CHAPTER 1

UNSYMMETRICAL BENDING

Summary

The second moments of area of a section are given by

$$I_{xx} = \int y^2\,dA \quad \text{and} \quad I_{yy} = \int x^2\,dA$$

The product second moment of area of a section is defined as

$$I_{xy} = \int xy\,dA$$

which reduces to $I_{xy} = Ahk$ for a rectangle of area A and centroid distance h and k from the X and Y axes.

The *principal second moments of area* are the maximum and minimum values for a section and they occur about the principal axes. *Product second moments of area about principal axes are zero.*

With a knowledge of I_{xx}, I_{yy} and I_{xy} for a given section, the principal values may be determined using either Mohr's or Land's circle construction.

The following relationships apply between the second moments of area about different axes:

$$I_u = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta$$

$$I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta$$

where θ is the angle between the U and X axes, and is given by

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})}$$

Then

$$I_u + I_v = I_{xx} + I_{yy}$$

The second moment of area about the neutral axis is given by

$$I_{\text{N.A.}} = \tfrac{1}{2}(I_u + I_v) + \tfrac{1}{2}(I_u - I_v)\cos 2\alpha_u$$

where α_u is the angle between the neutral axis (N.A.) and the U axis.

Also

$$I_{xx} = I_u \cos^2\theta + I_v \sin^2\theta$$

$$I_{yy} = I_v \cos^2\theta + I_u \sin^2\theta$$

$$I_{xy} = \tfrac{1}{2}(I_v - I_u)\sin 2\theta$$

$$I_{xx} - I_{yy} = (I_u - I_v)\cos 2\theta$$

Stress determination

For skew loading and other forms of bending about principal axes

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

where M_u and M_v are the components of the applied moment about the U and V axes.
Alternatively, with $\sigma = Px + Qy$

$$M_{xx} = PI_{xy} + QI_{xx}$$

$$M_{yy} = -PI_{yy} - QI_{xy}$$

Then the inclination of the N.A. to the X axis is given by

$$\tan\alpha = -\frac{P}{Q}$$

As a further alternative,

$$\sigma = \frac{M'n}{I_{\text{N.A.}}}$$

where M' is the component of the applied moment about the N.A., $I_{N.A.}$ is determined either from the momental ellipse or from the Mohr or Land constructions, and n is the perpendicular distance from the point in question to the N.A.

Deflections of unsymmetrical members are found by applying standard deflection formulae to bending about either the principal axes or the N.A. taking care to use the correct component of load and the correct second moment of area value.

Introduction

It has been shown in Chapter 4 of *Mechanics of Materials 1*† that the simple bending theory applies when bending takes place about an axis which is perpendicular to a plane of symmetry. If such an axis is drawn through the centroid of a section, and another mutually perpendicular to it also through the centroid, then these axes are principal axes. Thus a plane of symmetry is automatically a principal axis. Second moments of area of a cross-section about its principal axes are found to be maximum and minimum values, while the product second moment of area, $\int xy\,dA$, is found to be zero. All plane sections, whether they have an axis of symmetry or not, have two perpendicular axes about which the product second moment of area is zero. *Principal axes are thus defined as the axes about which the product second moment of area is zero.* Simple bending can then be taken as bending which takes place about a principal axis, moments being applied in a plane parallel to one such axis.

In general, however, moments are applied about a convenient axis in the cross-section; the plane containing the applied moment may not then be parallel to a principal axis. Such cases are termed "unsymmetrical" or "asymmetrical" bending.

The most simple type of unsymmetrical bending problem is that of "skew" loading of sections containing at least one axis of symmetry, as in Fig. 1.1. This axis and the axis

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

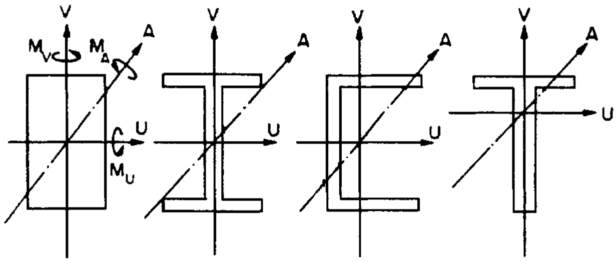

Fig. 1.1. Skew loading of sections containing one axis of symmetry.

perpendicular to it are then principal axes and the term skew loading implies load applied at some angle to these principal axes. The method of solution in this case is to resolve the applied moment M_A about some axis A into its components about the principal axes. Bending is then assumed to take place simultaneously about the two principal axes, the total stress being given by

$$\sigma = \frac{M_u v}{I_u} + \frac{M_v u}{I_v}$$

With at least one of the principal axes being an axis of symmetry the second moments of area about the principal axes I_u and I_v can easily be determined.

With unsymmetrical sections (e.g. angle-sections, Z-sections, etc.) the principal axes are not easily recognized and the second moments of area about the principal axes are not easily found except by the use of special techniques to be introduced in §§1.3 and 1.4. In such cases an easier solution is obtained as will be shown in §1.8. Before proceeding with the various methods of solution of unsymmetrical bending problems, however, it is advisable to consider in some detail the concept of principal and product second moments of area.

1.1. Product second moment of area

Consider a small element of area in a plane surface with a centroid having coordinates (x, y) relative to the X and Y axes (Fig. 1.2). The second moments of area of the surface about the X and Y axes are defined as

$$I_{xx} = \int y^2\,dA \quad \text{and} \quad I_{yy} = \int x^2\,dA \tag{1.1}$$

Similarly, the product second moment of area of the section is defined as follows:

$$I_{xy} = \int xy\,dA \tag{1.2}$$

Since the cross-section of most structural members used in bending applications consists of a combination of rectangles the value of the product second moment of area for such sections is determined by the addition of the I_{xy} value for each rectangle (Fig. 1.3),

i.e. $$I_{xy} = Ahk \tag{1.3}$$

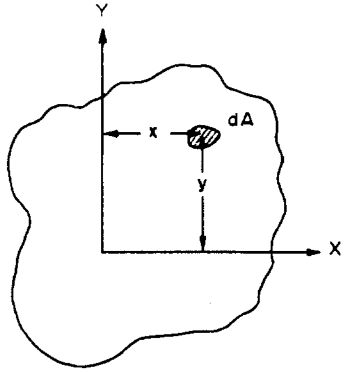

Fig. 1.2.

where h and k are the distances of the centroid of each rectangle from the X and Y axes respectively (taking account of the normal sign convention for x and y) and A is the area of the rectangle.

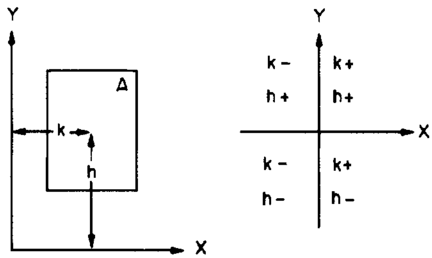

Fig. 1.3.

1.2. Principal second moments of area

The principal axes of a section have been defined in the introduction to this chapter. Second moments of area about these axes are then termed principal values and these may be related to the standard values about the conventional X and Y axes as follows.

