SLOPE AND DEFLECTION OF BEAMS

Summary

The following relationships exist between loading, shearing force (S.F.), bending moment (B.M.), slope and deflection of a beam:

$$\text{deflection} = y \quad (\text{or } \delta)$$

$$\text{slope} = i \text{ or } \theta = \frac{dy}{dx}$$

$$\text{bending moment} = M = EI\frac{d^2y}{dx^2}$$

$$\text{shearing force} = Q = EI\frac{d^3y}{dx^3}$$

$$\text{loading} = w = EI\frac{d^4y}{dx^4}$$

In order that the above results should agree mathematically the sign convention illustrated in Fig. 5.4 must be adopted.

Using the above formulae the following standard values for *maximum slopes* and *deflections* of simply supported beams are obtained. (These assume that the beam is uniform, i.e. EI is constant throughout the beam.)

MAXIMUM SLOPE AND DEFLECTION OF SIMPLY SUPPORTED BEAMS

Loading condition	Maximum slope	Deflection (y)	Max. deflection (y_{max})
Cantilever with concentrated load W at end	$\dfrac{WL^2}{2EI}$	$\dfrac{W}{6EI}[2L^3 - 3L^2x + x^3]$	$\dfrac{WL^3}{3EI}$
Cantilever with u.d.l. across the complete span	$\dfrac{wL^3}{6EI}$	$\dfrac{w}{24EI}[3L^4 - 4L^3x + x^4]$	$\dfrac{wL^4}{8EI}$
Simply supported beam with concentrated load W at the centre	$\dfrac{WL^2}{16EI}$	$\dfrac{Wx}{48EI}[3L^2 - 4x^2]$	$\dfrac{WL^3}{48EI}$
Simply supported beam with u.d.l. across complete span	$\dfrac{wL^3}{24EI}$	$\dfrac{wx}{24EI}[L^3 - 2Lx^2 + x^3]$	$\dfrac{5wL^4}{384EI}$
Simply supported beam with concentrated load W offset from centre (distance a from one end b from the other)	$0.062\dfrac{WL^2}{EI}$		$\dfrac{Wa}{3EIL}\left[\dfrac{L^2 - a^2}{3}\right]^{3/2}$

Here L is the length of span, EI is known as the flexural rigidity of the member and x for the cantilevers is measured from the free end.

The determination of beam slopes and deflections by simple integration or Macaulay's methods requires a knowledge of certain conditions for various loading systems in order that the constants of integration can be evaluated. They are as follows:

(1) Deflections at supports are assumed zero unless otherwise stated.
(2) Slopes at built-in supports are assumed zero unless otherwise stated.
(3) Slope at the centre of symmetrically loaded and supported beams is zero.
(4) Bending moments at the free ends of a beam (i.e. those not built-in) are zero.

Mohr's theorems for slope and deflection state that if A and B are two points on the deflection curve of a beam and B is a point of zero slope, then

(1) $\qquad$ slope at A = area of $\dfrac{M}{EI}$ diagram between A and B

For a uniform beam, EI is constant, and the above equation reduces to

$$\text{slope at } A = \frac{1}{EI} \times \text{area of B.M. diagram between } A \text{ and } B$$

N.B.–If B is not a point of zero slope the equation gives the change of slope between A and B.

(2) $\quad$ Total deflection of A relative to B = first moment of area of $\dfrac{M}{EI}$ diagram about A

For a uniform beam

$$\text{total deflection of } A \text{ relative to } B = \frac{1}{EI} \times \text{first moment of area of B.M. diagram about } A$$

Again, if B is not a point of zero slope the equation only gives the deflection of A relative to the tangent drawn at B.

Useful quantities for use with uniformly distributed loads are shown in Fig. 5.1.

$$A_1 = \frac{bd}{3} \qquad \bar{x}_1 = \frac{3b}{4}$$
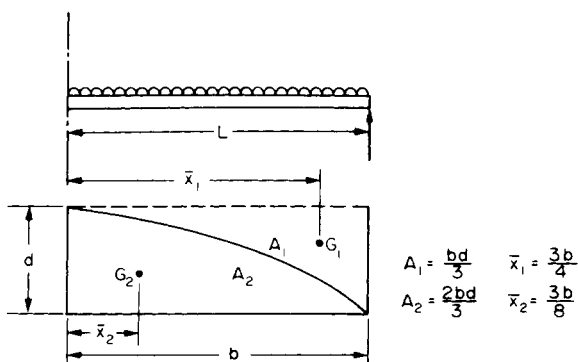$$A_2 = \frac{2bd}{3} \qquad \bar{x}_2 = \frac{3b}{8}$$

Fig. 5.1.

Both the straightforward integration method and Macaulay's method are based on the relationship $M = EI\dfrac{d^2 y}{dx^2}$ (see § 5.2 and § 5.3).

Clapeyron's equations of three moments for continuous beams in its simplest form states that for any portion of a beam on three supports 1, 2 and 3, with spans between of L_1 and L_2, the bending moments at the supports are related by

$$-M_1 L_1 - 2M_2(L_1 + L_2) - M_3 L_2 = 6\left[\frac{A_1 \bar{x}_1}{L_1} + \frac{A_2 \bar{x}_2}{L_2}\right]$$

where A_1 is the area of the B.M. diagram, assuming span L_1 simply supported, and $\bar{x}_1$ is the distance of the centroid of this area from the left-hand support. Similarly, A_2 refers to span L_2, with $\bar{x}_2$ the centroid distance from the right-hand support (see Examples 5.6 and 5.7). The following standard results are useful for $\dfrac{6A\bar{x}}{L}$:

(a) Concentrated load W, distance a from the nearest outside support

$$\frac{6A\bar{x}}{L} = \frac{Wa}{L}(L^2 - a^2)$$

(b) Uniformly distributed load w

$$\frac{6A\bar{x}}{L} = \frac{wL^3}{4} \quad \text{(see Example 5.6)}$$

Introduction

In practically all engineering applications limitations are placed upon the performance and behaviour of components and normally they are expected to operate within certain set limits of, for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service. In certain structural or machine linkage designs, however, maximum stress levels may not be the most severe condition for the component in question. In such cases it is the limitation in the maximum deflection which places the most severe restriction on the operation or design of the component. It is evident, therefore, that methods are required to accurately predict the deflection of members under lateral loads since it is this form of loading which will generally produce the greatest deflections of beams, struts and other structural types of members.

5.1. Relationship between loading, S.F., B.M., slope and deflection

Consider a beam AB which is initially horizontal when unloaded. If this deflects to a new position $A'B'$ under load, the slope at any point C is

$$i = \frac{dy}{dx}$$

Slope and Deflection of Beams
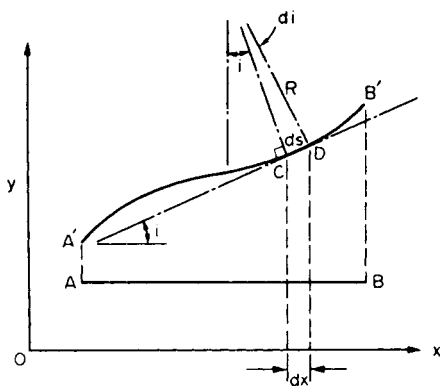
Fig. 5.2. Unloaded beam AB deflected to $A'B'$ under load.

This is usually very small in practice, and for small curvatures

$$ds = dx = R\,di \quad \text{(Fig. 5.2)}$$

$\therefore$
$$\frac{di}{dx} = \frac{1}{R}$$

But
$$i = \frac{dy}{dx}$$

$\therefore$
$$\frac{d^2y}{dx^2} = \frac{1}{R} \tag{5.1}$$

Now from the simple bending theory

$$\frac{M}{I} = \frac{E}{R}$$

$\therefore$
$$\frac{1}{R} = \frac{M}{EI}$$

Therefore substituting in eqn. (5.1)

$$M = EI\frac{d^2y}{dx^2} \tag{5.2}$$

This is the basic differential equation for the deflection of beams.

If the beam is now assumed to carry a distributed loading which varies in intensity over the length of the beam, then a small element of the beam of length dx will be subjected to the loading condition shown in Fig. 5.3. The parts of the beam on either side of the element $EFGH$ carry the externally applied forces, while reactions to these forces are shown on the element itself.

Thus for vertical equilibrium of $EFGH$,

$$Q - w\,dx = Q - dQ$$
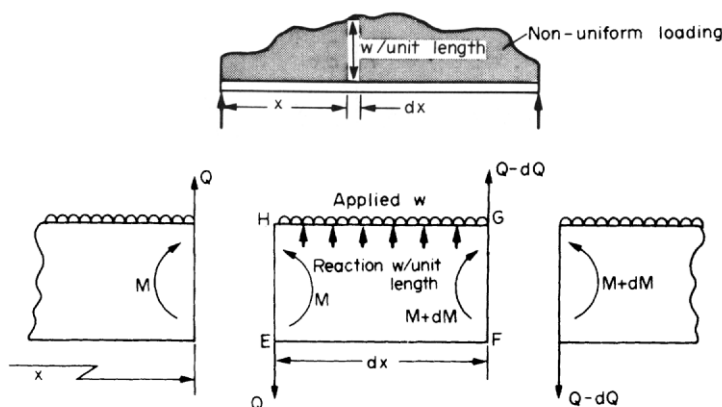$\therefore$
$$dQ = w\,dx$$

Fig. 5.3. Small element of beam subjected to non-uniform loading (effectively uniform over small length dx).

and integrating,
$$Q = \int w\,dx \qquad (5.3)$$

Also, for equilibrium, moments about any point must be zero.
Therefore taking moments about F,

$$(M + dM) + w\,dx\frac{dx}{2} = M + Q\,dx$$

Therefore neglecting the square of small quantities,

$$dM = Q\,dx$$

and integrating,
$$M = \int Q\,dx$$

The results can then be summarised as follows:

$$\text{deflection} = y$$

$$\text{slope} = \frac{dy}{dx}$$

$$\text{bending moment} = EI\frac{d^2y}{dx^2}$$

$$\text{shear force} = EI\frac{d^3y}{dx^3}$$

$$\text{load distribution} = EI\frac{d^4y}{dx^4}$$

In order that the above results should agree algebraically, i.e. that positive slopes shall have the normal mathematical interpretation of the positive sign and that B.M. and S.F. conventions are consistent with those introduced earlier, it is imperative that the sign convention illustrated in Fig. 5.4 be adopted.
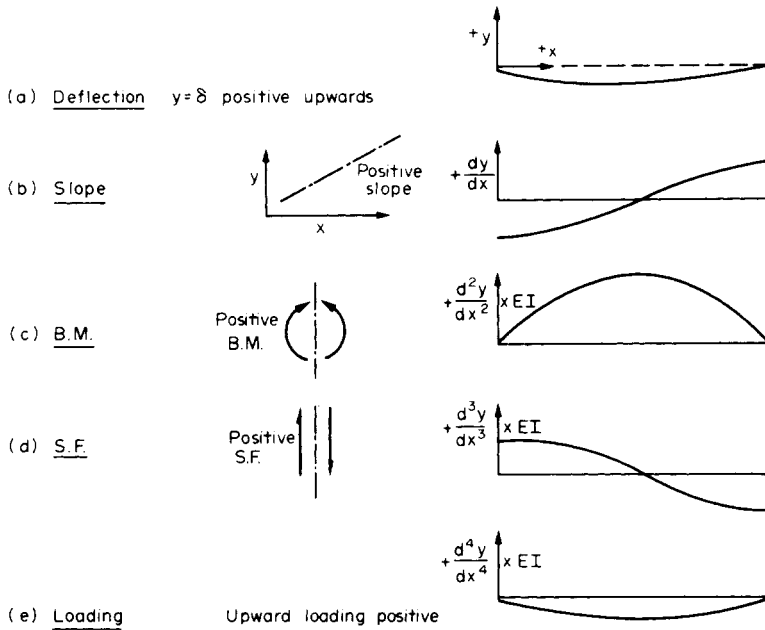
(a) Deflection   y = δ positive upwards

(b) Slope

(c) B.M.

(d) S.F.

(e) Loading   Upward loading positive

Fig. 5.4. Sign conventions for load, S.F., B.M., slope and deflection.

N /η'

5.2. Direct integration method

If the value of the B.M. at any point on a beam is known in terms of x, the distance along the beam, and provided that the equation applies along the complete beam, then integration of eqn. (5.4a) will yield slopes and deflections at any point,

i.e.

$$M = EI\frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI}dx + A$$

or

$$y = \int\int\left(\frac{M}{EI}dx\right)dx + Ax + B$$

where A and B are constants of integration evaluated from known conditions of slope and deflection for particular values of x.

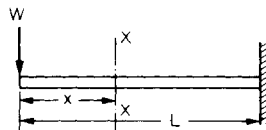(a) *Cantilever with concentrated load at the end* (Fig. 5.5)

Fig. 5.5.

$$M_{xx} = EI\frac{d^2y}{dx^2} = -Wx$$

$$\therefore \qquad EI\frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

assuming *EI* is constant.

$$EIy = -\frac{Wx^3}{6} + Ax + B$$

Now when $\qquad x = L, \quad \frac{dy}{dx} = 0 \qquad \therefore \quad A = \frac{WL^2}{2}$

and when $\qquad x = L, y = 0 \qquad \therefore \quad B = \frac{WL^3}{6} - \frac{WL^2}{2}L = -\frac{WL^3}{3}$

$$\therefore \qquad y = \frac{1}{EI}\left[-\frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3} \right] \qquad (5.5)$$

This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when $x = 0$,

i.e. $\qquad$ **Maximum deflection** $= y_{max} = -\frac{WL^3}{3EI} \qquad (5.6)$

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

Similarly $\qquad \frac{dy}{dx} = \frac{1}{EI}\left[-\frac{Wx^2}{2} + \frac{WL^2}{2} \right] \qquad (5.7)$

and produces a maximum value again when $x = 0$.

$$\textbf{Maximum slope} = \left(\frac{dy}{dx}\right)_{max} = \frac{WL^2}{2EI} \quad \textbf{(positive)} \qquad (5.8)$$

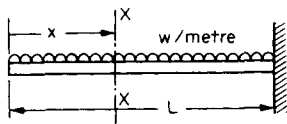*(b) Cantilever with uniformly distributed load* (Fig. 5.6)

Fig. 5.6.

$$M_{xx} = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A$$

$$EIy = -\frac{wx^4}{24} + Ax + B$$

Again, when
$$x = L, \quad \frac{dy}{dx} = 0 \quad \text{and} \quad A = \frac{wL^3}{6}$$

$$x = L, \quad y = 0 \quad \text{and} \quad B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$$

$$\therefore \qquad y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8} \right] \tag{5.9}$$

At $x = 0$,
$$y_{max} = -\frac{wL^4}{8EI} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{max} = \frac{wL^3}{6EI} \tag{5.10}$$

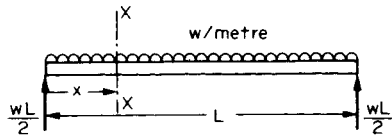(c) *Simply-supported beam with uniformly distributed load* (Fig. 5.7)

Fig. 5.7.

