \quad \sigma_3 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

with σ_y zero,

i.e.
$$\sigma_1 \quad \text{or} \quad \sigma_3 = \frac{1}{2}\left\{\frac{80}{\pi d^2} \pm \sqrt{\left[\left(\frac{80}{\pi d^2}\right)^2 + 4\left(\frac{40}{\pi d^2}\right)^2\right\}}\right.$$

$$= \frac{40}{\pi d^2}(1 \pm \sqrt{2})$$

∴
$$\sigma_1 = \frac{40 \times 2.414}{\pi d^2} = \frac{30.7}{d^2} \quad \text{kN/m}^2$$

$$\sigma_3 = -\frac{40 \times 0.414}{\pi d^2} = -\frac{5.27}{d^2} \quad \text{kN/m}^2$$

and
$$\sigma_2 = 0$$

Since the elastic limit in tension is 270 MN/m^2 and the factor of safety is 4, the working stress or effective yield stress is

$$\sigma_y = \frac{270}{4} = 67.5 \, \text{MN/m}^2$$

(a) *Maximum principal stress theory*

Failure is assumed to occur when

$$\sigma_1 = \sigma_y$$

∴
$$\frac{30.7 \times 10^3}{d^2} = 67.5 \times 10^6$$

∴
$$d^2 = \frac{30.7}{67.5} \times 10^{-3} = 4.55 \times 10^{-4} \, \text{m}^2$$

∴
$$d = 2.13 \times 10^{-2} \, \text{m} = \mathbf{21.3\,mm}$$

(b) *Maximum shear strain energy*

From eqn. (15.4) the criterion of failure is

$$2\sigma_y^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

Therefore taking account of the safety factor

$$2(67.5 \times 10^6)^2 = \left[\left(\frac{30.7}{d^2}\right)^2 + \left(-\frac{5.27}{d^2}\right)^2 + \left(\frac{-5.27 - 30.7}{d^2}\right)^2\right] \times 10^6$$

$$= \frac{2264 \times 10^6}{d^4}$$

$$\therefore \quad d^4 = \frac{1132 \times 10^6}{(67.5 \times 10^6)^2}$$

$$\therefore \quad d^2 = \frac{33.6 \times 10^3}{67.5 \times 10^6} = 4.985 \times 10^{-4}\,\text{m}^2$$

$$\therefore \quad d = \mathbf{22.3\,mm}$$

Example 15.4

Assuming the formulae for the principal stresses and the maximum shear stress induced in a material owing to combined stresses and the fundamental formulae for pure bending, derive a formula in terms of the bending moment M and the twisting moment T for the equivalent twisting moment on a shaft subjected to combined bending and torsion for

(a) the maximum principal stress criterion;
(b) the maximum shear stress criterion.

Solution

The *equivalent torque*, or turning moment, is defined as that torque which, acting alone, will produce the same conditions of stress as the combined bending and turning moments.

At failure the stress produced by the equivalent torque T_E is given by the torsion theory

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\therefore \quad \tau_{max} = \frac{T_E R}{J} = \frac{T_E \times D}{2J}$$

The direct stress owing to bending is

$$\sigma_x = \frac{M y_{max}}{I} = \frac{MD}{2I} = \frac{MD}{J}$$

and the shear stress due to torsion is

$$\tau = \frac{TD}{2J}$$

The principal stresses are then given by

$$\sigma_{1,3} = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \quad \text{with} \quad \sigma_y = 0 \quad \text{and} \quad \sigma_2 = 0$$

$$= \frac{1}{2}\left(\frac{MD}{J}\right) \pm \frac{1}{2}\sqrt{\left[\left(\frac{MD}{J}\right)^2 + 4\left(\frac{TD}{2J}\right)^2\right]}$$

