1 Mathematical Preliminaries

1.1 Probability Theory Elements

1.1.1 Introduction

Probability theory [326,357,365] is a part of theoretical and applied mathematics, which is engaged in establishing the rules governing random events – random games or experimental testing. The definitions, theorems and lemmas given below are necessary to understand the basic equations and computer implementation aspects used in the later numerical analyses presented in the book. They can also be used to calculate many of the closed−form equations applied frequently in applied sciences and engineering practice [19,37,150,201,202,253].

Definition
The variations with n elements for k elements are k elements series where each number 1,2,...,k corresponds to the single element from the initial set. The variations can differ in the elements or their order. The total number of all variations with n elements for k is described by the relation

$$V_n^k = \frac{n!}{(n-k)!} = \underbrace{n(n-1)...(n-k+1)}_{k-times} \tag{1.1}$$

Example
Let us consider the three-element set A{X,Y,Z}. Two−element variations of this set are represented as $V_3^2 = 6$: XY, YZ, XZ, YX, ZY, ZX.

Definition
Permutations with n elements are n−element series where each number 1,2,...,n corresponds to the single element from the initial n−element set. The difference between permutations is in the element order. The total number of all permutations with n different elements is given by the formula:

$$P_n = V_n^n = 1 \cdot 2 \cdot ... \cdot n = n! \tag{1.2}$$

If among n elements X, Y, Z,... there are identical elements, where X repeats a times, Y appears b times, while Z repeats c times etc., then

$$P_n = \frac{n!}{a!b!c!} \tag{1.3}$$

Example
Let us consider the three element set A{X,Y,Z}. The following permutations of the set A are available: $P_3 = 6$: XYZ, XZY, YZX, YXZ, ZXY, ZYX.

Definition
The combinations with n elements for k elements are $k-$elements sets, which can be created by choosing any k elements from the given $n-$element set, where the order does not play any role. The combinations can differ in the elements only. The total number of all combinations with n for k elements is described by the formula

$$C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{1.4}$$

In specific cases it is found that

$$C_n^1 = \binom{n}{1} = n\,,\; C_n^n = \binom{n}{n} = 1 \tag{1.5}$$

where

$$\binom{n}{k} = \binom{n}{n-k},\; \binom{n}{n} = \binom{n}{0} = 1 \tag{1.6}$$

Example
Let us consider a set A{X,Y,Z} as before. Two−element variations of this set are the following: XY, XZ and YZ.

The fundamental concepts of probability theory are random experiments and random events resulting from them. A single event, which can result from some random experiment is called elementary event, an and for the single die throw is equivalent to any sum of the dots on a die taken from the set {1,…,6}. Further, it is concluded that all elementary events corresponding to the random experiment form the elementary events space defined usually as Ω, which various subsets like A and/or B belong to (favouring the specified event or not, for instance).

Definition
A formal notation $\omega \in A$ denotes that the elementary event ω belongs to the event A and is understood in the following way – if ω results from some experiment, then the event A happened too, which ω belongs to. The notation means that the elementary event ω favours the event A.

Definition
The formal notation $A \subset B$, which means that event A is included in the event B is understood such that event A results in the event B since the following implication holds true: if the elementary event ω favours event A, then event ω favours event B, too.

Definition
An alternative of the events $A_1, A_2, ..., A_n$ is the following sum:

$$A_1 \cup A_2 \cup ... \cup A_n = \bigcup_{i=1}^{n} A_i \tag{1.7}$$

which is a random event consisting of all random elementary events belonging to at least one of the events $A_1, A_2, ..., A_n$.

Definition
A conjunction of the events $A_1, A_2, ..., A_n$ is a product

$$A_1 \cap A_2 \cap ... \cap A_n = \bigcap_{i=1}^{n} A_i \tag{1.8}$$

which proceeds if and only if any of the events $A_1, A_2, ..., A_n$ proceed.

Definition
Probability is a function P which is defined on the subsets of the elementary events and having real values in closed interval [0,1] such that
(1) P(Ω)=1, P(∅)=0;
(2) for any finite and/or infinite series of the excluding events $A_1, A_2, ..., A_n, ...$ $A_i \cap A_j = \varnothing$, there holds for $i \neq j$

$$P\left(\bigcup_i A_i\right) = \sum_i P(A_i) \tag{1.9}$$

Starting from the above definitions one can demonstrate the following lemmas:

Lemma
The probability of the alternative of the events is equal to the sum of the probabilities of these events.

Lemma
If event B results from event A then

$$P(A) \leq P(B) \tag{1.10}$$

The definition of probability does not reflect however a natural very practical need of its value determination and that is why the simplified Laplace definition is frequently used for various random events.

Definition
If n trials forms the random space of elementary events where each experiment has the same probability equal to $1/n$, then the probability of the $m-$element event A is equal to

$$P(A) = \frac{m}{n} \tag{1.11}$$

Next, we will explain the definition, meaning and basic properties of the probability spaces. The probability space (Ω,F,P) is uniquely defined by the space of elementary random events Ω, the events field F and probabilistic measure P. The field of events F is the relevant family of subsets of the space of elementary random events Ω. This field F is a non−empty, complementary and countable additive set having σ-algebra structure.

Definition
The probabilistic measure P is a function

$$P : F \rightarrow [0,1] \tag{1.12}$$

which is a nonnegative, countable additive and normalized function defined on the fields of random events. The pair (Ω,F) is a countable space, while the events are countable subsets of Ω. The value $P(A)$ assigned by the probabilistic measure P to event A is called a probability of this event.

Definition
Two events A and B are independent if they fulfil the following condition:

$$P(A \cap B) = P(A) \cdot P(B) \tag{1.13}$$

while the events $\{A_1, A_2, ..., A_n\}$ are pair independent, if this condition holds true for any pair from this set.

Definition
Let us consider the probability space (Ω,F,P) and measurable space $\{\Re^n, B_n\}$, where B_n is a class of the Borelian sets. Then, the representation

$$X : \Omega \to \Re^n \tag{1.14}$$

is an $n-$dimensional random variable or n$-$dimensional random vector.

Definition
The probability distribution of the random variable X is a function $P_X : B \to [0,1]$ such that

$$\underset{b \in B}{\forall} P_X(b) = P(X \in B) \tag{1.15}$$

The probability distribution of the random variable is a probabilistic measure.

Definition
Let us consider the following probability space $(\Re, B, P_X)$. The function $F_X : \Re \to [0,1]$ defined as

$$F(x) = P_X[(-\infty, x)] \tag{1.16}$$

is called the cumulative distribution function of the variable X.

Definition
The function $f : \Re \to \Re_+$ has the following properties:
(1) there holds almost everywhere (in each point of the cumulative distribution function differentiability):

$$\frac{dF(x)}{dx} = f(x) \tag{1.17}$$

(2)

$$f(x) \geq 0 \tag{1.18}$$

(3)

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \tag{1.19}$$

(4) for any Borelian set $b \in B$ the integral $\int_b f(x)dx = P(X \in b)$ is a probability density function (PDF) of the variable X.

Definition
Let us consider the random variable $X : \Omega \to \Re$ defined on the probabilistic space (Ω, F, P). The expected value of the random variable X is defined as

$$E[X] = \int_{-\infty}^{+\infty} X(\omega)dP(\omega) \tag{1.20}$$

if only the Lesbegue integral with respect to the probabilistic measure exists and converges.