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2} \, .$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$$

At $x = 0, \quad y = 0 \qquad \therefore \quad B = 0$

At $x = L, \quad y = 0 \qquad \therefore \quad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL$

$$\therefore \qquad A = -\frac{wL^3}{24}$$

$$\therefore \qquad y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} \right] \tag{5.11}$$

In this case the maximum deflection will occur at the centre of the beam where $x = L/2$.

$$\therefore \qquad y_{max} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right) \right]$$

$$= -\frac{5wL^4}{384EI} \tag{5.12}$$

Similarly $\left(\frac{dy}{dx}\right)_{max} = \pm\frac{wL^3}{24EI}$ at the ends of the beam.   (5.13)

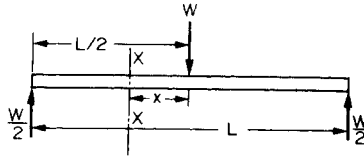(*d*) *Simply supported beam with central concentrated load* (Fig. 5.8)

Fig. 5.8.

In order to obtain a single expression for B.M. which will apply across the complete beam in this case it is convenient to take the origin for x at the centre, then:

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) = \frac{WL}{4} - \frac{Wx}{2}$$

$$EI\frac{dy}{dx} = \frac{WL}{4}x - \frac{Wx^2}{4} + A$$

$$EIy = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

At $\qquad x = 0, \quad \dfrac{dy}{dx} = 0 \qquad \therefore \qquad A = 0$

$$x = \frac{L}{2}, \quad y = 0 \qquad \therefore \qquad 0 = \frac{WL^3}{32} - \frac{WL^3}{96} + B$$

$$\therefore \qquad B = -\frac{WL^3}{48}$$

$\therefore$
$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right] \qquad (5.14)$$

$\therefore$
$$y_{max} = -\frac{WL^3}{48EI} \quad \text{at the centre} \qquad (5.15)$$

and
$$\left(\frac{dy}{dx}\right)_{max} = \pm\frac{WL^2}{16EI} \quad \text{at the ends} \qquad (5.16)$$

In some cases it is not convenient to commence the integration procedure with the B.M. equation since this may be difficult to obtain. In such cases it is often more convenient to commence with the equation for the loading at the general point XX on the beam. A typical example follows:

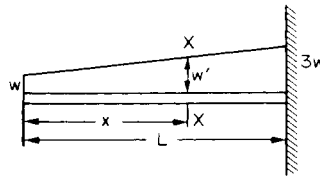*(e) Cantilever subjected to non-uniform distributed load* (Fig. 5.9)

Fig. 5.9.

The loading at section XX is

$$w' = EI\frac{d^4y}{dx^4} = -\left[w + (3w - w)\frac{x}{L}\right] = -w\left(1 + \frac{2x}{L}\right)$$

Integrating,

$$EI\frac{d^3y}{dx^3} = -w\left(x + \frac{x^2}{L}\right) + A \tag{1}$$

$$EI\frac{d^2y}{dx^2} = -w\left(\frac{x^2}{2} + \frac{x^3}{3L}\right) + Ax + B \tag{2}$$

$$EI\frac{dy}{dx} = -w\left(\frac{x^3}{6} + \frac{x^4}{12L}\right) + \frac{Ax^2}{2} + Bx + C \tag{3}$$

$$EIy = -w\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \tag{4}$$

Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At $x = 0$,   S.F. is zero
∴   from (1) $A = 0$
At $x = 0$,   B.M. is zero
∴   from (2) $B = 0$
At $x = L$, slope $dy/dx = 0$ (slope normally assumed zero at a built-in support)

∴   from (3) $0 = -w\left(\dfrac{L^3}{6} + \dfrac{L^3}{12}\right) + C$

$$C = \frac{wL^3}{4}$$

At $x = L$, $y = 0$
∴   from (4) $0 = -w\left(\dfrac{L^4}{24} + \dfrac{L^4}{60}\right) + \dfrac{wL^4}{4} + D$

∴ $D = -\dfrac{23wL^4}{120}$

$\therefore$
$$EIy = -\frac{wx^4}{24} - \frac{wx^5}{60L} + \frac{wL^3x}{4} - \frac{23wL^4}{120}$$

Then, for example, the deflection at the tip of the cantilever, where $x = 0$, is

$$y = -\frac{23wL^4}{120EI}$$

5.3. Macaulay's method

The simple integration method used in the previous examples can only be used when a single expression for B.M. applies along the complete length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions.

Consider, therefore, a small portion of a beam in which, at a particular section A, the shearing force is Q and the B.M. is M, as shown in Fig. 5.10. At another section B, distance a along the beam, a concentrated load W is applied which will change the B.M. for points beyond B.
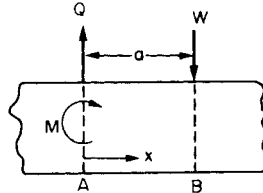

Fig. 5.10.

Between A and B,

$$M = EI\frac{d^2y}{dx^2} = M + Qx \tag{1}$$

$\therefore$
$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} + C_1 \tag{2}$$

and
$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} + C_1x + C_2 \tag{3}$$

Beyond B

$$M = EI\frac{d^2y}{dx^2} = M + Qx - W(x - a) \tag{4}$$

$\therefore$
$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_3 \tag{5}$$

and
$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{x^3}{6} + Wa\frac{x^2}{2} + C_3x + C_4 \tag{6}$$

Now for the same slope at B, equating (2) and (5),

$$Mx + Q\frac{x^2}{2} + C_1 = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_3$$

But at $B, x = a$

$\therefore$ $$C_1 = -\frac{Wa^2}{2} + Wa^2 + C_3$$

$\therefore$ $$C_3 = C_1 - \frac{Wa^2}{2}$$

Substituting in (5),

$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_1 - \frac{Wa^2}{2}$$

$\therefore$ $$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - \frac{W}{2}(x-a)^2 + C_1 \qquad (7)$$

Also, for the same deflection at B equating (3) and (6), with $x = a$

$$\frac{Ma^2}{2} + \frac{Qa^3}{6} + C_1 a + C_2 = \frac{Ma^2}{2} + \frac{Qa^3}{6} - \frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4$$

$\therefore$ $$C_1 a + C_2 = -\frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3 a + C_4$$

$$= -\frac{Wa^3}{6} + \frac{Wa^3}{2} + \left(C_1 - \frac{Wa^2}{2}\right)a + C_4$$

$\therefore$ $$C_4 = C_2 + \frac{Wa^3}{6}$$

Substituting in (6),

$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{x^3}{6} + Wa\frac{x^2}{2}\left(C_1 - \frac{Wa^2}{2}\right)x + W\frac{a^3}{6} + C_2$$

$$= M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{(x-a)^3}{6} + C_1 x + C_2 \qquad (8)$$

Thus, inspecting (4), (7) and (8), we can see that the general method of obtaining slopes and deflections (i.e. integrating the equation for M) will still apply provided that the term $W(x-a)$ is integrated with respect to $(x-a)$ and not x. Thus, when integrated, the term becomes

$$W\frac{(x-a)^2}{2} \quad \text{and} \quad W\frac{(x-a)^3}{6}$$

successively.

In addition, since the term $W(x-a)$ applies only after the discontinuity, i.e. when $x > a$, it *should be considered only when $x > a$ or when $(x-a)$ is positive*. For these reasons such terms are conventionally put into square or curly brackets and called *Macaulay terms*.

Thus Macaulay terms must be (a) integrated with respect to themselves and (b) neglected when negative.

For the whole beam, therefore,
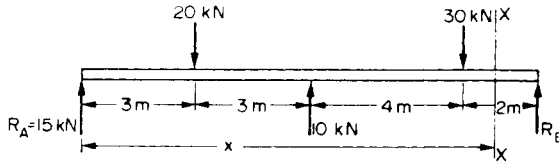
$$EI\frac{d^2y}{dx^2} = M + Qx - W[(x-a)]$$

Fig. 5.11.

As an illustration of the procedure consider the beam loaded as shown in Fig. 5.11 for which the central deflection is required. Using the Macaulay method the equation for the B.M. at any general section XX is then given by

$$\text{B.M.}_{XX} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

Care is then necessary to ensure that the terms inside the square brackets (Macaulay terms) are treated in the special way noted on the previous page.

Here it must be emphasised that all loads in the right-hand side of the equation are in units of kN (i.e. newtons × 10³). *In subsequent working, therefore, it is convenient to carry through this factor as a denominator on the left-hand side in order that the expressions are dimensionally correct.*

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 15\frac{x^2}{2} - 20\left[\frac{(x-3)^2}{2}\right] + 10\left[\frac{(x-6)^2}{2}\right] - 30\left[\frac{(x-10)^2}{2}\right] + A$$

and $$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] + Ax + B$$

where A and B are two constants of integration.

Now when $x = 0$, $y = 0$ ∴ $B = 0$
and when $x = 12$, $y = 0$

∴ $$0 = \frac{15 \times 12^3}{6} - 20\left[\frac{9^3}{6}\right] + 10\left[\frac{6^3}{6}\right] - 30\left[\frac{2^3}{6}\right] + 12A$$

$$= 4320 - 2430 + 360 - 40 + 12A$$

∴ $$12A = -4680 + 2470 = -2210$$

∴ $$A = -184.2$$

The deflection at any point is given by

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2x$$

The deflection at mid-span is thus found by substituting $x = 6$ in the above equation, bearing in mind that the dimensions of the equation are kN m³.

N.B. – Two of the Macaulay terms then vanish since one becomes zero and the other negative and therefore neglected.

∴ $$\text{central deflection} = \frac{10^3}{EI}\left[\frac{15 \times 6^3}{6} - \frac{20 \times 3^3}{6} - 184.2 \times 6\right]$$

$$= -\frac{655.2 \times 10^3}{EI}$$

With typical values of $E = 208 \text{ GN/m}^2$ and $I = 82 \times 10^{-6} \text{ m}^4$

$$\text{central deflection} = 38.4 \times 10^{-3} \text{ m} = \textbf{38.4 mm}$$

5.4. Macaulay's method for u.d.l.s

If a beam carries a uniformly distributed load over the complete span as shown in Fig. 5.12a the B.M. equation is

$$\text{B.M.}_{XX} = EI \frac{d^2 y}{dx^2} = R_A x - \frac{wx^2}{2} - W_1 [(x-a)] - W_2 [(x-b)]$$
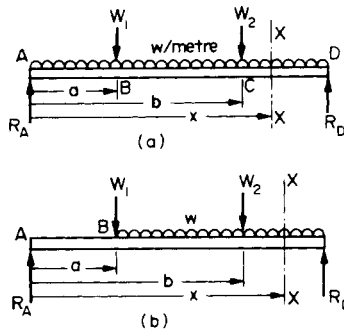

Fig. 5.12.

The u.d.l. term applies across the complete span and does not require the special treatment associated with the Macaulay terms. If, however, the u.d.l. starts at B as shown in Fig. 5.12b the B.M. equation is modified and the u.d.l. term becomes a Macaulay term and is written inside square brackets.

$$\text{B.M.}_{XX} = EI \frac{d^2 y}{dx^2} = R_A x - W_1 [(x-a)] - w \left[\frac{(x-a)^2}{2} \right] - W_2 [(x-b)]$$

Integrating,

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} - W_1 \left[\frac{(x-a)^2}{2} \right] - w \left[\frac{(x-a)^3}{6} \right] - W_2 \left[\frac{(x-b)^2}{2} \right] + A$$

$$EIy = R_A \frac{x^3}{6} - W_1 \left[\frac{(x-a)^3}{6} \right] - w \left[\frac{(x-a)^4}{24} \right] - W_2 \left[\frac{(x-b)^3}{6} \right] + Ax + B$$

Note that Macaulay terms are integrated with respect to, for example, $(x-a)$ and they must be ignored when negative. Substitution of end conditions will then yield the values of the constants A and B in the normal way and hence the required values of slope or deflection.

It must be appreciated, however, that once a term has been entered in the B.M. expression it will apply across the complete beam. The modifications to the procedure required for cases when u.d.l.s. are applied over part of the beam only are introduced in the following theory.

5.5. Macaulay's method for beams with u.d.l. applied over part of the beam

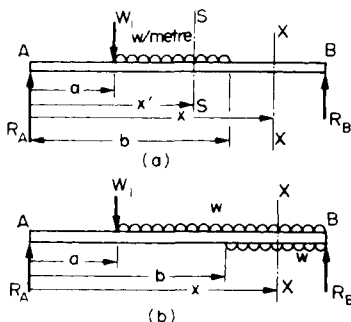Consider the beam loading case shown in Fig. 5.13a.

Fig. 5.13.

