CHAPTER 7

CIRCULAR PLATES AND DIAPHRAGMS

Summary

The slope and deflection of circular plates under various loading and support conditions are given by the fundamental deflection equation

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{Q}{D}$$

where y is the deflection at radius r; dy/dr is the slope θ at radius r; Q is the applied load or shear force per unit length, usually given as a function of r; D is a constant termed the "flexural stiffness" or "flexural rigidity" $= Et^3/[12(1-\nu^2)]$ and t is the plate thickness.

For applied uniformly distributed load (i.e. pressure q) the equation becomes

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{qr}{2D}$$

For central concentrated load F

$$Q = \frac{F}{2\pi r} \text{ and the right-hand-side becomes } -\frac{F}{2\pi r D}$$

For axisymmetric non-uniform pressure (e.g. impacting gas or water jet)

$$q = K/r \text{ and the right-hand-side becomes } -K/2D$$

The *bending moments per unit length* at any point in the plate are:

$$M_r = M_{xy} = D\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right]$$

$$M_z = M_{yz} = D\left[\nu\frac{d\theta}{dr} + \frac{\theta}{r}\right]$$

Similarly, the *radial and tangential stresses* at any radius r are given by:

$$\text{radial stress } \sigma_r = \frac{Eu}{(1-\nu^2)}\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right]$$

$$\text{tangential stress } \sigma_z = \frac{Eu}{(1-\nu^2)}\left[\nu\frac{d\theta}{dr} + \frac{\theta}{r}\right]$$

Alternatively, $$\sigma_r = \frac{12u}{t^3}M_r \quad \text{and} \quad \sigma_z = \frac{12u}{t^3}M_z$$

For a **circular plate**, radius R, *freely supported* at its edge and subjected to a *load F distributed around a circle radius R_1*

$$y_{max} = \frac{F}{8\pi D}\left[\frac{(3+\nu)}{2(1+\nu)}(R^2 - R_1^2) - R_1^2 \log_e \frac{R}{R_1}\right]$$

and

$$\sigma_{r_{max}} = \frac{3F}{4\pi t^2}\left[2(1+\nu)\log_e \frac{R}{R_1} + (1-\nu)\frac{(R^2 - R_1^2)}{R^2}\right]$$

$$= \sigma_{z_{max}}$$

Table 7.1. Summary of maximum deflections and stresses.

Loading condition	Maximum deflection (y_{max})	Maximum stresses	
		$\sigma_{r_{max}}$	$\sigma_{z_{max}}$
Uniformly loaded, edges clamped	$\frac{3qR^4}{16Et^3}(1-\nu^2)$	$\frac{3qR^2}{4t^2}$	$\frac{3qR^2}{8t^2}(1+\nu)$
Uniformly loaded, edges freely supported	$\frac{3qR^4}{16Et^3}(5+\nu)(1-\nu)$	$\frac{3qR^2}{8t^2}(3+\nu)$	$\frac{3qR^2}{8t^2}(3+\nu)$
Central load F, edges clamped	$\frac{3FR^2}{4\pi Et^3}(1-\nu^2)$	$\frac{3F}{2\pi t^2}$	$\frac{3\nu F}{2\pi t^2}$
Central load F, edges freely supported	$\frac{3FR^2}{4\pi Et^3}(3+\nu)(1-\nu)$	From $\frac{3F}{2\pi t^2}(1+\nu)\log_e \frac{R}{r}$	From $\frac{3F}{2\pi t^2}\left[(1+\nu)\log_e \frac{R}{r} + (1-\nu)\right]$

For an **annular ring**, *freely supported* at its outside edge, with total *load F applied around the inside radius R_1*, the maximum stress is tangential at the inside radius,

i.e.

$$\sigma_{z_{max}} = \frac{3F(1+\nu)}{\pi t^2}\left[\frac{R^2}{(R-R_1)}\log_e \frac{R}{R_1}\right]$$

If the outside edge is *clamped* the maximum stress becomes

$$\sigma_{max} = \frac{3F}{2\pi t^2}\left[\frac{(R^2 - R_1^2)}{R^2}\right]$$

For **thin membranes** subjected to *uniform pressure q* the maximum deflection is given by

$$y_{max} = 0.662\, R\left[\frac{qR}{Et}\right]^{1/3}$$

For **rectangular plates** subjected to *uniform loads* the maximum deflection and bending moments are given by equations of the form

$$y_{max} = \alpha\frac{qb^4}{Et^3}$$

$$M = \beta q b^2$$

the constants α and β depending on the method of support and plate dimensions. Typical values are listed later in Tables 7.3 and 7.4.

A. CIRCULAR PLATES

7.1. Stresses

Consider the portion of a thin plate or diaphragm shown in Fig. 7.1 bent to a radius R_{XY} in the XY plane and R_{YZ} in the YZ plane. The relationship between stresses and strains in a three-dimensional strain system is given by eqn. (7.2),†

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y - \nu\sigma_z]$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu\sigma_x - \nu\sigma_y]$$

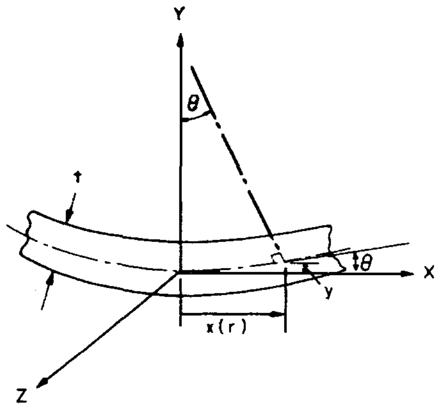

Fig. 7.1.

Now for thin plates, provided deflections are restricted to no greater than half the plate thickness,‡ the direct stress in the Y direction may be assumed to be zero and the above equations give

$$\sigma_x = \frac{E}{(1-\nu^2)}[\varepsilon_x + \nu\varepsilon_z] \tag{7.1}$$

$$\sigma_z = \frac{E}{(1-\nu^2)}[\varepsilon_z + \nu\varepsilon_x] \tag{7.2}$$

† E.J. Hearn, *Mechanics of Materials 1*, Butterworth-Heinemann, 1997.
‡ S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, 1959.

If u is the distance of any fibre from the neutral axis, then, for pure bending in the XY and YZ planes,

$$\frac{M}{I}=\frac{\sigma}{y}=\frac{E}{R} \quad \text{and} \quad \frac{\sigma}{E}=\frac{u}{R}=\varepsilon$$

$$\therefore \qquad \varepsilon_x=\frac{u}{R_{XY}} \quad \text{and} \quad \varepsilon_z=\frac{u}{R_{YZ}}$$

Now $\dfrac{1}{R}=\dfrac{d^2y}{dx^2}$ and, for small deflections, $\dfrac{du}{dx}=\tan\theta=\theta$ (radians).

$$\therefore \qquad \frac{1}{R_{XY}}=\frac{d^2y}{dx^2}=\frac{d\theta}{dx}$$

and

$$\varepsilon_x=u\frac{d\theta}{dx} \quad (= \text{radial strain}) \qquad (7.3)$$

Consider now the diagram Fig. 7.2 in which the radii of the concentric circles through C_1 and D_1 on the unloaded plate increase to $[(x+dx)+(\theta+d\theta)u]$ and $[x+u\theta]$, respectively, when the plate is loaded.

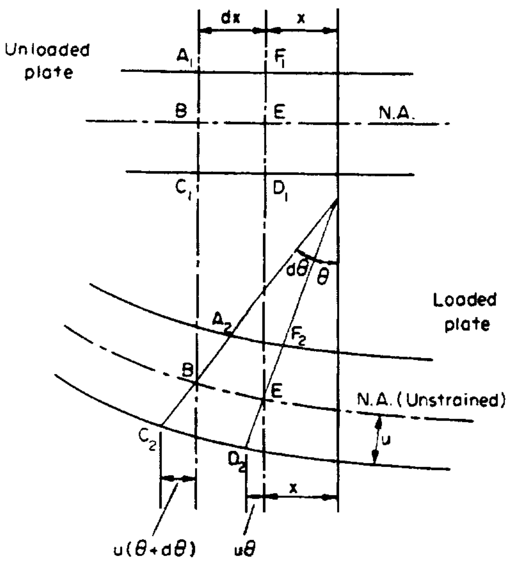

Fig. 7.2.