Lemma

$$\underset{c\in\Re}{\forall} E[c] = c \tag{1.21}$$

Lemma
There holds for any random numbers X_i and the real numbers $c_i \in \Re$

$$E\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} c_i E[X_i] \tag{1.22}$$

Lemma
There holds for any independent random variables X_i

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i] \tag{1.23}$$

Definition
Let us consider the following random variable $X : \Omega \to \Re$ defined on the probabilistic space (Ω, F, P). The variance of the variable X is defined as

$$Var(X) = \int_{\Omega} (X(\omega) - E[X])^2 dP(\omega) \tag{1.24}$$

and the standard deviation is called the quantity

$$\sigma(X) = \sqrt{Var(X)} \tag{1.25}$$

Lemma

$$\underset{c\in\Re}{\forall} Var(c) = 0 \tag{1.26}$$

Lemma

$$\underset{c\in\Re}{\forall} Var(cX) = c^2 Var(X) \tag{1.27}$$

Lemma
There holds for any two independent random variables X and Y

$$Var(X \pm Y) = Var(X) + Var(Y) \tag{1.28}$$

$$Var(X \cdot Y) = E^2[X] \cdot Var(Y) + Var(X) \cdot Var(Y) + Var(X) \cdot E^2[Y] \tag{1.29}$$

Definition
Let us consider the random variable $X:\Omega \to \Re$ defined on the probabilistic space (Ω, F, P). A complex function of the real variable $\varphi:\Re \to Z$ such that

$$\varphi(t) = E[\exp(itX)] \tag{1.30}$$

stands for the characteristic function of the variable X.

1.1.2 Gaussian and Quasi-Gaussian Random Variables

Let us consider the random variable *X* having a Gaussian probability distribution function with *m* being the expected value and $\sigma > 0$ the standard deviation. The distribution function of this variable is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-t^2}{2}\right) dt \tag{1.31}$$

where the probability density function is calculated as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) \tag{1.32}$$

The characteristic function for this variable is denoted as

$$\varphi(t) = E[\exp(itX)] = \exp\left(mit - \tfrac{1}{2}\sigma^2 t^2\right). \tag{1.33}$$

If the variable *X* with the parameters (m,σ) is Gaussian, then its linear transform $Y = AX + B$ with $A, B \in \Re$ is Gaussian, too, and its parameters are equal to $Am+B$ and $|A|\sigma$ for $A \neq 0$, respectively.

Problem
Let us consider the random variable *X* with the first two moments $E[X]$ and $Var(X)$. Let us determine the corresponding moments of the new variable $Y = X^2$.

Solution
The problem has been solved using three different ways illustrating various methods applicable in this and in analogous cases. The generality of these methods make them available in the determination of probabilistic moments and their parameters for most random variables and their transforms for given or unknown

probability density functions of the input frequently takes place in which numerous engineering problems.

I method
Starting from the definition of the variance of a ny random variable one can write

$$Var(Y) = E(Y^2) - E^2(Y) \tag{1.34}$$

Let $Y = X^2$, then

$$Var(X^2) = E((X^2)^2) - E^2(X^2) \tag{1.35}$$

The value of $E\left[X^4\right]$ will be determined through integration of the characteristic function for the Gaussian probability density function

$$E\left[X^4\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx \tag{1.36}$$

where $m=E[X]$ and $\sigma = \sqrt{Var(X)}$ denote the expected value and standard deviation of the considered distribution, respectively. Next, the following standardised variable is introduced

$$t = \frac{x-m}{\sigma}, \text{ where } x = t\sigma + m, dx = \sigma dt \tag{1.37}$$

which gives

$$E\left[X^4\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (t\sigma + m)^4 \exp\left(-\frac{t^2}{2}\right) dt \tag{1.38}$$

After some algebraic transforms of the integrand function it is obtained that

$$E\left[X^4\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\sigma^4 t^4 + 4\sigma^3 m t^3 + 6\sigma^2 m^2 t^2 + 4\sigma m^3 t + m^4)\, e^{-\frac{t^2}{2}} dt \tag{1.39}$$

and, dividing into particular integrals, there holds

$$E\left[X^4\right] = \frac{1}{\sqrt{2\pi}} (\sigma^4 I_1 + 4\sigma^3 m I_2 + 6\sigma^2 m^2 I_3 + 4\sigma m^3 I_4 + m^4 I_5)\, e^{-\frac{t^2}{2}} \tag{1.40}$$

where the components denote

$$I_1 = \int_{-\infty}^{+\infty} t^4 e^{-\frac{t^2}{2}} dt \; ; \; I_2 = \int_{-\infty}^{+\infty} t^3 e^{-\frac{t^2}{2}} dt \; ; \; I_3 = \int_{-\infty}^{+\infty} t^2 e^{-\frac{t^2}{2}} dt \; ;$$
$$I_4 = \int_{-\infty}^{+\infty} te^{-\frac{t^2}{2}} dt \; ; \; I_5 = \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt \tag{1.41}$$

It should be mentioned that the values of the odd integrals on the real domain are equal to 0 in the following calculation

$$\int_{-\infty}^{+\infty} f(x)g(x)dx = \int_{-\infty}^{0} f(x)g(x)dx + \int_{0}^{+\infty} f(x)g(x)dx \tag{1.42}$$

If the function $f(x)$ is odd and $g(x)$ is even

$$f(-x)=-f(x),\ g(-x)=g(x), \tag{1.43}$$

then it can be written

$$\int_{-\infty}^{0} f(x)g(x)dx = \int_{0}^{+\infty} f(-x)g(x)dx = -\int_{0}^{+\infty} f(x)g(x)dx \, . \tag{1.44}$$

Considering that the odd indices integrals are calculated; this results in

$$I_5 = \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi} \tag{1.45}$$

$$I_3 = \int_{-\infty}^{+\infty} t^2 e^{-\frac{t^2}{2}} dt = -\int_{-\infty}^{+\infty} t(te^{-\frac{t^2}{2}})dt = -\int_{-\infty}^{+\infty} td(e^{-\frac{t^2}{2}})$$
$$= -te^{-\frac{t^2}{2}}\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2\pi} \tag{1.46}$$

$$I_1 = \int_{-\infty}^{+\infty} t^4 e^{-\frac{t^2}{2}} dt = -\int_{-\infty}^{+\infty} t^3 de^{-\frac{t^2}{2}} = -\left[t^3 e^{-\frac{t^2}{2}}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt^3 \right]$$
$$= 3\int_{-\infty}^{+\infty} t^2 e^{-\frac{t^2}{2}} dt = -3\int_{-\infty}^{+\infty} tde^{-\frac{t^2}{2}} = -3\left[te^{-\frac{t^2}{2}}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt \right] = 3\sqrt{2\pi} \, . \tag{1.47}$$

After simplification the result is

$$E\left[X^4\right] = 3\sigma^4 + 6\sigma^2 m^2 + m^4 = E^4[X] + 6Var(X)E^2[X] + 3Var^2(X) \tag{1.48}$$

$$E\left[X^2\right] = \sigma^2 + m^2 = E^2[X] + Var(X) \tag{1.49}$$

$$\begin{aligned} Var(X^2) &= E\left[X^4\right] - E^2\left[X^2\right] = 2\sigma^2(\sigma^2 + 2m^2) \\ &= 2Var(X)(Var(X) + 2E^2\left[X\right]) \end{aligned} \tag{1.50}$$

II method
Initial algebraic rules can be proved following the method shown below. Using a modified algebraic definition of the variance

$$Var(X^2) = E\left[X^4\right] - E^2\left[X^2\right] \tag{1.51}$$

and the expected value

$$E\left[X^2\right] = Var(X) + E^2\left[X\right] \tag{1.52}$$

subtracted from the following equation

$$E^2\left[X^2\right] = \left(Var(X) + E^2\left[X\right]\right)^2 = Var^2 X + 2Var(X)E^2\left[X\right] + E^4\left[X\right] \tag{1.53}$$

we can demonstrate the following desired result:

$$Var(X^2) = E\left[X^4\right] - Var^2(X) - 2Var(X)E^2[X] - E^4\left[X\right] \tag{1.54}$$

III method
The characteristic function for the Gaussian PDF has the following form:

$$\varphi(t) = \exp\left(mit - \tfrac{1}{2}\sigma^2 t^2\right) \tag{1.55}$$

where

$$\varphi^{(k)}(0) = i^k E\left[X^k\right];\ k \geq 0 \tag{1.56}$$

and

$$\varphi^{(0)} = \varphi\ ;\ \varphi'(0) = im \tag{1.57}$$

The mathematical induction rule leads us to the conclusion that

$$\varphi^{(n)}(t) = \left(im - t\sigma^2\right) \cdot \varphi^{(n-1)}(t) - (n-1)\sigma^2 \cdot \varphi^{(n-2)}(t),\ n \geq 2 \tag{1.58}$$

which results in the equations

$$\varphi^{(2)}(0) = -m^2 - \sigma^2 \tag{1.59}$$

$$\varphi^{(3)}(0) = -mi\left(m^2 + 3\sigma^2\right) \tag{1.60}$$

$$\varphi^{(4)}(0) = m^4 + 6m^2\sigma^2 + 3\sigma^4 \tag{1.61}$$

giving the same input equations.