Consider Fig. 1.4 in which GX and GY are any two mutually perpendicular axes inclined at θ to the principal axes GV and GU. A small element of area A will then have coordinates (u, v) to the principal axes and (x, y) referred to the axes GX and GY. The area will thus have a product second moment of area about the principal axes given by $uv\,dA$.
$\therefore$ total product second moment of area of a cross-section

$$I_{uv} = \int uv\,dA$$

$$= \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA$$

$$= \int (xy\cos^2\theta + y^2\sin\theta\cos\theta - x^2\cos\theta\sin\theta - xy\sin^2\theta)\,dA$$

$$= (\cos^2\theta - \sin^2\theta)\int xy\,dA + \sin\theta\cos\theta\left[\int y^2\,dA - \int x^2\,dA\right]$$

$$= I_{xy}\cos 2\theta + \tfrac{1}{2}(I_{xx} - I_{yy})\sin 2\theta$$

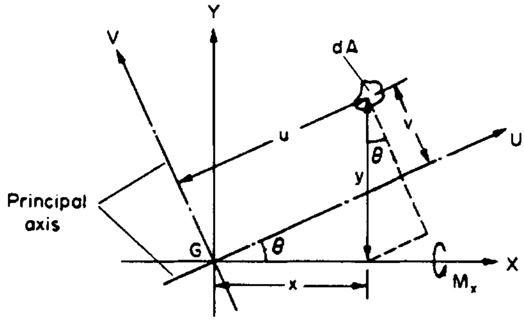

Fig. 1.4.

Now for principal axes the product second moment of area is zero.

$\therefore$
$$0 = I_{xy}\cos 2\theta + \tfrac{1}{2}(I_{xx} - I_{yy})\sin 2\theta$$

$$\mathbf{\tan 2\theta} = \frac{-2I_{xy}}{(I_{xx} - I_{yy})} = \frac{\mathbf{2I_{xy}}}{\mathbf{(I_{yy} - I_{xx})}} \tag{1.4}$$

This equation, therefore, gives the direction of the principal axes.
To determine the second moments of area about these axes,

$$I_u = \int v^2\,dA = \int (y\cos\theta - x\sin\theta)^2\,dA$$

$$= \cos^2\theta\int y^2\,dA + \sin^2\theta\int x^2\,dA - 2\cos\theta\sin\theta\int xy\,dA$$

$$= I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin 2\theta \tag{1.5}$$

Substituting for I_{xy} from eqn. (1.4),

$$I_u = \tfrac{1}{2}(1 + \cos 2\theta)I_{xx} + \tfrac{1}{2}(1 - \cos 2\theta)I_{yy} - \tfrac{1}{2}\frac{\sin^2 2\theta}{\cos 2\theta}(I_{yy} - I_{xx})$$

$$= \tfrac{1}{2}(1 + \cos 2\theta)I_{xx} + \tfrac{1}{2}(1 - \cos 2\theta)I_{yy} - \tfrac{1}{2}\left[\frac{(1 - \cos^2 2\theta)}{\cos 2\theta}(I_{yy} - I_{xx})\right]$$

$$= \tfrac{1}{2}(1 + \cos 2\theta)I_{xx} + \tfrac{1}{2}(1 - \cos 2\theta)I_{yy} - \tfrac{1}{2}\sec 2\theta(I_{yy} - I_{xx}) + \tfrac{1}{2}\cos 2\theta(I_{yy} - I_{xx})$$

$$= \tfrac{1}{2}(I_{xx} + I_{yy}) + (I_{xx} - I_{yy})\cos 2\theta - (I_{yy} - I_{xx})\sec 2\theta + (I_{yy} - I_{xx})\cos 2\theta$$

i.e.

$$\boldsymbol{I_u = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta} \tag{1.6}$$

Similarly,

$$I_v = \int u^2\, dA = \int (x\cos\theta + y\sin\theta)^2\, dA$$

$$= \boldsymbol{\tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta} \tag{1.7}$$

N.B.–Adding the above expressions,

$$I_u + I_v = I_{xx} + I_{yy}$$

Also from eqn. (1.5),

$$I_u = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\sin 2\theta$$

$$= \tfrac{1}{2}(1 + \cos 2\theta)I_{xx} + \tfrac{1}{2}(1 - \cos 2\theta)I_{yy} - I_{xy}\sin 2\theta$$

$$\boldsymbol{I_u = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta} \tag{1.8}$$

Similarly,

$$\boldsymbol{I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta} \tag{1.9}$$

These equations are then identical in form with the complex-stress eqns. (13.8) and (13.9)† with I_{xx}, I_{yy}, and I_{xy} replacing σ_x, σ_y and τ_{xy} and Mohr's circle can be drawn to represent I values in exactly the same way as Mohr's stress circle represents stress values.

1.3. Mohr's circle of second moments of area

The construction is as follows (Fig. 1.5):

(1) Set up axes for second moments of area (horizontal) and product second moments of area (vertical).

(2) Plot the points A and B represented by (I_{xx}, I_{xy}) and $(I_{yy}, -I_{xy})$.

(3) Join AB and construct a circle with this as diameter. *This is then the Mohr's circle.*

(4) Since the principal moments of area are those about the axes with a zero product second moment of area they are given by the points where the circle cuts the horizontal axis.

Thus OC and OD are the principal second moments of area I_v and I_u.

The point A represents values on the X axis and B those for the Y axis. Thus, in order to determine the second moment of area about some other axis, e.g. the N.A., at some angle α counterclockwise to the X axis, construct a line from G at an angle 2α counterclockwise to GA on the Mohr construction to cut the circle in point N. The horizontal coordinate of N then gives the value of $I_{N.A.}$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

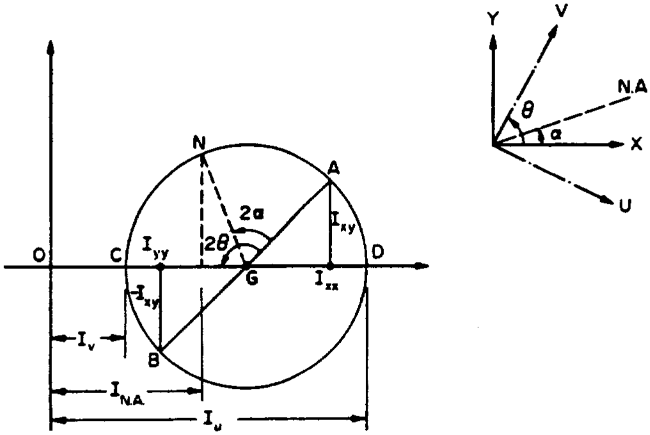

Fig. 1.5. Mohr's circle of second moments of area.

The procedure is therefore identical to that for determining the direct stress on some plane inclined at α to the plane on which σ_x acts in Mohr's stress circle construction, i.e. **angles are DOUBLED** on Mohr's circle.

1.4. Land's circle of second moments of area

An alternative graphical solution to the Mohr procedure has been developed by Land as follows (Fig. 1.6):

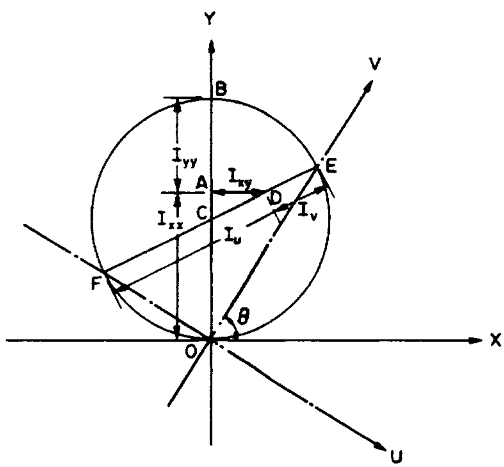

Fig. 1.6. Land's circle of second moments of area.

(1) From O as origin of the given XY axes mark off lengths $OA = I_{xx}$ and $AB = I_{yy}$ on the vertical axis.

(2) Draw a circle with OB as diameter and centre C. *This is then Land's circle of second moment of area.*

(3) From point A mark off $AD = I_{xy}$ parallel with the X axis.

(4) Join the centre of the circle C to D, and produce, to cut the circle in E and F. Then $ED = I_v$ and $DF = I_u$ are the principal moments of area about the principal axes OV and OU the positions of which are found by joining OE and OF. The principal axes are thus inclined at an angle θ to the OX and OY axes.

1.5. Rotation of axes: determination of moments of area in terms of the principal values

Figure 1.7 shows any plane section having coordinate axes XX and YY and principal axes UU and VV, each passing through the centroid O. Any element of area dA will then have coordinates (x, y) and (u, v), respectively, for the two sets of axes.