Circumferential strain at D_2

$$=\varepsilon_z=\frac{2\pi(x+u\theta)-2\pi x}{2\pi x}$$

$$=\frac{u\theta}{x} \quad (= \text{circumferential strain}) \qquad (7.4)$$

Substituting eqns. (7.3) and (7.4) in eqns. (7.1) and (7.2) yields

$$\sigma_x = \frac{E}{(1-\nu^2)}\left[u\frac{d\theta}{dx} + \nu\frac{u\theta}{x}\right]$$

i.e.
$$\sigma_x = \frac{Eu}{(1-\nu^2)}\left[\frac{d\theta}{dx} + \nu\frac{\theta}{x}\right] \tag{7.5}$$

Similarly,
$$\sigma_z = \frac{Eu}{(1-\nu^2)}\left[\frac{\theta}{x} + \nu\frac{d\theta}{dx}\right] \tag{7.6}$$

Thus we have equations for the stresses in terms of the slope θ and rate of change of slope $d\theta/dx$. We shall now proceed to evaluate the bending moments in the two planes in similar form and hence to the procedure for determination of θ and $d\theta/dx$ from a knowledge of the applied loading.

7.2. Bending moments

Consider the small section of plate shown in Fig. 7.3, which is of unit length. Defining the moments M as *moments per unit length* and applying the simple bending theory,

$$M = \frac{\sigma I}{y} = \frac{\sigma}{u}\left[\frac{1\times t^3}{12}\right] = \frac{\sigma t^3}{12u}$$

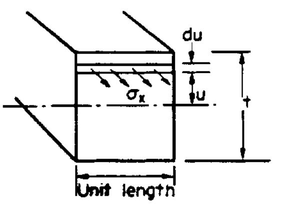

Fig. 7.3.

Substituting eqns. (7.5) and (7.6),

$$M_{XY} = \frac{Et^3}{12(1-\nu^2)}\left[\frac{d\theta}{dx} + \nu\frac{\theta}{x}\right] \tag{7.7}$$

Now $\dfrac{Et^3}{12(1-\nu^2)} = D$ is a constant and termed the *flexural stiffness*

so that
$$M_{XY} = D\left[\frac{d\theta}{dx} + \nu\frac{\theta}{x}\right] \tag{7.8}$$

and, similarly,
$$M_{YZ} = D\left[\frac{\theta}{x} + \nu\frac{d\theta}{dx}\right] \tag{7.9}$$

It is now possible to write the stress equations in terms of the applied moments,

i.e.
$$\sigma_x = M_{XY}\frac{12u}{t^3} \tag{7.10}$$

$$\sigma_z = M_{YZ}\frac{12u}{t^3} \tag{7.11}$$

7.3. General equation for slope and deflection

Consider now Fig. 7.4 which shows the forces and moments *per unit length* acting on a small element of the plate subtending an angle $\delta\phi$ at the centre. Thus M_{XY} and M_{YZ} are the moments per unit length in the two planes as described above and Q is the shearing force per unit length in the direction OY.

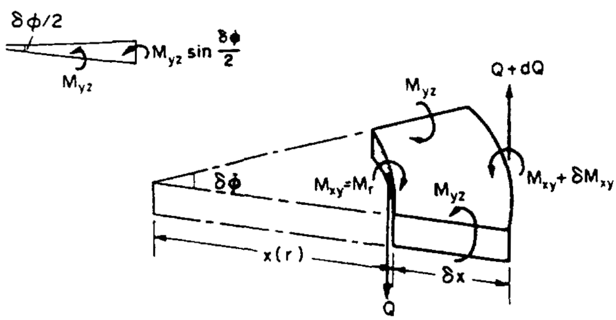

Fig. 7.4. Small element of circular plate showing applied moments and forces per unit length.

For equilibrium of moments in the radial XY plane, taking moments about the outside edge,

$$(M_{XY} + \delta M_{XY})(x + \delta x)\delta\phi - M_{XY}x\delta\phi - 2M_{YZ}\delta x \sin \tfrac{1}{2}\delta\phi + Qx\delta\phi\delta x = 0$$

which, neglecting squares of small quantities, reduces to

$$M_{XY}\delta x + \delta M_{XY}x - M_{YZ}\delta x + Qx\delta x = 0$$

In the limit, therefore,

$$M_{XY} + x\frac{dM_{XY}}{dx} - M_{YZ} = -Qx$$

Substituting eqns. (7.8) and (7.9), and simplifying

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \frac{\theta}{x^2} = -\frac{Q}{D}$$

This may be re-written in the form

$$\frac{d}{dx}\left[\frac{1}{x}\frac{d(x\theta)}{dx}\right] = -\frac{Q}{D} \tag{7.12}$$

This is then the general equation for slopes and deflections of circular plates or diaphragms. Provided that the applied loading Q is known as a function of x the expression can be treated in a similar manner to the equation

$$M = EI\frac{d^2y}{dx^2}$$

used in the Macaulay beam method, i.e. it may be successively integrated to determine θ, and hence y, in terms of constants of integration, and these can then be evaluated from known end conditions of the plate.

It will be noted that the expressions have been derived using cartesian coordinates (X, Y and Z). For circular plates, however, it is convenient to replace the variable x with the general radius r when the equations derived above may be re-written as follows:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{Q}{D} \tag{7.13}$$

radial stress

$$\sigma_r = \frac{Eu}{(1-\nu^2)}\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right] \tag{7.14}$$

tangential stress

$$\sigma_z = \frac{Eu}{(1-\nu^2)}\left[\nu\frac{d\theta}{dr} + \frac{\theta}{r}\right] \tag{7.15}$$

moments

$$M_r = D\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right] \tag{7.16}$$

$$M_z = D\left[\nu\frac{d\theta}{dr} + \frac{\theta}{r}\right] \tag{7.17}$$

In the case of applied uniformly distributed loads, i.e. pressures q, the effective shear load Q per unit length for use in eqn. (7.13) is found as follows.

At any radius r, for equilibrium,

$$Q \times 2\pi r = q \times \pi r^2$$

i.e. $$Q = \frac{qr}{2}$$

Thus *for applied pressures* eqn. (7.13) may be re-written

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{qr}{2D} \tag{7.18}$$

7.4. General case of a circular plate or diaphragm subjected to combined uniformly distributed load q (pressure) and central concentrated load F

For this general case the equivalent shear Q per unit length is given by

$$Q \times 2\pi r = q \times \pi r^2 + F$$

$\therefore$ $$Q = \frac{qr}{2} + \frac{F}{2\pi r}$$

Substituting in eqn. (7.18)

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = \left[-\frac{qr}{2} - \frac{F}{2\pi r}\right]\frac{1}{D}$$

Integrating,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) = -\frac{1}{D}\int\left[\frac{qr}{2} + \frac{F}{2\pi r}\right]dr$$

$$= -\frac{1}{D}\left[\frac{qr^2}{4} + \frac{F}{2\pi}\log_e r\right] + C_1$$

$$\therefore \quad \frac{d}{dr}\left(r\frac{dy}{dr}\right) = -\frac{1}{D}\left[\frac{qr^3}{4} + \frac{Fr}{2\pi}\log_e r\right] + C_1 r$$

Integrating,

$$r\frac{dy}{dr} = -\frac{1}{D}\left[\frac{qr^4}{16} + \frac{F}{2\pi}\left\{\frac{r^2}{2}\log_e r - \frac{r^2}{4}\right\}\right] + \frac{C_1 r^2}{2} + C_2$$

$$\therefore \quad \textbf{slope } \boldsymbol{\theta} = \frac{dy}{dr} = -\frac{qr^3}{16D} - \frac{Fr}{8\pi D}[2\log_e r - 1] + C_1\frac{r}{2} + \frac{C_2}{r} \tag{7.19}$$

Integrating again and simplifying,

$$\textbf{deflection } \boldsymbol{y} = -\frac{qr^4}{64D} - \frac{Fr^2}{8\pi D}[\log_e r - 1] + C_1\frac{r^2}{4} + C_2\log_e r + C_3 \tag{7.20}$$

The values of the constants of integration will be determined from known end conditions of the plate; slopes and deflections at any radius can then be evaluated. As an example of the procedure used it is now convenient to consider a number of standard loading cases and to determine the maximum deflections and stresses for each.