Problem 2

Let us determine the value of the Gaussian integral $\int_{-\infty}^{+\infty} \exp(-x^2)dx$.

Solution

Starting from the obvious fact that

$$\begin{aligned} &\iint_{K_1} \exp\left(-\left(x^2 + y^2\right)\right)dxdy \\ &\leq \iint_{K} \exp\left(-\left(x^2 + y^2\right)\right)dxdy \leq \iint_{K_2} \exp\left(-\left(x^2 + y^2\right)\right)dxdy \end{aligned} \tag{1.62}$$

with K_1 being a circle with radius t and located in the centre of Cartesian coordinates, K denotes the square contour with edge equal to $t\sqrt{2}$, while K_2 stands for a circle with radius $t\sqrt{2}$. The coordinates transform to the polar system given by

$$x = r\cos\varphi \;\;,\;\; y = r\sin\varphi \tag{1.63}$$

returns

$$\begin{aligned} &\iint_{\substack{0\le r\le t \\ 0\le\varphi\le 2\pi}} \exp\left(-r^2\right)rdrd\varphi \\ &\leq \int_{-t}^{t} \exp\left(-x^2\right)dx \int_{-t}^{t} \exp\left(-y^2\right)dy \leq \iint_{\substack{0\le r\le t\sqrt{2} \\ 0\le\varphi\le 2\pi}} \exp\left(-r^2\right)drd\varphi \end{aligned} \tag{1.64}$$

Using the observation that

$$\int_{-t}^{t} \exp\left(-x^2\right)dx = \int_{-t}^{t} \exp\left(-y^2\right)dy \tag{1.65}$$

one can determine

$$\int_0^{2\pi} d\varphi \int_0^{t} \exp\left(-r^2\right)rdr \leq \left[\int_{-t}^{t} \exp\left(-x^2\right)dx\right]^2 \leq \int_0^{2\pi} d\varphi \int_0^{t\sqrt{t}} \exp\left(-r^2\right)rdr \tag{1.66}$$

Next, considering the rule

$$\int \exp\left(-r^2\right) r dr = -\tfrac{1}{2}\exp\left(-r^2\right) \tag{1.67}$$

and the symmetry

$$\int_{-t}^{t} \exp\left(-x^2\right) dx = 2\int_{0}^{t} \exp\left(-x^2\right) dx \tag{1.68}$$

it is obtained finally that

$$\pi\left(1-\exp\left(-t^2\right)\right) \le 4\left[\int_{0}^{t} \exp\left(-x^2\right) dx\right]^2 \le \pi\left(1-\exp\left(-2t^2\right)\right) \tag{1.69}$$

Then, a square rooting procedure gives

$$\tfrac{1}{2}\sqrt{\pi}\left(1-\exp\left(-t^2\right)\right) \le \int_{0}^{t} \exp\left(-x^2\right) dx \le \tfrac{1}{2}\sqrt{\pi}\left(1-\exp\left(-2t^2\right)\right) \tag{1.70}$$

The three functions theorem and the limiting procedure for $t \to \infty$ allow us to show

$$\lim_{t\to\infty} \int_{0}^{t} \exp\left(-x^2\right) dx = \tfrac{1}{2}\sqrt{\pi} \tag{1.71}$$

with

$$\lim_{t\to\infty} \sqrt{1-\exp\left(-t^2\right)} = \lim_{t\to\infty} \sqrt{1-\exp\left(-2t^2\right)} = 1 \tag{1.72}$$

Lemma (Central Limit Theorem)
For any independent random variables X_i for i=1,2,...,n the following sum $X = \sum_i X_i$ is asymptotically Gaussian where the parameters are equal $m = \sum_i m_i$ and $\sigma^2 = \sum_i \sigma_i^2$, respectively.

Further X be the random variable with P and F being the probability density and distribution functions, respectively and S any given Borelian set such that

$$P = P(S) = P(X \subset S) \tag{1.73}$$

Now, let us consider the following subset $S_0 \subset S$ such that

$$S_0 = \{X \in S : a \le X \le b\}; \quad a < b \le +\infty \tag{1.74}$$

The probability density function defined on a domain S_0 is called a truncated Gaussian and its distribution function is given as

$$F(x|a \le X \le b) = \begin{cases} 0, & x < a \\ \frac{F(x)-F(a)}{F(b)-F(a)}; & a \le x \le b \\ 1, & x > b \end{cases} \tag{1.75}$$

and its probability density function is equal to

$$f(x|a \le X \le b) = \frac{f(x)}{\int_a^b f(t)dt} \tag{1.76}$$

The considered problem of cutting off the probability density function in case of Gaussian or related random variable is very important for engineering applications in probabilistic calculus. Most engineering parameters, being random, must have nonnegative values as Young modulus or heat conductivity coefficients for instance [132]. Other parameters, like the Poisson ratio, are restricted to small intervals only. Then, let us focus on modifications of the presented formula describing the expected values and variances demonstrated for classical Gaussian variates in the case of bounded real domains.

Let us consider the Gaussian variable $N(m,\sigma)$ restricted to the positive values only. According to the above formulae, there holds

$$\lambda = \frac{f\left(-\frac{m}{\sigma}\right)}{1 - F\left(-\frac{m}{\sigma}\right)} \tag{1.77}$$

Then, the first two probabilistic moments for the so modified Gaussian PDF are obtained as

$$E[X] = m + \lambda\sigma \tag{1.78}$$

$$Var(X) = m^2 + \lambda\sigma m + \sigma^2 \tag{1.79}$$

Starting from the derived equations one can calculate the expected values and the variances of the quasi−Gaussian random variables, whose domains are restricted to the specific and bounded intervals resulting from physical interpretation of a specific equilibrium problem.