$$= \frac{D}{2J}(M \pm \sqrt{[M^2 + T^2]})$$

$$\therefore \qquad \sigma_1 = \frac{D}{2J}(M + \sqrt{[M^2 + T^2]})$$

$$\sigma_3 = \frac{D}{2J}(M - \sqrt{[M^2 + T^2]})$$

(a) *For maximum principal stress criterion*

$$\frac{T_E D}{2J} = \sigma_1 = \frac{D}{2J}(M + \sqrt{[M^2 + T^2]})$$

$$\therefore \qquad T_E = M + \sqrt{(M^2 + T^2)}$$

(b) *For maximum shear stress criterion*

$$\frac{T_E D}{2J} = \tfrac{1}{2}(\sigma_1 - \sigma_3)$$

$$= \frac{1}{2}\left\{\frac{D}{2J}(M + \sqrt{[M^2 + T^2]}) - \frac{D}{2J}(M - \sqrt{[M^2 + T^2]})\right\}$$

$$T_E = \sqrt{(M^2 + T^2)}$$

Example 15.5

The test strengths of a material under pure compression and pure tension are $\sigma_{y_c} = 350\ \mathrm{MN/m^2}$ and $\sigma_{y_t} = 300\ \mathrm{MN/m^2}$. In a certain design of component the material may be subjected to each of the five biaxial stress states shown in Fig. 15.16. Assuming that failure is deemed to occur when yielding takes place, arrange the five stress states in order of diminishing factor of safety according to the maximum principal or normal stress, maximum shear stress, maximum shear strain energy (or distortion energy) and modified Mohr's (or internal friction) theories.

Solution

A graphical solution of this problem can be employed by constructing the combined yield loci for the criteria mentioned in the question. Since σ_1 the maximum principal stress is $+100\ \mathrm{MN/m^2}$ in each of the stress states only half the combined loci diagram is required, i.e. the positive σ_1 half.

Here it must be remembered that for stress condition (e) pure shear is exactly equivalent to two mutually perpendicular direct stresses – one tensile, the other compressive, acting on 45° planes and of equal value to the applied shear, i.e. for condition (e) $\sigma_1 = 100\ \mathrm{MN/m^2}$ and $\sigma_2 = -100\ \mathrm{MN/m^2}$ (see §13.2).

It is now possible to construct the "load lines" for each stress state with slopes of σ_2/σ_1. An immediate solution is then obtained by considering the intersection of each load line with the failure envelopes.

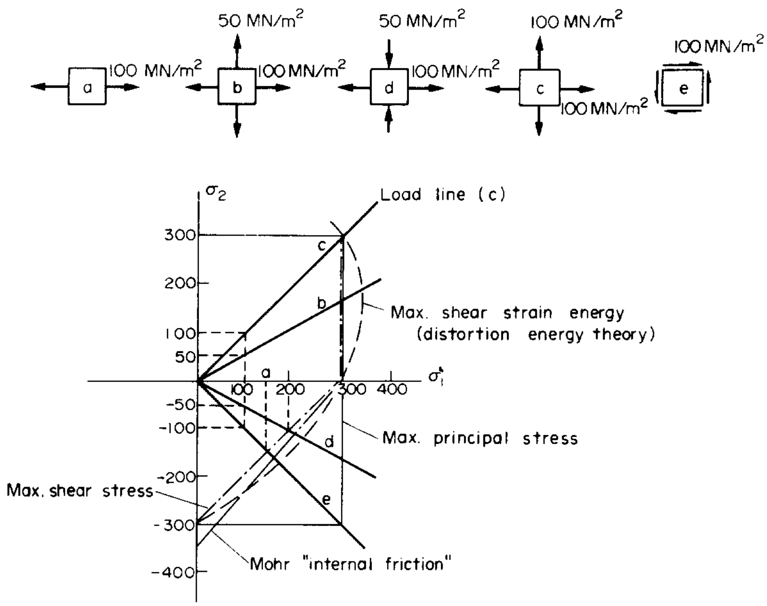

Fig. 15.16.

Maximum principal stress theory

All five load lines cut the failure envelope for this theory at $\sigma_1 = 300\ \text{MN/m}^2$. According to this theory, therefore, all the stress states will produce failure when the maximum direct stress reaches $300\ \text{MN/m}^2$. Since the maximum principal stress present in each stress state is $100\ \text{MN/m}^2$ it therefore follows that the safety factor for each state according to the maximum principal stress theory is $\dfrac{300}{100} = 3$.