7.5. Uniformly loaded circular plate with edges clamped

The relevant fundamental equation for this loading condition has been shown to be

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{qr}{2D}$$

Integrating,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) = -\frac{qr^2}{4D} + C_1$$

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) = -\frac{qr^3}{4D} + C_1 r$$

Integrating,

$$r\frac{dy}{dr} = -\frac{qr^4}{16D} + C_1\frac{r^2}{2} + C_2$$

$$\therefore \quad \text{slope } \theta = \frac{dy}{dr} = -\frac{qr^3}{16D} + C_1\frac{r}{2} + \frac{C_2}{r} \tag{7.21}$$

Integrating,

$$\text{deflection } y = \frac{-qr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \tag{7.22}$$

Now if the slope θ is not to be infinite at the centre of the plate, $C_2 = 0$.
Taking the origin at the centre of the deflected plate, $y = 0$ when $r = 0$.
Therefore, from eqn. (7.22), $C_3 = 0$.
At the outside, clamped edge where $r = R$, $\theta = dy/dr = 0$.

Therefore substituting in the slope eqn. (6.21),

$$-\frac{qR^3}{16D} + \frac{C_1 R}{2} = 0$$

$$\therefore \qquad C_1 = \frac{qR^2}{8D}$$

The maximum deflection of the plate will be at the centre, but since this has been used as the origin the deflection equation will yield $y = 0$ at $r = 0$; indeed, this was one of the conditions used to evaluate the constants. We must therefore determine the equivalent amount by which the end supports are assumed to move up relative to the "fixed" centre.

Substituting $r = R$ in the deflection eqn. (7.22) yields

$$\text{maximum deflection} = -\frac{qR^4}{64D} + \frac{qR^4}{32D} = \frac{qR^4}{64D}$$

The positive value indicates, as usual, upwards deflection of the ends relative to the centre, i.e. along the positive y direction. The central deflection of the plate is thus, as expected, in the same direction as the loading, along the negative y direction (downwards).

Substituting for D,

$$y_{\max} = \frac{qR^4}{64}\left[\frac{12(1-\nu^2)}{Et^3}\right]$$

$$= \mathbf{\frac{3qR^4}{16Et^3}(1-\nu^2)} \tag{7.23}$$

Similarly, from eqn. (7.21),

$$\text{slope } \theta = -\frac{qr^3}{16D} + \frac{qR^2 r}{16D} = -\frac{qr}{16D}[r^2 - R^2]$$

$$\therefore \qquad \frac{d\theta}{dr} = -\frac{3qr^2}{16D} + \frac{qR^2}{16D} = -\frac{q}{16D}[3r^2 - R^2]$$

Now, from eqn. (7.14)

$$\sigma_r = \frac{Eu}{(1-\nu^2)}\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right]$$

$$= \frac{Eu}{(1-\nu^2)}\left[-\frac{qr^2}{16D}(3+\nu) + \frac{qR^2}{16D}(1+\nu)\right]$$

The maximum stress for the clamped edge condition will thus be obtained at the edge where $r = R$ and at the surface of the plate where $u = t/2$,

i.e.
$$\sigma_{r_{max}} = \frac{E}{(1-\nu^2)}\frac{t}{2}\frac{2qR^2}{16D} = \frac{\mathbf{3qR^2}}{\mathbf{4t^2}} \tag{7.24}$$

N.B.–It is not possible to determine the maximum stress by equating $d\sigma_r/dr$ to zero since this only gives the point where the slope of the σ_r curve is zero (see Fig. 7.7). The value of the stress at this point is not as great as the value at the edge.

Similarly,

$$\sigma_z = \frac{Eu}{(1-\nu^2)}\left[\frac{\theta}{r} + \nu\frac{d\theta}{dr}\right]$$

$$= \frac{Eu}{(1-\nu^2)}\left[-\frac{qr^2}{16D}(3\nu+1) + \frac{qR^2}{16D}(1+\nu)\right]$$

Unlike σ_r, this has a maximum value when $r = 0$, i.e. at the centre.

$$\sigma_{z_{max}} = \frac{E}{(1-\nu^2)}\frac{t}{2}\frac{qR^2}{16D}(1+\nu)$$

$$= \frac{\mathbf{3qR^2}}{\mathbf{8t^2}}(\mathbf{1+\nu}) \tag{7.25}$$

7.6. Uniformly loaded circular plate with edges freely supported

Since the loading, and hence fundamental equation, is the same as for §7.4, the slope and deflection equations will be of the same form, i.e. eqns (7.21) and (7.22) will apply. Further, the constants C_2 and C_3 will again be zero for the same reasons as before and only one new condition to solve for the constant C_1 is required.

Here we must make use of the fact that the bending moment is always zero at any free support,

i.e. at $r = R$. $\quad M_r = 0$

Therefore from eqn. (7.16),

$$D\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right] = 0$$

$$\therefore \quad \frac{d\theta}{dr} = -\nu\frac{\theta}{r}$$

Substituting from eqn. (7.21) with $r = R$ and $C_2 = 0$,

$$-\frac{3qR^2}{16D} + \frac{C_1}{2} = -\nu\left[-\frac{qR^2}{16D} + \frac{C_1}{2}\right]$$

$$\therefore \quad C_1 = \frac{qR^2}{8D}\left[\frac{(3+\nu)}{(1+\nu)}\right]$$

The maximum deflection is at the centre and again equal to the deflection of the supports relative to the centre.

Substituting for the constants with $r = R$ in eqn. (7.22),

$$\text{maximum deflection} = -\frac{qR^4}{64D} + \frac{qR^2}{8D}\frac{(3+\nu)}{(1+\nu)}\frac{R^2}{4}$$

$$= \frac{qR^4}{64D}\left[\frac{(5+\nu)}{(1+\nu)}\right]$$

i.e. substituting for D,

$$\mathbf{y_{max} = \frac{3qR^4}{16Et^3}(5+\nu)(1-\nu)} \qquad (7.26)$$

With $\nu = 0.3$ this value is approximately *four times* that for the clamped edge condition.

As before, the stresses are obtained from eqns. (7.14) and (7.15) by substituting for $d\theta/dr$ and θ/r from eqn. (7.21),

$$\sigma_r = \frac{Eu}{(1-\nu^2)}\left[-\frac{qr^2}{16D}(3+\nu) + \frac{qR^2}{16D}(3+\nu)\right]$$

This gives a maximum stress at the centre where $r = 0$

$$\sigma_{r_{max}} = \frac{E}{(1-\nu^2)}\frac{t}{2}\frac{qR^2}{16D}(3+\nu)$$

$$= \frac{3qR^2}{8t^2}(3+\nu)$$

Similarly, $\sigma_{z_{max}} = \dfrac{3qR^2}{8t^2}(3+\nu)$ also at the centre

i.e. for a uniformly loaded circular plate with edges freely supported,

$$\mathbf{\sigma_{r_{max}} = \sigma_{z_{max}} = \frac{3qR^2}{8t^2}(3+\nu)} \qquad (7.27)$$

7.7. Circular plate with central concentrated load F and edges clamped

For a central concentrated load,

$$Q \times 2\pi r = F$$

$$\therefore \qquad Q = \frac{F}{2\pi r}$$

The fundamental equation for slope and deflection is, therefore,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{F}{2\pi r D}$$

Integrating, $\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dy}{dr}\right) = -\dfrac{F}{2\pi D}\log_e r + C_1$

$$\frac{d}{dr}\left(r\frac{dy}{dr}\right) = -\frac{Fr}{2\pi D}\log_e r + C_1 r$$

Integrating,
$$r\frac{dy}{dr} = -\frac{F}{2\pi D}\left[\frac{r^2}{2}\log_e r - \frac{r^2}{4}\right] + \frac{C_1 r^2}{2} + C_2$$

$$\therefore \quad \theta = \frac{dy}{dr} = -\frac{F}{2\pi D}\left[\frac{r}{2}\log_e r - \frac{r}{4}\right] + C_1\frac{r}{2} + \frac{C_2}{r} \tag{7.28}$$

Integrating,
$$y = -\frac{Fr^2}{8\pi D}\left[\log_e r - 1\right] + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \tag{7.29}$$

Again, taking the origin at the centre of the deflected plate as shown in Fig. 7.5, the following conditions apply:

For a non-infinite slope at the centre $C_2 = 0$ and at $r = 0$, $y = 0$, $\therefore C_3 = 0$.