1.2 Monte Carlo Simulation Method

Monte Carlo simulation is a numerical method based in general on random sampling and statistical estimation [39,44,125] and now there are multiple numerical realizations of the latters as crude simulation, stratified and importance sampling as well as Latin Hypercube Sampling methodology. Nevertheless, the most important part of the method is a reliable random number generator. Monte Carlo simulation (MCS) is, in fact, a numerical method based on random sampling via a random number generator. The most important applications of the MCS technique in engineering of composite materials are: (a) fatigue and/or failure modeling [10,243], (b) modeling of random material properties [73,171,174,175, 191,196,306] and (c) reliability analysis [79]. Random nature of the effective properties calculated in homogenisation problem follows usually randomness of material properties of a composite, which are defined as the input Gaussian variables. To obtain the random sequences of this variable it is necessary to produce first numerically uniform deviates. They are random numbers, which lie within a specified range ([0,1] typically), and each number is as likely to occur as any other in the range. Generation of the uniform distributions is done using a standard FORTRAN library routine, which can be implemented as a linear congruential generator, which generates a sequence of integer numbers $I_1, I_2, I_3, ...$, each between 0 and m-1, by using the recurrence relation [39]

$$I_{j+1} = a \cdot I_j + c \qquad (\text{mod } m) \tag{1.80}$$

where m is called the modulus and a, c are positive integers called the multiplier and the increment, respectively. The recurrence (1.80) will possibly repeat itself with a period that is obviously no greater than m. If m, a and c are properly chosen, then the period of recurrence is of maximal length m. The sequence of real numbers between 0 and 1 is returned here by dividing I_{j+1} by m, so that it is strictly less than 1, but occasionally (once in m calls). The linear congruential method is very fast and requires only a few operations per call, but it is not free of sequential correlation on successive calls and the special shuffling routine has to be added to eliminate this disadvantage. Next, the Box–Muller method is implemented to transform these variables to the normalized Gaussian distribution– let us consider for this purpose the transformation between two uniform deviates on (0,1) denoted by x_1, x_2 and two quantities y_1 and y_2 defined as follows

$$y_1 = \sqrt{-2 \cdot \ln x_1} \cdot \cos 2\pi x_2 \tag{1.81}$$

$$y_2 = \sqrt{-2 \cdot \ln x_1} \cdot \sin 2\pi x_2 \tag{1.82}$$

Equivalently it can be written that

$$x_1 = \exp\left[-\tfrac{1}{2}\left(y_1^2 + y_2^2\right)\right] \tag{1.83}$$

$$x_2 = \frac{1}{2\pi} \cdot arctg\,\frac{y_2}{y_1} \tag{1.84}$$

with the Jacobian determinant of the form

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = -\tfrac{1}{2\pi}\exp\left(-\tfrac{1}{2}(y_1^2 + y_2^2)\right) \tag{1.85}$$

since it is a product of the functions of y_2 and y_1 separately. Finally, we obtain each y is returned as the independent Gaussian variable.

The second part of the simulation procedure is a statistical estimation procedure [29], which enables approximation of probabilistic moments and the relevant coefficients for the given series of output variables and for the specified number of random trials. The equations listed below are implemented in the statistical estimation procedure to compute the probabilistic moments with respect to M, which denotes here the total number of Monte Carlo random trials.

Statistical estimation theory is devoted to determination and verification of statistical estimators computed on a basis of the random trials sets. These estimators are necessary for efficient approximation of the analysed random variable and they are introduced for the random variables, fields and processes to assure their stochastic convergence.

Definition
If there exist a random variable X such that

$$\forall_{\varepsilon>0} \lim_{n\to\infty} P\left(\left|X_n - X\right| < \varepsilon\right) = 1 \tag{1.86}$$

then the series of random variables X_n stochastically converges to X. Let us note that the consistent, unbiased, most effective and asymptotically most effective estimators are available in statistical estimation theory.

Definition
The consistent estimator is each estimator stochastically convergent to the estimated parameter.

Definition
The unbiased estimator fulfils the following condition:

$$E\left[\hat{Q}_n\right] = Q \tag{1.87}$$

Definition
The most effective estimator is the unbiased estimator with the minimal variance.

Definition
The asymptotically most effective estimator of the quantity $\hat{Q}_n$ is the following one:

$$\lim_{n\to\infty}\left(\frac{Var(\hat{Q}_0)}{Var(\hat{Q}_n)}\right)=1 \tag{1.88}$$

where $Var\left(\hat{Q}_0\right)$ is the most effective variance estimator.

Definition
The expected value estimator of the random variable $X(\omega)$ in an n−element random trial is the average value

$$E\left[X(\omega)\right]=\frac{1}{n}\sum_{i=1}^{n}X_i=\overline{X} \tag{1.89}$$

It can be proved that this is consistent, unbiased and the most effective estimator for the Gaussian, binomial and Poisson probability distribution.

Definition
The variance estimator for the random variable $X(\omega)$ in an n−element random event is the quantity

$$Var\left(X(\omega)\right)=\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 \tag{1.90}$$

It can be demonstrated that this estimator is consistent and unbiased. Using this estimator one can determine standard deviation estimator.

Definition
The standard deviation estimator is equal to

$$S\left(X(\omega)\right)=\sqrt{Var\left(X(\omega)\right)} \tag{1.91}$$

Comment
The variance estimator in the n−element random event can be defined as

$$Var\left(X(\omega)\right)=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\overline{X}\right)^2 \tag{1.92}$$

It can be demonstrated that

$$E[Var(X(\omega))] = \frac{n-1}{n}\sigma^2 \tag{1.93}$$

which gives the negative bias. The estimator bias is defined as the deviation of this estimator from its value to be determined. There holds

$$E[S_n^2] - \sigma^2 = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2 \tag{1.94}$$

which results in a negative and bias, which is irrelevant since the natural condition for the variance $VarY \geq 0$.

Definition
The estimator of the ordinary *k*th order probabilistic moment of the random variable $X(\omega)$ in the $n-$element random trial is given as

$$m_k(X(\omega)) = \frac{1}{n}\sum_{i=1}^{n} X_i^k \tag{1.95}$$

Definition
The estimator of the *k*th order central probabilistic moment is defined as

$$\mu_k(X(\omega)) = m_k[(X(\omega)) - m_1(X(\omega))] \tag{1.96}$$

Any central moments of odd order are equal to 0 in case of the normalized Gaussian PDF $N(m,\sigma)$, while the first three even moments are given below.

Definition
The estimator of the second order central moment is equal to

$$\mu_2(X(\omega)) = \sigma^2 \tag{1.97}$$

Definition
The estimator of the fourth order central moment is given as

$$\mu_4(X(\omega)) = 3\sigma^4 \tag{1.98}$$

Definition
The estimator of the sixth order central moment is equal to

$$\mu_6(X(\omega)) = \frac{15\sigma^6}{m^3} \tag{1.99}$$

Using the proposed estimators of the central moments of the random variable $X(\omega)$ valid for the $n-$element random event, the following probabilistic coefficients are usually calculated:

Definition
The coefficient of variation for $X(\omega)$ is equal to

$$\alpha(X(\omega)) = \frac{\sigma(X(\omega))}{E[X(\omega)]} \tag{1.100}$$

Definition
The coefficient of asymmetry for $X(\omega)$ equals to

$$\beta(X(\omega)) = \frac{\mu_3(X(\omega))}{\sigma^3(X(\omega))} \tag{1.101}$$

Definition
The coefficient of concentration for $X(\omega)$ is equal to

$$\gamma(X(\omega)) = \frac{\mu_4(X(\omega))}{\sigma^4(X(\omega))} \tag{1.102}$$

which results in $\beta \xrightarrow[n\to\infty]{} 0$ and $\gamma \xrightarrow[n\to\infty]{} 3$ for the Gaussian random variables.

Definition
The estimator of covariance for two random variables $X(\omega)$ and $Y(\omega)$ in a two dimensional $n-$element random trial is defined as

$$Cov(X(\omega), Y(\omega)) = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i(\omega) - \overline{X}\right)\left(Y_i(\omega) - \overline{Y}\right) \tag{1.103}$$

Definition
The coefficient of correlation for two variables $X(\omega)$ and $Y(\omega)$ in two dimensional $n-$element random event is equal to

$$\rho_{XY} = \frac{Cov(X(\omega), Y(\omega))}{\sqrt{Var(X(\omega))Var(Y(\omega))}} \tag{1.104}$$

Remark
Two random variables $X(\omega)$ and $Y(\omega)$ are fully correlated only if $\rho_{XY}=1$ and uncorrelated in case of $\rho_{XY}=0$.

Equations (1.101) and (1.102) are very useful together with the relevant PDF estimator in recognising of the probabilistic distribution function type for the output variables – using the Central Limit Theorem the Gaussian variables can be found. This is very important aspect considering the fact that theoretical considerations in this subject are rather complicated and not always possible.