Maximum shear stress theory

The load lines a, b and c cut the failure envelope for this theory at $\sigma_1 = 300\ \text{MN/m}^2$ whilst d and e cut it at $\sigma_1 = 200\ \text{MN/m}^2$ and $\sigma_1 = 150\ \text{MN/m}^2$ respectively as shown in Fig. 15.16. The safety factors are, therefore,

$$a, b, c = \frac{300}{100} = \mathbf{3}, \quad d = \frac{200}{100} = \mathbf{2}, \quad e = \frac{150}{100} = \mathbf{1.5}$$

Maximum shear strain energy theory

In decreasing order, the factors of safety for this theory, found as before from the points where each load line crosses the failure envelope, are

$$b = \frac{347}{100} = \mathbf{3.47}, \quad a, c = \frac{300}{100} = \mathbf{3}, \quad d = \frac{227}{100} = \mathbf{2.27}, \quad e = \frac{173}{100} = \mathbf{1.73}$$

Mohr's modified or internal friction theory (with $\sigma_{y_c} = 350\,MN/m^2$)

In this case the safety factors are:

$$a, b, c = \frac{300}{100} = 3, \quad d = \frac{210}{100} = \mathbf{2.1}, \quad e = \frac{162}{100} = \mathbf{1.62}$$

Example 15.6

The cast iron used in the manufacture of an engineering component has tensile and compressive strengths of $400\,MN/m^2$ and $1.20\,GN/m^2$ respectively.

(a) If the maximum value of the tensile principal stress is to be limited to one-quarter of the tensile strength, determine the maximum value and nature of the other principal stress using Mohr's modified yield theory for brittle materials.

(b) What would be the values of the principal stresses associated with a maximum shear stress of $450\,MN/m^2$ according to Mohr's modified theory?

(c) At some point in a component principal stresses of $100\,MN/m^2$ tensile and $100\,MN/m^2$ compressive are found to be present. Estimate the safety factor with respect to initial yield using the maximum principal stress, maximum shear stress, distortion energy and Mohr's modified theories of elastic failure.

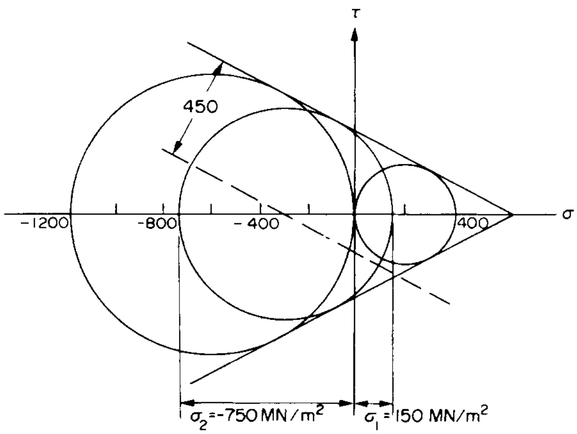

Fig. 15.17.

Solution

(a) Maximum principal stress $= \frac{400}{4} = 100\,MN/m^2$

According to Mohr's theory

$$\frac{\sigma_1}{\sigma_{y_t}} + \frac{\sigma_2}{\sigma_{y_c}} = 1$$

$$\therefore \quad \frac{100 \times 10^6}{400 \times 10^6} + \frac{\sigma_2}{-1.2 \times 10^9} = 1$$

$$\therefore \quad \sigma_2 = -1.2 \times 10^9(1 - \tfrac{1}{4}) = \mathbf{-900\ MN/m^2}$$

(b) In any Mohr circle construction the radius of the circle equals the maximum shear stress value. In order to answer this part of the question, therefore, it is necessary to draw the Mohr failure envelope on σ–τ axes as shown in Fig. 15.17 and to construct the circle which is tangential to the envelope and has a radius of 450 MN/m^2. This is achieved by drawing a line parallel to the failure envelope and a distance of 450 MN/m^2 (to scale) from it. Where this line cuts the σ axis is then the centre of the required circle. The desired principal stresses are then, as usual, the extremities of the horizontal diameter of the circle.