The B.M. at the section *SS* is given by the previously introduced procedure as

$$\text{B.M.}_{SS} = R_A x' - W_1 [(x' - a)] - W \left[\frac{(x' - a)^2}{2} \right]$$

Having introduced the last (u.d.l.) term, however, it will apply for all values of x' greater than a, i.e. across the rest of the span to the end of the beam. (Remember, Macaulay terms are only neglected when they are negative, e.g. with $x' < a$.) The above equation is *NOT* therefore the correct equation for the load condition shown. The Macaulay method requires that this continuation of the u.d.l. be shown on the loading diagram and the required loading condition can therefore only be achieved by introducing an equal and opposite u.d.l. over the last part of the beam to cancel the unwanted continuation of the initial distributed load. This procedure is shown in Fig. 5.13b.

The correct B.M. equation for any general section XX is then given by

$$\text{B.M.}_{XX} = EI \frac{d^2 y}{dx^2} = R_A x - W_1 [(x - a)] - w \left[\frac{(x - a)^2}{2} \right] + w \left[\frac{(x - b)^2}{2} \right]$$

This type of approach can be adopted for any beam loading cases in which u.d.l.s are stopped or added to.

A number of examples are shown in Figs. 5.14–17. In each case the required loading system is shown first, followed by the continuation and compensating load system and the resulting B.M. equation.

5.6. Macaulay's method for couple applied at a point

Consider the beam *AB* shown in Fig. 5.18 with a moment or couple *M* applied at some point *C*. Considering the equilibrium of moments about each end in turn produces reactions of

$$R_A = \frac{M}{L} \quad \text{upwards,} \quad \text{and} \quad R_B = \frac{M}{L} \quad \text{downwards}$$

These equal and opposite forces then automatically produce the required equilibrium of vertical forces.
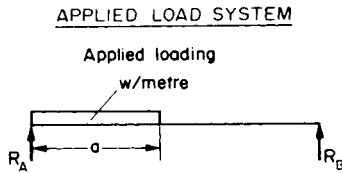
APPLIED LOAD SYSTEM   EQUIVALENT LOAD SYSTEM

Fig. 5.14.

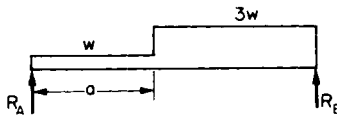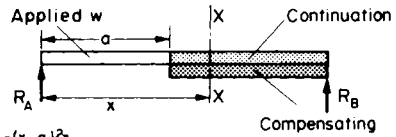$$\text{B.M.}_{xx} = R_A x - \frac{wx^2}{2} + w\left[\frac{(x-a)^2}{2}\right]$$

Fig. 5.15.

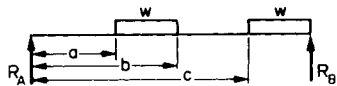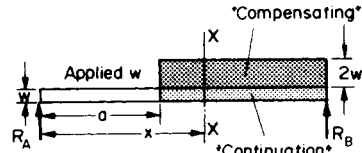$$\text{B.M.}_{xx} = R_A x - \frac{wx^2}{2} - 2w\left[\frac{(x-a)^2}{2}\right]$$

Fig. 5.16.

$$\text{B.M.}_{xx} = R_A x - w\left[\frac{(x-a)^2}{2}\right] + w\left[\frac{(x-b)^2}{2}\right] - w\left[\frac{(x-c)^2}{2}\right]$$
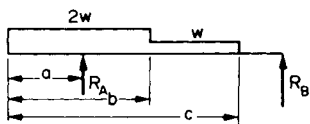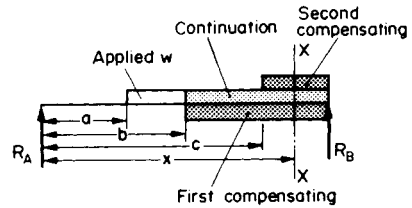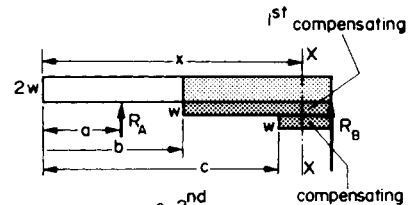
Fig. 5.17.

$$\text{B.M.}_{xx} = -2w\frac{x^2}{2} + R_A\left[(x-a)\right] + w\left[\frac{(x-b)^2}{2}\right] + w\left[\frac{(x-c)^2}{2}\right]$$

Figs. 5.14, 5.15, 5.16 and 5.17. Typical equivalent load systems for Macaulay method together with appropriate B.M. expressions.
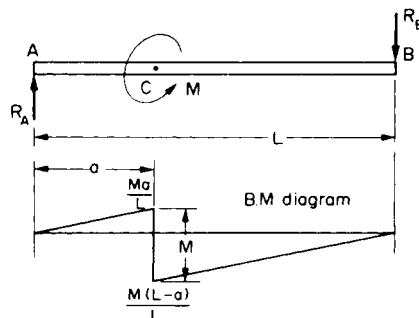
Fig. 5.18. Beam subjected to applied couple or moment M.

For sections between A and C the B.M. is $\dfrac{M}{L}x$.

For sections between C and B the B.M. is $\dfrac{Mx}{L} - M$.

The additional $(-M)$ term which enters the B.M. expression for points beyond C can be adequately catered for by the Macaulay method if written in the form

$$M[(x-a)^0]$$

This term can then be treated in precisely the same way as any other Macaulay term, integration being carried out with respect to $(x-a)$ and the term being neglected when x is less than a. The full B.M. equation for the beam is therefore

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{Mx}{L} - M[(x-a)^0] \qquad (5.17)$$

Then

$$EI\frac{dy}{dx} = \frac{Mx^2}{2L} - M[(x-a)] + A, \quad \text{etc.}$$

5.7. Mohr's "area–moment" method

In applications where the slope or deflection of beams or cantilevers is required at only one position the determination of the complete equations for slope and deflection at all points as obtained by Macaulay's method is rather laborious. In such cases, and in particular where loading systems are relatively simple, the Mohr moment–area method provides a rapid solution.
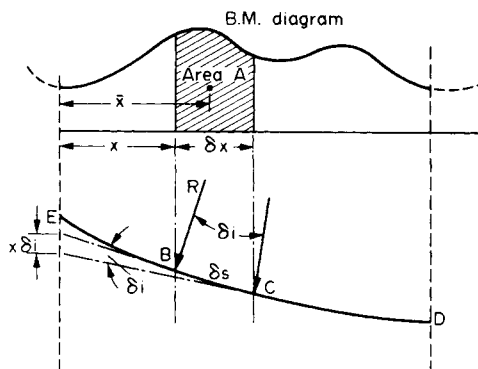
Fig. 5.19.

Figure 5.19 shows the deflected shape of part of a beam ED under the action of a B.M. which varies as shown in the B.M. diagram. Between any two points B and C the B.M. diagram has an area A and centroid distance $\bar{x}$ from E. The tangents at the points B and C give an intercept of $x\delta i$ on the vertical through E, where δi is the angle between the tangents.
    Now

$$\delta s = R\delta i$$

and
$$\delta x \simeq \delta s$$

if slopes are small.

$\therefore$
$$\delta i = \frac{\delta x}{R} = \frac{M}{EI} \delta x$$

$\therefore$ change of slope between E and $D = i = \int_E^D \frac{M}{EI} dx$

i.e.   *change of slope = area of M/EI diagram between E and D*   (5.18)

N.B.–For a uniform beam (EI constant) this equals $\dfrac{1}{EI} \times$ area of B.M. diagram.

Deflection at E resulting from the bending of $BC = x\delta i$

$\therefore$ total deflection resulting from bending of $ED = \int x\delta i$

$$= \int_E^D \frac{Mx}{EI} dx$$

The total deflection of E relative to the tangent at D is equal to the first moment of area of the M/EI diagram about E.   (5.19)

Again, if EI is constant this equals $1/EI \times$ first moment of area of the B.M. diagram about E.

The theorem is particularly useful when one point on the beam is a point of zero slope since the tangent at this point is then horizontal and deflections relative to the tangent are absolute values of vertical deflections. Thus if D is a point of zero slope the above equations yield the actual slope and deflection at E.

The Mohr area–moment procedure may be summarised in its most useful form as follows: if A and B are two points on the deflection curve of a beam, EI is constant and B is a point of zero slope, then Mohr's theorems state that:

(1) **Slope at A = $1/EI \times$ area of B.M. diagram between A and B.**   (5.20)

(2) **Deflection of A relative to B = $1/EI \times$ first moment of area of B.M. diagram between A and B about A.**   (5.21)

In many cases of apparently complicated load systems the loading can be separated into a combination of several simple systems which, by the application of the principle of superposition, will produce the same results. This procedure is illustrated in Examples 5.4 and 5.5.

The Mohr method will now be applied to the standard loading cases solved previously by the direct integration procedure.

(a) Cantilever with concentrated load at the end

In this case B is a point of zero slope and the simplified form of the Mohr theorems stated above can be applied.

Slope at $A = \dfrac{1}{EI}$ [area of B.M. diagram between A and B (Fig. 5.20)]

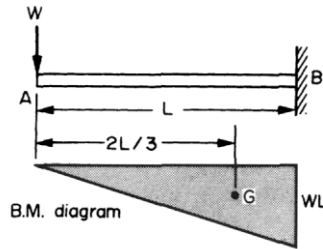$$= \frac{1}{EI}\left[\frac{L}{2}WL\right] = \frac{WL^2}{2EI}$$

Fig. 5.20.

Deflection at A (relative to B)

$$= \frac{1}{EI} \text{ [first moment of area of B.M. diagram between } A \text{ and } B \text{ about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{L}{2}WL\right)\frac{2L}{3}\right] = \frac{WL^3}{3EI}$$

(b) Cantilever with u.d.l.
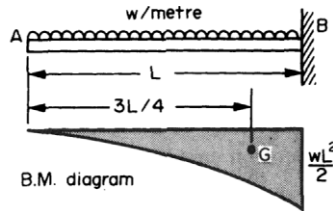
Fig. 5.21.

Again B is a point of zero slope.

$$\text{slope at } A = \frac{1}{EI} \text{ [area of B.M. diagram (Fig. 5.21)]}$$

$$= \frac{1}{EI}\left[\frac{1}{3}L\frac{wL^2}{2}\right]$$

$$= \frac{wL^3}{6EI}$$

$$\text{Deflection at } A = \frac{1}{EI} \text{ [moment of B.M. diagram about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{3}L\frac{wL^2}{2}\right)\frac{3L}{4}\right] = \frac{wL^4}{8EI}$$

(c) *Simply supported beam with u.d.l.*
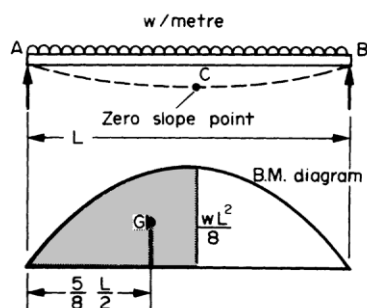
Fig. 5.22.

Here the point of zero slope is at the centre of the beam C. Working relative to C,

$$\text{slope at } A = \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.22)}]$$

$$= \frac{1}{EI} \left[\frac{2}{3} \frac{wL^2}{8} \frac{L}{2} \right] = \frac{wL^3}{24EI}$$

Deflection of A relative to C (= central deflection relative to A)

$$= \frac{1}{EI} [\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI} \left[\left(\frac{2}{3} \frac{wL^2}{8} \frac{L}{2} \right) \left(\frac{5L}{16} \right) \right] = \frac{5wL^4}{384EI}$$
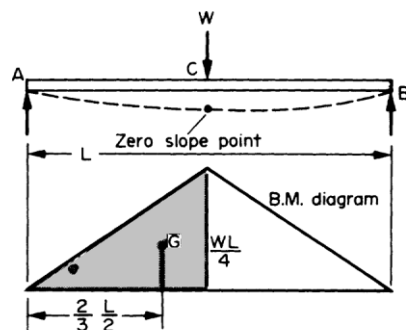
(d) *Simply supported beam with central concentrated load*

Fig. 5.23.

Again working relative to the zero slope point at the centre C,

$$\text{slope at } A = \frac{1}{EI} [\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.23)}]$$

$$= \frac{1}{EI} \left[\frac{1}{2} \frac{L}{2} \frac{WL}{4} \right] = \frac{WL^2}{16EI}$$

Deflection of A relative to C ($=$ central deflection of C)

$$= \frac{1}{EI} [\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI} \left[\left(\frac{1}{2} \frac{L}{2} \frac{WL}{4} \right) \left(\frac{2}{3} \frac{L}{2} \right) \right] = \frac{WL^3}{48EI}$$

5.8. Principle of superposition

The general statement for the principle of superposition asserts that the resultant stress or strain in a system subjected to several forces is the algebraic sum of their effects when applied separately. The principle can be utilised, however, to determine the deflections of beams subjected to complicated loading conditions which, in reality, are merely combinations of a number of simple systems. In addition to the simple standard cases introduced previously, numerous different loading conditions have been solved by various workers and their results may be found in civil or mechanical engineering handbooks or data sheets. Thus, the algebraic sum of the separate deflections caused by a convenient selection of standard loading cases will produce the total deflection of the apparently complex case.

It must be appreciated, however, that the principle of superposition is only valid whilst the beam material remains elastic and for small beam deflections. (Large deflections would produce unacceptable deviation of the lines of action of the loads relative to the beam axis.)

5.9. Energy method

A further, alternative, procedure for calculating deflections of beams or structures is based upon the application of strain energy considerations. This is introduced in detail in Chapter 11 and will not be considered further here.