1.3 Stochastic Second Moment Perturbation Approach

1.3.1 Transient Heat Transfer Problems

The main concept of stochastic second order perturbation technique [263] applied in the next chapters to various transient heat transfer computations can be explained on the example of the following equation [135]:

$$C \cdot \dot{T} + K \cdot T = Q \tag{1.105}$$

where K, C are some linear stochastic operators equivalent to the heat conductivity and capacity matrices, T is the random thermal response vector for the structure with $\dot{T}$ representing its time derivative, while Q is the admissible heat flux (due to the boundary conditions) applied on the system. To introduce a precise definition of K, for instance, let us consider the Hilbert space $\mathbf{H}$ defined on a real domain D and the probability space (Ω, σ, P), where $x \in D$, $\theta \in \Omega$ and $\Theta : \Omega \to R$. Then, the operator $K(\mathbf{x};\omega)$ is some stochastic operator defined on $\mathbf{H} \times \Theta$, which means that it is a differential operator with the coefficients varying randomly with respect to one or more independent design random variables of the system; the operator C can be defined analogously. As is known, the analytical solutions to such a class of partial differential equations are available for some specific cases and that is why quite different approximating numerical methods are used (simulation, perturbation or spectral methods as well).

Further, let us denote the vector of random variables of a problem as $\{b^r(x;\theta)\}$ and its probability density functions as $g(b^r)$ and $g(b^r, b^s)$, respectively; $r, s = 1,2,...,R$ are indexing input random variables. Next, let us introduce integral definition for the expected values of this vector as

$$E[b^r] = \int_{-\infty}^{+\infty} b^r g(b^r) db^r \tag{1.106}$$

with its covariance in the form

$$Cov\left(b^r, b^s\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(b^r - E\left[b^r\right]\right)\left(b^s - E\left[b^s\right]\right) g\left(b^r, b^s\right) db^r db^s \tag{1.107}$$

Next, all material and physical parameters of Ω as well as their state functions being random fields are extended by the use of stochastic expansion via the Taylor series as follows:

$$K(x;\theta) = K^0(x;\theta) + \sum_{n=1}^{N}\left\{\frac{\varepsilon^n}{n!} K^{(n)}(x;\theta)\prod_{r=1}^{n}\Delta b^r(\theta)\right\} \tag{1.108}$$

where ε is some given small perturbation parameter, $\varepsilon\Delta b^r$ denotes the first order variation of Δb^r about its expected value $E\left[b^r\right]$ and $K^{(n)}(x;\theta)$ represents the *n*th order partial derivatives with respect to the random variables determined at the expected values. The variable θ represents here the random event belonging to the corresponding probability space of admissible events (nonnegative, for instance). The second order perturbation approach is now analysed and then the random operator $K(x;\theta)$ is expanded as

$$K(x;\theta) = K^0(x;\theta) + \varepsilon K^{,r}(x;\theta)\Delta b^r + \tfrac{1}{2}\varepsilon^2 K^{,rs}(x;\theta)\Delta b^r \Delta b^s \tag{1.109}$$

It can be noted that the second order equation is obtained by multiplying the *R*-variate probability density function $p_R(b_1, b_2, ..., b_R)$ by the ε^2-terms and by integrating over the domain of $\mathbf{b}(\mathbf{x};\theta)$. Assuming that the small parameter ε of the expansion is equal to 1 and applying the stochastic second order perturbation methodology to the fundamental deterministic equation (1.105), we find

- zeroth order equations:

$$C^0(x;\theta)\cdot\dot{T}^0(x;\theta) + K^0(x;\theta)\cdot T^0(x;\theta) = Q^0(x;\theta) \tag{1.110}$$

- first order equations (for *r*=1,…,*R*):

$$\begin{aligned} &C^{,r}(x;\theta)\cdot\dot{T}^0(x;\theta) + C^0(x;\theta)\cdot\dot{T}^{,r}(x;\theta) \\ &\quad + K^{,r}(x;\theta)\cdot T^0(x;\theta) + K^0(x;\theta)\cdot T^{,r}(x;\theta) = Q^{,r}(x;\theta) \end{aligned} \tag{1.111}$$

- second order equations (for *r*,*s*=1,…,*R*):

$$
\begin{aligned}
&C^{,rs}(x;\theta)\cdot\dot{T}^{0}(x;\theta)+2C^{,r}(x;\theta)\cdot\dot{T}^{,s}(x;\theta)+C^{0}(x;\theta)\cdot\dot{T}^{,rs}(x;\theta)\\
&+K^{,rs}(x;\theta)\cdot T^{0}(x;\theta)+2K^{,r}(x;\theta)\cdot T^{,s}(x;\theta)+K^{0}(x;\theta)\cdot T^{,rs}(x;\theta)\\
&=Q^{,rs}(x;\theta)
\end{aligned}
\tag{1.112}
$$

It is clear that coefficients for the products of *K*, *C* and T are the successive orders of the initial basic deterministic eqn (1.110) and they are taken from the well−known Pascal triangle. As far as the *n*th order partial differential perturbation-based approach is concerned, then the general statement can be written out using the Leibniz differentiation rule in the following form:

$$
\begin{aligned}
&\binom{n}{0}C^{(n)}(x;\theta)\cdot\dot{T}^{0}(x;\theta)+\binom{n}{1}C^{(n-1)}(x;\theta)\cdot\dot{T}^{(1)}(x;\theta)+\ldots\\
&+\binom{n}{n}C^{(0)}(x;\theta)\cdot\dot{T}^{(n)}(x;\theta)+\binom{n}{0}K^{(n)}(x;\theta)\cdot T^{0}(x;\theta)\\
&+\binom{n}{1}K^{(n-1)}(x;\theta)\cdot T^{(1)}(x;\theta)+\ldots\\
&+\binom{n}{n-1}K^{(1)}(x;\theta)\cdot T^{(n-1)}(x;\theta)+\binom{n}{n}K^{(0)}(x;\theta)\cdot T^{(n)}(x;\theta)\\
&+\binom{n}{n}K^{(0)}(x;\theta)\cdot T^{(n)}(x;\theta)=Q^{(n)}(x;\theta)
\end{aligned}
\tag{1.113}
$$

The equations from *m*=0 to the specific value of *n* should be generated to introduce all hierarchical equations system for the *n*th order perturbation approach. Usually, it is assumed that higher than second order perturbations can be neglected, the system of equations (1.110) − (1.112) constitutes the given equilibrium problem. The detailed convergence studies should be carried out in further extensions of the model with respect to perturbation order, parameter ε and the coefficient of variation of input random variables.

Furthermore, it can be noted that system (1.111) is rewritten for all random parameters of the problem indexed by $r=1,\ldots,R$ (*R* equations), while system (1.112) gives us generally R^2 equations. The unnecessary equations are eliminated here through multiplying both sides of the highest order equation by the appropriate covariance matrix of input random parameters. There holds

- zeroth order equations:

$$C^{0}(x;\theta)\cdot\dot{T}^{0}(x;\theta)+K^{0}(x;\theta)\cdot T^{0}(x;\theta)=Q^{0}(x;\theta) \tag{1.114}$$

- 1st order equations (for $r=1,\ldots,R$):

$$C^{,r}(x;\theta)\cdot\dot{T}^{0}(x;\theta)+C^{0}(x;\theta)\cdot\dot{T}^{,r}(x;\theta)+K^{,r}(x;\theta)\cdot T^{0}(x;\theta)+K^{0}(x;\theta)\cdot T^{,r}(x;\theta)=Q^{,r}(x;\theta) \tag{1.115}$$