Also, at $r = R$, slope $\theta = dy/dr = 0$.

Therefore from eqn. (7.28),

$$\frac{F}{2\pi D}\left[\frac{R}{2}\log_e R - \frac{R}{4}\right] = \frac{C_1 R}{2}$$

$$\therefore \quad \frac{F}{\pi D}\left[\frac{\log_e R}{2} - \frac{1}{4}\right] = C_1$$

The maximum deflection will be at the centre and again equivalent to that obtained when $r = R$, i.e. from eqn. (7.29),

$$\begin{aligned}\text{maximum deflection} &= -\frac{FR^2}{8\pi D}[\log_e R - 1] + \frac{FR^2}{4\pi D}\left[\frac{\log_e R}{2} - \frac{1}{4}\right] \\ &= \frac{FR^2}{16\pi D}[-2\log_e R + 2 + 2\log_e R - 1] \\ &= \frac{FR^2}{16\pi D}\end{aligned}$$

Substituting for D,

$$\begin{aligned} y_{\max} &= \frac{FR^2}{16\pi}\frac{12(1-\nu^2)}{Et^3} \\ &= \mathbf{\frac{3FR^2}{4\pi Et^3}(1-\nu^2)} \end{aligned} \tag{7.30}$$

Again substituting for $d\theta/dr$ and θ/r from eqn. (7.28) into eqns (7.14) and (7.15) yields

$$\sigma_{r_{\max}} = \mathbf{\frac{3F}{2\pi t^2}} \tag{7.31}$$

$$\sigma_{z_{\max}} = \mathbf{\frac{3\nu F}{2\pi t^2}} \tag{7.32}$$

7.8. Circular plate with central concentrated load F and edges freely supported

The fundamental equation and hence the slope and deflection expressions will be as for the previous section (§7.7),

i.e.
$$\theta = \frac{dy}{dr} = -\frac{F}{2\pi D}\left[\frac{r}{2}\log_e r - \frac{r}{4}\right] + \frac{C_1 r}{2} \tag{7.33}$$

$$y = -\frac{Fr^2}{8\pi D}[\log_e r - 1] + \frac{C_1 r^2}{4} \tag{7.34}$$

constants C_2 and C_3 being zero as before.

As for the uniformly loaded plate with freely supported edges, the constant C_1 is determined from the knowledge that the bending moment M_r is zero at the free support,

i.e. at $r = R$, $\quad M_r = 0$

Therefore from eqn. (7.16),

$$D\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right] = 0 \quad \text{and} \quad \frac{d\theta}{dr} = -\nu\frac{\theta}{r}$$

and, substituting from eqn. (7.33) with $r = R$,

$$-\frac{F}{8\pi D}[2\log_e R - 1] + \frac{C_1}{2} = -\frac{\nu F}{8\pi D}[2\log_e R - 1] - \frac{\nu C_1}{2}$$

$$\therefore \quad \frac{C_1}{2}(1+\nu) = \frac{F}{8\pi D}[2(1+\nu)\log_e R - (1-\nu)]$$

$$C_1 = \frac{F}{4\pi D}\left[2\log_e R + \frac{(1-\nu)}{(1+\nu)}\right]$$

As before, the maximum deflection is at the centre and equivalent to that obtained with $r = R$.

Substituting in eqn. (7.34),

$$\text{maximum deflection} = \frac{FR^2}{8\pi D}[\log_e R - 1] + \frac{FR^2}{16\pi D}\left[2\log_e R + \frac{(1-\nu)}{(1+\nu)}\right]$$

$$= \frac{FR^2}{16\pi D}\frac{(3+\nu)}{(1+\nu)}$$

Substituting for D

$$\mathbf{y_{max} = \frac{3FR^2}{4\pi Et^3}(3+\nu)(1-\nu)} \tag{7.35}$$

For $\nu = 0.3$ this is approximately 2.5 times that for the clamped edge condition.

From eqn. (7.14),

$$\sigma_r = \frac{Eu}{(1-\nu^2)}\left[\frac{d\theta}{dr} + \nu\frac{\theta}{r}\right]$$

Substituting for $d\theta/dr$ and θ/r as above,

$$\sigma_r = \frac{Eu}{(1-\nu^2)}\left[\frac{F}{4\pi D}(1+\nu)\log_e\frac{R}{r}\right]$$

$$= \frac{3F}{2\pi t^2}(1+\nu)\log_e \frac{R}{r} \tag{7.36}$$

Thus the radial stress σ_r will be zero at the edge and will rise to a maximum value (theoretically infinite) at the centre. However, in practice, load cannot be applied strictly at a point but must contact over a finite area. Provided this area is known the maximum stress can be calculated.

Similarly, from eqn. (7.15)

$$\sigma_z = \frac{Eu}{(1-\nu^2)}\left[\nu\frac{d\theta}{dr} + \frac{\theta}{r}\right]$$

and, again substituting for $d\theta/dr$ and θ/r,

$$\sigma_z = \frac{3F}{2\pi t^2}\left[(1+\nu)\log_e \frac{R}{r} + (1-\nu)\right] \tag{7.37}$$

7.9. Circular plate subjected to a load F distributed round a circle

Consider the circular plate of Fig. 7.5 subjected to a total load F distributed round a circle of radius R_1. A solution is obtained to this problem by considering the plate as consisting of two parts $r < R_1$ and $r > R_1$, bearing in mind that the values of θ, y and M_r must be the same for both parts at the common radius $r = R_1$.

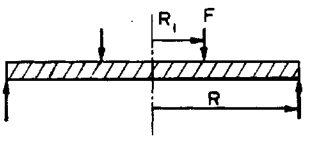

Fig. 7.5. Solid circular plate subjected to total load F distributed around a circle of radius R_1.

Thus, for $r < R_1$, we have a plate with zero distributed load and zero central concentrated load,

i.e. $$q = F = 0$$

Therefore from eqn. (7.20),

$$y = \frac{C_1 r^2}{4} + C_2 \log_e r + C_3$$

and from eqn. (7.19)

$$\theta = \frac{dy}{dr} = \frac{C_1 r}{2} + \frac{C_2}{r}$$

For non-infinite slope at the centre, $C_2 = 0$ and with the axis for deflections at the centre of the plate, $y = 0$ when $r = 0$, $\therefore\ C_3 = 0$.

Therefore for the inner portion of the plate

$$y = \frac{C_1 r^2}{4} \quad \text{and} \quad \theta = \frac{dy}{dr} = \frac{C_1 r}{2}$$

For the outer portion of the plate $r > R_1$ and eqn. (22.20) reduces to

$$y = -\frac{Fr^2}{8\pi D}[\log_e r - 1] + \frac{C_1' r^2}{4} + C_2' \log_e r + C_3' \tag{7.38}$$

and from eqn. (7.19)

$$\theta = \frac{dy}{dr} = -\frac{Fr}{8\pi D}[2\log_e r - 1] + \frac{C_1' r}{2} + \frac{C_2'}{r} \tag{7.39}$$

Equating these values with those obtained for the inner portions,

$$\frac{C_1 R_1^2}{4} = -\frac{FR_1^2}{8\pi D}[\log_e R_1 - 1] + \frac{C_1' R_1^2}{4} + C_2' \log_e R_1 + C_3'$$

and

$$\frac{C_1 R_1}{2} = -\frac{FR_1}{8\pi D}[2\log_e R_1 - 1] + \frac{C_1' R_1}{2} + \frac{C_2'}{R_1}$$

Similarly, from (7.16), equating the values of M_r at the common radius R_1 yields

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R_1 + (1-\nu)] + \frac{C_1'}{2}(1+\nu) - \frac{C_2'(1-\nu)}{R_1^2} = \frac{C_1}{2}(1+\nu)$$

Further, with $M_r = 0$ at $r = R$, the outside edge, from eqn. (7.16)

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R + (1-\nu)] + \frac{C_1'}{2}(1+\nu) - \frac{C_2'}{R^2}(1-\nu) = 0 \tag{7.40}$$

There are thus four equations with four unknowns C_1, C_1', C_2' and C_3' and a solution using standard simultaneous equation procedures is possible. Such a solution yields the following values:

$$C_1' = \frac{F}{4\pi D}\left[2\log_e R + \frac{(1-\nu)(R^2 - R_1^2)}{(1+\nu)\,R^2}\right]$$

$$C_2' = -\frac{FR_1^2}{8\pi D}$$

$$C_3' = \frac{FR_1^2}{8\pi D}[\log_e R_1 - 1]$$

The central deflection is found, as before, from the deflection of the edge, $r = R$, relative to the centre.