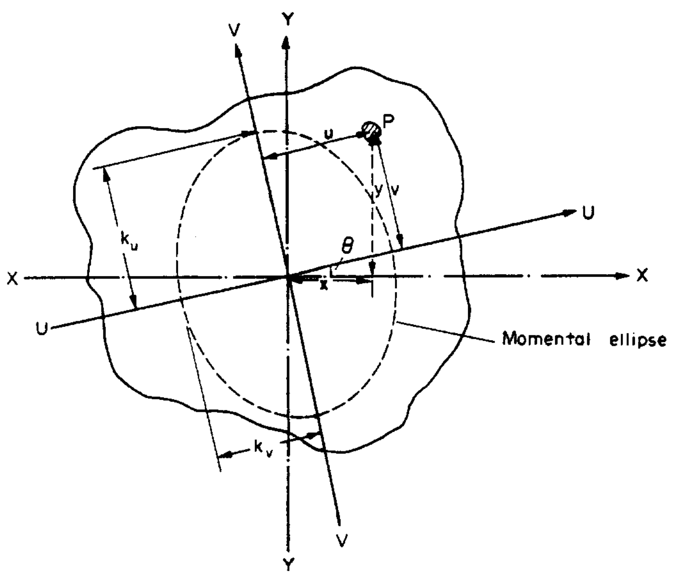

Fig. 1.7. The momental ellipse.

Now

$$I_{xx} = \int y^2\,dA = \int (v\cos\theta + u\sin\theta)^2\,dA$$

$$= \int v^2\cos^2\theta\,dA + \int 2uv\sin\theta\cos\theta\,dA + \int u^2\sin^2\theta\,dA$$

But UU and VV are the principal axes so that $I_{uv} = \int uv\,dA$ is zero.

$$\therefore \qquad \boldsymbol{I_{xx} = I_u\cos^2\theta + I_v\sin^2\theta} \qquad (1.10)$$

Similarly,

$$I_{yy} = \int x^2\, dA = \int (u\cos\theta - v\sin\theta)^2\, dA$$

$$= \int u^2\cos^2\theta\, dA - \int 2uv\sin\theta\cos\theta\, dA + \int v^2\sin^2\theta\, dA$$

and with $\int uv\, dA = 0$

$$\boldsymbol{I_{yy} = I_v\cos^2\theta + I_u\sin^2\theta} \tag{1.11}$$

Also

$$I_{xy} = \int xy\, dA = \int (u\cos\theta - v\sin\theta)(v\cos\theta + u\sin\theta)\, dA$$

$$= \int [uv(\cos^2\theta - \sin^2\theta) + (u^2 - v^2)\sin\theta\cos\theta]\, dA$$

$$= I_{uv}\cos 2\theta + \tfrac{1}{2}(I_v - I_u)\sin 2\theta \quad \text{and} \quad I_{uv} = 0$$

$$\therefore \qquad \boldsymbol{I_{xy} = \tfrac{1}{2}(I_v - I_u)\sin 2\theta} \tag{1.12}$$

From eqns. (1.10) and (1.11)

$$I_{xx} - I_{yy} = I_u\cos^2\theta + I_v\sin^2\theta - I_v\cos^2\theta - I_u\sin^2\theta$$

$$= (I_u - I_v)\cos^2\theta - (I_u - I_v)\sin^2\theta$$

$$\boldsymbol{I_{xx} - I_{yy} = (I_u - I_v)\cos 2\theta} \tag{1.13}$$

Combining eqns. (1.12) and (1.13) gives

$$\boldsymbol{\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}}} \tag{1.14}$$

and combining eqns. (1.10) and (1.11) gives

$$I_{xx} + I_{yy} = I_u + I_v \tag{1.15}$$

Substitution into eqns. (1.10) and (1.11) then yields

$$\boldsymbol{I_u = \tfrac{1}{2}[(I_{xx} + I_{yy}) + (I_{xx} - I_{yy})\sec 2\theta]} \qquad (1.16) \text{ as } (1.6)$$

$$\boldsymbol{I_v = \tfrac{1}{2}[(I_{xx} + I_{yy}) - (I_{xx} - I_{yy})\sec 2\theta]} \qquad (1.17) \text{ as } (1.7)$$

1.6. The ellipse of second moments of area

The above relationships can be used as the basis for construction of the moment of area ellipse proceeding as follows:

(1) Plot the values of I_u and I_v on two mutually perpendicular axes and draw concentric circles with centres at the origin, and radii equal to I_u and I_v (Fig. 1.8).

(2) Plot the point with coordinates $x = I_u\cos\theta$ and $y = I_v\sin\theta$, the value of θ being given by eqn. (1.14).

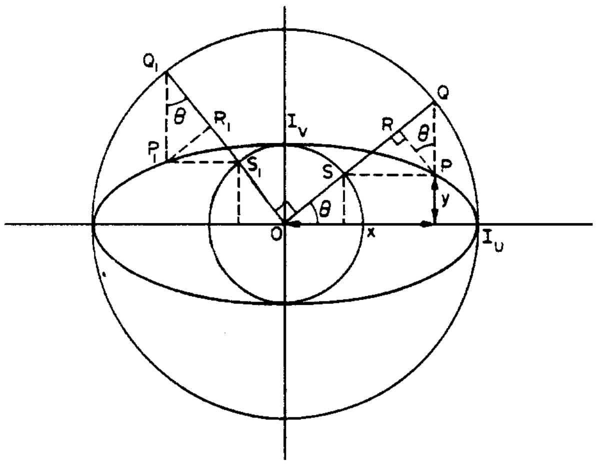

Fig. 1.8. The ellipse of second moments of area.

It then follows that

$$\frac{x^2}{(I_u)^2} + \frac{y^2}{(I_v)^2} = 1$$

This equation is the locus of the point P and represents the equation of an ellipse – the *ellipse of second moments of area.*

(3) Draw OQ at an angle θ to the I_u axis, cutting the circle through I_v in point S and join SP which is then parallel to the I_u axis. Construct a perpendicular to OQ through P to meet OQ in R.

Then

$$\begin{aligned} OR &= OQ - RQ \\ &= I_u - (I_u \sin\theta - I_v \sin\theta)\sin\theta \\ &= I_u - (I_u - I_v)\sin^2\theta \\ &= I_u \cos^2\theta + I_v \sin^2\theta \\ &= I_{xx} \end{aligned}$$

Similarly, repeating the process with OQ_1 perpendicular to OQ gives the result

$$OR_1 = I_{yy}$$

Further,

$$\begin{aligned} PR &= PQ\cos\theta \\ &= (I_u \sin\theta - I_v \sin\theta)\cos\theta \\ &= \tfrac{1}{2}(I_u - I_v)\sin 2\theta = I_{xy} \end{aligned}$$

Thus the construction shown in Fig. 1.8 can be used to determine the second moments of area and the product second moment of area about any set of perpendicular axis at a known orientation to the principal axes.

1.7. Momental ellipse

Consider again the general plane surface of Fig. 1.7 having radii of gyration k_u and k_v about the U and V axes respectively. An ellipse can be constructed on the principal axes with semi-major and semi-minor axes k_u and k_v, respectively, as shown.

Thus the perpendicular distance between the axis UU and a tangent to the ellipse which is parallel to UU is equal to the radius of gyration of the surface about UU. Similarly, the radius of gyration k_v is the perpendicular distance between the tangent to the ellipse which is parallel to the VV axis and the axis itself. Thus if the radius of gyration of the surface is required about any other axis, e.g. the N.A., then it is given by the distance between the N.A. and the tangent AA which is parallel to the N.A. (see Fig. 1.11). Thus

$$k_{\text{N.A.}} = h$$

The ellipse is then termed the *momental ellipse* and is extremely useful in the solution of unsymmetrical bending problems as described in §1.10.