- second order equations (for $r,s=1,\ldots,R$):

$$\begin{aligned}&C^{0}(x;\theta)\cdot\dot{T}^{(2)}(x;\theta)+K^{0}(x;\theta)\cdot T^{(2)}(x;\theta)\\&=\left\{Q^{,rs}(x;\theta)-K^{,rs}(x;\theta)\cdot T^{0}(x;\theta)+2K^{,r}(x;\theta)\cdot T^{,s}(x;\theta)\right.\\&\left.-C^{,rs}(x;\theta)\cdot\dot{T}^{0}(x;\theta)+2C^{,r}(x;\theta)\cdot\dot{T}^{,s}(x;\theta)\right\}Cov\left(b^{r},b^{s}\right)\end{aligned} \tag{1.116}$$

It is observed that solving for the nth order perturbation equations system, the closure of the entire hierarchical system is obtained by nth order correlation of input random vector components b^{r} and b^{s}, respectively; for this purpose nth order statistical information about input random variables is however necessary. To obtain the probabilistic solution for the analysed heat flow problem, eqn (1.114) is solved for T^{0}, eqn (1.115) for first order terms $T^{,r}$ and, finally, eqn (1.116) for $T^{(2)}$. Therefore, using the definition of expected value and applying the second order expansion, it is derived that

$$\begin{aligned}E\left[T\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]\right]&=\int_{-\infty}^{+\infty}T\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]p_{R}\left(\mathbf{b}(\mathbf{x};\theta)\right)d\mathbf{b}\\&=\int_{-\infty}^{+\infty}\left\{T^{0}\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]+T^{,r}\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]\Delta b_{r}(\mathbf{x})\right.\\&\left.+\tfrac{1}{2}T^{,rs}\left[\mathbf{b}(\mathbf{x};\theta);x\right]\Delta b_{r}(\mathbf{x})\Delta b_{s}(\mathbf{x})\right\}p_{R}\left(\mathbf{b}(\mathbf{x};\theta)\right)d\mathbf{b}\end{aligned} \tag{1.117}$$

and further

$$\begin{aligned}&T^{0}(\mathbf{x};\theta)\int_{-\infty}^{+\infty}p_{R}\left(\mathbf{b}(\mathbf{x};\theta)\right)d\mathbf{b}+T^{,r}(\mathbf{x};\theta)\int_{-\infty}^{+\infty}\Delta b_{r}(\mathbf{x};\theta)\,p_{R}\left(\mathbf{b}(\mathbf{x};\theta)\right)d\mathbf{b}\\&+\tfrac{1}{2}T^{,rs}(\mathbf{x};\theta)\int_{-\infty}^{+\infty}\Delta b_{r}(\mathbf{x};\theta)\Delta b_{s}(\mathbf{x};\theta)\,p_{R}\left(\mathbf{b}(\mathbf{x};\theta)\right)d\mathbf{b}\end{aligned} \tag{1.118}$$

This result leads us to the following relation for the expected values [135,190]:

$$E\left[T\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]\right]=T^{0}\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]+\tfrac{1}{2}T^{,rs}\left[\mathbf{b}(\mathbf{x};\theta);\mathbf{x}\right]S_{b}^{rs} \tag{1.119}$$

Now, using the perturbation approach, both spatial and temporal cross-covariances can be determined separately. There holds for spatial cross−covariance computed at the specific time moment τ

$$Cov\left(T\left[\mathbf{b}\left(\mathbf{x}^{(1)};\theta\right),\mathbf{x}^{(1)};\tau\right];T\left[\mathbf{b}\left(\mathbf{x}^{(2)};\theta\right),\mathbf{x}^{(2)};\tau\right]\right)=S_T^{ij}\left(\mathbf{x}^{(1)};\mathbf{x}^{(2)};\tau\right)$$
$$=\int_{-\infty}^{+\infty}\left\{T\left[\mathbf{b}\left(\mathbf{x}^{(1)};\theta\right),\mathbf{x}^{(1)};\tau\right]-E\left[T\left[\mathbf{b}\left(\mathbf{x}^{(1)};\theta\right),\mathbf{x}^{(1)};\tau\right]\right]\right\} \tag{1.120}$$
$$\times\left\{T\left[\mathbf{b}\left(\mathbf{x}^{(2)};\theta\right),\mathbf{x}^{(2)};\tau\right]-E\left[T\left[\mathbf{b}\left(\mathbf{x}^{(2)};\theta\right),\mathbf{x}^{(2)};\tau\right]\right]\right\}p_R\left(\mathbf{b}\left(\mathbf{x};\theta\right)\right)d\mathbf{b}$$

which gives as a result

$$S_T^{ij}\left(\mathbf{x}^{(1)};\mathbf{x}^{(2)};\theta;\tau\right)=T^{,r}\left(\mathbf{x}^{(1)};\theta;\tau\right)T^{,s}\left(\mathbf{x}^{(2)};\theta;\tau\right)S_b^{rs}\left(\mathbf{x}^{(1)};\mathbf{x}^{(2)};\theta;\tau\right) \tag{1.121}$$

Alternatively, one can compute the time cross–covariances in the case where the input random process varies in time (and does not depend on spatial variables). It is obtained for time moments τ_1 and τ_2 by the use of analogous definitions that

$$Cov\left(T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_1\right];T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_2\right]\right)=S_T^{ij}\left(\mathbf{x};\tau_1;\tau_2\right)$$
$$=\int_{-\infty}^{+\infty}\left\{T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_1\right]-E\left[T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_1\right]\right]\right\} \tag{1.122}$$
$$\times\left\{T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_2\right]-E\left[T\left[\mathbf{b}\left(\mathbf{x};\theta\right),\mathbf{x};\tau_2\right]\right]\right\}p_R\left(\mathbf{b}\left(\mathbf{x};\theta\right)\right)d\mathbf{b}$$

which yields

$$S_T^{ij}\left(\mathbf{x};\theta;\tau_1;\tau_2\right)=T^{,r}\left(\mathbf{x};\theta;\tau_1\right)T^{,s}\left(\mathbf{x};\theta;\tau_2\right)S_b^{rs}\left(\mathbf{x};\theta;\tau_1;\tau_2\right) \tag{1.123}$$

It is important to underline that the perturbation methodology at the present stage does not allow for computational modeling of the boundary–initial problems where the input parameters are full stochastic processes varying in space and time.

1.3.2 Elastodynamics with Random Parameters

Generally, the following problem is solved now [56,181,198]:

$$M\ddot{u}+C\dot{u}+Ku=f \tag{1.124}$$

where M, C and K are linear stochastic operators, u is the random structural response, while f is the admissible excitation of this system. The definitions of the matrices as random operators are introduced analogously to the considerations included in Sec. 1.3.1. Usually, such operators are identified as mass, damping and stiffness matrices in structural dynamics applications. As is known, the analytical solutions for such a class of partial differential equations are available for some specific cases, since quite different approximating numerical methods are used;

various mathematical approaches to the solution of that problem are reported and presented in [233,249,324,326]. However the second order perturbation second central probabilistic moment approach is documented below.