Thus from Fig. 15.17

$$\boldsymbol{\sigma_1} = \mathbf{150\ MN/m^2} \quad \text{and } \boldsymbol{\sigma_2} = \mathbf{-750\ MN/m^2}$$

(c) The solution here is similar to that used for Example 15.5. The yield loci are first plotted for the given failure theories and the required safety factors determined from the points of intersection of the loci and the load line with a slope of $100/-100 = -1$.

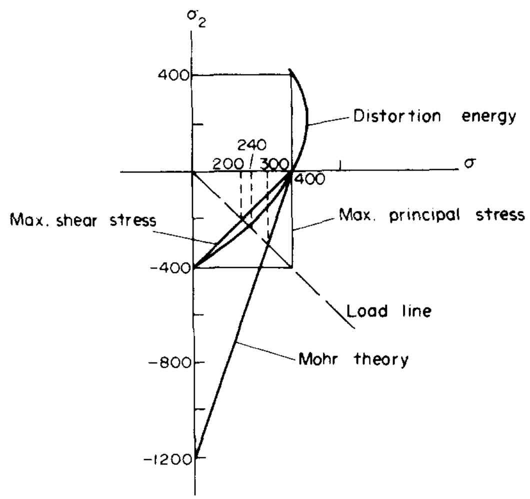

Fig. 15.18.

Thus from Fig. 15.18 the safety factors are:

$$\text{Maximum principal stress} = \frac{400}{100} = \mathbf{4}$$

$$\text{Maximum shear stress} = \frac{200}{100} = \mathbf{2}$$

Distortion energy $= \frac{240}{100} = \mathbf{2.4}$

Mohr theory $= \frac{300}{100} = \mathbf{3}$

Problems

15.1 (B). If the principal stresses at a point in an elastic material are 120 MN/m^2 tensile, 180 MN/m^2 tensile and 75 MN/m^2 compressive, find the stress at the limit of proportionality expected in a simple tensile test assuming:
(a) the maximum shear stress theory;
(b) the maximum shear strain energy theory;
(c) the maximum principal strain theory.
Assume $\nu = 0.294$. [255, 230.9, 166.8 MN/m^2.]

15.2 (B). A horizontal shaft of 75 mm diameter projects from a bearing, and in addition to the torque transmitted the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tensile test, is 135 MN/m^2, find the safe torque to which the shaft may be subjected using as the criterion (a) the maximum shearing stress; (b) the maximum strain energy. Poisson's ratio $\nu = 0.29$. [U.L.] [5.05, 6.3 kN m.]

15.3 (B). Show that the strain energy per unit volume of a material under a single direct stress is given by $\frac{1}{2}$ (stress × strain). Hence show that for a material under the action of the principal stresses σ_1, σ_2 and σ_3 the strain energy per unit volume becomes

$$\frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)]$$

A thin cylinder 1 m diameter and 3 m long is filled with a liquid to a pressure of 2 MN/m^2. Assuming a yield stress for the material of 240 MN/m^2 in simple tension and a safety factor of 4, determine the necessary wall thickness of the cylinder, taking the maximum shear strain energy as the criterion of failure.
For the cylinder material, $E = 207$ GN/m^2 and $\nu = 0.286$. [14.4 mm.]

15.4 (B). An aluminium-alloy tube of 25 mm outside diameter and 22 mm inside diameter is to be used as a shaft. It is 500 mm long, in self-aligning bearings, and supports a load of 0.5 kN at mid-span. In order to find the maximum allowable shear stress a length of tube was tested in tension and reached the limit of proportionality at 21 kN. Assuming the criterion for elastic failure to be the maximum shear stress, find the greatest torque to which the shaft could be subjected. [98.2 N m.]

15.5 (B). A bending moment of 4 kN m is found to cause elastic failure of a solid circular shaft. An exactly similar shaft is now subjected to a torque T. Determine the value of T which will cause failure of the shaft according to the following theories:
(a) maximum principal stress;
(b) maximum principal strain;
(c) maximum shear strain energy. ($\nu = 0.3$.)
Which of these values would you expect to be the most reliable and why? [8, 6.15, 4.62 kN m.]