5.10. Maxwell's theorem of reciprocal displacements

Consider a beam subjected to two loads W_A and W_B at points A and B respectively as shown in Fig. 5.24. Let W_A be gradually applied first, producing a deflection a at A.

$$\text{Work done} = \tfrac{1}{2} W_A a$$

When W_B is applied it will produce a deflection b at B and an additional deflection δ_{ab} at A (the latter occurring in the presence of a now constant load W_A).

$$\text{Extra work done} = \tfrac{1}{2} W_B b + W_A \delta_{ab}$$

$$\therefore \qquad \text{total work done} = \tfrac{1}{2} W_A a + \tfrac{1}{2} W_B b + W_A \delta_{ab}$$

δ_{ab} = deflection at A with load at B
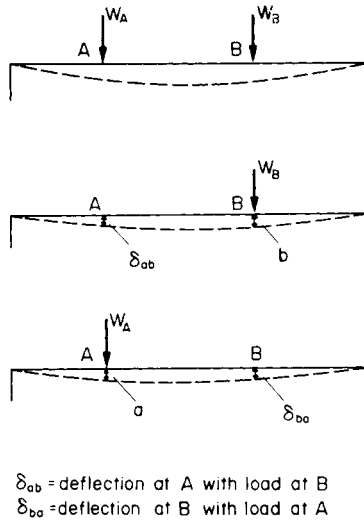δ_{ba} = deflection at B with load at A

Fig. 5.24. Maxwell's theorem of reciprocal displacements.

Similarly, if the loads were applied in reverse order and the load W_A at A produced an additional deflection δ_{ba} at B, then

$$\text{total work done} = \tfrac{1}{2}W_B b + \tfrac{1}{2}W_A a + W_B \delta_{ba}$$

It should be clear that, regardless of the order in which the loads are applied, the total work done must be the same. Inspection of the above equations thus shows that

$$W_A \delta_{ab} = W_B \delta_{ba}$$

If the two loads are now made equal, then

$$\delta_{ab} = \delta_{ba} \tag{5.22}$$

i.e. *the deflection at A produced by a load at B equals the deflection at B produced by the same load at A.* This is Maxwell's theorem of reciprocal displacements.

As a typical example of the application of this theorem to beams consider the case of a simply supported beam carrying a single concentrated load off-set from the centre (Fig. 5.25).
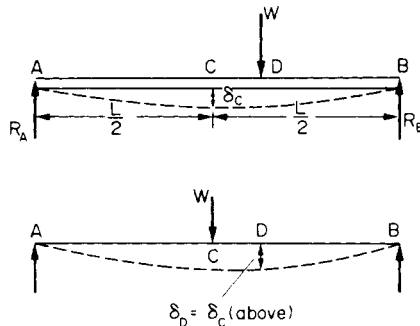

Fig. 5.25.

The central deflection of the beam for this loading condition would be given by the reciprocal displacement theorem as the deflection at D if the load is moved to the centre. Since the deflection equation for a central point load is one of the standard cases treated earlier the required deflection value can be readily obtained.

Maxwell's theorem of reciprocal displacements can also be applied if one or both of the loads are replaced by moments or couples. In this case it can be shown that the theorem is modified to the relevant one of the following forms (a), (b):

(a) *The angle of rotation at A due to a concentrated force at B is numerically equal to the deflection at B due to a couple at A provided that the force and couple are also numerically equal* (Fig. 5.26).

(a)

(b)
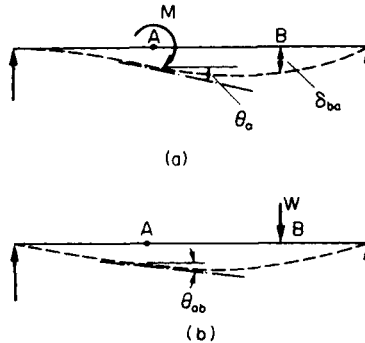
θ_a = slope at A with moment (or load) at A
θ_{ab} = slope at a with load at B

Fig. 5.26.

(b) *The angle of rotation at A due to a couple at B is equal to the rotation at B due to the same couple applied at A* (Fig. 5.27).

(a)

(b)

Fig. 5.27.

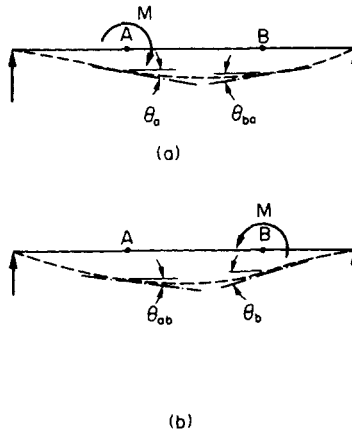All three forms of the theorem are quite general in application and are not restricted to beam problems. Any type of component or structure subjected to bending, direct load, shear or torsional deformation may be considered provided always that linear elastic conditions prevail, i.e. Hooke's law applies, and deflections are small enough not to significantly affect the undeformed geometry.

5.11. Continuous beams – Clapeyron's "three-moment" equation

When a beam is supported on more than two supports it is termed *continuous*. In cases such as these it is not possible to determine directly the reactions at the three supports by the normal equations of static equilibrium since there are too many unknowns. An extension of Mohr's area–moment method is therefore used to obtain a relationship between the B.M.s at the supports, from which the reaction values can then be determined and the B.M. and S.F. diagrams drawn.

Consider therefore the beam shown in Fig. 5.28. The areas A_1 and A_2 are the "free" B.M. diagrams, treating the beam as simply supported over two separate spans L_1 and L_2. In general the B.M.s at the three supports will not be zero as this diagram suggests, but will have some values M_1, M_2 and M_3. Thus a *fixing-moment diagram* must be introduced as shown, the actual B.M. diagram then being the algebraic sum of the two diagrams.
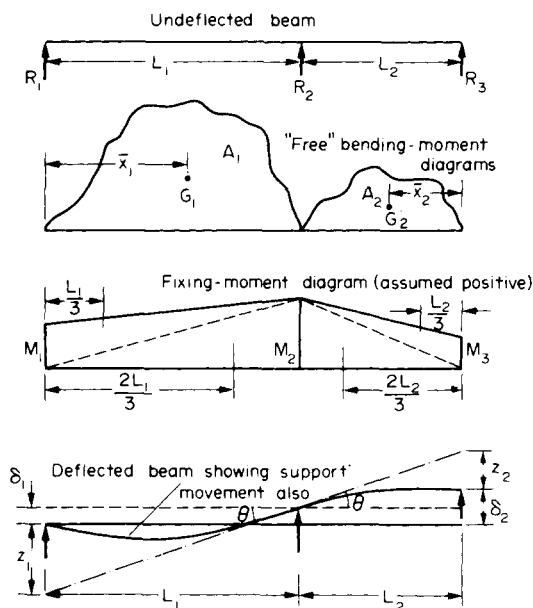
Fig. 5.28. Continuous beam over three supports showing "free" and "fixing" moment diagrams together with the deflected beam form including support movement.

The bottom figure shows the deflected position of the beam, the deflections δ_1 and δ_2 being *relative to the left-hand support*. If a tangent is drawn at the centre support then the intercepts at the end of each span are z_1 and z_2 and θ is the slope of the tangent, and hence the beam, at the centre support.

Now, assuming deflections are small,

$$\theta \text{ (radians)} = \frac{z_1 + \delta_1}{L_1} = \frac{z_2 + \delta_2 - \delta_1}{L_2}$$

$$\therefore \qquad \frac{z_1}{L_1} + \frac{\delta_1}{L_1} = \frac{z_2}{L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

But from Mohr's area–moment method,

$$z = \frac{A\bar{x}}{EI}$$

where A is the area of the B.M. diagram over the span to which z refers.

$$\therefore \qquad z_1 = -\frac{1}{EI_1}\left[A_1\bar{x}_1 + \left(\frac{M_1 L_1}{2} \times \frac{L_1}{3}\right) + \left(\frac{M_2 L_1}{2} \times \frac{2L_1}{3}\right)\right]$$

$$= -\frac{1}{EI_1}\left[A_1\bar{x}_1 + \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3}\right]$$

and

$$z_2 = \frac{1}{EI_2}\left[A_2\bar{x}_2 + \left(\frac{M_3 L_2}{2} \times \frac{L_2}{3}\right) + \left(\frac{M_2 L_2}{2} \times \frac{2L_2}{3}\right)\right]$$

$$= \frac{1}{EI_2}\left[A_2\bar{x}_2 + \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3}\right]$$

N.B. – Since the intercepts are in opposite directions, they are of opposite sign.

$$\therefore \qquad -\frac{\left[A_1\bar{x}_1 + \dfrac{M_1 L_1^2}{6} + \dfrac{M_2 L_1^2}{3}\right]}{EI_1 L_1} + \frac{\delta_1}{L_1} = \frac{\left[A_2\bar{x}_2 + \dfrac{M_3 L_2^2}{6} + \dfrac{M_2 L_2^2}{3}\right]}{EI_2 L_2} + \frac{(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \qquad -\frac{A_1\bar{x}_1}{I_1 L_1} - \frac{M_1 L_1}{6I_1} - \frac{M_2 L_1}{3I_1} + \frac{E\delta_1}{L_1} = \frac{A_2\bar{x}_2}{I_2 L_2} + \frac{M_3 L_2}{6I_2} + \frac{M_2 L_2}{3I_2} + \frac{E(\delta_2 - \delta_1)}{L_2}$$

$$\therefore \qquad -\frac{M_1 L_1}{I_1} - 2M_2\left[\frac{L_1}{I_1} + \frac{L_2}{I_2}\right] - \frac{M_3 L_2}{I_2} = 6\left[\frac{A_1\bar{x}_1}{I_1 L_1} + \frac{A_2\bar{x}_2}{I_2 L_2}\right] + 6E\left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1}\right]$$

$$(5.23)$$

This is the full three-moment equation; it can be greatly simplified **if the beam is uniform**, i.e. $I_1 = I_2 = I$, as follows:

$$-M_1 L_1 - 2M_2[L_1 + L_2] - M_3 L_2 = 6\left[\frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2}\right] + 6EI\left[\frac{(\delta_2 - \delta_1)}{L_2} - \frac{\delta_1}{L_1}\right]$$

If the supports are on the same level, i.e. $\delta_1 = \delta_2 = 0$,

$$-M_1 L_1 - 2M_2[L_1 + L_2] - M_3 L_2 = 6\left[\frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2}\right] \qquad (5.24)$$

This is the form in which Clapeyron's three-moment equation is normally used.

The following standard results for $\dfrac{6A\bar{x}}{L}$ are very useful:

(1) *Concentrated loads* (Fig. 5.29)
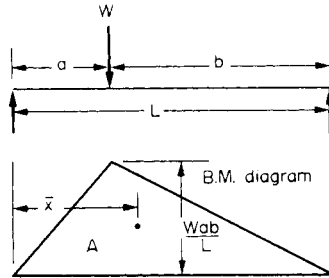
Fig. 5.29.

$$\frac{6A\bar{x}}{L} = \frac{6}{L}\left[\frac{Wab}{L} \times \frac{a}{2} \times \frac{2a}{3} + \frac{Wab}{L} \times \frac{b}{2}\left(a + \frac{b}{3}\right)\right]$$

$$= \frac{6Wab}{L^2}\left[\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{6}\right]$$

$$= \frac{Wab}{L^2}[2a^2 + 3ab + b^2] = \frac{Wab}{L^2}(2a+b)(a+b)$$

$$= \frac{Wab}{L}(2a+b) \tag{5.25}$$

But $$\qquad\qquad b = L - a$$

$$\therefore \qquad\qquad \frac{6A\bar{x}}{L} = \frac{Wa}{L}(L-a)(2a+L-a)$$

$$= \frac{Wa}{L}(L^2 - a^2) \tag{5.26}$$

(2) *Uniformly distributed loads* (Fig. 5.30)

Fig. 5.30.

Here the B.M. diagram is a parabola for which

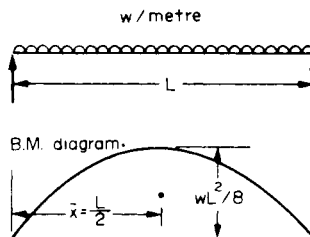$$\text{area} = \tfrac{2}{3} \text{ base} \times \text{height}$$

$$\therefore \quad \frac{6A\bar{x}}{L} = \frac{6}{L} \times \frac{2}{3} \times L \times \frac{wL^2}{8} \times \frac{L}{2}$$

$$= \frac{wL^3}{4} \tag{5.27}$$

5.12. Finite difference method

A numerical method for the calculation of beam deflections which is particularly useful for non-prismatic beams or for cases of irregular loading is the so-called *finite difference* method.

The basic principle of the method is to replace the standard differential equation (5.2) by its finite difference approximation, obtain equations for deflections in terms of moments at various points along the beam and solve these simultaneously to yield the required deflection values.

Consider, therefore, Fig. 5.31 which shows part of a deflected beam with the *x* axis divided into a series of equally spaced intervals. By convention, the ordinates are numbered with respect to the central ordinate *B*.
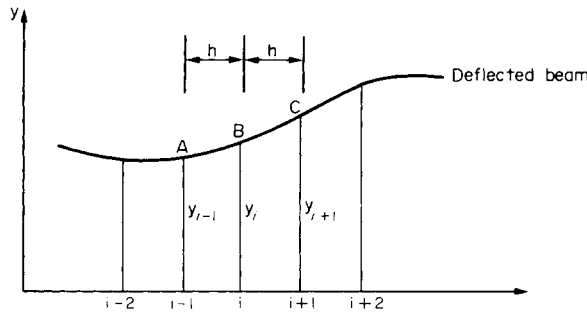
Fig. 5.31. Deflected beam curve divided into a convenient number of equally spaced intervals.

If the equation for the deflected curve of the beam is taken to be $y = f(x)$ then the first derivative dy/dx at *B* is the slope of the curve at *B*. Approximately (provided *h* is small) this can be taken to be the slope of the chord joining *A* and *C* so that:

$$\left(\frac{dy}{dx}\right)_i \simeq \frac{(y_{i+1} - y_{i-1})}{2h} = \frac{1}{2h}(y_{i+1} - y_{i-1}) \tag{5.28}$$

The rate of change of the first derivative, i.e. the rate of change of the slope $\left(= \dfrac{d^2y}{dx^2}\right)$ is given in the same way approximately as the slope to the right of *i* minus the slope to the left of *i* divided by the interval between them.