Substituting in eqn. (7.38) yields

$$y_{max} = \frac{F}{8\pi D}\left[\frac{(3+\nu)}{2(1+\nu)}(R^2 - R_1^2) - R_1^2) - R_1^2 \log_e \frac{R}{R_1}\right] \tag{7.41}$$

The maximum radial bending moment and hence radial stress occurs at $r = R_1$, giving

$$\sigma_{r_{max}} = \frac{3F}{4\pi t^2}\left[2(1+\nu)\log_e \frac{R}{R_1} + (1-\nu)\frac{(R^2 - R_1^2)}{R^2}\right] \tag{7.42}$$

It can also be shown similarly that the maximum tangential stress is of equal value to the maximum radial stress.

7.10. Application to the loading of annular rings

The general eqns. (7.38) and (7.39) derived above apply also for annular rings with a total load F applied around the inner edge of radius R_1 as shown in Fig. 7.6.

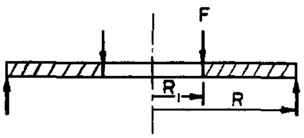

Fig. 7.6. Annular ring with total load F distributed around inner radius.

Here, however, the radial bending M_r is zero at both $r = R_1$ and $r = R$. Thus, applying the condition of eqn. (7.40) for both these radii yields

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{R^2}(1-\nu) = 0$$

and

$$-\frac{F}{8\pi D}[2(1+\nu)\log_e R_1 + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{R_1^2}(1-\nu) = 0$$

Subtracting to eliminate C_1 gives

$$C_2 = \frac{F}{4\pi D}\left[\frac{(1+\nu)}{(1-\nu)}\frac{R^2R_1^2}{(R^2-R_1^2)}\log_e\frac{R}{R_1}\right]$$

and hence

$$C_1 = \frac{F}{4\pi D}\left[\frac{2(R^2\log_e R - R_1^2\log_e R_1)}{(R^2-R_1^2)} + \frac{(1-\nu)}{(1+\nu)}\right]$$

It can then be shown that the maximum stress set up is the tangential stress at $r = R_1$ of value

$$\sigma_{z_{\max}} = \frac{3F(1+\nu)}{\pi t^2}\left[\frac{R^2}{(R^2-R_1^2)}\right]\log_e\frac{R}{R_1} \tag{7.43}$$

If the **outside edge of the plate is clamped** instead of freely supported the maximum stress becomes

$$\sigma_{\max} = \frac{3F}{2\pi t^2}\left[\frac{(R^2-R_1^2)}{R^2}\right]$$

7.11. Summary of end conditions

Axes can be selected to move with the plate as shown in Fig. 7.7(a) or stay at the initial, undeflected position Fig. 7.7(b).

For the former case, i.e. axes origin at the centre of the deflected plate, the end conditions which should be used for solution of the constants of integration are:

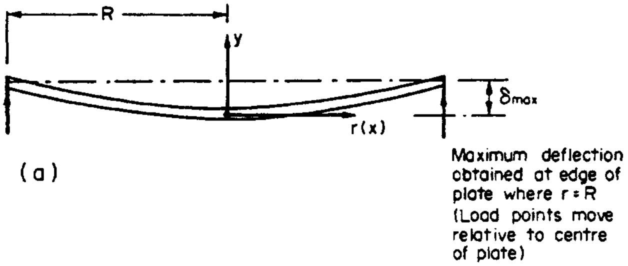

Fig. 7.7(a). Origin of reference axes taken to move with the plate.

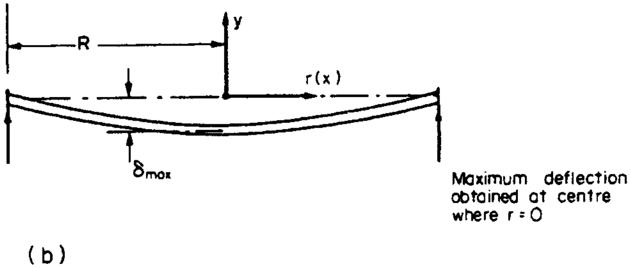

Fig. 7.7(b). Origin of reference axes remaining in the undeflected plate position.

Edges freely supported:

(i) Slope θ and deflection y non-infinite at the centre. $\therefore C_2 = 0$.
(ii) At $x = 0$, $y = 0$ giving $C_3 = 0$.
(iii) At $x = R$, $M_{xy} = 0$; hence C_1.

The maximum deflection is then that given at $x = R$.

Edges clamped:

(i) Slope θ and deflection y non-infinite at the centre. $\therefore C_2 = 0$.
(ii) At $x = 0$, $y = 0$ $\quad \therefore C_3 = 0$.
(iii) At $x = R$, $\dfrac{dy}{dx} = 0$; hence C_1.

Again the maximum deflection is that given at $x = R$.

7.12. Stress distributions in circular plates and diaphragms subjected to lateral pressures

It is now convenient to consider the stress distribution in plates subjected to lateral, uniformly distributed loads or pressures in more detail since this represents the loading condition encountered most often in practice.

Figures 7.8(a) and 7.8(b) show the radial and tangential stress distributions on the lower surface of a thin plate subjected to uniform pressure as given by the equations obtained in §§7.5 and 7.6.

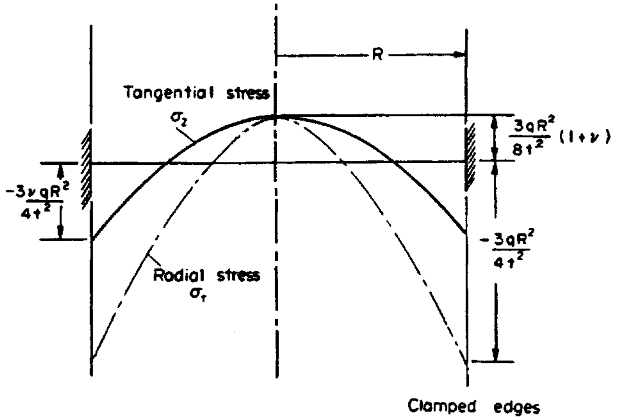

Fig. 7.8(a). Radial and tangential stress distributions in circular plates with clamped edges.

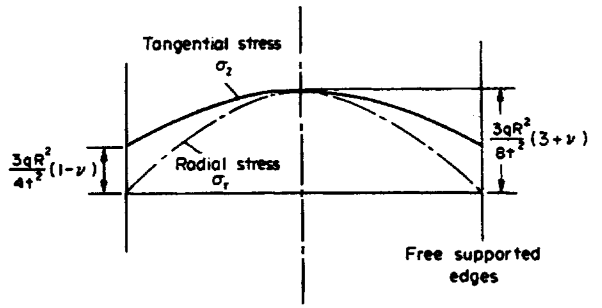

Fig. 7.8(b). Radial and tangential stress distributions in circular plates with freely supported edges.