15.6 (B). A thin cylindrical pressure vessel with closed ends is required to withstand an internal pressure of 4 MN/m^2. The inside diameter of the vessel is to be 500 mm and a factor of safety of 4 is required. A sample of the proposed material tested in simple tension gave a yield stress of 360 MN/m^2.
Find the thickness of the vessel, assuming the criterion of elastic failure to be (a) the maximum shear stress, (b) the shear strain energy. [E.M.E.U.] [11.1, 9.62 mm.]

15.7 (B). Derive an expression for the strain energy stored in a material when subjected to three principal stresses.
A material is subjected to a system of three mutually perpendicular stresses as follows: f tensile, $2f$ tensile and f compressive. If this material failed in simple tension at a stress of 180 MN/m^2, determine the value of f if the criterion of failure is:

(a) maximum principal stress;
(b) maximum shear stress;
(c) maximum strain energy.
Take Poisson's ratio $\nu = 0.3$. [90, 60, 70 MN/m^2.]

15.8 (B). The external and internal diameters of a hollow steel shaft are 150 mm and 100 mm. A power transmission test with a torsion dynamometer showed an angle of twist of 0.13 degree on a 250 mm length when the speed was 500 rev/min. Find the power being transmitted and the torsional strain energy per metre length.

If, in addition to this torque, a bending moment of 15 kN m together with an axial compressive force of 80 kN also acted upon the shaft, find the value of the equivalent stress in simple tension corresponding to the maximum shear strain energy theory of elastic failure. Take $G = 80\,\text{GN/m}^2$.

[I.Mech.E.] [1.52 MW; 13.13 J/m; 113 MN/m^2.]

15.9 (B/C). A close-coiled helical spring has a wire diameter of 2.5 mm and a mean coil diameter of 40 mm. The spring is subjected to a combined axial load of 60 N and a torque acting about the axis of the spring. Determine the maximum permissible torque if (a) the material is brittle and ultimate failure is to be avoided, the criterion of failure is the maximum tensile stress, and the ultimate tensile stress is $1.2\,\text{GN/m}^2$, (b) the material is ductile and failure by yielding is to be avoided, the criterion of failure is the maximum shear stress, and the yield in tension is $0.9\,\text{GN/m}^2$.

[I.Mech.E.] [1.645, 0.68 N m.]

15.10 (C). A closed-ended thick-walled steel cylinder with a diameter ratio of 2 is subjected to an internal pressure. If yield occurs at a pressure of $270\,\text{MN/m}^2$ find the yield strength of the steel used and the diametral strain at the bore at yield. Yield can be assumed to occur at a critical value of the maximum shear stress. It can be assumed that the stresses in a thick-walled cylinder are:

$$\text{hoop stress } \sigma_H = A + \frac{B}{r^2}$$

$$\text{radial stress } \sigma_r = A - \frac{B}{r^2}$$

$$\text{axial stress } \sigma_L = \tfrac{1}{2}(\sigma_H + \sigma_r)$$

where A and B are constants and r is any radius.

For the cylinder material $E = 210\,\text{GN/m}^2$ and $\nu = 0.3$. [I.Mech.E.] [721 MN/m^2; 2.4×10^{-3}.]

15.11 (C). For a certain material subjected to plane stress it is assumed that the criterion of elastic failure is the shear strain energy per unit volume. By considering co-ordinates relative to two axes at 45° to the principal axes, show that the limiting values of the two principal stresses can be represented by an ellipse having semi-diameters $\sigma_e\sqrt{2}$ and $\sigma_e\sqrt{\tfrac{2}{3}}$, where σ_e is the equivalent simple tension. Hence show that for a given value of the major principal stress the elastic factor of safety is greatest when the minor principal stress is half the major, both stresses being of the same sign. [U.L.]

15.12 (C). A horizontal circular shaft of diameter d and second moment of area I is subjected to a bending moment $M\cos\theta$ in a vertical plane and to an axial twisting moment $M\sin\theta$. Show that the principal stresses at the ends of a vertical diameter are $\tfrac{1}{2}Mk(\cos\theta \pm 1)$, where

$$k = \frac{d}{2I}$$

If strain energy is the criterion of failure, show that

$$\tau_{max} = \frac{\tau_0\sqrt{2}}{[\cos^2\theta(1-\nu) + (1+\nu)]^{\frac{1}{2}}}$$

where τ_{max} = maximum shearing stress,
τ_0 = maximum shearing stress in the special case when $\theta = 0$,
ν = Poisson's ratio. [U.L.]