Thus:

$$\left(\frac{d^2y}{dx^2}\right)_i = \frac{\dfrac{(y_{i+1} - y_i)}{h} - \dfrac{(y_i - y_{i-1})}{h}}{h} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \tag{5.29}$$

Equations 5.28 and 5.29 are the *finite difference approximations* of the standard beam deflection differential equations and, because they are written in terms of ordinates on either side of the central point *i*, they are known as *central differences*. Alternative expressions which can be formed to contain only ordinates at, or to the right of *i*, or ordinates at, or to the left of *i* are known as forward and backward differences, respectively but these will not be considered here.

Now from eqn. (5.2)

$$M = EI\frac{d^2y}{dx^2}$$

∴ At position *i*, combining eqn. (5.2) and (5.29).

$$\left(\frac{M}{EI_i}\right) = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \tag{5.30}$$

A solution for any of the deflection (*y*) values can then be obtained by applying the finite difference equation at a series of points along the beam and solving the resulting simultaneous equations – see Example 5.8.

The higher the number of points selected the greater the accuracy of solution but the more the number of equations which are required to be solved. The method thus lends itself to computer-assisted evaluation.

In addition to the solution of statically determinate beam problems of the type treated in Example 5.8 the method is also applicable to the analysis of statically indeterminate beams, i.e. those beam loading conditions with unknown (or redundant) quantities such as prop loads or fixing moments – see Example 5.9.

The method is similar in that the bending moment is written in terms of the applied loads and the redundant quantities and equated to the finite difference equation at selected points. Since each redundancy is usually associated with a known (or assumed) condition of slope or deflection, e.g. zero deflection at a propped support, there will always be sufficient equations to allow solution of the unknowns.

The principal advantages of the finite difference method are thus:

(a) that it can be applied to statically determinate or indeterminate beams,
(b) that it can be used for non-prismatic beams,
(c) that it is amenable to computer solutions.

5.13. Deflections due to temperature effects

It has been shown in §2.3 that a uniform temperature increase *t* on an unconstrained bar of length *L* will produce an increase in length

$$\Delta L = \alpha Lt$$

where α is the coefficient of linear expansion of the material of the bar. Provided that the bar remains unconstrained, i.e. is free to expand, no stresses will result.

Similarly, in the case of a beam supported in such a way that longitudinal expansion can occur freely, no stresses are set up and there will be no tendency for the beam to bend. If, however, the beam is constrained then stresses will result, their values being calculated using

the procedure of §2.3 provided that the temperature change is uniform across the whole beam section.

If the temperature is not constant across the beam then, again, stresses and deflections will result and the following procedure must be adopted:
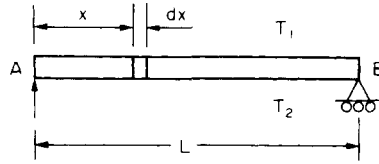

Fig. 5.32(a). Beam initially straight before application of temperature T_1 on the top surface and T_2 on the lower surface. (Beam supported on rollers at B to allow "free" lateral expansion).
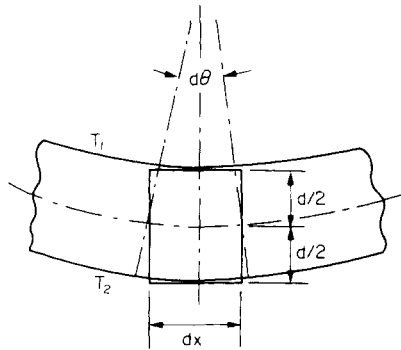

Fig. 5.32(b). Beam after application of temperatures T_1 and T_2, showing distortions of element dx.

Consider the initially straight, simply-supported beam shown in Fig. 5.32(a) with an initial uniform temperature T_0. If the temperature changes to a value T_1 on the upper surface and T_2 on the lower surface with, say, $T_2 > T_1$ then an element dx on the bottom surface will expand to $\alpha(T_2 - T_0).dx$ whilst the same length on the top surface will only expand to $\alpha(T_1 - T_0).dx$. As a result the beam will bend to accommodate the distortion of the element dx, the sides of the element rotating relative to one another by the angle $d\theta$, as shown in Fig. 5.32(b). For a depth of beam d:

$$d.d\theta = \alpha(T_2 - T_0)dx - \alpha(T_1 - T_0)dx$$

or $\qquad\qquad \dfrac{d\theta}{dx} = \dfrac{\alpha(T_2 - T_1)}{d} \qquad\qquad\qquad\qquad (5.31)$

The differential equation gives the rate of change of slope of the beam and, since $\theta = dy/dx$,

then $\qquad\qquad \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = \dfrac{\alpha(T_2 - T_1)}{d}$

Thus the standard differential equation for bending of the beam due to temperature gradient

across the beam section is:

$$\frac{d^2y}{dx^2} = \frac{a(T_2 - T_1)}{d}$$ (5.32)

This is directly analogous to the standard deflection equation $\frac{d^2y}{dx^2} = \frac{M}{EI}$ so that integration of this equation in exactly the same way as previously for bending moments allows a solution for slopes and deflections produced by the thermal effects.

NB. If the temperature gradient across the beam section is linear, the average temperature $\frac{1}{2}(T_1 + T_2)$ will occur at the mid-height position and, in addition to the bending, the beam will change in overall length by an amount $\alpha L[\frac{1}{2}(T_1 + T_2) - T_0]$ in the absence of any constraint.

Application to cantilevers

Consider the cantilever shown in Fig. 5.33 subjected to temperature T_1 on the top surface and T_2 on the lower surface. In the absence of external loads, and because the cantilever is free to bend, there will be no moment or reaction set up at the built-in end.
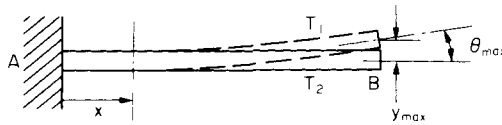
Fig. 5.33. Cantilever with temperature T_1 on the upper surface, T_2 on the lower surface ($T_2 > T_1$).

Applying the differential equation (5.32) we have:

$$\frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d}.$$

Integrating:

$$\frac{dy}{dx} = \frac{\alpha(T_2 - T_1)}{d}x + C_1$$

But at $x = 0$, $\frac{dy}{dx} = 0$, $\therefore C_1 = 0$ and:

$$\frac{dy}{dx} = \frac{\alpha(T_2 - T_1)}{d}x = \theta$$

$\therefore$ The slope at the end of the cantilever is:

$$\theta_{max} = \frac{a(T_2 - T_1)}{d}L.$$ (5.33)

Integrating again to find deflections:

$$y = \frac{\alpha(T_2 - T_1)}{d}\frac{x^2}{2} + C_2$$

and, since $y = 0$ at $x = 0$, then $C_2 = 0$, and:

$$y = \frac{\alpha(T_2 - T_1)}{2d} x^2$$

At the end of the cantilever, therefore, the deflection is:

$$y_{max} = \frac{a(T_2 - T_1)}{2d} L^2 \tag{5.34}$$

Application to built-in beams
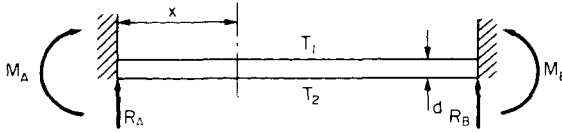
Fig. 5.34. Built-in beam subjected to thermal gradient with temperature T_1 on the upper surface, T_2 on the lower surface.

Consider the built-in beam shown in Fig. 5.34. Using the principle of superposition the differential equation for the beam is given by the combination of the equations for applied bending moment and thermal effects.

$$\text{For bending } EI\frac{d^2y}{dx^2} = M_A + R_A x.$$

$$\text{For thermal effects } \frac{d^2y}{dx^2} = \frac{\alpha(T_2 - T_1)}{d}$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = EI\frac{\alpha(T_2 - T_1)}{d}$$

$\therefore$ The combined differential equation is:

$$EI\frac{d^2y}{dx^2} = M_A + R_A x + EI\frac{\alpha(T_2 - T_1)}{d}.$$

However, in the absence of applied loads and from symmetry of the beam:

$$R_A = R_B = 0,$$

and
$$M_A = M_B = M.$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = M + EI\frac{\alpha(T_2 - T_1)}{d}.$$

Integrating:
$$EI\frac{dy}{dx} = M_x + EI\frac{\alpha(T_2 - T_1)}{d}x + C_1$$

Now at $x = 0, \dfrac{dy}{dx} = 0 \quad \therefore \; C_1 = 0,$

and at $x = L \; \dfrac{dy}{dx} = 0 \quad \therefore \; M = -EI\frac{a(T_2 - T_1)}{d}$ \hfill (5.35)

Integrating again to find the deflection equation we have:

$$EIy = M \cdot \frac{x^2}{2} + EI\frac{\alpha(T_2-T_1)}{d} \cdot \frac{x^2}{2} + C_2$$

When $x = 0$, $y = 0$ $\therefore$ $C_2 = 0$,

and, since $M = -EI\frac{\alpha(T_2-T_1)}{d}$ then $y = 0$ for all values of x.

Thus a rather surprising result is obtained whereby the beam will remain horizontal in the presence of a thermal gradient. It will, however, be subject to residual stresses arising from the constraint on overall expansion of the beam under the average temperature $\frac{1}{2}(T_1+T_2)$. i.e. from §2.3

$$\text{residual stress} = E\alpha[\tfrac{1}{2}(T_1+T_2)]$$

$$= \tfrac{1}{2}E\alpha(T_1+T_2). \tag{5.36}$$

Examples

Example 5.1

(a) A uniform cantilever is 4 m long and carries a concentrated load of 40 kN at a point 3 m from the support. Determine the vertical deflection of the free end of the cantilever if $EI = 65$ MN m^2.

(b) How would this value change if the same total load were applied but uniformly distributed over the portion of the cantilever 3 m from the support?

Solution
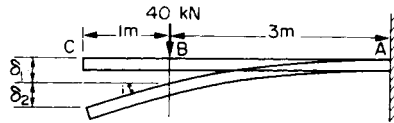
Fig. 5.35.

(a) With the load in the position shown in Fig. 5.35 the cantilever is effectively only 3 m long, the remaining 1 m being unloaded and therefore not bending. Thus, the standard equations for slope and deflections apply between points A and B only.

$$\text{Vertical deflection of } B = -\frac{WL^3}{3EI} = -\frac{40 \times 10^3 \times 3^3}{3 \times 65 \times 10^6} = -5.538 \times 10^{-3} \text{ m} = \delta_1$$

$$\text{Slope at } B = \frac{WL^2}{2EI} = \frac{40 \times 10^3 \times 3^2}{2 \times 65 \times 10^6} = 2.769 \times 10^{-3} \text{ rad} = i$$

Now BC remains straight since it is not subject to bending.

$$\therefore \qquad \delta_2 = -iL = -2.769 \times 10^{-3} \times 1 = -2.769 \times 10^{-3}\,\text{m}$$

$$\therefore \qquad \text{vertical deflection of } C = \delta_1 + \delta_2 = -(5.538 + 2.769)10^{-3} = -\textbf{8.31 mm}$$

The negative sign indicates a deflection in the negative y direction, i.e. downwards.
(b) With the load uniformly distributed,

$$w = \frac{40 \times 10^3}{3} = 13.33 \times 10^3\,\text{N/m}$$

Again using standard equations listed in the summary

$$\delta_1' = -\frac{wL^4}{8EI} = \frac{13.33 \times 10^3 \times 3^4}{8 \times 65 \times 10^6} = -2.076 \times 10^{-3}\,\text{m}$$

$$\text{and slope } i = \frac{wL^3}{6EI} = \frac{13.33 \times 10^3 \times 3^3}{6 \times 65 \times 10^6} = 0.923 \times 10^3\,\text{rad}$$

$$\therefore \qquad \delta_2' = -0.923 \times 10^{-3} \times 1 = 0.923 \times 10^{-3}\,\text{m}$$

$$\therefore \qquad \text{vertical deflection of } C = \delta_1' + \delta_2' = -(2.076 + 0.923)10^{-3} = -\textbf{3 mm}$$

There is thus a considerable (63.9 %) reduction in the end deflection when the load is uniformly distributed.

Example 5.2

Determine the slope and deflection under the 50 kN load for the beam loading system shown in Fig. 5.36. Find also the position and magnitude of the maximum deflection. $E = 200\,\text{GN/m}^2$; $I = 83 \times 10^{-6}\,\text{m}^4$.
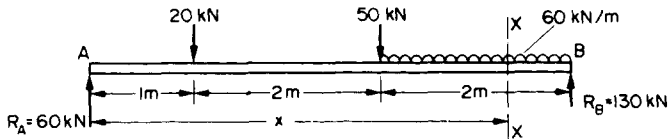
Fig. 5.36.

Solution

Taking moments about either end of the beam gives

$$R_A = 60\,\text{kN} \quad \text{and} \quad R_B = 130\,\text{kN}$$

Applying Macaulay's method,

$$BM_{XX} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = 60x - 20[(x-1)] - 50[(x-3)] - 60\left[\frac{(x-3)^2}{2}\right] \qquad (1)$$

The load unit of kilonewton is accounted for by dividing the left-hand side of (1) by 10^3 and the u.d.l. term is obtained by treating the u.d.l. to the left of XX as a concentrated load of $60(x-3)$ acting at its mid-point of $(x-3)/2$ from XX.