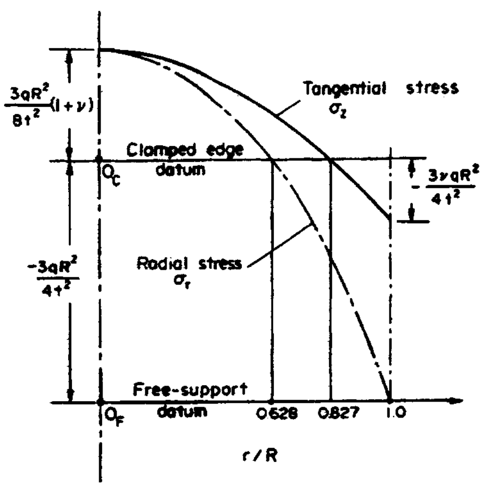

Fig. 7.9. Comparison of bending stresses in circular plates for clamped and freely supported edge conditions.

The two figures may be combined on to common axes as in Fig. 7.9 to facilitate comparison of the stress distributions for freely supported and clamped-edge conditions. Then if ordinates are measured from the horizontal axis through origin O_c, the curves give the values of radial and tangential stress for clamped-edge conditions.

Alternatively, measuring the ordinates from the horizontal axis passing through origin O_F in Fig. 7.9, i.e. adding to the clamped-edges stresses the constant value $\frac{3}{4}qR^2/t^2$, we obtain the stresses for a simply supported edge condition. The combined diagram clearly illustrates that a more favourable stress distribution is obtained when the edges of a plate are clamped.

7.13. Discussion of results – limitations of theory

The results of the preceding paragraphs are summarised in Table 7.1 at the start of the chapter. From this table the following approximate relationships are seen to apply:

(1) The maximum deflection of a uniformly loaded circular plate with freely supported edges is approximately four times that for the clamped-edge condition.
(2) Similarly, for a central concentrated load, the maximum deflection in the freely supported edge condition is 2.5 times that for clamped edges.
(3) With clamped edges the maximum deflection for a central concentrated load is four times that for the equivalent u.d.l. (i.e. $F = q \times \pi R^2$) and the maximum stresses are doubled.
(4) With freely supported edges, the maximum deflection for a central concentrated load is 2.5 times that for the equivalent u.d.l.

It must be remembered that the theory developed in this chapter has been based upon the assumption that deflections are small in comparison with the thickness of the plate. If deflections exceed half the plate thickness, then stretching of the middle surface of the plate must be considered. Under these conditions deflections are no longer proportional to the loads applied, e.g. for circular plates with clamped edges deflections δ can be determined from the equation

$$\delta + 0.58\frac{\delta^3}{t^2} = \frac{qR^4}{64D} \qquad (7.44)$$

For very thin diaphragms or membranes subjected to uniform pressure, stresses due to stretching of the middle surface may far exceed those due to bending and under these conditions the central deflection is given by

$$y_{\max} = 0.0662\ R\left[\frac{qR}{Et}\right]^{1/3} \qquad (7.45)$$

In the design of circular plates subjected to central concentrated loading, the maximum tensile stress on the lower surface of the plate is of prime interest since the often higher compressive stresses in the upper surface are generally much more localised. Local yielding of ductile materials in these regions will not generally affect the overall deformation of the plate provided that the lower surface tensile stresses are kept within safe limits. The situation is similar for plates constructed from brittle materials since their compressive strengths far exceed their strength in tension so that a limit on the latter is normally a safe design procedure. The theory covered in this text has involved certain simplifying assumptions; a full treatment of the problem shows that the limiting tensile stress is more accurately given by the equation

$$\sigma_{r_{\max}} = \frac{F}{t^2}(1+\nu)(0.485\log_e R/t + 0.52) \tag{7.46}$$

7.14. Other loading cases of practical importance

In addition to the standard cases covered in the previous sections there are a number of other loading cases which are often encountered in practice; these are illustrated in Fig. 7.10†. The method of solution for such cases is introduced briefly below.‡

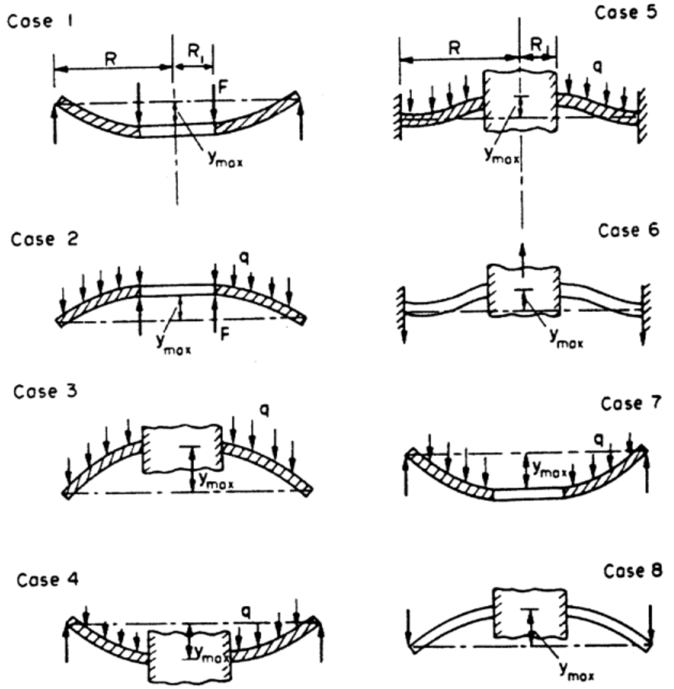

Fig. 7.10. Circular plates and diaphragms: various loading cases encountered in practice.

In all the cases illustrated the **maximum stress** is obtained from the following standard form of equations:

For *uniformly distributed loads*

$$\sigma_{\max} = k_1 \frac{qR^2}{t^2}$$

For *loads concentrated around the edge of the central hole*,

$$\sigma_{\max} = \frac{k_1 F}{t^2}$$

† S. Timoshenko, *Strength of Materials*, Part II, *Advanced Theory and Problems*, Van Nostrand.

‡ A.M. Wahl and G. Lobo, *Trans. ASME* 52 (1929).

Similarly, the **maximum deflections** in each case are given by the following equations:
For *uniformly distributed loads*,

$$y_{\max} = k_2 \frac{qR^4}{Et^3} \tag{7.49}$$

For *loads concentrated around the central hole*,

$$y_{\max} = k_2 \frac{FR^2}{Et^3} \tag{7.50}$$

The values of the factors k_1 and k_2 for the loading cases of Fig. 7.10 are given in Table 7.2, assuming a Poisson's ratio ν of 0.3.

Table 7.2. Coefficients k_1 and k_2 for the eight cases shown in Fig. 7.10[a].

$\frac{R}{R_1}$	1.25		1.5		2		3		4		5	
Case	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2
1	1.10	0.341	1.26	0.519	1.48	0.672	1.88	0.734	2.17	0.724	2.34	0.704
2	0.66	0.202	1.19	0.491	2.04	0.902	3.34	1.220	4.30	1.300	5.10	1.310
3	0.135	0.00231	0.410	0.0183	1.04	0.0938	2.15	0.293	2.99	0.448	3.69	0.564
4	0.122	0.00343	0.336	0.0313	0.74	0.1250	1.21	0.291	1.45	0.417	1.59	0.492
5	0.090	0.00077	0.273	0.0062	0.71	0.0329	1.54	0.110	2.23	0.179	2.80	0.234
6	0.115	0.00129	0.220	0.0064	0.405	0.0237	0.703	0.062	0.933	0.092	1.13	0.114
7	0.592	0.184	0.976	0.414	1.440	0.664	1.880	0.824	2.08	0.830	2.19	0.813
8	0.227	0.00510	0.428	0.0249	0.753	0.0877	1.205	0.209	1.514	0.293	1.745	0.350

[a] S. Timoshenko, *Strength of Materials*, Part II, *Advanced Theory and Problems*, Van Nostrand, p. 113.

B. BENDING OF RECTANGULAR PLATES

The theory of bending of rectangular plates is beyond the scope of this text and will not be introduced here. The standard formulae obtained from the theory,† however, may be presented in simple form and are relatively easy to apply. The results for the two most frequently used loading conditions are therefore summarised below.

7.15. Rectangular plates with simply supported edges carrying uniformly distributed loads

For a rectangular plate length d, shorter side b and thickness t, the *maximum deflection* is found to occur at the centre of the plate and given by

$$y_{\max} = \alpha \frac{qb^4}{Et^3} \tag{7.51}$$

the value of the factor α depending on the ratio d/b and given in Table 7.3.