1.8. Stress determination

Having determined both the values of the principal second moments of area I_u and I_v and the inclination of the principal axes U and V from the equations listed below,

$$I_u = \tfrac{1}{2}(I_{xx} + I_{yy}) + \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta \tag{1.16}$$

$$I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) - \tfrac{1}{2}(I_{xx} - I_{yy})\sec 2\theta \tag{1.17}$$

and

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})} \tag{1.14}$$

the stress at any point is found by application of the simple bending theory simultaneously about the principal axes,

i.e.

$$\sigma = \frac{\boldsymbol{M_v u}}{\boldsymbol{I_v}} + \frac{\boldsymbol{M_u v}}{\boldsymbol{I_u}} \tag{1.18}$$

where M_v and M_u are the moments of the applied loads about the V and U axes, e.g. if loads are applied to produce a bending moment M_x about the X axis (see Fig. 1.14), then

$$M_v = M_x \sin\theta$$

$$M_u = M_x \cos\theta$$

the maximum value of M_x, and hence M_u and M_v, for cantilevers such as that shown in Fig. 1.10, being found at the root of the cantilever. The maximum stress due to bending will then occur at this position.

1.9. Alternative procedure for stress determination

Consider any unsymmetrical section, represented by Fig. 1.9. The assumption is made initially that the stress at any point on the unsymmetrical section is given by

$$\sigma = Px + Qy \tag{1.19}$$

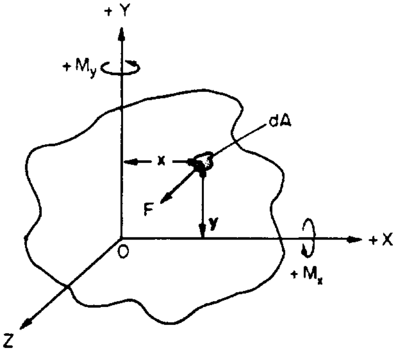

Fig. 1.9. Alternative procedure for stress determination.

where P and Q are constants; in other words it is assumed that bending takes place about the X and Y axes at the same time, stresses resulting from each effect being proportional to the distance from the respective axis of bending.

Now let there be a tensile stress σ on the element of area dA. Then

$$\text{force } F \text{ on the element} = \sigma\, dA$$

the direction of the force being parallel to the Z axis. The moment of this force about the X axis is then $\sigma\, dA\, y$.

$$\therefore \quad \text{total moment} = M_x = \int \sigma\, dA\, y$$

$$= \int (Px + Qy) y\, dA = \int Pxy\, dA + \int Qy^2\, dA$$

Now, by definition,

$$I_{xx} = \int y^2\, dA, \quad I_{yy} = \int x^2\, dA \quad \text{and} \quad I_{xy} = \int xy\, dA$$

the latter being termed the product second moment of area (see §1.1):

$$\therefore \quad \boldsymbol{M_x = PI_{xy} + QI_{xx}} \tag{1.20}$$

Similarly, considering moments about the Y axis,

$$\therefore \quad M_y = -\int \sigma\, dAx = -\int (Px + Qy) x\, dA$$

$$\therefore \quad \boldsymbol{M_y = -PI_{yy} - QI_{xy}} \tag{1.21}$$

The sign convention used above for bending moments is the *corkscrew rule*. A positive moment is the direction in which a corkscrew or screwdriver has to be turned in order to produce motion of a screw in the direction of positive X or Y, as shown in Fig. 1.9. Thus with a knowledge of the applied moments and the second moments of area about any two perpendicular axes, P and Q can be found from eqns. (1.20) and (1.21) and hence the stress at any point (x, y) from eqn. (1.19).

Since stresses resulting from bending are zero on the N.A. the equation of the N.A. is

$$Px + Qy = 0$$

$$\frac{y}{x} = -\frac{P}{Q} = \tan \alpha_{N.A.} \tag{1.22}$$

where $\alpha_{N.A.}$ is the inclination of the N.A. to the X axis.

If the unsymmetrical member is drawn to scale and the N.A. is inserted through the centroid of the section at the above angle, the points of maximum stress can be determined quickly by inspection as the points most distant from the N.A., e.g. for the angle section of Fig. 1.10, subjected to the load shown, the maximum tensile stress occurs at R while the maximum compressive stress will arise at either S or T depending on the value of α.

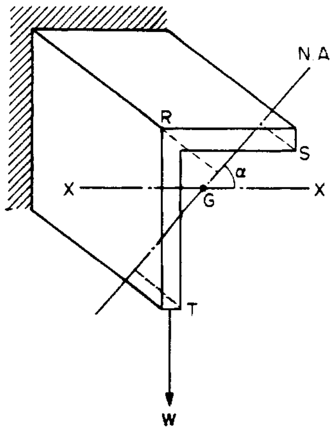

Fig. 1.10.

1.10. Alternative procedure using the momental ellipse

Consider the unsymmetrical section shown in Fig. 1.11 with principal axes UU and VV. Any moment applied to the section can be resolved into its components about the principal axes and the stress at any point found by application of eqn. (1.18).

For example, if vertical loads only are applied to the section to produce moments about the OX axis, then the components will be $M \cos\theta$ about UU and $M \sin\theta$ about VV. Then

$$\text{stress at P} = \frac{M\cos\theta}{I_u} v - \frac{M\sin\theta}{I_v} u \tag{1.23}$$

the value of θ having been obtained from eqn. (1.14).

Alternatively, however, the problem may be solved by realising that the N.A. and the plane of the external bending moment are conjugate diameters of an ellipse† – the *momental*

† *Conjugate diameters of an ellipse:* two diameters of an ellipse are conjugate when each bisects all chords parallel to the other diameter.

Two diameters $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $m_1 m_2 = -\frac{b^2}{a^2}$.

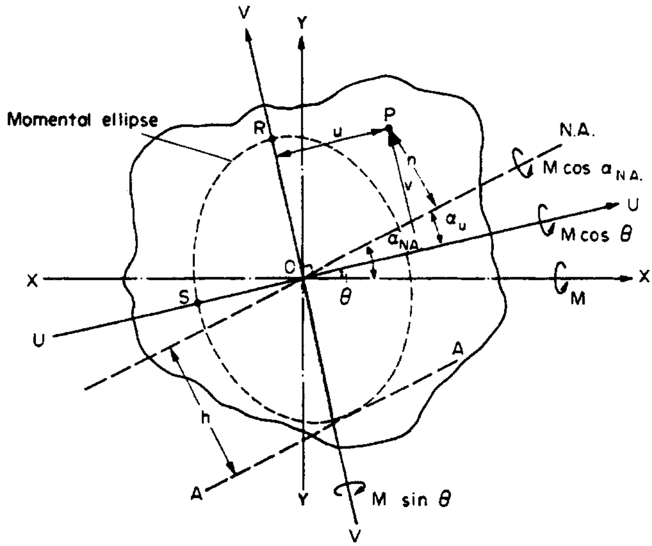

Fig. 1.11. Determination of stresses using the momental ellipse.

ellipse. The actual plane of resultant bending will then be perpendicular to the N.A., the inclination of which, relative to the U axis (α_u), is obtained by equating the above formula for stress at P to zero,

i.e.
$$\frac{M\cos\theta}{I_u}v = \frac{M\sin\theta}{I_v}u$$

so that
$$\tan\alpha_u = \frac{v}{u} = \frac{I_u}{I_v}\tan\theta$$
$$= \frac{k_u^2}{k_v^2}\tan\theta \qquad (1.24)$$

where k_u and k_v are the radii of gyration about the principal axes and hence the semi-axes of the momental ellipse.

The N.A. can now be added to the diagram to scale. The second moment of area of the section about the N.A. is then given by Ah^2, where h is the perpendicular distance between the N.A. and a tangent AA to the ellipse drawn parallel to the N.A. (see Fig. 1.11 and §1.7).