Integrating (1),

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] + A \qquad (2)$$

and

$$\frac{EI}{10^3}y = \frac{60x^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] + Ax + B \quad (3)$$

Now when $x = 0$, $y = 0$ $\therefore B = 0$
when $x = 5$, $y = 0$ $\therefore$ substituting in (3)

$$0 = \frac{60 \times 5^3}{6} - \frac{20 \times 4^3}{6} - \frac{50 \times 2^3}{6} - \frac{60 \times 2^4}{24} + 5A$$

$$0 = 1250 - 213.3 - 66.7 - 40 + 5A$$

$\therefore$ $\qquad 5A = -930 \qquad A = -186$

Substituting in (2),

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60x^2}{2} - 20\left[\frac{(x-1)^2}{2}\right] - 50\left[\frac{(x-3)^2}{2}\right] - 60\left[\frac{(x-3)^3}{6}\right] - 186$$

$\therefore$ slope at $x = 3$ m (i.e. under the 50 kN load)

$$= \frac{10^3}{EI}\left[\frac{60 \times 3^2}{2} - \frac{20 \times 2^2}{2} - 186\right] = \frac{10^3 \times 44}{200 \times 10^9 \times 83 \times 10^{-6}}$$

$$= \mathbf{0.00265\,rad}$$

And, substituting in (3),

$$\frac{EI}{10^3}y = \frac{60 \times 3^3}{6} - 20\left[\frac{(x-1)^3}{6}\right] - 50\left[\frac{(x-3)^3}{6}\right] - 60\left[\frac{(x-3)^4}{24}\right] - 186x$$

$\therefore$ deflection at $x = 3$ m

$$= \frac{10^3}{EI}\left[\frac{60 \times 3^3}{6} - \frac{20 \times 2^3}{6} - 186 \times 3\right]$$

$$= \frac{10^3}{EI}[270 - 26.67 - 558] = -\frac{10^3 \times 314.7}{200 \times 10^9 \times 83 \times 10^{-6}}$$

$$= -0.01896\,m = \mathbf{-19\,mm}$$

In order to determine the maximum deflection, its position must first be estimated. In this case, as the slope is positive under the 50 kN load it is reasonable to assume that the maximum deflection point will occur somewhere between the 20 kN and 50 kN loads. For this position, from (2),

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60x^2}{2} - 20\frac{(x-1)^2}{2} - 186$$

$$= 30x^2 - 10x^2 + 20x - 10 - 186$$

$$= 20x^2 + 20x - 196$$

But, where the deflection is a maximum, the slope is zero.

$\therefore$ $0 = 20x^2 + 20x - 196$

$\therefore$ $x = \dfrac{-20 \pm (400 + 15680)^{1/2}}{40} = \dfrac{-20 \pm 126.8}{40}$

i.e. $x = \mathbf{2.67\,m}$

Then, from (3), the maximum deflection is given by

$$\delta_{max} = -\frac{10^3}{EI}\left[\frac{60 \times 2.67^3}{6} - \frac{20 \times 1.67^3}{6} - 186 \times 2.67\right]$$

$$= -\frac{10^3 \times 321.78}{200 \times 10^9 \times 83 \times 10^{-6}} = -0.0194 = -\mathbf{19.4\,mm}$$

In loading situations where this point lies within the portion of a beam covered by a uniformly distributed load the above procedure is cumbersome since it involves the solution of a cubic equation to determine x.

As an alternative procedure it is possible to obtain a reasonable estimate of the position of zero slope, and hence maximum deflection, by sketching the slope diagram, commencing with the slope at either side of the estimated maximum deflection position; slopes will then be respectively positive and negative and the point of zero slope thus may be estimated. Since the slope diagram is generally a curve, the accuracy of the estimate is improved as the points chosen approach the point of maximum deflection.

As an example of this procedure we may re-solve the final part of the question.

Thus, selecting the initial two points as $x = 2$ and $x = 3$,
when $x = 2$,

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60 \times 2^2}{2} - \frac{20(1^2)}{2} - 186 = -76$$

when $x = 3$,

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60 \times 3^2}{2} - \frac{20(2^2)}{2} - 186 = +44$$

Figure 5.37 then gives a first estimate of the zero slope (maximum deflection) position as $x = 2.63$ on the basis of a straight line between the above-determined values. Recognising the inaccuracy of this assumption, however, it appears reasonable that the required position can
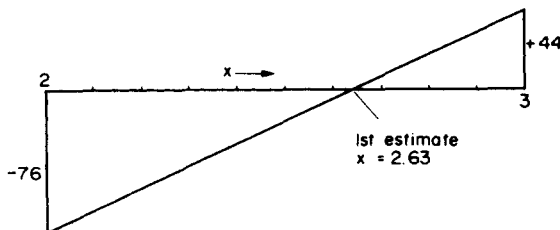

Fig. 5.37.

be more closely estimated as between $x = 2.5$ and $x = 2.7$. Thus, refining the process further, when $x = 2.5$,

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60 \times 2.5^2}{2} - \frac{20 \times 1.5^2}{2} - 186 = -21$$

when $x = 2.7$,

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{60 \times 2.7^2}{2} - \frac{20 \times 1.7^2}{2} - 186 = +3.8$$

Figure 5.38 then gives the improved estimate of

$$x = 2.669$$

which is effectively the same value as that obtained previously.

Fig. 5.38.

Example 5.3

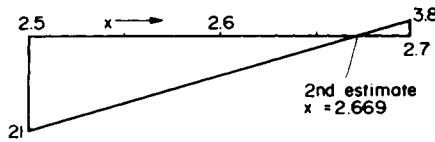Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in Fig. 5.39a using Macaulay's method. $EI = 0.65 \, \text{MN m}^2$.

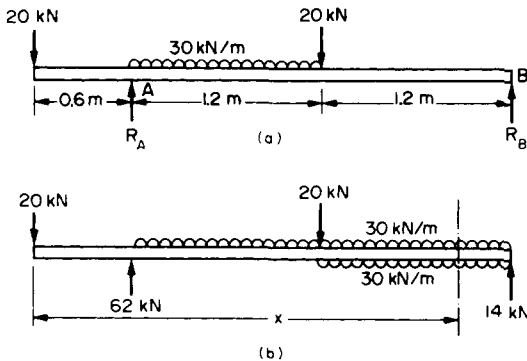
Fig. 5.39.

Solution

Taking moments about B

$$(3 \times 20) + (30 \times 1.2 \times 1.8) + (1.2 \times 20) = 2.4R_A$$

$$\therefore \quad R_A = 62 \, \text{kN} \quad \text{and} \quad R_B = 20 + (30 \times 1.2) + 20 - 62 = 14 \, \text{kN}$$

Using the modified Macaulay approach for distributed loads over part of a beam introduced in § 5.5 (Fig. 5.39b),

$$M_{xx} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = -20x + 62[(x-0.6)] - 30\left[\frac{(x-0.6)^2}{2}\right] + 30\left[\frac{(x-1.8)^2}{2}\right] - 20[(x-1.8)]$$

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{-20x^2}{2} + 62\left[\frac{(x-0.6)^2}{2}\right] - 30\left[\frac{(x-0.6)^3}{6}\right] + 30\left[\frac{(x-1.8)^3}{6}\right]$$
$$-20\left[\frac{(x-1.8)^2}{2}\right] + A$$

$$\frac{EI}{10^3}y = \frac{-20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right]$$
$$-20\left[\frac{(x-1.8)^3}{6}\right] + Ax + B$$

Now when $x = 0.6$, $y = 0$,

$$\therefore \qquad 0 = -\frac{20 \times 0.6^3}{6} + 0.6A + B$$

$$0.72 = 0.6A + B \tag{1}$$

and when $x = 3$, $y = 0$,

$$\therefore \qquad 0 = -\frac{20 \times 3^3}{6} + \frac{62 \times 2.4^3}{6} - \frac{30 \times 2.4^4}{24} + \frac{30 \times 1.2^4}{24} - \frac{20 \times 1.2^3}{6} + 3A + B$$

$$= -90 + 142.848 - 41.472 + 2.592 - 5.76 + 3A + B$$

$$-8.208 = 3A + B \tag{2}$$

$(2) - (1)$

$$-8.928 = 2.4A \qquad \therefore \quad A = -3.72$$

Substituting in (1),

$$B = 0.72 - 0.6(-3.72) \qquad B = 2.952$$

Substituting into the Macaulay deflection equation,

$$\frac{EI}{10^3}y = -\frac{20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right]$$
$$-20\left[\frac{(x-1.8)^3}{6}\right] - 3.72x + 2.952$$

At $\qquad x = 1$

$$y = \frac{10^3}{EI}\left[-\frac{20}{6} + \frac{62}{6} \times 0.4^3 - \frac{30 \times 0.4^4}{24} - 3.72 \times 1 + 2.952\right]$$

$$= \frac{10^3}{EI}[-3.33 + 0.661 - 0.032 - 3.72 + 2.952]$$

$$= -\frac{10^3 \times 3.472}{0.65 \times 10^6} = -5.34 \times 10^{-3}\,\text{m} = -5.34\,\text{mm}$$

The beam therefore is deflected *downwards* at the given position.

Example 5.4

Calculate the slope and deflection of the beam loaded as shown in Fig. 5.40 at a point 1.6 m from the left-hand end. $EI = 1.4\,\text{MN}\,\text{m}^2$.
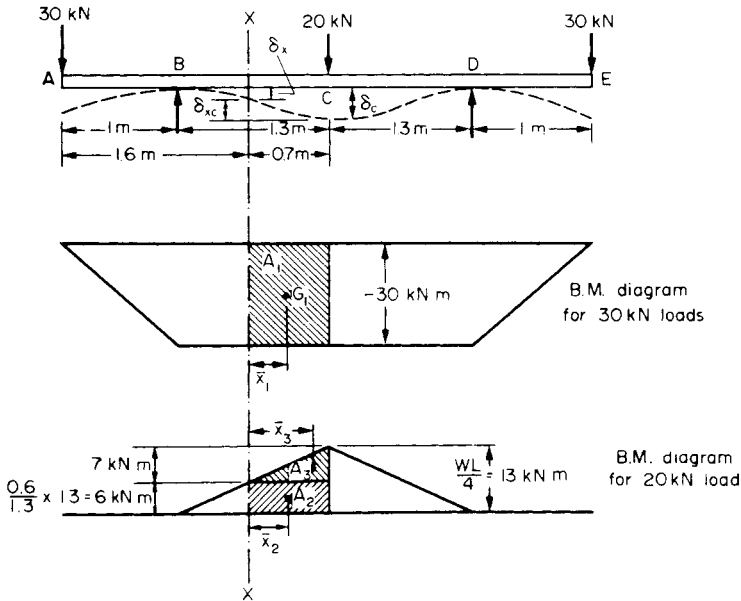
Fig. 5.40.

Solution

Since, by symmetry, the point of zero slope can be located at C a solution can be obtained conveniently using Mohr's method. This is best applied by drawing the B.M. diagrams for the separate effects of (a) the 30 kN loads, and (b) the 20 kN load as shown in Fig. 5.40. Thus, using the zero slope position C as the datum for the Mohr method, from eqn. (5.20)

slope at $X = \dfrac{1}{EI}[\text{area of B.M. diagram between } X \text{ and } C]$

$$= \frac{10^3}{EI}[(-30 \times 0.7) + (6 \times 0.7) + (\tfrac{1}{2} \times 7 \times 0.7)]$$

$$= \frac{10^3}{EI}[-21 + 4.2 + 2.45] = -\frac{14.35 \times 10^3}{1.4 \times 10^6}$$

$$= -10.25 \times 10^{-3}\,\text{rad}$$

and from eqn. (5.21)

deflection at X relative to the tangent at C

$$= \frac{1}{EI} [\text{first moment of area of B.M. diagram between } X \text{ and } C \text{ about } X]$$

$$\delta_{XC} = \frac{10^3}{EI} [(-30 \times 0.7 \times 0.35) + \underbrace{(6 \times 0.7 \times 0.35)}_{A_1 \bar{x}_1} + \underbrace{(7 \times 0.7 \times \tfrac{1}{2} \times \tfrac{2}{3} \times 0.7)}_{A_3 \bar{x}_3}]$$

(with $A_1\bar{x}_1$, $A_2\bar{x}_2$, $A_3\bar{x}_3$ labels under the three terms)

$$= \frac{10^3}{EI} [-7.35 + 1.47 + 1.143] = -\frac{10^3 \times 4.737}{1.4 \times 10^6}$$

$$= -3.38 \times 10^{-3} \text{ m} = -3.38 \text{ mm}$$

This must now be subtracted from the deflection of C relative to the support B to obtain the actual deflection at X.

Now deflection of C relative to B

$$= \text{deflection of } B \text{ relative to } C$$

$$= \frac{1}{EI} [\text{first moment of area of B.M. diagram between } B \text{ and } C \text{ about } B]$$

$$= \frac{10^3}{EI} [(-30 \times 1.3 \times 0.65) + (13 \times 1.3 \times \tfrac{1}{2} \times 1.3 \times \tfrac{2}{3})]$$

$$= \frac{10^3}{EI} [-25.35 + 7.323] = -\frac{18.027 \times 10^3}{1.4 \times 10^6} = -12.88 \times 10^{-3} = -12.88 \text{ mm}$$

$\therefore$ required deflection of $X = -(12.88 - 3.38) = -\mathbf{9.5\,mm}$

Example 5.5

(a) Find the slope and deflection at the tip of the cantilever shown in Fig. 5.41.
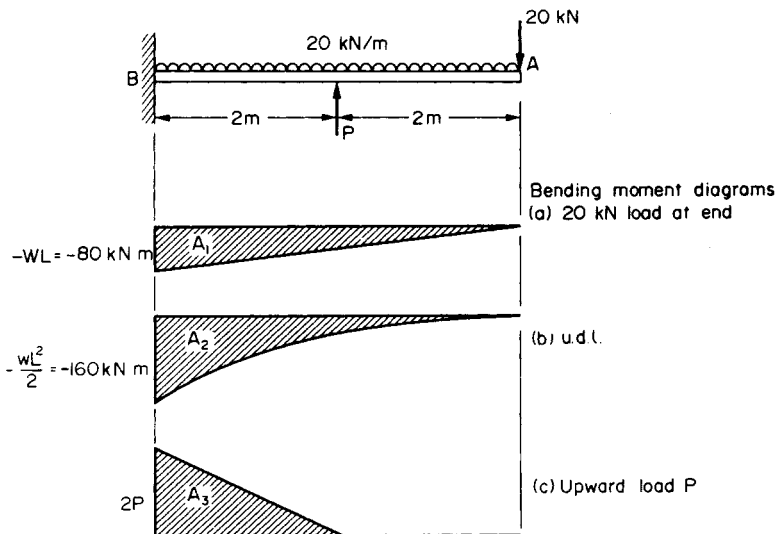

Fig. 5.41.