15.13 (C). What are meant by the terms "yield criterion" and "yield locus" as related to ductile metals and why, in general, are principal stresses involved?

Define the maximum shear stress and shear strain energy theories of yielding. Describe the three-dimensional loci and sketch the plane stress loci for the above theories. [C.E.I.]

15.14 (B). The maximum shear stress theory of elastic failure is sometimes criticised because it makes no allowance for the magnitude of the intermediate principal stress. On these grounds a theory is preferred which predicts that yield will not occur provided that

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 < 2\sigma_y^2$$

What is the criterion of failure implied here?

Assuming that σ_1 and σ_3 are fixed and unequal, find the value of σ_2 which will be most effective in preventing failure according to this theory. If this theory is correct, by what percentage does the maximum shear stress theory underestimate the strength of a material in this case? [City U.] [$\tfrac{1}{2}(\sigma_1 + \sigma_3)$; 13.4%.]

15.15 (B) The cast iron used in the manufacture of an engineering component has tensile and compressive strengths of 400 MN/m^2 and 1.20 GN/m^2 respectively.

(a) If the maximum value of the tensile principal stress is to be limited to one-quarter of the tensile strength, determine the maximum value and nature of the other principal stress using Mohr's modified yield theory for brittle materials.

(b) What would be the values of the principal stresses associated with a maximum shear stress of 450 MN/m^2 according to Mohr's modified theory?

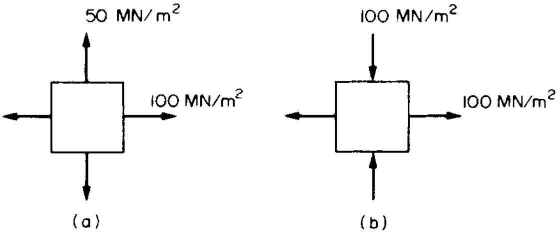

Fig. 15.19.

(c) Estimate the safety factor with respect to initial failure for the stress conditions of Fig. 15.19 using the maximum principal stress, maximum shear stress, distortion energy and Mohr's modified theories of elastic failure. [B.P.] [−900 MN/m^2; 150, −750 MN/m^2; 4, 4, 4.7, 4 and 4, 2, 2.4, 3.]

15.16 (B). Show that for a material subjected to two principal stresses, σ_1 and σ_2, the strain energy per unit volume is

$$\frac{1}{2E}(\sigma_1{}^2 + \sigma_2{}^2 - 2\sigma_1\sigma_2)$$

A thin-walled steel tube of internal diameter 150 mm, closed at its ends, is subjected to an internal fluid pressure of 3 MN/m^2. Find the thickness of the tube if the criterion of failure is the maximum strain energy. Assume a factor of safety of 4 and take the elastic limit in pure tension as 300 MN/m^2. Poisson's ratio $\nu = 0.28$.

[I.Mech.E.] [2.95 mm]

15.17 (B). A circular shaft, 100 mm diameter is subjected to combined bending moment and torque, the bending moment being 3 times the torque. If the direct tension yield point of the material is 300 MN/m^2 and the factor of safety on yield is to be 4, calculate the allowable twisting moment by the three following theories of failure:

(a) Maximum principal stress theory
(b) Maximum shear stress theory
(c) Maximum shear strain energy theory. [U.L.] [2.86, 2.79, 2.83 kNm]

15.18 (B). A horizontal shaft of 75 mm diameter projects from a bearing and, in addition to the torque transmitted, the shaft carries a vertical load of 8 kN at 300 mm from the bearing. If the safe stress for the material, as determined in a simple tension test, is 135 MN/m^2 find the safe torque to which the shaft may be subjected using as a criterion

(a) the maximum shearing, stress,
(b) the maximum strain energy per unit volume.

Poisson's ratio $\nu = 0.29$. [U.L.] [5.59, 8.3 kNm.]