The bending moment about the N.A. is $M\cos\alpha_{\text{N.A.}}$ where $\alpha_{\text{N.A.}}$ is the angle between the N.A. and the axis XX about which the moment is applied.

The stress at P is now given by the simple bending formula

$$\sigma = \frac{M\cos\alpha_{\text{N.A.}}}{I_{\text{N.A.}}}n \qquad (1.25)$$

the distance n being measured perpendicularly from the N.A. to the point P in question.

As for the procedure introduced in §1.7, this method has the advantage of immediate indication of the points of maximum stress once the N.A. has been drawn. The solution does, however, involve the use of principal moments of area which must be obtained by calculation or graphically using Mohr's or Land's circle.

1.11. Deflections

The deflections of unsymmetrical members in the directions of the principal axes may always be determined by application of the standard deflection formulae of §5.7.†

For example, the deflection at the free end of a cantilever carrying an end-point-load is

$$\frac{WL^3}{3EI}$$

With the appropriate value of I and the correct component of the load perpendicular to the principal axis used, the required deflection is obtained.

Thus $$\delta_v = \frac{W_u L^3}{3EI_u} \quad \text{and} \quad \delta_u = \frac{W_v L^3}{3EI_v} \tag{1.26}$$

where W_u and W_v are the components of the load *perpendicular* to the U and V principal axes respectively.

The total resultant deflection is then given by combining the above values vectorially as shown in Fig. 1.12,

i.e. $$\delta = \sqrt{(\delta_u^2 + \delta_v^2)}$$

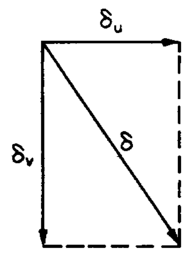

Fig. 1.12.

Alternatively, since bending always occurs about the N.A., the deflection equation can be written in the form

$$\delta = \frac{W'L^3}{3EI_{\text{N.A.}}} \tag{1.27}$$

where $I_{\text{N.A.}}$ is the second moment of area about the N.A. and W' is the component of the load perpendicular to the N.A. The value of $I_{\text{N.A.}}$ may be found either graphically using Mohr's circle or the momental ellipse, or by calculation using

$$I_{\text{N.A.}} = \tfrac{1}{2}[(I_u + I_v) + (I_u - I_v)\cos 2\alpha_u] \tag{1.28}$$

where α_u is the angle between the N.A. and the principal U axis.

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

Examples

Example 1.1

A rectangular-section beam 80 mm × 50 mm is arranged as a cantilever 1.3 m long and loaded at its free end with a load of 5 kN inclined at an angle of 30° to the vertical as shown in Fig. 1.13. Determine the position and magnitude of the greatest tensile stress in the section. What will be the vertical deflection at the end? $E = 210$ GN/m^2.

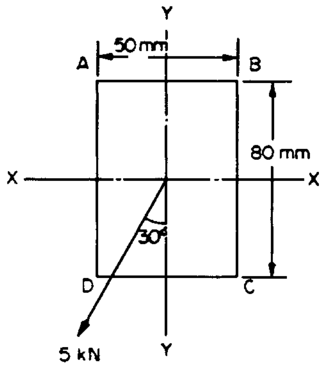

Fig. 1.13.

Solution

In the case of symmetrical sections such as this, subjected to skew loading, a solution is obtained by resolving the load into its components parallel to the two major axes and applying the bending theory simultaneously to both axes, i.e.

$$\sigma = \frac{M_{xx}y}{I_{xx}} \pm \frac{M_{yy}x}{I_{yy}}$$

Now the most highly stressed areas of the cantilever will be those at the built-in end where

$$M_{xx} = 5000\cos 30^\circ \times 1.3 = 5629 \text{ Nm}$$

$$M_{yy} = 5000\sin 30^\circ \times 1.3 = 3250 \text{ Nm}$$

The stresses on the short edges AB and DC resulting from bending about XX are then

$$\frac{M_{xx}}{I_{xx}}y = \frac{5629 \times 40 \times 10^{-3} \times 12}{50 \times 80^3 \times 10^{-12}} = 105.5 \text{ MN/m}^2$$

tensile on AB and compressive on DC.

The stresses on the long edges AD and BC resulting from bending about YY are

$$\frac{M_{yy}}{I_{yy}}x = \frac{3250 \times 25 \times 10^{-3} \times 12}{80 \times 50^3 \times 10^{-12}} = 97.5 \text{ MN/m}^2$$

tensile on BC and compressive on AD.

The maximum tensile stress will therefore occur at point B where the two tensile stresses add, i.e.

$$\text{maximum tensile stress} = 105.5 + 97.5 = \mathbf{203\ MN/m^2}$$

The deflection at the free end of the cantilever is then given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore deflection vertically (i.e. along the YY axis) is

$$\delta_v = \frac{(W\cos 30^\circ)L^3}{3EI_{xx}} = \frac{5000 \times 0.866 \times 1.3^3 \times 12}{3 \times 210 \times 10^9 \times 50 \times 80^3 \times 10^{-12}}$$
$$= 0.0071 = \mathbf{7.1\ mm}$$

Example 1.2

A cantilever of length 1.2 m and of the cross section shown in Fig. 1.14 carries a vertical load of 10 kN at its outer end, the line of action being parallel with the longer leg and arranged to pass through the shear centre of the section (i.e. there is no twisting of the section, see §7.5†). Working from first principles, find the stress set up in the section at points A, B and C, given that the centroid is located as shown. Determine also the angle of inclination of the N.A.

$$I_{xx} = 4 \times 10^{-6}\ \text{m}^4, \quad I_{yy} = 1.08 \times 10^{-6}\ \text{m}^4$$

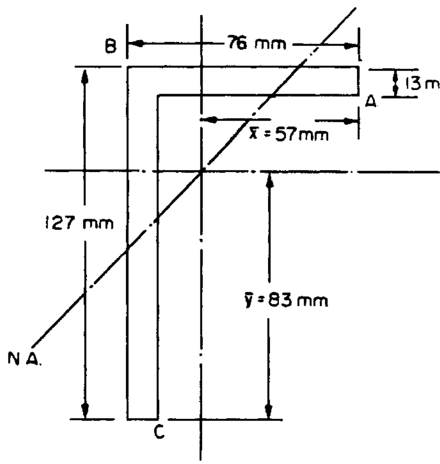

Fig. 1.14.

Solution

The product second moment of area of the section is given by eqn. (1.3).

$$I_{xy} = \Sigma Ahk$$
$$= \{76 \times 13(\tfrac{1}{2} \times 76 - 19)(44 - \tfrac{1}{2} \times 13)$$
$$+ 114 \times 13[-(83 - \tfrac{1}{2} \times 114)][-(19 - \tfrac{1}{2} \times 13)]\}10^{-12}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.

$$= (0.704 + 0.482)10^{-6} = 1.186 \times 10^{-6}\ \text{m}^4$$

From eqn. (1.20) $\quad M_x = PI_{xy} + QI_{xx} = 10\,000 \times 1.2 = 12\,000$

i.e. $\quad 1.186P + 4Q = 12\,000 \times 10^6 \qquad (1)$

Since the load is vertical there will be no moment about the Y axis and eqn. (1.21) gives

$$M_y = -PI_{yy} - QI_{xy} = 0$$

$$\therefore \quad -1.08P - 1.186Q = 0$$

$$\therefore \quad \frac{P}{Q} = -\frac{1.186}{1.08} = -1.098$$

But the angle of inclination of the N.A. is given by eqn. (1.22) as

$$\tan \alpha_{\text{N.A.}} = -\frac{P}{Q} = 1.098$$

i.e. $\quad \alpha_{\text{N.A.}} = \mathbf{47°41'}$

Substituting $P = -1.098Q$ in eqn. (1),

$$1.186(-1.098Q) + 4Q = 12\,000 \times 10^6$$

$$\therefore \quad Q = \frac{12\,000 \times 10^6}{2.69} = 4460 \times 10^6$$

$$\therefore \quad P = -4897 \times 10^6$$

If the N.A. is drawn as shown in Fig. 1.14 at an angle of 47°41′ to the XX axis through the centroid of the section, then this is the axis about which bending takes place. The points of maximum stress are then obtained by inspection as the points which are the maximum perpendicular distance from the N.A.