(b) What load P must be applied upwards at mid-span to reduce the deflection by half? $EI = 20 \, \text{MN m}^2$.

Solution

Here again the best approach is to draw separate B.M. diagrams for the concentrated and uniformly distributed loads. Then, since B is a point of zero slope, the Mohr method may be applied.

(a) Slope at $A = \dfrac{1}{EI}$ [area of B.M. diagram between A and B]

$$= \frac{1}{EI}[A_1 + A_2] = \frac{10^3}{EI}[\{\tfrac{1}{2} \times 4 \times (-80)\} + \{\tfrac{1}{3} \times 4 \times (-160)\}]$$

$$= \frac{10^3}{EI}[-160 - 213.3] = \frac{373.3 \times 10^3}{20 \times 10^6}$$

$$= \mathbf{18.67 \times 10^{-3} \, rad}$$

Deflection of $A = \dfrac{1}{EI}$ [first moment of area of B.M. diagram between A and B about A]

$$= \frac{10^3}{EI}\left[\left(\frac{-80 \times 4 \times \tfrac{2}{3} \times 4}{2}\right) + \left(\frac{-160 \times 4 \times \tfrac{3}{4} \times 4}{3}\right)\right]$$

$$= \frac{-10^3}{EI}[426.6 + 640] = -\frac{1066.6 \times 10^3}{20 \times 10^6} = -53.3 \times 10^{-3} \, \text{m} = \mathbf{-53 \, mm}$$

(b) When an extra load P is applied upwards at mid-span its effect on the deflection is required to be $\tfrac{1}{2} \times 53.3 = 26.67 \, \text{mm}$. Thus

$$26.67 \times 10^{-3} = \frac{1}{EI} \text{ [first moment of area of B.M. diagram for } P \text{ about } A]$$

$$= \frac{10^3}{EI}[\tfrac{1}{2} \times 2P \times 2(2 + \tfrac{2}{3} \times 2)]$$

$$\therefore \qquad P = \frac{26.67 \times 20 \times 10^6}{10^3 \times 6.66} = 80 \times 10^3 \, \text{N}$$

The required load at mid-span is 80 kN.

Example 5.6

The uniform beam of Fig. 5.42 carries the loads indicated. Determine the B.M. at B and hence draw the S.F. and B.M. diagrams for the beam.
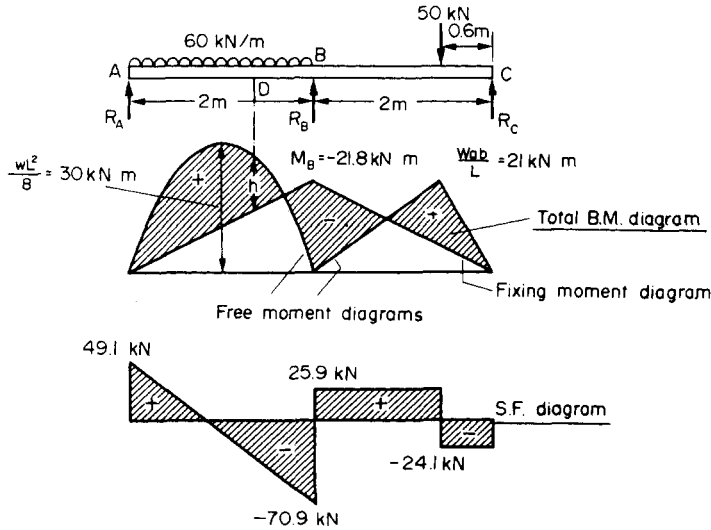
Fig. 5.42.

Solution

Applying the three-moment equation (5.24) to the beam we have,

$$-M_A L_1 - 2M_B(L_1 + L_2) - M_C L_2 = \frac{wL_1^3}{4} + \frac{Wa}{L_2}(L_2^2 - a^2)$$

(Note that the dimension a is always to the "outside" support of the particular span carrying the concentrated load.)

Now with A and C simply supported

$$M_A = M_C = 0$$

$$\therefore \qquad -2M_B(2+2) = \frac{60 \times 10^3 \times 2^3}{4} + \frac{50 \times 10^3 \times 0.6}{2}(2^2 - 0.6^2)$$

$$-8M_B = (120 + 54.6)10^3 = 174.6 \times 10^3$$

$$M_B = -\mathbf{21.8\,kN\,m}$$

With the normal B.M. sign convention the B.M. at B is therefore $-21.8\,\text{kN m}$.

Taking moments about B (forces to left),

$$2R_A - (60 \times 10^3 \times 2 \times 1) = -21.8 \times 10^3$$

$$R_A = \tfrac{1}{2}(-21.8 + 120)10^3 = \mathbf{49.1\,kN}$$

Taking moments about B (forces to right),

$$2R_C - (50 \times 10^3 \times 1.4) = -21.8 \times 10^3$$

$$R_C = \tfrac{1}{2}(-21.8 + 70) = \mathbf{24.1\,kN}$$

and, since the total load $= R_A + R_B + R_C = 50 + (60 \times 2) = 170 \,\text{kN}$

∴ $R_B = 170 - 49.1 - 24.1 = \textbf{96.8 kN}$

The B.M. and S.F. diagrams are then as shown in Fig. 5.42. The fixing moment diagram can be directly subtracted from the free moment diagrams since M_B is negative. The final B.M. diagram is then as shown shaded, values at any particular section being measured from the fixing moment line as datum,

e.g. B.M. at $D = +h$ (to scale)

Example 5.7

A beam *ABCDE* is continuous over four supports and carries the loads shown in Fig. 5.43. Determine the values of the fixing moment at each support and hence draw the S.F. and B.M. diagrams for the beam.
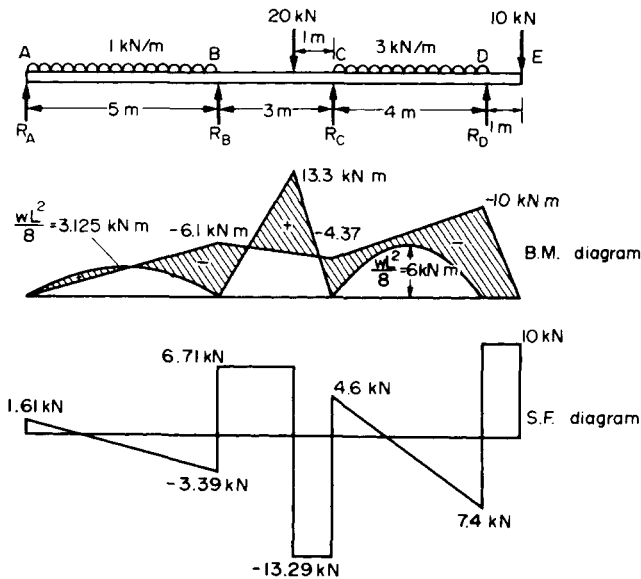
Fig. 5.43.

Solution

By inspection, $M_A = 0$ and $M_D = -1 \times 10 = -10 \,\text{kN m}$
Applying the three-moment equation for the first two spans,

$$-M_A L_1 - 2M_B(L_1 + L_2) - M_C L_2 = \frac{wL_1^3}{4} + \frac{wa}{L_2}(L_2^2 - a^2)$$

$$0 - 2M_B(5+3) - 3M_C = \left[\frac{1 \times 5^3}{4} + \frac{20 \times 1}{3}(3^2 - 1^2) \right] 10^3$$

$$-16M_B - 3M_C = (31.25 + 53.33)10^3$$

$$-16M_B - 3M_C = 84.58 \times 10^3 \tag{1}$$

and, for the second and third spans,

$$-M_B L_2 - 2M_C(L_2 + L_3) - M_D L_3 = \frac{wa}{L_2}(L_2^2 - a^2) + \frac{wL_3^3}{4}$$

$$-3M_B - 2M_C(3 + 4) - (-10 \times 10^3)4 = \left[\frac{20 \times 2}{3}(3^2 - 2^2) + \frac{(3 \times 4^3)}{4}\right]10^3$$

$$-3M_B - 14M_C + (40 \times 10^3) = (66.67 + 48)10^3$$

$$-3M_B - 14M_C = 74.67 \times 10^3 \tag{2}$$

$(2) \times 16/3$ $\qquad\qquad -16M_B - 74.67M_C = 398.24 \times 10^3 \tag{3}$

$(3) - (1)$ $\qquad\qquad\qquad -71.67M_C = 313.66 \times 10^3$

$$M_C = -4.37 \times 10^3 \text{ Nm}$$

Substituting in (1),

$$-16M_B - 3(-4.37 \times 10^3) = 84.58 \times 10^3$$

$$M_B = -\frac{(84.58 - 13.11)10^3}{16}$$

$$= -4.47 \text{ kN m}$$

Moments about B (to left),

$$R_A \times 5 - \left(\frac{1 \times 10^3}{2} \times 5^2\right) = -4.47 \times 10^3$$

$$5R_A = (-4.47 + 12.5)10^3$$

$$R_A = 1.61 \text{ kN}$$

Moments about C (to left),

$$R_A \times 8 - (1 \times 10^3 \times 5 \times 5.5) + (R_B \times 3) - (20 \times 10^3 \times 1) = -4.37 \times 10^3$$

$$3R_B = -4.37 \times 10^3 + 27.5 \times 10^3 + 20 \times 10^3 - 8 \times 1.61 \times 10^3$$

$$3R_B = 30.3 \times 10^3$$

$$R_B = 10.1 \text{ kN}$$

Moments about C (to right),

$$(-10 \times 10^3 \times 5) + 4R_D - (3 \times 10^3 \times 4 \times 2) = -4.37 \times 10^3$$

$$4R_D = (-4.37 + 50 + 24)10^3$$

$$R_D = 17.4 \text{ kN}$$

Then, since

$$R_A + R_B + R_C + R_D = 47 \text{ kN}$$

$$1.61 + 10.1 + R_C + 17.4 = 47$$

$$R_C = 17.9 \text{ kN}$$

This value should then be checked by taking moments to the right of B,

$$(-10 \times 10^3 \times 8) + 7R_D + 3R_C - (3 \times 10^3 \times 4 \times 5) - (20 \times 10^3 \times 2) = -4.47 \times 10^3$$
$$3R_C = (-4.47 + 40 + 60 + 80 - 121.8)10^3 = 53.73 \times 10^3$$
$$R_C = \textbf{17.9 kN}$$

The S.F. and B.M. diagrams for the beam are shown in Fig. 5.43.

Example 5.8

Using the finite difference method, determine the central deflection of a simply-supported beam carrying a uniformly distributed load ω over its complete span. The beam can be assumed to have constant flexural rigidity EI throughout.

Solution
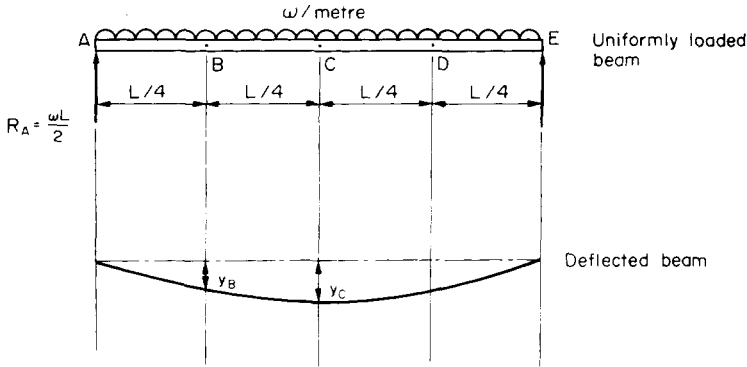
Fig. 5.44.

As a simple demonstration of the finite difference approach, assume that the beam is divided into only four equal segments (thus reducing the accuracy of the solution from that which could be achieved with a greater number of segments).

Then,
$$\text{B.M. at B} = \frac{\omega L}{2} \times \frac{L}{4} - \frac{\omega L}{4} \cdot \frac{L}{8} = \frac{3\omega L^2}{32} = M_B$$

but, from eqn. (5.30):

$$\frac{M_B}{EI} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$\therefore \qquad \frac{I}{EI}\left(\frac{3\omega L^2}{32}\right) = \frac{1}{(L/4)^2}(y_c - 2y_B + y_A)$$

and, since $y_A = 0$,

$$\frac{3\omega L^2}{512\,EI} = y_c - 2y_B. \tag{1}$$

Similarly   B.M. at $C = \dfrac{\omega L}{2} \cdot \dfrac{L}{2} - \dfrac{\omega L}{2} \cdot \dfrac{L}{4} = \dfrac{\omega L^2}{8} = M_c.$

and, from eqn. (5.30)

$$\frac{1}{EI}\left(\frac{\omega L^2}{8}\right) = \frac{1}{(L/4)^2}(y_B - 2y_c + y_D)$$

Now, from symmetry, $y_D = y_B$

$\therefore$ $\dfrac{\omega L^4}{128EI} = 2y_B - 2y_c$ (2)

Adding eqns. (1) and (2);

$$-y_c = \frac{\omega L^4}{128EI} + \frac{3\omega L^4}{512EI}$$

$\therefore$ $y_c = \dfrac{-7\omega L^4}{512EI} = -0.0137\dfrac{\omega L^4}{EI}$

the negative sign indicating a downwards deflection as expected. This value compares with the "exact" value of:

$$y_c = \frac{5\omega L^4}{384EI} = -0.01302\frac{\omega L^4}{EI}$$

a difference of about 5%. As stated earlier, this comparison could be improved by selecting more segments but, nevertheless, it is remarkably accurate for the very small number of segments chosen.