† S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

Table 7.3. Constants for uniformly loaded rectangular plates with simply supported edges[a].

d/b	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
α	0.0443	0.0530	0.0616	0.0697	0.0770	0.0843	0.0906	0.0964
β_1	0.0479	0.0553	0.0626	0.0693	0.0753	0.0812	0.0862	0.0908
β_2	0.0479	0.0494	0.0501	0.0504	0.0506	0.0500	0.0493	0.0486
d/b	1.8	1.9	2.0	3.0	4.0	5.0	∞	
α	0.1017	0.1064	0.1106	0.1336	0.1400	0.1416	0.1422	
β_1	0.0948	0.0985	0.1017	0.1189	0.1235	0.1246	0.1250	
β_2	0.0479	0.0471	0.0464	0.0404	0.0384	0.0375	0.0375	

[a] S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

The *maximum bending moments*, per unit length, also occur at the centre of the plate and are given by

$$M_{XY_{\max}} = \beta_1 q b^2 \tag{7.52}$$

$$M_{YZ_{\max}} = \beta_2 q b^2 \tag{7.53}$$

the factors β_1 and β_2 being given in Table 7.4 for an assumed value of Poisson's ratio ν equal to 0.3.

It will be observed that for length ratios d/b in excess of 3 the values of the factors α, β_1, and β_2 remain practically constant as also will the corresponding maximum deflections and bending moments.

7.16. Rectangular plates with clamped edges carrying uniformly distributed loads

Here again the *maximum deflection* takes place at the centre of the plate, the value being given by an equation of similar form to eqn. (7.51) for the simply-supported edge case but with different values of α,

i.e.
$$y_{\max} = \alpha \frac{qb^4}{Et^3}$$

The *bending moment* equations are also similar in form, the *numerical maximum occurring at the middle of the longer side* and given by

$$M_{\max} = \beta q b^2$$

Typical values for α and β are given in Table 7.4. In this case values are practically constant for $d/b > 2$.

Table 7.4. Constants for uniformly loaded rectangular plates with clamped edges.[a].

d/b	1.00	1.25	1.50	1.75	2.00	∞
α	0.0138	0.0199	0.0240	0.0264	0.0277	0.0284
β	0.0513	0.0665	0.0757	0.0806	0.0829	0.0833

[a] S. Timoshenko, *Theory of Plates and Shells*, 2nd edn., McGraw-Hill, New York, 1959.

It will be observed, by comparison of the values of the factors in Tables 7.3 and 7.4, that when the edges of a plate are clamped the maximum deflection is considerably reduced from the freely supported condition but the maximum bending moments, and hence maximum stresses, are not greatly affected.

Examples

Example 7.1

A circular flat plate of diameter 120 mm and thickness 10 mm is constructed from steel with $E = 208\ \text{GN/m}^2$ and $\nu = 0.3$. The plate is subjected to a uniform pressure of $5\ \text{MN/m}^2$ on one side only. If the plate is clamped at the edges determine:

(a) the maximum deflection;
(b) the position and magnitude of the maximum radial stress.

What percentage change in the results will be obtained if the edge conditions are changed such that the plate can be assumed to be freely supported?

Solution

(a) From eqn. (7.23) the maximum deflection with clamped edges is given by

$$\begin{aligned} y_{max} &= \frac{3qR^4}{16Et^3}(1-\nu^2) \\ &= \frac{3 \times 5 \times 10^6 \times (60 \times 10^{-3})^4(1-0.3^2)}{16 \times 208 \times 10^9 \times (10 \times 10^{-3})^3} \\ &= 0.053 \times 10^{-3} = \mathbf{0.053\ mm} \end{aligned}$$

(b) From eqn. (7.24) the maximum radial stress occurs at the outside edge and is given by

$$\begin{aligned} \sigma_{r_{max}} &= \frac{3qR^2}{4t^2} \\ &= \frac{3 \times 5 \times 10^6 \times (60 \times 10^{-3})^2}{4 \times (10 \times 10^{-3})^2} \\ &= 135 \times 10^6 = \mathbf{135\ MN/m^2} \end{aligned}$$

When the edges are freely supported, eqn. (7.26) gives

$$\begin{aligned} y'_{max} &= \frac{3qR^4}{16Et^3}(5+\nu)(1-\nu) \\ &= \frac{(5+\nu)(1-\nu)}{(1-\nu^2)} y_{max} \\ &= \frac{(5.3 \times 0.7)}{0.91} \times 0.053 = \mathbf{0.216\ mm} \end{aligned}$$

and eqn. (7.27) gives

$$\sigma'_{r_{max}} = \frac{3qR^2}{8t^2}(3+\nu)$$

$$\sigma'_{r_{max}} = \frac{(3+\nu)}{2}\sigma_{r_{max}}$$

$$= \frac{3.3}{2} \times 135 = \mathbf{223\ MN/m^2}$$

Thus the percentage increase in maximum deflection

$$= \frac{(0.216 - 0.053)}{0.053}100 = \mathbf{308\%}$$

and the percentage increase in maximum radial stress

$$= \frac{(223 - 135)}{135}100 = \mathbf{65\%}$$

Example 7.2

A circular disc 150 mm diameter and 12 mm thickness is clamped around the periphery and built into a piston of diameter 60 mm at the centre. Assuming that the piston remains rigid, determine the maximum deflection of the disc when the piston carries a load of 5 kN. For the material of the disc $E = 208$ GN/m^2 and $\nu = 0.3$.

Solution

From eqn. (7.29) the deflection of the disc is given by

$$y = \frac{-Fr^2}{8\pi D}[\log_e r - 1] + \frac{C_1 r^2}{4} + C_2 \log_e r + C_3 \qquad (1)$$

and from eqn. (7.28)

$$\text{slope } \theta = \frac{-Fr}{8\pi D}[2\log_e r - 1] + \frac{C_1 r}{2} + \frac{C_2}{r} \qquad (2)$$

Now slope = 0 at $r = 0.03$ m.
Therefore from eqn. (2)

$$0 = \frac{-5000 \times 0.03}{8\pi D}[2\log_e 0.03 - 1] + 0.015C_1 + 33.3C_2$$

But

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{208 \times 10^9 \times (12 \times 10^{-3})^3}{12(1 - 0.09)}$$

$$= \frac{208 \times 1728}{12 \times 0.91} = 32900$$

$$\therefore \quad 0 = \frac{-5000 \times 0.03}{8\pi \times 32900}[2(-3.5066) - 1] + 0.015C_1 + 33.3C_2$$

$$\therefore \quad -1.45 \times 10^{-3} = 0.015C_1 + 33.3C_2 \qquad (3)$$

Also the slope $= 0$ at $r = 0.075$.
Therefore from eqn. (2) again,

$$0 = \frac{-5000 \times 0.075}{8\pi \times 32900}[2\log_e 0.075 - 1] + 0.0375C_1 + \frac{C_2}{0.075}$$

$$= -4.54 \times 10^{-4}[2(-2.5903) - 1] + 0.0375C_1 + 13.33C_2$$

$$-2.8 \times 10^{-3} = 0.0375C_1 + 13.33C_2 \tag{4}$$

(3) $\times \dfrac{0.0375}{0.015}$,

$$-3.625 \times 10^{-3} = 0.0375C_1 + 83.25C_2 \tag{5}$$

(5) − (4),

$$-0.825 \times 10^{-3} = 69.92C_2$$

$$\therefore \quad C_2 = -11.8 \times 10^{-6}$$

Substituting in (5),

$$-3.625 \times 10^{-3} = 0.0375C_1 - 9.82 \times 10^{-4}$$

$$C_1 = -\frac{(3.625 - 0.982)}{0.0375}10^{-3}$$

$$= -7.048 \times 10^{-2}$$

Now taking $y = 0$ at $r = 0.075$, from eqn. (1)

$$0 = \frac{-5000 \times (0.075)^2}{8\pi \times 32900}[\log_e 0.075 - 1] - \frac{7.048 \times 10^{-2}}{4}(0.075)^2 - 11.8 \times 10^{-6}\log_e 0.075 + C_3$$

$$= -3.4 \times 10^{-5}(-3.5903) - 99.1 \times 10^{-6} + 30.6 \times 10^{-6} + C_3$$

$$= 10^{-6}(122 - 99.1 + 30.6) + C_3$$

$$\therefore \quad C_3 = -53.5 \times 10^{-6}$$

Therefore deflection at $r = 0.03$ is given by eqn. (1),

$$\delta_{max} = \frac{-5000 \times (0.03)^2}{8\pi \times 32900}[\log_e 0.03 - 1] - \frac{7.048 \times 10^{-2}}{4}(0.03)^2 - 11.8 \times 10^{-6}\log_e 0.03 - 53.5 \times 10^{-6}$$

$$= 10^{-6}[+24.5 - 15.9 + 41.4 - 53.5] = \mathbf{-3.5 \times 10^{-6}\ m}$$

Problems

In the following examples assume that

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right)\right] = -\frac{Q}{D} \quad \text{or} \quad = -\frac{qr}{2D}$$

with conventional notations.