The stochastic second order Taylor series based extension [208] of the basic deterministic equation (1.124) of the problem leads by equating of the same order terms for $\tau \in [0,\infty)$ to

- zeroth order equations:

$$M^0\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)+K^0\left(b^0\right)u^0\left(b^0;\tau\right)=f^0\left(b^0;\tau\right) \tag{1.125}$$

- first order equations (for r=1,…,R):

$$\begin{aligned}&M^{,r}\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)+C^{,r}\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)+K^{,r}\left(b^0\right)u^0\left(b^0;\tau\right)\\&+M^0\left(b^0\right)\ddot{u}^{,r}\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^{,r}\left(b^0;\tau\right)+K^0\left(b^0\right)u^{,r}\left(b^0;\tau\right)\\&=f^{,r}\left(b^0;\tau\right)\end{aligned} \tag{1.126}$$

- second order equations (for r,s=1,…,R):

$$\begin{aligned}&M^{,rs}\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)+C^{,rs}\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)+K^{,rs}\left(b^0\right)u^0\left(b^0;\tau\right)\\&+2M^{,r}\left(b^0\right)\ddot{u}^{,s}\left(b^0;\tau\right)+2C^{,r}\left(b^0\right)\dot{u}^{,s}\left(b^0;\tau\right)+2K^{,r}\left(b^0\right)u^{,s}\left(b^0;\tau\right)\\&+M^0\left(b^0\right)\ddot{u}^{,rs}\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^{,rs}\left(b^0;\tau\right)+K^0\left(b^0\right)u^{,rs}\left(b^0;\tau\right)\\&=f^{,rs}\left(b^0;\tau\right)\end{aligned} \tag{1.127}$$

Therefore, the generalized nth order partial differential perturbation–based equation of motion can be proposed as

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}M^{,n-k}\left(b^0(x;\theta)\right)\ddot{u}^{,k}\left(b^0(x;\theta);\tau\right)\\&+\sum_{k=0}^{n}\binom{n}{k}C^{,n-k}\left(b^0(x;\theta)\right)\dot{u}^{,k}\left(b^0(x;\theta);\tau\right)\\&+\sum_{k=0}^{n}\binom{n}{k}K^{,n-k}\left(b^0(x;\theta)\right)u^{,k}\left(b^0(x;\theta);\tau\right)=f^{,n}\left(b^0(x;\theta);\tau\right)\end{aligned} \tag{1.128}$$

where the operators $M^{,n},C^{,n},K^{,n}$ denote nth order partial derivatives of mass, damping and stiffness matrices with respect to the input random variables determined at the expected values of these variables, respectively. The vectors $f^{,n}\left(b^0;\tau\right)$, $\ddot{u}^{,n}\left(b^0;\tau\right)$, $\dot{u}^{,n}\left(b^0;\tau\right)$, $u^{,n}\left(b^0;\tau\right)$ represent analogous nth order partial derivatives of external excitation, accelerations, velocities as well as displacements of the system.

Let us note that the stochastic hierarchical equations of motion for desired perturbation order m can be obtained from eqn (1.128) by successive expansion and substitution of n by the natural numbers 0,1,…,m, which returns the system of (m+1) equations. Then zeroth order solution is obtained from the first equation; then, inserting the zeroth order solution into the second equation (of the first order), the first order solution can be determined. An analogous procedure is repeated to determine all orders of the structural response, which are finally used in the calculation of the response probabilistic moments.

Assuming that higher than second order perturbations can be neglected, this equation system constitutes the equilibrium problem. The detailed convergence studies should be carried out in further extensions of the model with respect to perturbation order, parameter θ and coefficient of variation of input random variables. If higher than the second probabilistic moment approach is considered, then the coefficients of assymetry, concentration, etc., also influence final effectiveness of the perturbation–based solution.

Analogously to the stochastic expansion of (1.105), the first and second order equations are modified and it is found that

- zeroth order equations:

$$M^0\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)+K^0\left(b^0\right)u^0\left(b^0;\tau\right)=f^0\left(b^0;\tau\right) \tag{1.129}$$

- first order equations (for r=1,…,R):

$$\begin{aligned} &M^0\left(b^0\right)\ddot{u}^{,r}\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^{,r}\left(b^0;\tau\right)+K^0\left(b^0\right)u^{,r}\left(b^0;\tau\right)\\ &=f^{,r}\left(b^0;\tau\right)-\left\{M^{,r}\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)+C^{,r}\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)+K^{,r}\left(b^0\right)u^0\left(b^0;\tau\right)\right\} \end{aligned} \tag{1.130}$$

- second order equations (for r,s=1,…,R):

$$\begin{aligned} &M^0\left(b^0\right)\ddot{u}^{(2)}\left(b^0;\tau\right)+C^0\left(b^0\right)\dot{u}^{(2)}\left(b^0;\tau\right)+K^0\left(b^0\right)u^{(2)}\left(b^0;\tau\right)\\ &=\left\{f^{,rs}\left(b^0;\tau\right)-M^{,rs}\left(b^0\right)\ddot{u}^0\left(b^0;\tau\right)-C^{,rs}\left(b^0\right)\dot{u}^0\left(b^0;\tau\right)-K^{,rs}\left(b^0\right)u^0\left(b^0;\tau\right)\right.\\ &\left.-2M^{,r}\left(b^0\right)\ddot{u}^{,s}\left(b^0;\tau\right)-2C^{,r}\left(b^0\right)\dot{u}^{,s}\left(b^0;\tau\right)-2K^{,r}\left(b^0\right)u^{,s}\left(b^0;\tau\right)\right\}Cov\left(b^r,b^s\right) \end{aligned} \tag{1.131}$$

Let us observe that looking for the nth order perturbation approach, the closure of hierarchical equations is obtained by the nth order correlation of input random process components b^r and b^s, respectively; nth order statistical information about input random variables is however necessary for this purpose.

To obtain the probabilistic solution for the considered equilibrium problem, (1.129) is solved for u^0 (and its time derivatives $\dot{u}^0$ and $\ddot{u}^0$, respectively), next (1.130) for first order terms of $u^{,r}$ and, finally, (1.131) for $u^{(2)}$. Two probabilistic moment characterisations of all the state functions for the boundary value problem starts from the expected value of the structural displacement vector components. Using its definition

$$E[u(t)] = \int_{-\infty}^{+\infty} u(t) p_R(\mathbf{b}(\mathbf{x};\theta))\, d\mathbf{b} \tag{1.132}$$

the second order accurate expectations are equal to

$$E[u(t)] = u^0(t) + \tfrac{1}{2} u^{,rs} S_b^{rs} = u^0(t) + \tfrac{1}{2} u^{(2)} \tag{1.133}$$

In quite a similar manner the second moment probabilistic characteristics are obtained. Defining the time cross–correlation function as

$$Cov(u(t_1), u(t_2)) = \int_{-\infty}^{+\infty} \{u(t_1) - E[u(t_1)]\}\{u(t_2) - E[u(t_2)]\} p_R(\mathbf{b}(\mathbf{x};\theta))\, d\mathbf{b} \tag{1.134}$$

it is found that

$$Cov(u(t_1); u(t_2)) = u^{,r}(t_1)\, u^{,s}(t_2)\, Cov(b^r, b^s) \tag{1.135}$$

which completes the two–moment characterization of the perturbation–based solution for the dynamic equilibrium problem (1.124). The entire solution simplifies in the case of free vibrations when the following equations are to be solved:

$$[K - \Omega_{(\alpha)} M]\, \Phi = 0 \tag{1.136}$$

$\Omega_{(\alpha)}$ and Φ are the eigenvalues and eigenvectors, respectively and α=1,...,N denotes the total number of degrees of freedom of a structure. The second order expansion leads to the following equation system:

$$[K^0 - \Omega_{(\alpha)}^0 M^0]\, \Phi^0 = 0 \tag{1.137}$$

$$[K^0 - \Omega_{(\alpha)}^0 M^0]\, \Phi^{,r} = -[K^{,r} - \Omega_{(\alpha)}^{,r} M^0 - \Omega_{(\alpha)}^0 M^{,r}]\, \Phi^0 \tag{1.138}$$

$$\begin{aligned} &[K^0 - \Omega_{(\alpha)}^0 M^0]\, \Phi^{(2)} = \\ &-\{[K^{,rs} - \Omega_{(\alpha)}^{,rs} M^0 - 2\Omega_{(\alpha)}^{,r} M^{,s} - \Omega_{(\alpha)}^0 M^{,rs}]\, \Phi^0 \\ &-2[K^{,r} - \Omega_{(\alpha)}^{,r} M^0 - \Omega_{(\alpha)}^0 M^{,r}]\, \Phi^{,s}\} Cov(b^r, b^s) \end{aligned} \tag{1.139}$$

To determine the probabilistic moments of the eigenvectors, up to the second order derivatives with respect to input random variables are to be determined first. While zeroth order quantities are obtained directly from the relation (1.137), the methodology of first order terms calculation is definitely more complicated.