Thus B is the point of maximum tensile stress and C the point of maximum compressive stress.

Now from eqn (1.19) the stress at any point is given by

$$\sigma = Px + Qy$$

$$\therefore \quad \text{stress at } A = -4897 \times 10^6(57 \times 10^{-3}) + 4460 \times 10^6(31 \times 10^{-3})$$

$$= \mathbf{-141\ MN/m^2\ (compressive)}$$

$$\text{stress at } B = -4897 \times 10^6(-19 \times 10^{-3}) + 4460 \times 10^6(44 \times 10^{-3})$$

$$= \mathbf{289\ MN/m^2\ (tensile)}$$

$$\text{stress at } C = -4897 \times 10^6(-6 \times 10^{-3}) + 4460 \times 10^6(-83 \times 10^{-3})$$

$$= \mathbf{-341\ MN/m\ (compressive)}$$

Example 1.3

(a) A horizontal cantilever 2 m long is constructed from the Z-section shown in Fig. 1.15. A load of 10 kN is applied to the end of the cantilever at an angle of 60° to the horizontal as shown. Assuming that no twisting moment is applied to the section, determine the stresses at points A and B. ($I_{xx} \times 48.3 \times 10^{-6}$ m^4, $I_{yy} = 4.4 \times 10^{-6}$ m^4.)

(b) Determine the principal second moments of area of the section and hence, by applying the simple bending theory about each principal axis, check the answers obtained in part (a).

(c) What will be the deflection of the end of the cantilever? $E = 200$ GN/m^2.

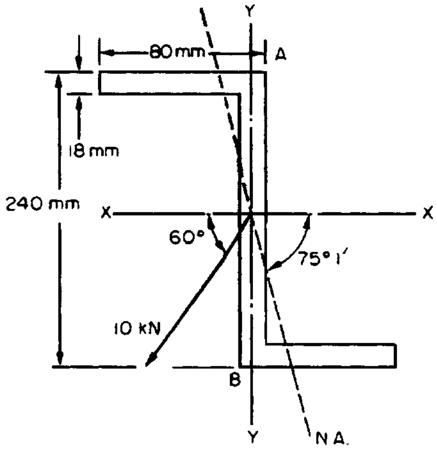

Fig. 1.15.

Solution

(a) For this section I_{xy} for the web is zero since its centroid lies on both axes and hence h and k are both zero. The contributions to I_{xy} of the other two portions will be negative since in both cases either h or k is negative.

$$\therefore \qquad I_{xy} = -2(80 \times 18)(40 - 9)(120 - 9)10^{-12}$$

$$= -9.91 \times 10^{-6} \text{ m}^4$$

Now, at the built-in end,

$$M_x = +10\,000 \sin 60^\circ \times 2 = +17\,320 \text{ Nm}$$

$$M_y = -10\,000 \cos 60^\circ \times 2 = -10\,000 \text{ Nm}$$

Substituting in eqns. (1.20) and (1.21),

$$17\,320 = PI_{xy} + QI_{xx} = (-9.91P + 48.3Q)10^{-6}$$

$$-10\,000 = -PI_{yy} - QI_{xy} = (-4.4P + 9.91Q)10^{-6}$$

$$\therefore \qquad 1.732 \times 10^{10} = -9.91P + 48.3Q \qquad (1)$$

$$-1 \times 10^{10} = -4.4P + 9.91Q \qquad (2)$$

$(1) \times \dfrac{4.4}{9.91}$,

$$0.769 \times 10^{10} = -4.4P + 21.45Q \qquad (3)$$

(3) − (2),

$$1.769 \times 10^{10} = 11.54Q$$

$$\therefore \qquad Q = 1533 \times 10^6$$

and substituting in (2) gives

$$P = 5725 \times 10^6$$

The inclination of the N.A. relative to the X axis is then given by

$$\tan \alpha_{\text{N.A.}} = -\frac{P}{Q} = -\frac{5725}{1533} = -3.735$$

$$\alpha_{\text{N.A.}} = \mathbf{-75°1'}$$

This has been added to Fig. 1.15 and indicates that the points A and B are on either side of the N.A. and equidistant from it. Stresses at A and B are therefore of equal magnitude but opposite sign.

Now

$$\sigma = Px + Qy$$

$$\therefore \qquad \text{stress at A} = 5725 \times 10^6 \times 9 \times 10^{-3} + 1533 \times 10^6 \times 120 \times 10^{-3}$$

$$= \mathbf{235 \ \ MN/m^2 \ (tensile)}$$

Similarly,

$$\text{stress at } B = \mathbf{235 \ MN/m^2 \ (compressive)}$$

(b) The principal second moments of area may be found from Mohr's circle as shown in Fig. 1.16 or from eqns. (1.6) and (1.7),

i.e. $$I_u, I_v = \tfrac{1}{2}(I_{xx} + I_{yy}) \pm \tfrac{1}{2}(I_{xx} - I_{yy}) \sec 2\theta$$

with $$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{-2 \times 9.91 \times 10^{-6}}{(4.4 - 48.3)10^{-6}}$$

$$= 0.451$$

$$\therefore \qquad 2\theta = 24°18', \ \ \theta = 12°9'$$

$$\therefore \qquad I_u, I_v = \tfrac{1}{2}[(48.3 + 4.4) \pm (48.3 - 4.4)1.0972]10^{-6}$$

$$= \tfrac{1}{2}[52.7 \pm 48.17]10^{-6}$$

$$\therefore \qquad I_u = 50.43 \times 10^{-6} \text{ m}^4$$

$$I_v = 2.27 \times 10^{-6} \text{ m}^4$$

The required stresses can now be obtained from eqn. (1.18).

$$\sigma = \frac{M_v u}{I_v} + \frac{M_u v}{I_u}$$

Now $$M_u = 10\,000 \sin(60° - 12°9') \times 2$$

$$= 10\,000 \sin 47°51' \times 2 = 14\,828 \text{ Nm}$$

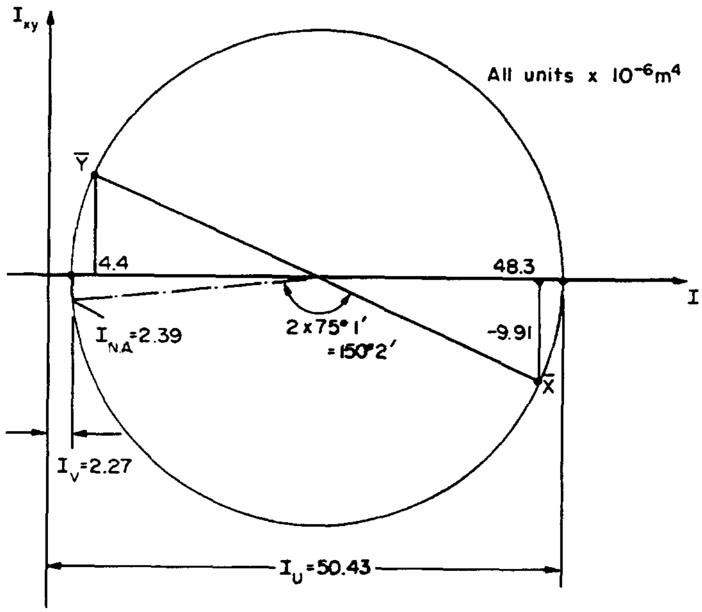

Fig. 1.16.

and $$M_v = 10\,000 \cos 47°51' \times 2 = 13\,422 \text{ Nm}$$

and, for A,

$$u = x\cos\theta + y\sin\theta = (9 \times 0.9776) + (120 \times 0.2105)$$
$$= 34.05 \text{ mm}$$
$$v = y\cos\theta - x\sin\theta = (120 \times 0.9776) - (9 \times 0.2105)$$
$$= 115.4 \text{ mm}$$

$$\therefore \quad \sigma = \frac{14\,828 \times 115.4 \times 10^{-3}}{50.43 \times 10^{-6}} + \frac{13\,422 \times 34.05 \times 10^{-3}}{2.27 \times 10^{-6}}$$

$= \mathbf{235\ MN/m^2}$ as before.