Example 5.9

The statically indeterminate propped cantilever shown in Fig. 5.45 is propped at B and carries a central load W. It can be assumed to have a constant flexural rigidity EI throughout.
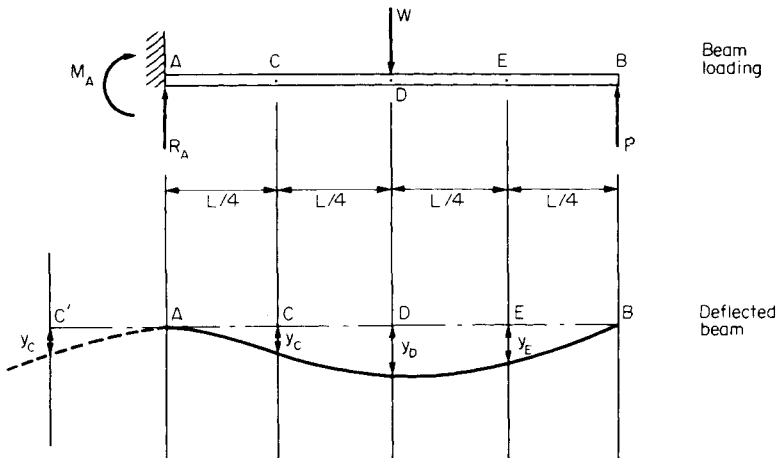

Fig. 5.45.

Determine, using a finite difference approach, the values of the reaction at the prop and the central deflection.

Solution

Whilst at first sight, perhaps, there appears to be a number of redundancies in the cantilever loading condition, in fact the problem reduces to that of a single redundancy, say the unknown prop load P, since with a knowledge of P the other "unknowns" M_A and R_A can be evaluated easily.

Thus, again for simplicity, consider the beam divided into four equal segments giving three unknown deflections y_C, y_D and y_E (assuming zero deflection at the prop B) and one redundancy. Four equations are thus required for solution and these may be obtained by applying the difference equation at four selected points on the beam:

From eqn. (5.30)

$$\text{B.M. at } E = M_E = \frac{PL}{4} = \frac{EI}{(L/4)^2}(y_B - 2y_E + y_D)$$

but $y_B = 0$
$$y_D - 2y_E = \frac{PL^3}{64EI} \tag{1}$$

$$\text{B.M. at } D = M_D = \frac{PL}{2} = \frac{EI}{(L/4)^2}(y_E - 2y_D + y_C)$$

∴
$$y_E - 2y_D + y_C = \frac{PL^3}{32EI} \tag{2}$$

$$\text{B.M. at } C = M_C = P.\frac{3L}{4} - \frac{WL}{4} = \frac{EI}{(L/4)^2}(y_A - 2y_C + y_D)$$

But $y_A = 0$

∴
$$y_D - 2y_C = \frac{3PL^3}{64} - \frac{WL^3}{64} \tag{3}$$

At point A it is necessary to introduce the mirror image of the beam giving point C' to the left of A with a deflection $y'_C = y_C$ in order to produce the fourth equation. Then:

$$\text{B.M. at } A = M_A = PL - \frac{WL}{2} = \frac{EI}{(L/4)^2}(y'_C - 2y_A + y_C)$$

and again since $y_A = 0$
$$y_C = \frac{PL^3}{32} - \frac{WL^3}{64} \tag{4}$$

Solving equations (1) to (4) simultaneously gives the required prop load:

$$P = \frac{7W}{22} = 0.318 \text{ W},$$

and the central deflection:

$$y_D = -\frac{17WL^3}{1408EI} = -0.0121\frac{WL^3}{EI}$$

Problems

5.1 (A/B). A beam of length 10 m is symmetrically placed on two supports 7 m apart. The loading is 15 kN/m between the supports and 20 kN at each end. What is the central deflection of the beam? $E = 210 \, \text{GN/m}^2$; $I = 200 \times 10^{-6} \, \text{m}^4$. [6.8 mm.]

5.2 (A/B). Derive the expression for the maximum deflection of a simply supported beam of negligible weight carrying a point load at its mid-span position. The distance between the supports is L, the second moment of area of the cross-section is I and the modulus of elasticity of the beam material is E.

The maximum deflection of such a simply supported beam of length 3 m is 4.3 mm when carrying a load of 200 kN at its mid-span position. What would be the deflection at the free end of a cantilever of the same material, length and cross-section if it carries a load of 100 kN at a point 1.3 m from the free end? [13.4 mm.]

5.3 (A/B). A horizontal beam, simply supported at its ends, carries a load which varies uniformly from 15 kN/m at one end to 60 kN/m at the other. Estimate the central deflection if the span is 7 m, the section 450 mm deep and the maximum bending stress 100 MN/m². $E = 210 \, \text{GN/m}^2$. [U.L.] [21.9 mm.]

5.4 (A/B). A beam AB, 8 m long, is freely supported at its ends and carries loads of 30 kN and 50 kN at points 1 m and 5 m respectively from A. Find the position and magnitude of the maximum deflection. $E = 210 \, \text{GN/m}^2$; $I = 200 \times 10^{-6} \, \text{m}^4$. [14.4 mm.]

5.5 (A/B). A beam 7 m long is simply supported at its ends and loaded as follows: 120 kN at 1 m from one end A, 20 kN at 4 m from A and 60 kN at 5 m from A. Calculate the position and magnitude of the maximum deflection. The second moment of area of the beam section is $400 \times 10^{-6} \, \text{m}^4$ and E for the beam material is $210 \, \text{GN/m}^2$. [9.8 mm at 3.474 m.]

5.6 (B). A beam $ABCD$, 6 m long, is simply-supported at the right-hand end D and at a point B 1 m from the left-hand end A. It carries a vertical load of 10 kN at A, a second concentrated load of 20 kN at C, 3 m from D, and a uniformly distributed load of 10 kN/m between C and D. Determine the position and magnitude of the maximum deflection if $E = 208 \, \text{GN/m}^2$ and $I = 35 \times 10^{-6} \, \text{m}^4$. [3.553 m from A, 11.95 mm.]

5.7 (B). A 3 m long cantilever ABC is built-in at A, partially supported at B, 2 m from A, with a force of 10 kN and carries a vertical load of 20 kN at C. A uniformly distributed load of 5 kN/m is also applied between A and B. Determine a) the values of the vertical reaction and built-in moment at A and b) the deflection of the free end C of the cantilever.

Develop an expression for the slope of the beam at any position and hence plot a slope diagram. $E = 208 \, \text{GN/m}^2$ and $I = 24 \times 10^{-6} \, \text{m}^4$. [20 kN, 50 kN m, −15 mm.]

5.8 (B). Develop a general expression for the slope of the beam of question 5.6 and hence plot a slope diagram for the beam. Use the slope diagram to confirm the answer given in question 5.6 for the position of the maximum deflection of the beam.

5.9 (B). What would be the effect on the end deflection for question 5.7, if the built-in end A were replaced by a simple support at the same position and point B becomes a full simple support position (i.e. the force at B is no longer 10 kN). What general observation can you make about the effect of built-in constraints on the stiffness of beams? [5.7 mm.]

5.10 (B). A beam AB is simply supported at A and B over a span of 3 m. It carries loads of 50 kN and 40 kN at 0.6 m and 2 m respectively from A, together with a uniformly distributed load of 60 kN/m between the 50 kN and 40 kN concentrated loads. If the cross-section of the beam is such that $I = 60 \times 10^{-6} \, \text{m}^4$ determine the value of the deflection of the beam under the 50 kN load. $E = 210 \, \text{GN/m}^2$. Sketch the S.F. and B.M. diagrams for the beam. [3.7 mm.]

5.11 (B). Obtain the relationship between the B.M., S.F., and intensity of loading of a laterally loaded beam.

A simply supported beam of span L carries a distributed load of intensity kx^2/L^2 where x is measured from one support towards the other. Find:

(a) the location and magnitude of the greatest bending moment;
(b) the support reactions. [U.Birm.] [0.63L, 0.0393kL², kL/12, kL/4.]

5.12 (B). A uniform beam 4 m long is simply supported at its ends, where couples are applied, each 3 kN m in magnitude but opposite in sense. If $E = 210 \, \text{GN/m}^2$ and $I = 90 \times 10^{-6} \, \text{m}^4$ determine the magnitude of the deflection at mid-span.

What load must be applied at mid-span to reduce the deflection by half? [0.317 mm, 2.25 kN.]

5.13 (B). A 500 mm × 175 mm steel beam of length 8 m is supported at the left-hand end and at a point 1.6 m from the right-hand end. The beam carries a uniformly distributed load of 12 kN/m on its whole length, an additional uniformly distributed load of 18 kN/m on the length between the supports and a point load of 30 kN at the right-hand end. Determine the slope and deflection of the beam at the section midway between the supports and also at the right-hand end. EI for the beam is $1.5 \times 10^8 \, \text{Nm}^2$. [U.L.] [$1.13 \times 10^{-4}$, 3.29 mm, 9.7×10^{-4}, 1.71 mm.]

5.14 (B). A cantilever, 2.6 m long, carrying a uniformly distributed load w along the entire length, is propped at its free end to the level of the fixed end. If the load on the prop is then 30 kN, calculate the value of w. Determine also the slope of the beam at the support. If any formula for deflection is used it must first be proved. $E = 210 \, \text{GN/m}^2$; $I = 4 \times 10^{-6} \, \text{m}^4$. [U.E.I.] [30.8 kN/m, 0.014 rad.]

5.15 (B). A beam *ABC* of total length *L* is simply supported at one end *A* and at some point *B* along its length. It carries a uniformly distributed load of w per unit length over its whole length. Find the optimum position of *B* so that the greatest bending moment in the beam is as low as possible. [U.Birm.] [*L*/2.]

5.16 (B). A beam *AB*, of constant section, depth 400 mm and $I_{max} = 250 \times 10^{-6} \, \text{m}^4$, is hinged at *A* and simply supported on a non-yielding support at *C*. The beam is subjected to the given loading (Fig. 5.46). For this loading determine (a) the vertical deflection of *B*; (b) the slope of the tangent to the bent centre line at *C*. $E = 80 \, \text{GN/m}^2$. [I.Struct.E.] [27.3 mm, 0.0147 rad.]
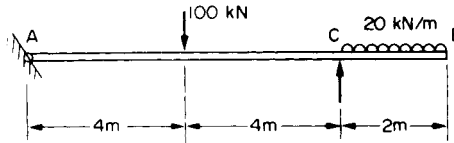

Fig. 5.46.

5.17 (B). A simply supported beam *AB* is 7 m long and carries a uniformly distributed load of 30 kN/m run. A couple is applied to the beam at a point *C*, 2.5 m from the left-hand end, *A*, the couple being clockwise in sense and of magnitude 70 kNm. Calculate the slope and deflection of the beam at a point *D*, 2 m from the left-hand end. Take $EI = 5 \times 10^7 \, \text{Nm}^2$. [E.M.E.U.] [$5.78 \times 10^{-3}$ rad, 16.5 mm.]

5.18 (B). A uniform horizontal beam *ABC* is 0.75 m long and is simply supported at *A* and *B*, 0.5 m apart, by supports which can resist upward or downward forces. A vertical load of 50 N is applied at the free end *C*, which produces a deflection of 5 mm at the centre of span *AB*. Determine the position and magnitude of the maximum deflection in the span *AB*, and the magnitude of the deflection at *C*. [E.I.E.] [5.12 mm (upwards), 20.1 mm.]

5.19 (B). A continuous beam *ABC* rests on supports at *A*, *B* and *C*. The portion *AB* is 2 m long and carries a central concentrated load of 40 kN, and *BC* is 3 m long with a u.d.l. of 60 kN/m on the complete length. Draw the S.F. and B.M. diagrams for the beam. [-3.25, 148.75, 74.5 kN (Reactions); $M_B = -46.5$ kN m.]

5.20 (B). State Clapeyron's theorem of three moments. A continuous beam *ABCD* is constructed of built-up sections whose effective flexural rigidity *EI* is constant throughout its length. Bay lengths are $AB = 1$ m, $BC = 5$ m, $CD = 4$ m. The beam is simply supported at *B*, *C* and *D*, and carries point loads of 20 kN and 60 kN at *A* and midway between *C* and *D* respectively, and a distributed load of 30 kN/m over *BC*. Determine the bending moments and vertical reactions at the supports and sketch the B.M. and S.F. diagrams. [U.Birm.] [-20, -66.5, 0 kN m; 85.7, 130.93, 13.37 kN.]

5.21 (B). A continuous beam *ABCD* is simply supported over three spans $AB = 1$ m, $BC = 2$ m and $CD = 2$ m. The first span carries a central load of 20 kN and the third span a uniformly distributed load of 30 kN/m. The central span remains unloaded. Calculate the bending moments at *B* and *C* and draw the S.F. and B.M. diagrams. The supports remain at the same level when the beam is loaded. [1.36, -7.84 kN m; 11.36, 4.03, 38.52, 26.08 kN (Reactions).]

5.22 (B). A beam, simply supported at its ends, carries a load which increases uniformly from 15 kN/m at the left-hand end to 100 kN/m at the right-hand end. If the beam is 5 m long find the equation for the rate of loading and, using this, the deflection of the beam at mid-span if $E = 200 \, \text{GN/m}^2$ and $I = 600 \times 10^{-6} \, \text{m}^4$. [$w = -(15 + 85x/L)$; 3.9 mm.]

5.23 (B). A beam 5 m long is firmly fixed horizontally at one end and simply supported at the other by a prop. The beam carries a uniformly distributed load of 30 kN/m run over its whole length together with a concentrated load of 60 kN at a point 3 m from the fixed end. Determine:
(a) the load carried by the prop if the prop remains at the same level as the end support;
(b) the position of the point of maximum deflection. [B.P.] [82.16 kN; 2.075 m.]

5.24 (B/C). A continuous beam *ABCDE* rests on five simple supports *A*, *B*, *C*, *D* and *E*. Spans *AB* and *BC* carry a u.d.l. of 60 kN/m and are respectively 2 m and 3 m long. *CD* is 2.5 m long and carries a concentrated load of 50 kN at 1.5 m from *C*. *DE* is 3 m long and carries a concentrated load of 50 kN at the centre and a u.d.l. of 30 kN/m. Draw the B.M. and S.F. diagrams for the beam.
[Fixing moments: 0, -44.91, -25.1, -38.95, 0 kN m. Reactions: 37.55, 179.1, 97.83, 118.5, 57.02 kN.]