Unless otherwise stated, $E = 207$ GN/m^2 and $\nu = 0.3$.

7.1 (B/C). A circular flat plate of 120 mm diameter and 6.35 mm thickness is clamped at the edges and subjected to a uniform lateral pressure of 345 kN/m^2. Evaluate (a) the central deflection, (b) the position and magnitude of the maximum radial stress. [1.45×10^{-5} m, 23.1 MN/m^2; $r = 60$ mm.]

7.2 (B/C). The plate of Problem 7.1 is subjected to the same load but is simply supported round the edges. Calculate the central deflection. [58×10^{-6} m.]

7.3 (B/C). An aluminium plate diaphragm is 500 mm diameter and 6 mm thick. It is clamped around its periphery and subjected to a uniform pressure q of 70 kN/m^2. Calculate the values of maximum bending stress and deflection.

Take $Q = qR/2$, $E = 70$ GN/m^2 and $\nu = 0.3$. [91, 59.1 MN/m^2; 3.1 mm.]

7.4 (B/C). A circular disc of uniform thickness 1.5 mm and diameter 150 mm is clamped around the periphery and built into a piston, diameter 50 mm, at the centre. The piston may be assumed rigid and carries a central load of 450 N. Determine the maximum deflection. [0.21 mm.]

7.5 (C). A circular steel plate 5 mm thick, outside diameter 120 mm, inside diameter 30 mm, is clamped at its outer edge and loaded by a ring of edge moments $M_r = 8$ kN/m of circumference at its inner edge. Calculate the deflection at the inside edge. [4.68 mm.]

7.6 (C). A solid circular steel plate 5 mm thick, 120 mm outside diameter, is clamped at its outer edge and loaded by a ring of loads at $r = 20$ mm. The total load on the plate is 10 kN. Calculate the central deflection of the plate. [0.195 mm.]

7.7 (C). A pressure vessel is fitted with a circular manhole 600 mm diameter, the cover of which is 25 mm thick. If the edges are clamped, determine the maximum allowable pressure, given that the maximum principal strain in the cover plate must not exceed that produced by a simple direct stress of 140 MN/m^2. [1.19 MN/m^2.]

7.8 (B/C). The crown of a gas engine piston may be treated as a cast-iron diaphragm 300 mm diameter and 10 mm thick, clamped at its edges. If the gas pressure is 3 MN/m^2, determine the maximum principal stresses and the central deflection.

$\nu = 0.3$ and $E = 100$ GN/m^2. [506, 329 MN/m^2; 2.59 mm.]

7.9 (B/C). How would the values for Problem 7.8 change if the edges are released from clamping and freely supported? [835,835 MN/m^2; 10.6 mm.]

7.10 (B/C). A circular flat plate of diameter 305 mm and thickness 6.35 mm is clamped at the edges and subjected to a uniform lateral pressure of 345 kN/m^2.

Evaluate: (a) the central deflection, (b) the position and magnitude of the maximum radial stress. [6.1×10^{-4} m; 149.2 MN/m^2.]

7.11 (B/C). The plate in Problem 7.10 is subjected to the same load, but simply supported round the edges. Evaluate the central deflection. [24.7×10^{-4} m.]

7.12 (B/C). The flat end-plate of a 2 m diameter container can be regarded as clamped around its edge. Under operating conditions the plate will be subjected to a uniformly distributed pressure of 0.02 MN/m^2. Calculate from first principles the required thickness of the end plate if the bending stress in the plate should not exceed 150 MN/m^2. For the plate material $E = 200$ GN/m^2 and $\nu = 0.3$. [C.E.I.] [10 mm.]

7.13 (C). A cylinder head valve of diameter 38 mm is subjected to a gas pressure of 1.4 MN/m^2. It may be regarded as a uniform thin circular plate simply supported around the periphery. Assuming that the valve stem applies a concentrated force at the centre of the plate, calculate the movement of the stem necessary to lift the valve from its seat. The flexural rigidity of the valve is 260 Nm and Poisson's ratio for the material is 0.3. [C.E.I.] [0.067 mm.]

7.14 (C). A diaphragm of light alloy is 200 mm diameter, 2 mm thick and firmly clamped around its periphery before and after loading. Calculate the maximum deflection of the diaphragm due to the application of a uniform pressure of 20 kN/m^2 normal to the surface of the plate.

Determine also the value of the maximum radial stress set up in the material of the diaphragm.

Assume $E = 70$ GN/m^2 and Poisson's ratio $\nu = 0.3$. [B.P.] [0.61 mm; 37.5 MN/m^2.]

7.15 (C). A thin plate of light alloy and 200 mm diameter is firmly clamped around its periphery. Under service conditions the plate is to be subjected to a uniform pressure p of 20 kN/m^2 acting normally over its whole surface area.

Determine the required minimum thickness t of the plate if the following design criteria apply;

(a) the maximum deflection is not to exceed 6 mm;

(b) the maximum radial stress is not to exceed 50 MN/m^2.

Take $E = 70$ GN/m^2 and $\nu = 0.3$. [B.P.] [1.732 mm.]

7.16 (C). Determine equations for the maximum deflection and maximum radial stress for a circular plate, radius R, subjected to a distributed pressure of the form $q = K/r$. Assume simply supported edge conditions:

$$\left[\delta_{max} = \frac{-KR^3(4+\nu)}{36D(1+\nu)}, \sigma_{max} = \frac{EtRK(2+\nu)}{12D(1-\nu^2)}\right]$$

7.17 (C). The cover of the access hole for a large steel pressure vessel may be considered as a circular plate of 500 mm diameter which is firmly clamped around its periphery. Under service conditions the vessel operates with an internal pressure of 0.65 MN/m^2.

Determine the minimum thickness of plate required in order to achieve the following design criteria:

(a) the maximum deflection is limited to 5 mm;

(b) the maximum radial stress is limited to 200 MN/m^2.

For the steel, $E = 208$ GN/m^2 and $\nu = 0.3$.

You may commence your solution on the assumption that the deflection y at radius r for a uniform circular plate under the action of a uniform pressure q is given by:

$$\frac{d}{dr}\left[\frac{1}{r}\cdot\frac{d}{dr}\left(r\cdot\frac{dy}{dr}\right)\right] = -\frac{qr}{2D}$$

where D is the "flexural stiffness" of the plate. [9.95 mm.]

7.18 (C). A circular plate, 300 mm diameter and 5 mm thick, is built-in at its periphery. In order to strengthen the plate against a concentrated central axial load P the plate is stiffened by radial ribs and a prototype is found to have a stiffness of 11300 N per mm central deflection.

(a) Check that the equation:

$$y = \frac{Pr^2}{8\pi D}\left[l_n\left(\frac{r}{R_2}\right) - \frac{1}{2}\right] + \frac{PR_2^2}{16\pi D}$$

satisfies the boundary conditions for the *unstiffened* plate.

(b) Hence determine the stiffness of the plate without the ribs in terms of central deflection and calculate the relative stiffening effect of the ribs.

(c) What additional thickness would be required for an unstiffened plate to produce the same effect? For the plate material $E = 200$ GN/m^2 and $\nu = 0.28$. [5050 N/mm; 124%; 1.54 mm.]