(c) The deflection at the free end of a cantilever is given by

$$\delta = \frac{WL^3}{3EI}$$

Therefore component of deflection perpendicular to the V axis

$$\delta_v = \frac{W_v L^3}{3EI_v} = \frac{10\,000 \cos 47°51' \times 2^3}{3 \times 200 \times 10^9 \times 2.27 \times 10^{-6}}$$
$$= 39.4 \times 10^{-3} = 39.4 \text{ mm}$$

and component of deflection perpendicular to the U axis

$$\delta_u = \frac{W_u L^3}{3EI_u} = \frac{10\,000 \sin 47°51' \times 2^3}{3 \times 200 \times 10^9 \times 50.43 \times 10^{-6}}$$

$$= 1.96 \times 10^{-3} = 1.96 \text{ mm}$$

The total deflection is then given by

$$= \sqrt{(\delta_u^2 + \delta_v^2)} = 10^{-3}\sqrt{(39.4^2 + 1.96^2)} = 39.45 \times 10^{-3}$$

$$= \mathbf{39.45\ mm}$$

Alternatively, since bending actually occurs about the N.A., the deflection can be found from

$$\delta = \frac{W_{\text{N.A.}} L^3}{3EI_{\text{N.A.}}}$$

its direction being normal to the N.A.

From Mohr's circle of Fig. 1.16, $I_{\text{N.A.}} = 2.39 \times 10^{-6}$ m^4

$$\therefore \qquad \delta = \frac{10\,000 \sin(30° + 14°59') \times 2^3}{3 \times 200 \times 10^9 \times 2.39 \times 10^{-6}} = 39.44 \times 10^{-3}$$

$$= \mathbf{39.44\ mm}$$

Example 1.4

Check the answer obtained for the stress at point B on the angle section of Example 1.2 using the momental ellipse procedure.

Solution

The semi-axes of the momental ellipse are given by

$$k_u = \sqrt{\frac{I_u}{A}} \quad \text{and} \quad k_v = \sqrt{\frac{I_v}{A}}$$

The ellipse can then be constructed by setting off the above dimensions on the principal axes as shown in Fig. 1.17 (The inclination of the N.A. can be determined as in Example 1.2 or from eqn. (1.24).) The second moment of area of the section about the N.A. is then obtained from the momental ellipse as

$$I_{\text{N.A.}} = Ah^2$$

Thus for the angle section of Fig. 1.14

$$I_{xy} = 1.186 \times 10^{-6} \text{ m}^4, \quad I_{xx} = 4 \times 10^{-6} \text{ m}^4, \quad I_{yy} = 1.08 \times 10^{-6} \text{ m}^4$$

The principal second moments of area are then given by Mohr's circle of Fig. 1.18 or from the equation

$$I_u, I_v = \tfrac{1}{2}[(I_{xx} + I_{yy}) \pm (I_{xx} - I_{yy}) \sec 2\theta]$$

where

$$\tan 2\theta = \frac{2I_{xy}}{(I_{yy} - I_{xx})} = \frac{2 \times 1.186 \times 10^{-6}}{(1.08 - 4)10^{-6}} = -0.8123$$

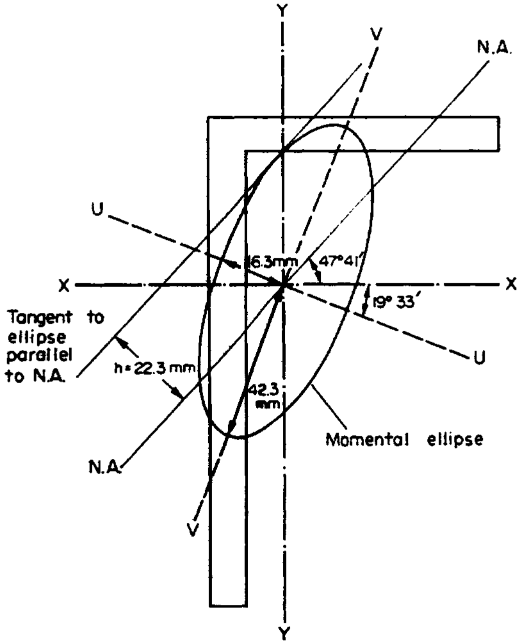

Fig. 1.17.

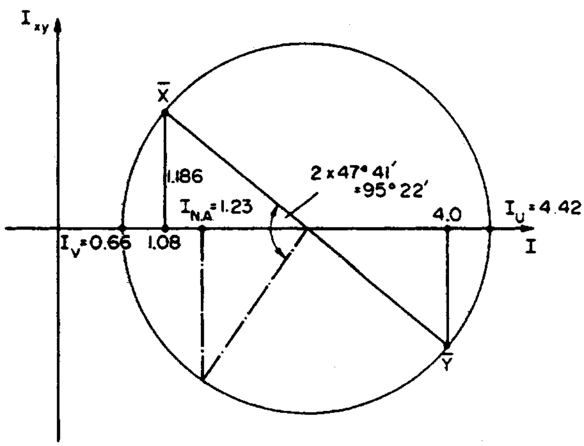

Fig. 1.18.

$$\therefore \quad 2\theta = -39^\circ 5', \;\; \theta = -19^\circ 33'$$

and $$\sec 2\theta = -1.2883$$

$$\therefore \quad I_u, I_v = \tfrac{1}{2}[(4 + 1.08) \pm (4 - 1.08)(-1.2883)]10^{-6}$$

$$= \tfrac{1}{2}[5.08 \pm 3.762]10^{-6}$$

$$I_u = 4.421 \times 10^{-6}, \quad I_v = 0.659 \times 10^{-6}\ \text{m}^4$$

and $$A = [(76 \times 13) + (114 \times 13)]10^{-6} = 2.47 \times 10^{-3}\ \text{m}^2$$

$\therefore$ $$k_u = \sqrt{\left(\frac{4.421 \times 10^{-6}}{2.47 \times 10^{-3}}\right)} = 0.0423 = 42.3\ \text{mm}$$

$$k_v = \sqrt{\left(\frac{0.659 \times 10^{-6}}{2.47 \times 10^{-3}}\right)} = 0.0163 = 16.3\ \text{mm}$$

The momental ellipse can now be constructed as described above and drawn in Fig. 1.17 and by measurement

$$h = 22.3\ \text{mm}$$

Then $$I_{\text{N.A.}} = Ah^2 = 2.47 \times 10^{-3} \times 22.3^2 \times 10^{-6}$$
$$= 1.23 \times 10^{-6}\ \text{m}^4$$

(This value may also be obtained from Mohr's circle of Fig. 1.18.)
The stress at B is then given by

$$\sigma = \frac{M_{N.A.} n}{I_{N.A.}}$$

where

$$n = \text{perpendicular distance from } B \text{ to the N.A.}$$
$$= 44\ \text{mm}$$

and $$M_{\text{N.A.}} = 10\,000 \cos 47°41' \times 1.2 = 8079\ \text{Nm}$$

$\therefore$ $$\text{stress at } B = \frac{8079 \times 44 \times 10^{-3}}{1.23 \times 10^{-6}} = \mathbf{289\ MN/m^2}$$

This confirms the result obtained with the alternative procedure of Example 1.2.