Equation (1.138) is transformed for this purpose by multiplying by the transposed zeroth order eigenvector, which gives

$$\Phi^{0T}\left[K^0-\Omega^0_{(\alpha)}M^0\right]\Phi^{,r}-\Phi^{0T}\Omega^{,r}_{(\alpha)}M^0\Phi^0=-\Phi^{0T}\left[K^{,r}-\Omega^0_{(\alpha)}M^{,r}\right]\Phi^{,r} \tag{1.140}$$

Since Φ^0 is diagonal and K^0 and M^0 are symmetric, (1.140) is modified as

$$\left[\Phi^{0T}\left[K^0-\Omega^0_{(\alpha)}M^0\right]\Phi^{,r}\right]^T=\Phi^{,rT}\left[K^0-\Omega^0_{(\alpha)}M^0\right]\Phi^{,r}=0 \tag{1.141}$$

Let us observe that $\Omega^{,r}$ is diagonal and therefore

$$\Phi^{0T}\Omega^{,r}_{(\alpha)}M^0\Phi^0=\Omega^{,r}_{(\alpha)}\Phi^{0T}M^0\Phi^0 \tag{1.142}$$

which finally results in

$$\Omega^{,r}_{(\alpha)}=\Phi^{0T}\left[K^{,r}-\Omega^0_{(\alpha)}M^{,r}\right]\Phi^0 \tag{1.143}$$

Next, using an analogous technique in the case of the second order equation, it is derived that

$$\begin{aligned}&\Phi^{0T}\left[K^0-\Omega^0_{(\alpha)}M^0\right]\Phi^{(2)}-\Phi^{0T}\Omega^{(2)}_{(\alpha)}M^0\Phi^0\\&=-\Phi^{0T}\left[K^{,rs}-2\Omega^{,r}_{(\alpha)}M^{,s}-\Omega^0_{(\alpha)}M^{,rs}\right]\Phi^0Cov\left(b^r,b^s\right)\\&-\Phi^{0T}\left[K^{,r}-\Omega^{,r}_{(\alpha)}M^0-\Omega^0_{(\alpha)}M^{,r}\right]\Phi^{,s}Cov\left(b^r,b^s\right)\end{aligned} \tag{1.144}$$

which finally implies

$$\begin{aligned}\Phi^{(2)}=&\Phi^{0T}\left[K^{,rs}-2\Omega^{,r}_{(\alpha)}M^{,s}-\Omega^0_{(\alpha)}M^{,rs}\right]\Phi^0Cov\left(b^r,b^s\right)\\&+2\Phi^{0T}\left[K^{,r}-\Omega^{,r}_{(\alpha)}M^0-\Omega^0_{(\alpha)}M^{,r}\right]\Phi^{,s}Cov\left(b^r,b^s\right)\end{aligned} \tag{1.145}$$

The next problem is to determine the first and second order derivatives of the eigenvectors. Basically, the eigenvector derivative is expressed as a linear combination of all the eigenvectors in the original system. Equations describing the coefficients of the linear combination are formed using the $M-$orthonormality and $K-$orthogonality conditions. Starting from (1.138), the αth eigenpair is determined as

$$\left[K^0-\Omega^0_{(\alpha)}M^0\right]\Phi^{,r}_{(\alpha)}=-\left[K^{,r}-\Omega^{,r}_{(\alpha)}M^0-\Omega^0_{(\alpha)}M^{,r}\right]\Phi^0_{(\alpha)} \tag{1.146}$$

and (1.143) in the following form:

$$\Omega^{,r}_{(\alpha)} = \Phi^0_{(\alpha)}\left[K^{,r} - \omega^0_{(\alpha)}M^{,r}\right]\Phi^0_{(\alpha)} \tag{1.147}$$

It yields by substitution

$$\left[K^0 - \Omega^0_{(\alpha)}M^0\right]\Phi^{,r}_{(\alpha)} = R^r_{(\alpha)} \tag{1.148}$$

with $R^r_{(\alpha)}$ being equal to

$$R^r_{(\alpha)} = -\left[K^{,r} - \Phi^0_{(\alpha)}\left(K^{,r} - \Omega^0_{(\alpha)}M^{,r}\right)\Phi^0_{(\alpha)}M^0 - \Omega^0_{(\alpha)}M^{,r}\right]\Phi^0_{(\alpha)} \tag{1.149}$$

Further, it is assumed that the αth first order eigenvector $\Phi^{,r}_{(\alpha)}$ can be expressed as a linear combination of all the zeroth order eigenvectors as

$$\Phi^{,r}_{(\alpha)} = \Phi^0 a^r_{(\alpha)} \tag{1.150}$$

The complete description of the coefficients $a^r_{(\alpha)}$ is given by the following formula:

$$a^r_{(\alpha)} = \begin{cases} \dfrac{\Phi^0_{(\alpha)}R^r_{(\hat{\alpha})}}{\omega^0_{(\alpha)} - \omega^0_{(\hat{\alpha})}}, \; for\; \alpha \neq \hat{\alpha} \\ -\dfrac{1}{2}\Phi^0_{(\hat{\alpha})}M^{,r}\Phi^0_{(\hat{\alpha})}, \; for\; \alpha = \hat{\alpha} \end{cases} \tag{1.151}$$

Similarly as above, the second order eigenvector $\Phi^{(2)}_{(\alpha)}$ is approximated by a linear combination of all the zeroth order eigenvectors

$$\Phi^{(2)}_{(\alpha)} = \Phi^0 a^{(2)}_{(\alpha)} \tag{1.152}$$

Then, one can show the following result [208]:

$$a^{(2)}_{(\hat{\alpha})} = \begin{cases} \dfrac{\Phi^0_{(\alpha)}R^{(2)}_{(\hat{\alpha})}}{\omega^0_{(\alpha)} - \omega^0_{(\hat{\alpha})}}, \; for\; \alpha \neq \hat{\alpha} \\ -\left(\dfrac{1}{2}\Phi^0_{(\hat{\alpha})}M^{,rs}\Phi^0_{(\hat{\alpha})} + 2\Phi^0_{(\hat{\alpha})}M^{,r}\Phi^{,s}_{(\hat{\alpha})} + a^r_{(\hat{\alpha})}a^s_{(\hat{\alpha})}\right)Cov\left(b^r, b^s\right) \\ for\; \alpha = \hat{\alpha} \end{cases} \tag{1.153}$$

Finally, the first two probabilistic moments of the eigenvalues and eigenvectors are found in a typical way, which completes the solution of the second order second central probabilistic moment eigenvalue and eigenvector problem.

Summing up the applications of the stochastic perturbation methodology it should be pointed out that the main disadvantage is dependence between the assumed order of the expansion, interrelations between input probabilistic characteristics and overall precision of such a computational methodology. The method found its numerous applications in structural engineering [88,208,237], in homogenisation [162,164,192] as well as in fluid dynamics computations [184]. Computational implementation in conjunction with Finite Element Method both in displacement [208] and stress versions [186], Boundary Element Method [51,185] as well as with Finite Difference Method [187,198] are available now, whereas the scaled the Boundary–Finite Element Method has no such extension [369].

Nevertheless, the perturbation method can be very useful after successful implementation in symbolic computations programs, which will enable automatic perturbation–based extension of up to *n*th order [178] for any variational equation [25,297] as well as ordinary or partial differential equations solutions [68,90]. The application of the perturbation method in stochastic processes [319,326] modelling needs its essential improvements, because now the randomness of an input cannot be introduced both in space and time.