Problems

1.1 (B). A rectangular-sectioned beam of 75 mm × 50 mm cross-section is used as a simply supported beam and carries a uniformly distributed load of 500 N/m over a span of 3 m. The beam is supported in such a way that its long edges are inclined at 20° to the vertical. Determine:
(a) the maximum stress set up in the cross-section;
(b) the vertical deflection at mid-span.
$E = 208\ \text{GN/m}^2$. [17.4 MN/m^2; 1.76 mm.]

1.2 (B). An I-section girder 1.3 m long is rigidly built in at one end and loaded at the other with a load of 1.5 kN inclined at 30° to the web. If the load passes through the centroid of the section and the girder dimensions are: flanges 100 mm × 20 mm, web 200 mm × 12 mm, determine the maximum stress set up in the cross-section. How does this compare with the maximum stress set up if the load is vertical? [18.1, 4.14 MN/m^2.]

1.3 (B). A 75 mm × 75 mm × 12 mm angle is used as a cantilever with the face AB horizontal, as shown in Fig. 1.19. A vertical load of 3 kN is applied at the tip of the cantilever which is 1 m long. Determine the stress at A, B and C. [196.37, −207 MN/m^2.]

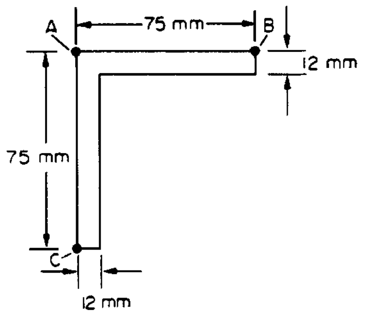

Fig. 1.19.

1.4 (B). A cantilever of length 2 m is constructed from 150 mm × 100 mm by 12 mm angle and arranged with its 150 mm leg vertical. If a vertical load of 5 kN is applied at the free end, passing through the shear centre of the section, determine the maximum tensile and compressive stresses set up across the section.

[B.P.] [169, − 204 MN/m^2.]

1.5 (B). A 180 mm × 130 mm × 13 mm unequal angle section is arranged with the long leg vertical and simply supported over a span of 4 m. Determine the maximum central load which the beam can carry if the maximum stress in the section is limited to 90 MN/m^2. Determine also the angle of inclination of the neutral axis.

$$I_{xx} = 12.8 \times 10^{-6}\ \text{m}^4,\ I_{yy} = 5.7 \times 10^{-6}\ \text{m}^4.$$

What will be the vertical deflection of the beam at mid-span? $E = 210$ GN/m^2. [8.73 kN, 41.6°, 7.74 mm.]

1.6 (B). The unequal-leg angle section shown in Fig. 1.20 is used as a cantilever with the 130 mm leg vertical. The length of the cantilever is 1.3 m. A vertical point load of 4.5 kN is applied at the free end, its line of action passing through the shear centre.

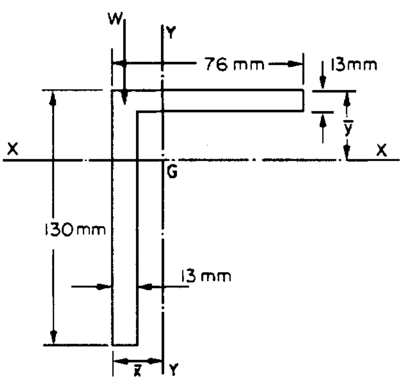

Fig. 1.20.

The properties of the section are as follows:

$$\bar{x} = 19\ \text{mm},\ \bar{y} = 45\ \text{mm},\ I_{xx} = 4 \times 10^{-6}\ \text{m}^4,\ I_{yy} = 1.1 \times 10^{-6}\ \text{m}^4,\ I_{xy} = 1.2 \times 10^{-6}\ \text{m}^4.$$

Determine:

(a) the magnitude of the principal second moments of area together with the inclination of their axes relative to XX;

(b) the position of the neutral plane ($N-N$) and the magnitude of I_{NN};

(c) the end deflection of the centroid G in magnitude, direction and sense.

Take $E = 207$ GN/m^2 (2.07 Mbar).

[444×10^{-8} m^4, 66×10^{-8} m^4, $-19°51'$ to XX, $47°42'$ to XX, 121×10^{-8} m^4, 8.85 mm at $-42°18'$ to XX.]

1.7 (B). An extruded aluminium alloy section having the cross-section shown in Fig. 1.21 will be used as a cantilever as indicated and loaded with a single concentrated load at the free end. This load F acts in the plane of the cross-section but may have any orientation within the cross-section. Given that $I_{xx} = 101.2 \times 10^{-8}$ m^4 and $I_{yy} = 29.2 \times 10^{-8}$ m^4:

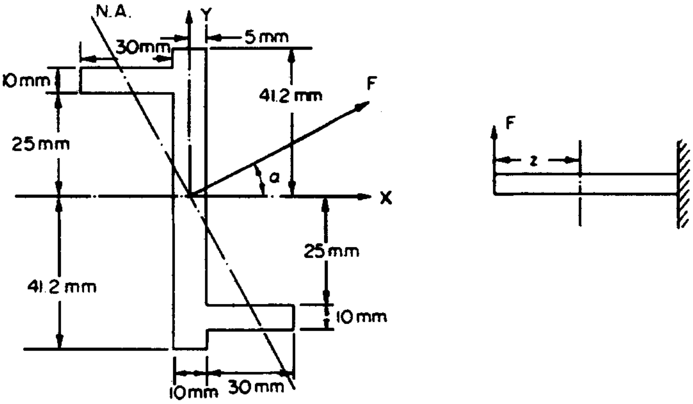

Fig. 1.21.

(a) determine the values of the principal second moments of area and the orientation of the principal axes;

(b) for such a case that the neutral axis is orientated at $-45°$ to the X-axis, as shown, find the angle α of the line of action of F to the X-axis and hence determine the numerical constant K in the expression $\sigma = KFz$, which expresses the magnitude of the greatest bending stress at any distance z from the free end.

[City U.] [116.1×10^{-8}, 14.3×10^{-8}, 22.5°, $-84°$, 0.71×10^5.]

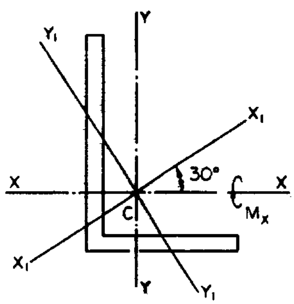

Fig. 1.22.

1.8 (B). A beam of length 2 m has the unequal-leg angle section shown in Fig. 1.22 for which $I_{xx} = 0.8 \times 10^{-6}$ m^4, $I_{yy} = 0.4 \times 10^{-6}$ m^4 and the angle between $X - X$ and the principal second moment of area axis $X_1 - X_1$ is 30°. The beam is subjected to a constant bending moment (M_x) of magnitude 1000 Nm about the $X - X$ axis as shown.

Determine:

(a) the values of the principal second moments of area Ix_1 and Iy_1 respectively;

(b) the inclination of the N.A., or line of zero stress ($N - N$) relative to the axis $X_1 - X_1$ and the value of the second moment of area of the section about $N - N$, that is I_N;

(c) the magnitude, direction and sense of the resultant maximum deflection of the centroid C.

For the beam material, Young's modulus $E = 200$ GN/m^2. For a beam subjected to a constant bending moment M, the maximum deflection δ is given by the formula

$$\delta = \frac{ML^2}{8EI}$$

[1×10^{-6}, 0.2×10^{-6} m^4, $-70°54'$ to X_1X_1, 0.2847×10^{-6} m^4, 6.62 mm, 90° to N.A.]